J. Math. Sci. Univ. Tokyo
**23** (2016), 499–528.

# *On the Initial Value Problem for the Navier-Stokes Equations with the Initial Datum in Critical Sobolev and Besov Spaces*

By D. Q. Khai and N. M. Tri

**Abstract.** The existence of local unique mild solutions to the Navier-Stokes equations in the whole space with an initial tempered distribution datum in critical homogeneous or inhomogeneous Sobolev spaces is shown. Especially, the case when the integral-exponent is less than 2 is investigated. The global existence is also obtained for the initial datum in critical homogeneous Sobolev spaces with a norm small enough in suitable critical Besov spaces. The key lemma is to establish the bilinear estimates in these spaces, due to the point-wise decay of the kernel of the heat semigroup.

## §1. Introduction

We consider the Navier-Stokes equations (NSE) in $d$ dimensions in special setting of a viscous, homogeneous, incompressible fluid which fills the entire space and is not submitted to external forces. Thus, the equations we consider are the system:

$$\begin{cases} \partial_t u = \Delta u - \nabla.(u \otimes u) - \nabla p, \\ \operatorname{div}(u) = 0, \\ u(0, x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0, x) = u_{0k}. \end{cases}$$

2010 *Mathematics Subject Classification.* Primary 35Q30; Secondary 76D05, 76N10.
Key words: Navier-Stokes equations, existence and uniqueness of local and global mild solutions, critical Sobolev and Besov spaces.

The unknown quantities are the velocity $u(t,x) = (u_1(t,x), \dots, u_d(t,x))$ of the fluid element at time $t$ and position $x$ and the pressure $p(t,x)$.

A translation invariant Banach space of tempered distributions $\mathcal{E}$ is called a critical space for NSE if its norm is invariant under the action of the scaling $f(.) \longrightarrow \lambda f(\lambda.)$. One can take, for example, $\mathcal{E} = L^d(\mathbb{R}^d)$ or the smaller space $\mathcal{E} = \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$. In fact, one has the chain of critical spaces given by the continuous embeddings

$$(1) \qquad \begin{aligned} \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)_{(d\leq q<\infty)} \\ \hookrightarrow BMO^{-1}(\mathbb{R}^d) \hookrightarrow \dot{B}_\infty^{-1,\infty}(\mathbb{R}^d). \end{aligned}$$

It is remarkable feature that NSE are well-posed in the sense of Hadarmard (existence, uniqueness and continuous dependence on data) when the initial datum is divergence-free and belong to the critical function spaces (except $\dot{B}_\infty^{-1,\infty}$) listed in (1) (see [7] for $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, $L^d(\mathbb{R}^d)$, and $\dot{B}_q^{\frac{d}{q}-1,\infty}(\mathbb{R}^d)$, see [28] for $BMO^{-1}(\mathbb{R}^d)$. The recent ill-posedness result for $\dot{B}_\infty^{-1,\infty}(\mathbb{R}^d)$) with $d \geq 3$ was established in [3]. However, the ill-posedness in $\dot{B}_\infty^{-1,\infty}(\mathbb{R}^d)$ is still open when $d = 2$.

In the 1960s, mild solutions were first constructed by Kato and Fujita ([20], [16]) that are continuous in time and take values in the Sobolev space $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, say $u \in C([0,T]; H^s(\mathbb{R}^d))$. In 1992, a modern treatment for mild solutions in $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$ was given by Chemin [11]. In 1995, using the simplified version of the bilinear operator, Cannone proved the existence of mild solutions in $\dot{H}^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, see [7]. Results on the existence of mild solutions with value in $L^q(\mathbb{R}^d), (q > d)$ were established in the papers of Fabes, Jones and Rivière [14] and of Giga [17]. Concerning the initial datum in the space $L^\infty(\mathbb{R}^d)$, the existence of a mild solution was obtained by Cannone and Meyer in ([7], [10]). Moreover, in ([7], [10]), they also obtained theorems on the existence of mild solutions with value in Morrey-Campanato space $M_2^q(\mathbb{R}^d), (q > d)$ and Sobolev space $H_q^s(\mathbb{R}^d), (q < d, \frac{1}{q} - \frac{s}{d} < \frac{1}{d})$, and in general in the case of a so-called well-suited space $\mathcal{W}$ for NSE. NSE in the Morrey-Campanato spaces were also treated by Kato [22], Taylor [33], Kozono and Yamazaki [24].

In 1981, Weissler [34] gave the first existence result of mild solutions in the half space $L^3(\mathbb{R}^3_+)$. Then Giga and Miyakawa [18] generalized the result to $L^3(\Omega)$, where $\Omega$ is an open bounded domain in $\mathbb{R}^3$. Finally, in

1984, Kato [21] obtained, by means of a purely analytical tool (involving only the Hölder and Young inequalities and without using any estimate of fractional powers of the Stokes operator), an existence theorem in the whole space $L^3(\mathbb{R}^3)$. In ([7], [8], [9]), Cannone showed how to simplify Kato's proof. The idea is to take advantage of the structure of the bilinear operator in its scalar form. In particular, the divergence $\nabla$ and heat $e^{t\Delta}$ operators can be treated as a single convolution operator. In 1994, Kato and Ponce [23] showed that NSE are well-posed when the initial datum belongs to the homogeneous Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d), (d \leq q < \infty)$. Recently, the authors of this article have considered NSE in mixed-norm Sobolev-Lorentz spaces and Sobolev-Fourier-Lorentz spaces, see [25] and [26] respectively. In [27], we showed that the bilinear operator

$$B(u,v)(t) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,.)\otimes v(\tau,.)\big)\mathrm{d}\tau \tag{2}$$

is bicontinuous in $L^\infty([0,T];\dot{H}_q^s(\mathbb{R}^d))$ with super-critical, non-negative-regular indexes ($0 \leq s \leq d, q > 1$, and $\frac{s}{d} < \frac{1}{q} < \min\Big\{\frac{s+1}{d}, \frac{s+d}{2d}\Big\}$), and we obtain the inequality

$$\big\|B(u,v)\big\|_{L^\infty([0,T];\dot{H}_q^s)} \leq C_{s,q,d}T^{\frac{1}{2}(1+s-\frac{d}{q})}\big\|u\big\|_{L^\infty([0,T];\dot{H}_q^s)}\big\|v\big\|_{L^\infty([0,T];\dot{H}_q^s)}.$$

In this case existence and uniqueness theorems of local mild solutions can therefore be easily deduced.

In this paper, first, for $d \geq 3, s \geq 0, p > 1,$ and $r > 2$ be such that $\frac{s}{d} < \frac{1}{p} < \frac{1}{2} + \frac{s}{2d}$ and $\frac{2}{r} + \frac{d}{p} - s \leq 1$, we investigate mild solutions to NSE in the spaces $L^r\big([0,T];\dot{H}_p^s(\mathbb{R}^d)\big)$. We obtain the existence of local mild solutions with arbitrary initial tempered distribution datum in the Besov spaces $B_p^{s-\frac{2}{r},r}$. In the case of critical indexes $\frac{2}{r} - s + \frac{d}{p} = 1$, we obtain the existence of global mild solutions when the norm of the initial tempered distribution datum in the Besov space $\dot{B}_p^{s-\frac{2}{r},r}$ is small enough. The particular case of the above result, when $s = 0$, was presented in the book by Lemarie-Rieusset [29]. We also note that the Cauchy problem for an incompressible magneto-hydrodynamics system with positive viscosity and magnetic resistivity, in the framework of the Besov spaces was considered in [30].

Next, we present two different algorithms for constructing mild solutions in $C([0,T];\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d))$ or $C([0,T];H_q^{\frac{d}{q}-1}(\mathbb{R}^d))$ to the Cauchy problem for the Navier-Stokes equations when the initial datum belongs to the Sobolev spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ $\big($or $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)\big)$. We use the first algorithm to consider the case when the initial datum belongs to $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ or $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with $3 \leq d \leq 4$ and $2 \leq q \leq d$. Our results, when $q = d$, are a generalization the ones obtained in [29]. With the second algorithm, we can treat the case when the initial datum belongs to the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with $d \geq 3$ and $1 < q \leq d$. The cases $q = 2$ and $q = d$ were considered by many authors, see ([7], [9], [11], [12], [16], [20], [21], [29], [31]). A part of our results in the case when $2 < q < d$ can also be obtained by using the interpolation method of the results between the spaces $\dot{H}^{\frac{d}{2}}$ and $L^d$. So we will concentrate our efforts on the case $1 < q < 2$. To obtain the existence theorem in $C([0,T];\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d))$, we need to establish the continuity of the bilinear operator $B$ from

$$L^{2q}\Big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big) \times L^{2q}\Big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big) \text{ to } C([0,T];\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)),$$

and establishes the continuity of the bilinear operator $B$ from $L^r([0,T];H_p^s) \times L^r([0,T];H_p^s)$ into $L^r([0,T];H_p^s)$. In order to evaluate the norm of the bilinear operator $B$ in these spaces we use Lemma 7 which estimates the point-wise product of two functions in $\dot{H}_q^s(\mathbb{R}^d)$.

The paper is organized as follows. In Section 2 we recall some embedding theorems in the Triebel and Besov spaces and auxiliary lemmas. In Section 3 we present the main results of the paper.

In the sequence, for a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as $E$.

## §2. Some Imbedding Theorems

In this paper we use the definition of the Besov space $B_q^{s,p}$, the Triebel space $F_q^{s,p}$, and their homogeneous space $\dot{B}_q^{s,p}$ and $\dot{F}_q^{s,p}$ in [5, 6, 13, 32]. A known property of these spaces is the Riesz potential $\dot{\Lambda}^s = (-\Delta)^{s/2}$ which is an isomorphism from $\dot{B}_q^{s_0,p}$ onto $\dot{B}_q^{s_0-s,p}$ and from $\dot{F}_q^{s_0,p}$ to $\dot{F}_q^{s_0-s,p}$, see [4].

Let $1 < q < \infty$ and $s < d/q$, we define the homogeneous Sobolev space $\dot{H}^s_q$ as the closure of the space $S_0 = \{f \in \mathcal{S} :\ 0 \notin \mathrm{Supp}\hat{f}\}$ in the norm $\|f\|_{\dot{H}^s_q} = \|\dot{\Lambda}^s f\|_q$. Let us recall the following lemmas.

LEMMA 1. *Let $1 \le p, q \le \infty$ and $s \in \mathbb{R}$.*
(a) *If $s < 1$ then the two quantities*

$$\Big(\int_0^\infty \big(t^{-\frac{s}{2}}\big\|e^{t\Delta}t^{\frac{1}{2}}\dot{\Lambda}f\big\|_q\big)^p\frac{\mathrm{d}t}{t}\Big)^{1/p} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_q} \text{ are equivalent.}$$

(b) *If $s < 0$ then the two quantities*

$$\Big(\int_0^\infty \big(t^{-\frac{s}{2}}\big\|e^{t\Delta}f\big\|_q\big)^p\frac{\mathrm{d}t}{t}\Big)^{1/p} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_q} \text{ are equivalent.}$$

PROOF. See ([15], Proposition 1, p. 181 and Proposition 3, p. 182), or see ([29], Theorem 5.4, p. 45). □

The following lemma is a generalization of the above lemma.

LEMMA 2. *Let $1 \le p, q \le \infty$, $\alpha \ge 0$, and $s < \alpha$. Then the two quantities*

$$\Big(\int_0^\infty (t^{-\frac{s}{2}}\big\|e^{t\Delta}t^{\frac{\alpha}{2}}\dot{\Lambda}^\alpha f\big\|_{L^q})^p\frac{\mathrm{d}t}{t}\Big)^{\frac{1}{p}} \text{ and } \big\|f\big\|_{\dot{B}^{s,p}_q} \text{ are equivalent.}$$

PROOF. Note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}^{s,p}_q$ to $\dot{B}^{s-s_0,p}_q$, then we can easily prove the lemma. □

LEMMA 3. *For $1 \le p, q, r \le \infty$ and $s \in \mathbb{R}$, we have the following embedding mappings.*
(a) *If $1 < q \le 2$ then*

$$\dot{B}^{s,q}_q \hookrightarrow \dot{H}^s_q \hookrightarrow \dot{B}^{s,2}_q,\ B^{s,q}_q \hookrightarrow H^s_q \hookrightarrow B^{s,2}_q.$$

(b) *If $2 \le q < \infty$ then*

$$\dot{B}^{s,2}_q \hookrightarrow \dot{H}^s_q \hookrightarrow \dot{B}^{s,q}_q,\ B^{s,2}_q \hookrightarrow H^s_q \hookrightarrow B^{s,q}_q.$$

(c) *If* $1 \leq p_1 < p_2 \leq \infty$ *then*

$$\dot{B}_q^{s,p_1} \hookrightarrow \dot{B}_q^{s,p_2}, B_q^{s,p_1} \hookrightarrow B_q^{s,p_2},\ \dot{F}_q^{s,p_1} \hookrightarrow \dot{F}_q^{s,p_2}, F_q^{s,p_1} \hookrightarrow F_q^{s,p_2}.$$

(d) *If* $s_1 > s_2,\ 1 \leq q_1,\ q_2 \leq \infty$, *and* $s_1 - \frac{d}{q_1} = s_2 - \frac{d}{q_2}$ *then*

$$\dot{B}_{q_1}^{s_1,p} \hookrightarrow \dot{B}_{q_2}^{s_2,p}, B_{q_1}^{s_1,p} \hookrightarrow B_{q_2}^{s_2,p}, \dot{F}_{q_1}^{s_1,p} \hookrightarrow \dot{F}_{q_2}^{s_2,r},\ F_{q_1}^{s_1,p} \hookrightarrow F_{q_2}^{s_2,r}.$$

(e) *If* $p \leq q$ *then*

$$B_q^{s,p} \hookrightarrow F_q^{s,p},\ \dot{B}_q^{s,p} \hookrightarrow \dot{F}_q^{s,p}.$$

(f) *If* $q \leq p$ *then*

$$F_q^{s,p} \hookrightarrow B_q^{s,p},\ \dot{F}_q^{s,p} \hookrightarrow \dot{B}_q^{s,p}.$$

(g)

$$F_q^{s,q} = B_q^{s,q},\ \dot{F}_q^{s,q} = \dot{B}_q^{s,q}.$$

(h) *If* $1 < q < \infty$

$$H_q^s = F_q^{s,2},\ \dot{H}_q^s = \dot{F}_q^{s,2}.$$

Proof. For the proof of (a) and (b) see Theorem 6.4.4 ([2], p. 152). For the proof of (c) see [1] and [2]. For the proof of (d) see Theorem 6.5.1 ([2], p. 153) and [4]. For the proof of (e), (f), (g), and (h) see [1] and [4]. □

Lemma 4. *Let* $p \geq 1$ *and* $s \in \mathbb{R}$. *Then the following statements hold*
(1) *Assume that* $u_0 \in H_p^s$. *Then*

$$e^{t\Delta}u_0 \in L^\infty([0,\infty); H_p^s) \text{ and } \big\|e^{t\Delta}u_0\big\|_{L^\infty([0,\infty);H_p^s)} \leq \big\|u_0\big\|_{H_p^s}.$$

(2) *Assume that* $u_0 \in \dot{H}_p^s$. *Then*

$$e^{t\Delta}u_0 \in L^\infty([0,\infty); \dot{H}_p^s) \text{ and } \big\|e^{t\Delta}u_0\big\|_{L^\infty([0,\infty);\dot{H}_p^s)} \leq \big\|u_0\big\|_{\dot{H}_p^s}.$$

PROOF. (1) We have

$$\begin{aligned}
&\big\|e^{t\Delta}u_0\big\|_{H^s_p} = \big\|e^{t\Delta}(Id-\Delta)^{s/2}u_0\big\|_{L^p} = \\
&\frac{1}{(4\pi t)^{d/2}}\Big\|\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big((Id-\Delta)^{s/2}u_0\big)(\,.-\xi)\mathrm{d}\xi\Big\|_{L^p} \\
&\le \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big\|\big((Id-\Delta)^{s/2}u_0\big)(\,.-\xi)\big\|_{L^p}\mathrm{d}\xi \\
&= \frac{1}{(4\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{\frac{-|\xi|^2}{4t}}\big\|u_0\big\|_{H^s_p}\mathrm{d}\xi = \big\|u_0\big\|_{H^s_p},\ t\ge 0.
\end{aligned}$$

(2) The proof of (2) is similar to the proof of (1). □

THEOREM 1. *Let $E$ be an Banach space, and let $B : E \times E \to E$ be a continuous bilinear form such that there exists $\eta > 0$ so that*

$$\|B(x,y)\| \le \eta\|x\|\|y\|,$$

*for all $x$ and $y$ in $E$. Then for any fixed $y \in E$ such that $\|y\| \le \frac{1}{4\eta}$, the equation $x = y - B(x,x)$ has a unique solution $\overline{x} \in E$ satisfying $\|\overline{x}\| \le \frac{1}{2\eta}$.*

PROOF. See Theorem 22.4 ([29], p. 227). □

The following lemmas, in which we estimate the point-wise product of two functions in $\dot{H}^s_p(\mathbb{R}^d)$ is more general than the Hölder inequality. In the case when $s = 0, p \ge 2$, we get back the usual Hölder inequality.

LEMMA 5. *Assume that*

$$1 < p, q < d \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{1}{d}.$$

*Then there exists a constant $C$ independent of $u, v$ such that the following inequality holds*

$$\big\|uv\big\|_{\dot{H}^1_r} \le C\big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q},\ \forall u \in \dot{H}^1_p, v \in \dot{H}^1_q,$$

*where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{1}{d}$. In the subsequence the above kinds of conclusions will be shorten as*

$$\big\|uv\big\|_{\dot{H}^1_r} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q}.$$

Proof. By applying the Leibniz formula for the derivatives of a product of two functions, we have

$$\big\|uv\big\|_{\dot{H}^1_r} \simeq \sum_{|\alpha|=1} \big\|\partial^\alpha(uv)\big\|_{L^r} \leq \sum_{|\alpha|=1} \big\|(\partial^\alpha u)v\big\|_{L^r} + \sum_{|\alpha|=1} \big\|u(\partial^\alpha v)\big\|_{L^r}.$$

From the Hölder and Sobolev inequalities it follows that

$$\sum_{|\alpha|=1} \big\|(\partial^\alpha u)v\big\|_{L^r} \leq \sum_{|\alpha|=1} \big\|\partial^\alpha u\big\|_{L^p}\big\|v\big\|_{L^{q_1}} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q},$$

where

$$\frac{1}{q_1} = \frac{1}{q} - \frac{1}{d}.$$

Similar to the above proof, we have

$$\sum_{|\alpha|=1} \big\|u(\partial^\alpha v)\big\|_{L^r} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q}.$$

This gives the desired result

$$\big\|uv\big\|_{\dot{H}^1_r} \lesssim \big\|u\big\|_{\dot{H}^1_p}\big\|v\big\|_{\dot{H}^1_q}. \ \square$$

Lemma 6. *Assume that*

$$0 \leq s \leq 1, \frac{1}{p} > \frac{s}{d}, \frac{1}{q} > \frac{s}{d}, \text{ and } \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \tag{3}$$

*Then the following inequality holds*

$$\big\|uv\big\|_{\dot{H}^s_r} \lesssim \big\|u\big\|_{\dot{H}^s_p}\big\|v\big\|_{\dot{H}^s_q}, \ \forall u \in \dot{H}^s_p, v \in \dot{H}^s_q,$$

*where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$.

Proof. It is not difficult to show that if $p, q,$ and $s$ satisfy (3) then there exist numbers $p_1, p_2, q_1, q_2 \in (1, +\infty)$ (may be many of them) such that

$$\frac{1}{p} = \frac{1-s}{p_1} + \frac{s}{p_2}, \frac{1}{q} = \frac{1-s}{q_1} + \frac{s}{q_2}, \frac{1}{p_1} + \frac{1}{q_1} < 1,$$
$$p_2 < d, q_2 < d, \text{ and } \frac{1}{p_2} + \frac{1}{q_2} < 1 + \frac{1}{d}.$$

Setting

$$\frac{1}{r_1} = \frac{1}{p_1} + \frac{1}{q_1}, \frac{1}{r_2} = \frac{1}{p_2} + \frac{1}{q_2} - \frac{1}{d},$$

we have

$$\frac{1}{r} = \frac{1-s}{r_1} + \frac{s}{r_2}.$$

Therefore, applying Theorem 6.4.5 (p. 152) of [2] (see also [19] for $\dot{H}^s_p$), we get

$$\dot{H}^s_p = [L^{p_1}, \dot{H}^1_{p_2}]_s, \dot{H}^s_q = [L^{q_1}, \dot{H}^1_{q_2}]_s, \dot{H}^s_r = [L^{r_1}, \dot{H}^1_{r_2}]_s.$$

Applying the Hölder inequality and Lemma 5 in order to obtain

$$\big\|uv\big\|_{L^{r_1}} \lesssim \big\|u\big\|_{L^{p_1}} \big\|v\big\|_{L^{q_1}}, \ \forall u \in L^{p_1}, v \in L^{q_1},$$
$$\big\|uv\big\|_{\dot{H}^1_{r_2}} \lesssim \big\|u\big\|_{\dot{H}^1_{p_2}} \big\|v\big\|_{\dot{H}^1_{q_2}}, \ \forall u \in \dot{H}^1_{p_2}, v \in \dot{H}^1_{q_2}.$$

From Theorem 4.4.1 (p. 96) of [2] we get

$$\big\|uv\big\|_{\dot{H}^s_r} \lesssim \big\|u\big\|_{\dot{H}^s_p} \big\|v\big\|_{\dot{H}^s_q}. \ \square$$

LEMMA 7. *Assume that*

$$0 \le s < d, \frac{s}{d} < \frac{1}{p}, \frac{s}{d} < \frac{1}{q}, \ \textit{and} \ \frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}. \tag{4}$$

*Then we have the inequality*

$$\big\|uv\big\|_{\dot{H}^s_r} \lesssim \big\|u\big\|_{\dot{H}^s_p} \big\|v\big\|_{\dot{H}^s_q}, \ \forall u \in \dot{H}^s_p, v \in \dot{H}^s_q,$$

*where* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - \frac{s}{d}$.

PROOF. Denote by $[s]$ the integer part of $s$ and by $\{s\}$ the fraction part of $s$. Using formula for the derivatives of a product of two functions, we have

$$\big\|uv\big\|_{\dot{H}^s_r} = \big\|\dot{\Lambda}^s(uv)\big\|_{L^r} = \big\|\dot{\Lambda}^{\{s\}}(uv)\big\|_{\dot{H}^{[s]}_r} \simeq$$
$$\sum_{|\alpha|=[s]} \big\|\partial^\alpha \dot{\Lambda}^{\{s\}}(uv)\big\|_{L^r} = \sum_{|\alpha|=[s]} \big\|\dot{\Lambda}^{\{s\}} \partial^\alpha (uv)\big\|_{L^r}$$
$$= \sum_{|\alpha|=[s]} \big\|\partial^\alpha (uv)\big\|_{\dot{H}^{\{s\}}_r} \lesssim \sum_{|\gamma|+|\beta|=[s]} \big\|\partial^\gamma u \partial^\beta v\big\|_{\dot{H}^{\{s\}}_r}.$$

Set
$$\frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s - |\gamma| - \{s\}}{d}, \frac{1}{\tilde{q}} = \frac{1}{q} - \frac{s - |\beta| - \{s\}}{d}.$$
Applying Lemma 6 and the Sobolev inequality in order to obtain
$$\begin{aligned}\big\|\partial^\gamma u \partial^\beta v\big\|_{\dot{H}^{\{s\}}_r} &\lesssim \big\|\partial^\gamma u\big\|_{\dot{H}^{\{s\}}_{\tilde{p}}} \big\|\partial^\beta v\big\|_{\dot{H}^{\{s\}}_{\tilde{q}}} \\ &\lesssim \big\|u\big\|_{\dot{H}^{|\gamma|+\{s\}}_{\tilde{p}}} \big\|v\big\|_{\dot{H}^{|\beta|+\{s\}}_{\tilde{q}}} \lesssim \big\|u\big\|_{\dot{H}^s_p} \big\|v\big\|_{\dot{H}^s_q}.\end{aligned}$$
This gives the desired result
$$\big\|uv\big\|_{\dot{H}^s_r} \lesssim \big\|u\big\|_{\dot{H}^s_p} \big\|v\big\|_{\dot{H}^s_q}. \ \square$$

REMARK 1. Lemmas 5, 6, and 7 are still valid when the homogeneous space $\dot{H}^s_p$ is replaced by the inhomogeneous space $H^s_p$.

## §3. The Main Results

For $T > 0$, we say that $u$ is a mild solution of NSE on $[0, T]$ corresponding to a divergence-free initial data $u_0$ when $u$ satisfies the integral equation
$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau, .) \otimes u(\tau, .)\big)\mathrm{d}\tau.$$
Above we have used the following notation: For a tensor $F = (F_{ij})$ we define the vector $\nabla.F$ by $(\nabla.F)_i = \sum_{i=1}^d \partial_j F_{ij}$ and for vectors $u$ and $v$, we define their tensor product $(u \otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Helmholtz-Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1 \le k \le d} R_j R_k f_k, \tag{5}$$

where $R_j$ is the Riesz transforms defined on a scalar function $g$ as
$$\widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi).$$
The heat kernel $e^{t\Delta}$ is defined as
$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|.|^2/4t} * u)(x).$$

If $X$ is a normed space and $u = (u_1, u_2, ..., u_d), u_i \in X, 1 \le i \le d$, then we write
$$u \in X, \|u\|_X = \Big(\sum_{i=1}^{d} \|u_i\|_X^2\Big)^{1/2}.$$

### 3.1. On the continuity and regularity of the bilinear operator

In this subsection a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (2).

LEMMA 8. *Let*

$$d \ge 3,\ \ s \ge 0,\ \ p > 1,\ \ r > 2, \textit{and } T > 0 \tag{6}$$

*be such that*

$$\frac{s}{d} < \frac{1}{p} < \frac{1}{2} + \frac{s}{2d} \textit{ and } \frac{2}{r} + \frac{d}{p} - s \le 1. \tag{7}$$

*Then the bilinear operator $B(u,v)(t)$ is continuous from*
$$L^r([0,T]; H_p^s) \times L^r([0,T]; H_p^s)$$
*into*
$$L^r([0,T]; H_p^s),$$
*and the following inequality holds*

$$\big\|B(u,v)\big\|_{L^r([0,T];H_p^s)} \le CT^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}\big\|u\big\|_{L^r([0,T];H_p^s)}\big\|v\big\|_{L^r([0,T];H_p^s)}, \tag{8}$$

*where $C$ is a positive constant independent of $T$.*

PROOF. We have

$$\begin{aligned}\big\|B(u,v)(t)\big\|_{H_p^s} \le \int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{H_p^s} \mathrm{d}\tau = \\ \int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^p} \mathrm{d}\tau,\end{aligned} \tag{9}$$

where the operator $(Id-\Delta)^{\frac{s}{2}}$ is defined via the Fourier transform as
$$\big((Id-\Delta)^{\frac{s}{2}}g\big)^{\wedge}(\xi) = (1+|\xi|^2)^{\frac{s}{2}}\hat{g}(\xi).$$

We have

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big)_j =$$
$$e^{(t-\tau)\Delta}\sum_{l,k=1}^{d}\Big(\delta_{jk}-\frac{\partial_j\partial_k}{\Delta}\Big)\partial_l(Id-\Delta)^{s/2}\big(u_l(\tau,.)v_k(\tau,.)\big).$$

From the property of the Fourier transform we have

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big)^{\wedge}_j(\xi) =$$
$$e^{-(t-\tau)|\xi|^2}\sum_{l,k=1}^{d}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l)\Big((Id-\Delta)^{s/2}\big(u_l(\tau,.)v_k(\tau,.)\big)\Big)^{\wedge}(\xi),$$

and therefore

$$\Big(e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big)_j = \tag{10}$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\sum_{l,k=1}^{d}K_{l,k,j}\Big(\frac{.}{\sqrt{t-\tau}}\Big)*\Big((Id-\Delta)^{s/2}\big(u_l(\tau,.)v_k(\tau,.)\big)\Big),$$

where

$$\widehat{K_{l,k,j}}(\xi)=\frac{1}{(2\pi)^{d/2}}.e^{-|\xi|^2}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l).$$

Applying Proposition 11.1 ([29], p. 107) with $|\alpha|=1$ we obtain

$$|K_{l,k,j}(x)|\lesssim\frac{1}{(1+|x|)^{d+1}}.$$

Thus, the tensor $K(x)=\{K_{l,k,j}(x)\}$ satisfies

$$|K(x)|\lesssim\frac{1}{(1+|x|)^{d+1}}. \tag{11}$$

So, we can rewrite the equality (10) in the tensor form

$$e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big) =$$
$$\frac{1}{(t-\tau)^{\frac{d+1}{2}}}K\Big(\frac{.}{\sqrt{t-\tau}}\Big)*\Big((Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big).$$

Set

$$\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d},\ \frac{1}{h} = \frac{s}{d} - \frac{1}{p} + 1. \tag{12}$$

Note that from the inequalities (6) and (7), we can check that the following relations are satisfied

$$1 < h, \tilde{p} < \infty \text{ and } \frac{1}{p} + 1 = \frac{1}{h} + \frac{1}{\tilde{p}}.$$

Applying the Young inequality for convolution we obtain

$$\begin{gathered}\Big\| e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^p} \lesssim \\ \frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{.}{\sqrt{t-\tau}}\Big)\Big\|_{L^h}\Big\|(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^{\tilde{p}}}.\end{gathered} \tag{13}$$

Applying Lemma 7

$$\begin{aligned}\Big\|(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^{\tilde{p}}} &= \big\|u(\tau,.)\otimes v(\tau,.)\big\|_{H^s_{\tilde{p}}} \\ &\lesssim \big\|u(\tau,.)\big\|_{H^s_p}\big\|v(\tau,.)\big\|_{H^s_p}.\end{aligned} \tag{14}$$

From the estimate (11) and the equality (12), we have

$$\Big\|K\Big(\frac{.}{\sqrt{t-\tau}}\Big)\Big\|_{L^h} = (t-\tau)^{\frac{d}{2h}}\big\|K\big\|_{L^h} \simeq (t-\tau)^{\frac{s}{2}-\frac{d}{2p}+\frac{d}{2}}. \tag{15}$$

The inequalities (13), (14), and (15) imply that

$$\begin{gathered}\Big\| e^{(t-\tau)\Delta}\mathbb{P}\nabla.(Id-\Delta)^{s/2}\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^p} \lesssim \\ (t-\tau)^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|u(\tau,.)\big\|_{H^s_p}\big\|v(\tau,.)\big\|_{H^s_p}.\end{gathered} \tag{16}$$

From the inequalities (9) and (16), we get

$$\big\|B(u,v)(t)\big\|_{H^s_p} \lesssim \int_0^t (t-\tau)^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|u(\tau,.)\big\|_{H^s_p}\big\|v(\tau,.)\big\|_{H^s_p}\mathrm{d}\tau.$$

Applying of Proposition 2.4 (c) in ([29], p. 20) for the convolution in the Lorentz spaces, we have the following estimates

$$\begin{aligned}
(17)\qquad \Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^r_t(0,T)} &= \Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^{r,r}_t(0,T)} \\
&\leq \Big\|\big\|B(u,v)(t)\big\|_{H^s_p}\Big\|_{L^{r,\frac{r}{2}}_t(0,T)} \lesssim \\
&\big\|1_{[0,T]}t^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\big\|_{L^{r',\infty}}\Big\|\big\|u(t,.)\big\|_{H^s_p}\big\|v(t,.)\big\|_{H^s_p}\Big\|_{L^{\frac{r}{2},\frac{r}{2}}_t(0,T)},
\end{aligned}$$

where $\frac{1}{r'}+\frac{1}{r}=1$ and $1_{[0,T]}$ is the indicator function of set $[0,T]$ on $\mathbb{R}$.

By applying the Hölder inequality we get

$$\begin{aligned}
(18)\qquad \Big\|\big\|u(t,.)\big\|_{H^s_p}\big\|v(t,.)\big\|_{H^s_p}\Big\|_{L^{\frac{r}{2},\frac{r}{2}}_t(0,T)} &= \Big\|\big\|u(t,.)\big\|_{H^s_p}\big\|v(t,.)\big\|_{H^s_p}\Big\|_{L^{\frac{r}{2}}_t(0,T)} \\
&\leq \Big\|\big\|u(t,.)\big\|_{H^s_p}\Big\|_{L^r_t(0,T)}\Big\|\big\|v(t,.)\big\|_{H^s_p}\Big\|_{L^r_t(0,T)}.
\end{aligned}$$

Note that

$$(19)\qquad \Big\|1_{[0,T]}t^{\frac{s}{2}-\frac{d}{2p}-\frac{1}{2}}\Big\|_{L^{r',\infty}} \simeq T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}.$$

Therefore the inequality (8) can be deduced from the inequalities (17), (18), and (19). □

REMARK 2. Lemma 8 is still valid when the inhomogeneous space $H^s_p$ is replaced by the homogeneous space $\dot{H}^s_p$.

LEMMA 9. *Let*

$$d \geq 3,\ 0 \leq s < d,\ p > 1,\ r > 2, \textit{and } T > 0$$

*be such that*

$$\frac{1}{p} < \frac{1}{2} + \frac{s}{2d}, \frac{2}{p} \geq \frac{s+1}{d},\ \textit{and } \frac{2}{r} + \frac{d}{p} - s = 1.$$

*Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^r\big([0,T];\dot{H}^s_p\big) \times L^r\big([0,T];\dot{H}^s_p\big)$$

*into*

$$L^\infty\Big([0,T];\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}\Big),$$

*where*

$$\frac{1}{\tilde{p}}=\frac{2}{p}-\frac{s}{d},$$

*and we have the inequality*

$$(20)\qquad \big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}\big)}\le C\big\|u\big\|_{L^r([0,T];\dot{H}_p^s)}\big\|v\big\|_{L^r([0,T];\dot{H}_p^s)},$$

*where $C$ is a positive constant independent of $T$.*

Proof. To prove this lemma by duality (in the x-variable), (see Proposition 3.9 in ([29], p. 29)), let us consider an arbitrary test function $h(x)\in\mathcal{S}(\mathbb{R}^d)$ and evaluate the quantity

$$(21)\qquad I_t=\big\langle B(u,v)(t),h\big\rangle=\int_{\mathbb{R}^d}\big(B(u,v)(t)\big)(x)h(x)\mathrm{d}x.$$

We have

$$(22)\qquad \big\langle B(u,v)(t),h\big\rangle=\int_0^t\big\langle e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau,.)\otimes v(\tau,.)\big),h\big\rangle\mathrm{d}\tau=$$

$$\int_0^t\Big\langle e^{(t-\tau)\Delta}\dot{\Lambda}\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\big(u(\tau,.)\otimes v(\tau,.)\big),h\Big\rangle\mathrm{d}\tau=$$

$$\int_0^t\Big\langle\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\big(u(\tau,.)\otimes v(\tau,.)\big),e^{(t-\tau)\Delta}\dot{\Lambda}h\Big\rangle\mathrm{d}\tau=$$

$$\int_0^t\Big\langle\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\dot{\Lambda}^s\big(u(\tau,.)\otimes v(\tau,.)\big),e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\Big\rangle\mathrm{d}\tau.$$

By applying the Hölder inequality in the x-variable, from the equality (22) and the fact that (see [29])

$$\mathbb{P}\text{ and }\frac{\nabla}{\dot{\Lambda}}\text{ are continuous from }L^p\text{ into }L^p,1<p<\infty,$$

we get

$$(23)\qquad |I_t|\le\int_0^t\Big\|\mathbb{P}\frac{\nabla}{\dot{\Lambda}}.\dot{\Lambda}^s\big(u(\tau,.)\otimes v(\tau,.)\big)\Big\|_{L^{\tilde{p}}}\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\ \mathrm{d}\tau$$

$$\lesssim\int_0^t\big\|\dot{\Lambda}^s\big(u(\tau,.)\otimes v(\tau,.)\big)\big\|_{L^{\tilde{p}}}\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\ \mathrm{d}\tau,$$

where

$$\frac{1}{\tilde{p}}+\frac{1}{\tilde{p}'}=1.$$

Applying Lemma 7, we have

$$\begin{aligned}(24)\qquad \big\|\dot{\Lambda}^s\big(u(\tau,.)\otimes v(\tau,.)\big)\big\|_{L^{\tilde{p}}} &= \big\|u(\tau,.)\otimes v(\tau,.)\big\|_{\dot{H}^s_{\tilde{p}}}\\ &\lesssim \big\|u(\tau,.)\big\|_{\dot{H}^s_p}\big\|v(\tau,.)\big\|_{\dot{H}^s_p}.\end{aligned}$$

From the inequalities (23) and (24), applying the Hölder inequality in the t-variable, we deduce that

$$\begin{aligned}(25)\qquad |I_t| &\lesssim \int_0^t \big\|u(\tau,.)\big\|_{\dot{H}^s_p}\big\|v(\tau,.)\big\|_{\dot{H}^s_p}\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\,\mathrm{d}\tau \leq\\ &\Big(\int_0^t \big(\big\|u(\tau,.)\big\|_{\dot{H}^s_p}\big\|v(\tau,.)\big\|_{\dot{H}^s_p}\big)^{\frac{r}{2}}\mathrm{d}\tau\Big)^{\frac{2}{r}}\Big(\int_0^t \big(\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}d\tau\Big)^{\frac{r-2}{r}}\\ &\leq \big\|u\big\|_{L^r([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^r([0,T];\dot{H}^s_p)}\Big(\int_0^t \big(\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}\tau\Big)^{\frac{r-2}{r}}.\end{aligned}$$

From Lemma 1 and note that $\dot{\Lambda}^{s_0}$ is an isomorphism from $\dot{B}^{s,p}_q$ to $\dot{B}^{s-s_0,p}_q$ (see [4]), we have the following estimates

$$\begin{aligned}(26)\qquad &\Big(\int_0^t \big(\big\|e^{(t-\tau)\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}\tau\Big)^{\frac{r-2}{r}}\\ &\leq \Big(\int_0^\infty \big(\big\|e^{t\Delta}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\mathrm{d}t\Big)^{\frac{r-2}{r}}\\ &= \Big(\int_0^\infty \big(t^{\frac{r-4}{2r}}\big\|e^{t\Delta}t^{\frac{1}{2}}\dot{\Lambda}\dot{\Lambda}^{-s}h\big\|_{L^{\tilde{p}'}}\big)^{\frac{r}{r-2}}\frac{\mathrm{d}t}{t}\Big)^{\frac{r-2}{r}} \simeq \big\|\dot{\Lambda}^{-s}h\big\|_{\dot{B}^{\frac{4-r}{r},\frac{r}{r-2}}_{\tilde{p}'}}\\ &\simeq \big\|h\big\|_{\dot{B}^{\frac{4-r}{r}-s,\frac{r}{r-2}}_{\tilde{p}'}} = \big\|h\big\|_{\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}}.\end{aligned}$$

From the equality (21) and the inequalities (25) and (26), we get

$$\big|\big\langle B(u,v)(t),h\big\rangle\big| \lesssim \big\|u\big\|_{L^r([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^r([0,T];\dot{H}^s_p)}\Big\|h\Big\|_{\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}}.$$

However, $\dot{B}^{1-\frac{d}{\tilde{p}},\frac{r}{r-2}}_{\tilde{p}'}$ is exactly the dual of $\dot{B}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}_{\tilde{p}}$, (the restriction $\frac{2}{p}\geq\frac{s+1}{d}$ is mainly because we are interested in non-negative indexes), therefore we conclude that

$$\big\|B(u,v)(t)\big\|_{\dot{B}_{\tilde{p}}^{\frac{d}{\tilde{p}}-1,\frac{r}{2}}} \lesssim \|u\|_{L^r([0,T];\dot{H}^s_p)}\|v\|_{L^r([0,T];\dot{H}^s_p)},\ 0\le t\le T. \tag{27}$$

Finally, the estimate (20) can be deduced from the inequality (27). □

Combining Theorem 1 with Lemma 8, we get the following existence results, the particular case of which, when $s=0$, was obtained in [29].

Theorem 2. *Let*

$$d\ge 3, s\ge 0,\ p>1,\ \textit{and}\ r>2,$$

*be such that*

$$\frac{s}{d}<\frac{1}{p}<\frac{1}{2}+\frac{s}{2d}\ \textit{and}\ \frac{2}{r}+\frac{d}{p}-s\le 1.$$

(a) *There exists a positive constant $\delta_{s,p,r,d}$ such that for all $T>0$ and for all $u_0\in\mathcal{S}'(\mathbb{R}^d)$ with* $\operatorname{div}(u)=0$, *satisfying*

$$T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}\big\|e^{t\Delta}u_0\big\|_{L^r([0,T];\dot{H}^s_p)}\le\delta_{s,p,r,d}, \tag{28}$$

*there is a unique mild solution $u\in L^r\big([0,T];\dot{H}^s_p)\big)$ for NSE.*

*If*

$$e^{t\Delta}u_0\in L^r\big([0,1];\dot{H}^s_p\big),$$

*then the inequality* (28) *holds when $T(u_0)$ is small enough.*
(b) *If $\frac{2}{r}+\frac{d}{p}-s=1$ then there exists a positive constant $\delta_{s,p,d}$ such that we can take $T=\infty$ whenever $\big\|e^{t\Delta}u_0\big\|_{L^r([0,\infty];\dot{H}^s_p)}\le\delta_{s,p,d}$.*

Proof. (a) From Lemma 8, we use the estimate

$$\big\|B\big\|_{L^r([0,T];\dot{H}^s_p)}\le C_{s,p,r,d}T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})},$$

where $C_{s,p,r,d}$ is a positive constant independent of $T$. From Theorem 1 and the above inequality, we deduce the existence of a solution to the Navier-Stokes equations on the interval $(0,T)$ with

$$4C_{s,p,r,d}T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})}\big\|e^{t\Delta}u_0\big\|_{L^r([0,T];\dot{H}^s_p)}\le 1.$$

If $e^{t\Delta}u_0 \in L^r([0,1];\dot{H}^s_p)$ then this condition is fulfilled for $T = T(u_0)$ small enough, this is obvious for the case when $\frac{2}{r} + \frac{d}{p} - s < 1$ since $\lim\limits_{T\to 0} T^{\frac{1}{2}(1+s-\frac{2}{r}-\frac{d}{p})} = 0$. For the case when $\frac{2}{r} + \frac{d}{p} - s = 1$, the condition is fulfilled since we have $\lim\limits_{T\to 0}\big\|e^{t\Delta}u_0\big\|_{L^r([0,T];\dot{H}^s_p)} = 0$.
(b) This is obvious. □

Remark 3. From Theorem 5.3 ([29], p. 44), if $u_0 \in B_p^{s-\frac{2}{r},r} \cap \mathcal{S}'(\mathbb{R}^d)$ then $e^{t\Delta}u_0 \in L^r([0,1];\dot{H}^s_p)$. From Lemma 2, if $u_0 \in \mathcal{S}'(\mathbb{R}^d)$ the two quantities $\big\|u_0\big\|_{\dot{B}_p^{s-\frac{2}{r},r}}$ and $\big\|e^{t\Delta}u_0\big\|_{L^r([0,\infty];\dot{H}^s_p)}$ are equivalent.

### 3.2. Solutions to the Navier-Stokes equations with initial value in the critical spaces $H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ and $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $3 \leq d \leq 4,\ 2 \leq q \leq d$

Lemma 10. *Let $d \geq 3$ and $2 \leq q \leq d$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^4\Big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\Big) \times L^4\Big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\Big)$$

*into*

$$L^\infty\Big([0,T];\dot{B}_q^{\frac{d}{q}-1,2}\Big),$$

*and we have the inequality*

$$\begin{aligned} \big\|B(u,v)\big\|_{L^\infty([0,T];\dot{H}_q^{\frac{d}{q}-1})} &\lesssim \big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{B}_q^{\frac{d}{q}-1,2}\big)} \\ &\leq C\big\|u\big\|_{L^4\big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}\big\|v\big\|_{L^4\big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}, \end{aligned} \tag{29}$$

*where $C$ is a positive constant and independent of $T$.*

Proof. Applying Lemma 9 with $r = 4, p = \frac{2dq}{2d-q}$, and $s = \frac{d}{q} - 1$, we get

$$\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d} = \frac{2d-q}{dq} - \frac{\frac{d}{q}-1}{d} = \frac{1}{q},$$

$$\big\|B(u,v)\big\|_{L^\infty\big([0,T];\dot{B}_q^{\frac{d}{q}-1,2}\big)} \lesssim \big\|u\big\|_{L^4\big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}\big\|v\big\|_{L^4\big([0,T];\dot{H}^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}. \tag{30}$$

From (b) of Lemma 3, we have

$$\dot{B}_q^{\frac{d}{q}-1,2} \hookrightarrow \dot{H}_q^{\frac{d}{q}-1}. \tag{31}$$

Finally, the estimate (29) can be deduced from the inequality (30) and the imbedding (31). □

LEMMA 11. *Let $d \geq 3$ and $2 \leq q \leq d$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^4\Big([0,T]; H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\Big) \times L^4\Big([0,T]; H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\Big)$$

*into*

$$L^\infty\Big([0,T]; H_q^{\frac{d}{q}-1}\Big),$$

*and we have the inequality*

$$\big\|B(u,v)\big\|_{L^\infty\left([0,T];H_q^{\frac{d}{q}-1}\right)} \leq C\big\|u\big\|_{L^4\left([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\right)}\big\|v\big\|_{L^4\left([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\right)}, \tag{32}$$

*where $C$ is a positive constant and independent of $T$.*

PROOF. To prove this lemma by duality (in the x-variable), let us consider an arbitrary test function $h(x) \in \mathcal{S}(\mathbb{R}^d)$. Similar to the proof of Lemma 9, we have

$$\begin{aligned}&\Big|\Big\langle(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t), h\Big\rangle\Big|\\ &\lesssim \big\|u\big\|_{L^4\left([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\right)}\big\|v\big\|_{L^4\left([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\right)}\big\|h\big\|_{\dot{B}_{q'}^{0,2}},\end{aligned}$$

where

$$\frac{1}{q}+\frac{1}{q'}=1.$$

However the dual space of $\dot{B}_{q'}^{0,2}$ is $\dot{B}_q^{0,2}$, therefore we get

$$\begin{aligned}&\Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot{B}_q^{0,2}}\\ &\lesssim \big\|u\big\|_{L^4\left([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\right)}\big\|v\big\|_{L^4\left([0,T];H_{\frac{2dq}{2d-q}}^{\frac{d}{q}-1}\right)}.\end{aligned} \tag{33}$$

From (b) of Lemma 3 and the estimate (33), we have

$$\Big\|B(u,v)(t)\Big\|_{H_q^{\frac{d}{q}-1}} = \Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{L^q} = \tag{34}$$

$$\Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot{H}_q^0} \lesssim \Big\|(\sqrt{Id-\Delta})^{\frac{d}{q}-1}B(u,v)(t)\Big\|_{\dot{B}_q^{0,2}}$$

$$\lesssim \big\|u\big\|_{L^4\big([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)}\big\|v\big\|_{L^4\big([0,T];H^{\frac{d}{q}-1}_{\frac{2dq}{2d-q}}\big)},\ 0\le t\le T.$$

Finally, the estimate (32) can be deduced from the inequality (34). □

LEMMA 12. *Let $d \geq 3$ and $2 \leq q \leq 4$.*
(a) *If $u_0 \in H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ then*

$$\big\|e^{t\Delta}u_0\big\|_{L^4\big([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\big)} \lesssim \big\|u_0\big\|_{H_q^{d/q-1}}.$$

(b) *If $u_0 \in \dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ then*

$$\Big\|e^{t\Delta}u_0\Big\|_{L^4\big([0,\infty);\dot{H}^{d/q-1}_{2dq/(2d-q)}\big)} \simeq \big\|u_0\big\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{d/q-1}}.$$

PROOF. (a) From Lemma 1, we have the estimates

$$\begin{aligned} &\Big\|e^{t\Delta}u_0\Big\|_{L^4\big([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\big)} \\ &= \Big(\int_0^\infty \Big\|e^{t\Delta}(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|^4_{L^{2dq/(2d-q)}}\mathrm{d}t\Big)^{1/4} \\ &= \Big(\int_0^\infty \Big(t^{\frac14}\Big\|e^{t\Delta}(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|_{L^{2dq/(2d-q)}}\Big)^4\frac{\mathrm{d}t}{t}\Big)^{1/4} \\ &\simeq \Big\|(\sqrt{Id-\Delta})^{d/q-1}u_0\Big\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}}. \end{aligned} \tag{35}$$

Applying (b), (c), and (d) of Lemma 3 in order to obtain

$$L^q = \dot{H}_q^0 \hookrightarrow \dot{B}_q^{0,q} \hookrightarrow \dot{B}_q^{0,4} \hookrightarrow \dot{B}^{-1/2,4}_{2dq/(2d-q)}. \tag{36}$$

From the inequality (35) and the imbedding (36), we get

$$\left\|e^{t\Delta}u_0\right\|_{L^4\left([0,\infty);H^{d/q-1}_{2dq/(2d-q)}\right)} \simeq \left\|(\sqrt{Id-\Delta})^{d/q-1}u_0\right\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}}$$
$$\lesssim \left\|(\sqrt{Id-\Delta})^{d/q-1}u_0\right\|_{L^q} = \left\|u_0\right\|_{H_q^{d/q-1}}.$$

(b) Similar to the proof of (a) we have

$$\left\|e^{t\Delta}u_0\right\|_{L^4\left([0,\infty);\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \simeq \left\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\right\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}}$$
$$\lesssim \left\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\right\|_{L^q} = \left\|u_0\right\|_{\dot{H}_q^{d/q-1}},$$
$$\text{and} \quad \left\|\dot{\Lambda}^{\frac{d}{q}-1}u_0\right\|_{\dot{B}^{-1/2,4}_{2dq/(2d-q)}} \simeq \left\|u_0\right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}}. \ \Box$$

Combining Theorem 1 with Lemmas 4, 8, 10, and 12 we obtain the following existence result.

Theorem 3. *Let $3 \le d \le 4$ and $2 \le q \le d$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with* $\operatorname{div}(u_0) = 0$ *satisfying*

$$\left\|e^{t\Delta}u_0\right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \le \delta_{q,d}, \tag{37}$$

*NSE has a unique mild solution $u \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big) \cap C\big([0,T];\dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{t\Delta}u_0$, then we have*

$$w \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big) \cap L^\infty\big([0,T];\dot{B}_q^{d/q-1,2}\big).$$

*Finally, we have*

$$\left\|e^{t\Delta}u_0\right\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \lesssim \left\|u_0\right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \left\|u_0\right\|_{\dot{H}_q^{d/q-1}},$$

*in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ the inequality* (37) *holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,d}$ such that for all $\left\|u_0\right\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \le \sigma_{q,d}$ we can take $T = \infty$.*

PROOF. By applying Lemma 8 with $r = 4,\ p = \frac{2dq}{2d-q},\ s = \frac{d}{q} - 1$, and notice that $1 + s - \frac{2}{r} - \frac{d}{p} = 0$ we have

$$\big\|B\big\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \leq C_{q,d},$$

where $C_{q,d}$ is a positive constant independent of $T$. From Theorem 1 and the above inequality, we deduce that for any $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ such that

$$\operatorname{div}(u_0) = 0, \quad \big\|e^{t\Delta}u_0\big\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \leq \frac{1}{4C_{q,d}},$$

NSE has a mild solution $u$ on the interval $(0, T)$ so that

$$u \in L^4\big([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\big). \tag{38}$$

From Lemma 10 and (38), we have $B(u,u) \in L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. From (2) of Lemma 4, we have $e^{t\Delta}u_0 \in L^\infty\big([0,T];\dot{H}_q^{d/q-1}\big)$. Therefore

$$u = e^{t\Delta}u_0 - B(u,u) \in L^\infty([0,T];\dot{H}_q^{d/q-1}).$$

In the space $H^{d/2-1}$ or $L^d$ (see [29]), the solutions can also be constructed by a successive approximation via the integral equation and therefore they are continuous in time up to the initial time. Since $e^{t\Delta}$ is a $(C_0)$-semigroup in $H_q^s$ and $\dot{H}_q^s$ with finite integral-exponent $(q < \infty)$, by the same way as, we can easily show that the obtained mild solution $u \in C\big([0,T];\dot{H}_q^{d/q-1}\big)$.

From (b) of Lemma 12, we have

$$\begin{aligned}\big\|e^{t\Delta}u_0\big\|_{L^4\left([0,T];\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} &\lesssim \big\|e^{t\Delta}u_0\big\|_{L^4\left([0,\infty);\dot{H}^{d/q-1}_{2dq/(2d-q)}\right)} \\ &\simeq \Big\|u_0\Big\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{d/q-1}} < \infty.\end{aligned}$$

Hence, the left-hand side of the inequality (37) converges to 0 when $T$ tends to 0. Therefore, for arbitrary $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ there is $T(u_0)$ small enough such that the inequality (37) holds. Also, there exists a positive constants $\sigma_{q,d}$ such that for all $\Big\|u_0\Big\|_{\dot{B}^{d/q-3/2,4}_{2dq/(2d-q)}} \leq \sigma_{q,d}$ and $T = \infty$ the inequality (37) holds. □

REMARK 4. Theorem 3 in the particular case $q = d$ is Proposition 20.1 in [29].

THEOREM 4. *Let $3 \leq d \leq 4$ and $2 \leq q \leq d$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in H_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ with* $\operatorname{div}(u_0) = 0$ *satisfying*

$$\big\|e^{t\Delta}u_0\big\|_{L^4\left([0,T];H^{d/q-1}_{2dq/(2d-q)}\right)} \leq \delta_{q,d}, \tag{39}$$

*NSE has a unique mild solution $u \in L^4\big([0,T];H^{d/q-1}_{2dq/(2d-q)}\big) \cap C\big([0,T];H_q^{d/q-1}\big)$. Finally, we have*

$$\big\|e^{t\Delta}u_0\big\|_{L^4\left([0,T];H^{d/q-1}_{2dq/(2d-q)}\right)} \leq \big\|u_0\big\|_{H_q^{d/q-1}},$$

*in particular, for arbitrary $u_0 \in H_q^{\frac{d}{q}-1}$ the inequality* (39) *holds when $T(u_0)$ is small enough;*

PROOF. The proof of Theorem 4 is similar to the one of Theorem 3, by combining Theorem 1 with Lemmas 4, 8, 11, and 12. □

### 3.3. Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \geq 3$ and $1 < q \leq d$

We consider two cases $2 < q \leq d$ and $1 < q \leq 2$ separately.

#### *3.3.1 Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \geq 3$ and $2 < q \leq d$*

LEMMA 13. *Let $d \geq 3$ and $2 < q \leq d$. Then for all $p$ such that*

$$2 < p < \min\Big\{\frac{(d-2)q}{d-q},\ d+2\Big\}, (if\ q = d\ then\ \frac{(d-2)q}{d-q} = +\infty),$$

*the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^p([0,T];\dot{H}_p^{\frac{2+d-p}{p}}) \times L^p([0,T];\dot{H}_p^{\frac{2+d-p}{p}})$$

*into*

$$L^\infty\big([0,T];\dot{B}^{\frac{d+p-2}{p}-1,\frac{p}{2}}_{\frac{dp}{d+p-2}}\big),$$

*and we have the inequality*

$$\big\|B(u,v)\big\|_{L^\infty\left([0,T];\dot H_q^{\frac{d}{q}-1}\right)} \lesssim \big\|B(u,v)\big\|_{L^\infty\left([0,T];\dot B_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}}\right)} \tag{40}$$
$$\leq C\big\|u\big\|_{L^p\left([0,T];\dot H_p^{\frac{2+d-p}{p}}\right)}\big\|v\big\|_{L^p\left([0,T];\dot H_p^{\frac{2+d-p}{p}}\right)},$$

*where $C$ is a positive constant independent of $T$.*

Proof. Applying Lemma 9 with $r = p$ and $s = \frac{2+d-p}{p}$, we get

$$\frac{1}{\tilde p} = \frac{2}{p} - \frac{s}{d} = \frac{d+p-2}{dp},$$
$$\Big\|B(u,v)\Big\|_{L^\infty\left([0,T];\dot B_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}}\right)} \tag{41}$$
$$\lesssim \big\|u\big\|_{L^p\left([0,T];\dot H_p^{\frac{2+d-p}{p}}\right)}\big\|v\big\|_{L^p\left([0,T];\dot H_p^{\frac{2+d-p}{p}}\right)}.$$

Applying (e), (d), and (h) of Lemma 3 in order to obtain

$$\dot B_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}} \hookrightarrow \dot F_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}} \hookrightarrow \dot F_q^{\frac{d}{q}-1,2} = \dot H_q^{\frac{d}{q}-1}. \tag{42}$$

Therefore the estimate (40) is deduced from the inequality (41) and the imbedding (42).

Lemma 14. *Let $2 < q < p < +\infty$. Then for all $u_0 \in \dot H_q^{\frac{d}{q}-1}$ we have the estimates*

$$\big\|e^{t\Delta}u_0\big\|_{L^p\left([0,\infty);\dot H_p^{\frac{2+d-p}{p}}\right)} \simeq \big\|u_0\big\|_{\dot B_p^{\frac{d}{p}-1,p}} \lesssim \big\|u_0\big\|_{\dot H_q^{\frac{d}{q}-1}}.$$

Proof. From Lemma 1, we have the estimates

$$\big\|e^{t\Delta}u_0\big\|_{L^p\left([0,\infty);\dot H_p^{\frac{2+d-p}{p}}\right)} \simeq \big\|u_0\big\|_{\dot B_p^{\frac{d}{p}-1,p}}. \tag{43}$$

Applying (b), (d), and (c) of Lemma 3 in order to obtain

$$\dot H_q^{\frac{d}{q}-1} \hookrightarrow \dot B_q^{\frac{d}{q}-1,q} \hookrightarrow \dot B_p^{\frac{d}{p}-1,q} \hookrightarrow \dot B_p^{\frac{d}{p}-1,p}. \tag{44}$$

From the estimate (43) and the imbedding (44), we have

$$\Big\| e^{t\Delta} u_0 \Big\|_{L^p\left([0,\infty); \dot{H}_p^{\frac{2+d-p}{p}}\right)} \simeq \big\| u_0 \big\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \lesssim \big\| u_0 \big\|_{\dot{H}_q^{\frac{d}{q}-1}}. \ \Box$$

Theorem 5. *Let $d \geq 3$ and $2 < q \leq d$. Then for any $p$ be such that*

$$q < p < \min\Big\{ \frac{(d-2)q}{d-q},\ d+2 \Big\},$$

*there exists a constant $\delta_{q,p,d} > 0$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying*

$$\Big\| e^{t\Delta} u_0 \Big\|_{L^p\left([0,T]; \dot{H}_p^{\frac{2+d-p}{p}}\right)} \leq \delta_{q,p,d}, \tag{45}$$

*NSE has a unique mild solution $u \in L^p\big([0,T]; \dot{H}_p^{\frac{2+d-p}{p}}\big) \cap C\big([0,T]; \dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{t\Delta}u_0$, then we have*

$$w \in L^p\big([0,T]; \dot{H}_p^{\frac{2+d-p}{p}}\big) \cap L^\infty\Big([0,T]; \dot{B}_{\frac{dp}{d+p-2}}^{\frac{d+p-2}{p}-1,\frac{p}{2}}\Big).$$

*Finally, we have*

$$\Big\| e^{t\Delta} u_0 \Big\|_{L^p\left([0,T]; \dot{H}_p^{\frac{2+d-p}{p}}\right)} \leq \big\| u_0 \big\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \lesssim \big\| u_0 \big\|_{\dot{H}_q^{\frac{d}{q}-1}},$$

*in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}$ the inequality (45) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,p,d}$ such that for all $\big\|u_0\big\|_{\dot{B}_p^{\frac{d}{p}-1,p}} \leq \sigma_{q,p,d}$ we can take $T = \infty$.*

Proof. The proof of Theorem 5 is similar to the one of Theorem 3, by combining Theorem 1 with Lemmas 4, 8 (for $r = p,\ s = \frac{2+d-p}{p}$), 13, and 14. $\Box$

Remark 5. The case $q = d$ was treated by several authors, see for example ([7], [12], [21]). However their results are different from ours.

*3.3.2 Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}_q^{\frac{d}{q}-1}(\mathbb{R}^d)$ for $d \geq 3$ and $1 < q \leq 2$*

Lemma 15. *Let $d \geq 3$ and $1 < q \leq 2$. Then the bilinear operator $B(u,v)(t)$ is continuous from*

$$L^{2q}\Big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big) \times L^{2q}\Big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\Big)$$

*into*

$$L^{\infty}\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big),$$

*and we have the inequality*

$$\big\|B(u,v)\big\|_{L^{\infty}\big([0,T];\dot{H}_q^{\frac{d}{q}-1}\big)} \lesssim \big\|B(u,v)\big\|_{L^{\infty}\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big)}$$
$$\leq C\Big\|u\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}\Big\|v\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)},$$

*where $C$ is a positive constant independent of $T$.*

Proof. Applying Lemma 9 with $r = 2q$, $p = \frac{dq}{d+1-q}$, and $s = \frac{d+2-2q}{q}$, we get

$$\frac{1}{\tilde{p}} = \frac{2}{p} - \frac{s}{d} = \frac{1}{q},$$

and from (a) of Lemma 3, we have

$$\big\|B(u,v)\big\|_{L^{\infty}\big([0,T];\dot{H}_q^{\frac{d}{q}-1}\big)} \lesssim \big\|B(u,v)\big\|_{L^{\infty}\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big)}$$
$$\lesssim \Big\|u\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}\Big\|v\Big\|_{L^{2q}\big([0,T];\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)}. \qquad \square$$

Lemma 16. *Assume that $u_0 \in \dot{H}_q^{\frac{d}{q}-1}$ with $d \geq 3$ and $1 < q \leq 2$. Then*

$$\Big\|e^{t\Delta}u_0\Big\|_{L^{2q}\big([0,\infty);\dot{H}^{\frac{d+2-2q}{q}}_{\frac{dq}{d+1-q}}\big)} \simeq \big\|u_0\big\|_{\dot{B}^{(d+1)/q-2,2q}_{dq/(d+1-q)}} \lesssim \big\|u_0\big\|_{\dot{H}_q^{d/q-1}}.$$

PROOF. By using (a), (c), and (d) of Lemma 3 in order to obtain

$$\dot{H}_q^{\frac{d}{q}-1} \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,2} \hookrightarrow \dot{B}_q^{\frac{d}{q}-1,2q} \hookrightarrow \dot{B}_{dq/(d+1-q)}^{(d+1)/q-2,2q}. \tag{46}$$

Applying Lemma 1 and from the imbedding (46) we have the estimates

$$\begin{aligned}\left\|e^{t\Delta}u_0\right\|_{L^{2q}\left([0,\infty);\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\right)} &\simeq \left\|\dot{\Lambda}^{\frac{d+2-2q}{q}}u_0\right\|_{\dot{B}_{dq/(d+1-q)}^{-1/q,2q}} \\ &\simeq \left\|u_0\right\|_{\dot{B}_{dq/(d+1-q)}^{(d+1)/q-2,2q}} \lesssim \left\|u_0\right\|_{\dot{H}_q^{d/q-1}}. \ \square\end{aligned}$$

THEOREM 6. *Let $d \geq 3$ and $1 < q \leq 2$. There exists a positive constant $\delta_{q,d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ with $\operatorname{div}(u_0) = 0$ satisfying*

$$\left\|e^{t\Delta}u_0\right\|_{L^{2q}\left([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\right)} \leq \delta_{q,d}, \tag{47}$$

*NSE has a unique mild solution $u \in L^{2q}\big([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\big) \cap C\big([0,T];\dot{H}_q^{d/q-1}\big)$. Denoting $w = u - e^{t\Delta}u_0$, then we have*

$$w \in L^{2q}\big([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\big) \cap L^{\infty}\big([0,T];\dot{B}_q^{\frac{d}{q}-1,q}\big).$$

*Finally, we have*

$$\left\|e^{t\Delta}u_0\right\|_{L^{2q}\left([0,T];\dot{H}_{\frac{dq}{d+1-q}}^{\frac{d+2-2q}{q}}\right)} \leq \left\|u_0\right\|_{\dot{B}_{dq/(d+1-q)}^{(d+1)/q-2,2q}} \lesssim \left\|u_0\right\|_{\dot{H}_q^{d/q-1}},$$

*in particular, for arbitrary $u_0 \in \dot{H}_q^{d/q-1}(\mathbb{R}^d)$ the inequality (47) holds when $T(u_0)$ is small enough; and there exists a positive constant $\sigma_{q,d}$ such that for all $\|u_0\|_{\dot{B}_{dq/(d+1-q)}^{(d+1)/q-2,2q}} \leq \sigma_{q,d}$ we can take $T = \infty$.*

PROOF. The proof of Theorem 6 is similar to the one of Theorem 3, by combining Theorem 1 with Lemmas 4, 8 (for $r = 2q, p = \frac{dq}{d+1-q}, s = \frac{d+2-2q}{q}$), 15, and 16. $\square$

Remark 6. The case $q = 2$ was treated by several authors, see for example ([7],[16], [29]). However their results are different from ours.

*Acknowledgments*. This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.02-2014.50.

## References

[1] Adams, A. R., *Sobolev Spaces*, Academic press, Boston, MA, 1975, 268 p.

[2] Bergh, J. and J. Lofstrom, *Interpolation Spaces*, Springer-Verlag, 1976, 264 p.

[3] Bourgain, J. and N. Pavloviéc, Ill-posedness of the Navier-Stokes equations in a critical space in 3D, J. Funct. Anal. **255** (9) (2008), 2233–2247.

[4] Jawerth, B., Some observations on Besov and Lizorkin-Triebel space, Math. Scand. **40** (1977), 94–104.

[5] Bourdaud, G., Réalisation des espaces de Besov homogènes, Ark. Mat. **26** (1) (1988), 41–54.

[6] Bourdaud, G., Ce qu'il faut savoir sur les espaces de Besov, Prépublication de l'Universitéde Paris 7 (janvier 1993).

[7] Cannone, M., *Ondelettes, Paraproduits et Navier-Stokes*, Diderot Editeur, Paris, 1995, 191 p.

[8] Cannone, M., A generalization of a theorem by Kato on Navier-Stokes equations, Rev. Mat. Iberoamericana **13** (3) (1997), 515–541.

[9] Cannone, M. and F. Planchon, On the nonstationary Navier-Stokes equations with an external force, Adv. in Diff. Eq. **4** (5) (1999), 697–730.

[10] Cannone, M. and Y. Meyer, Littlewood-Paley decomposition and the Navier-Stokes equations, Meth. and Appl. of Anal. **2** (1995), 307–319.

[11] Chemin, J. M., Remarques sur l'existence globale pour le système de Navier-Stokes incompressible, SIAM J. Math. Anal. **23** (1992), 20–28.

[12] Dong, H. and D. Du, On the local smoothness of solutions of the Navier-Stokes equations, J. Math. Fluid Mech. **9** (2) (2007), 139–152.

[13] Frazier, M., Jawerth, B. and G. Weiss, *Littlewood-Paley Theory and the Study of Function Spaces*, CBMS Regional Conference Series in Mathematics, 79, AMS, Providence (1991).

[14] Fabes, E., Jones, B. and N. Riviere, The initial value problem for the Navier-Stokes equations with data in $L^p$, Arch. Rat. Mech. Anal. **45** (1972), 222–240.

[15] Friedlander, S. and D. Serre, *Handbook of Mathematical Fluid Dynamics*, Volume 3, Elsevier, 2004.

[16] Fujita, H. and T. Kato, On the Navier-Stokes initial value problem I, Arch. Rat. Mech. Anal. **16** (1964), 269–315.

[17] Giga, Y., Solutions of semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system, J. Differ. Eq. **62** (1986), 186–212.

[18] Giga, Y. and T. Miyakawa, Solutions in $L^r$ of the Navier-Stokes initial value problem, Arch. Rat. Mech. Anal. **89** (1985), 267–281.

[19] Kalton, N., Mayboroda, S. and M. Mitrea, *Interpolation of Hardy-Sobolev-Besov-Triebel-Lizorkin Spaces and Applications to Problems in Partial Differential Equations. Interpolation Theory and Applications*, Contemp. Math., **445**, Amer. Math. Soc., Providence, RI, 2007, 121–177.

[20] Kato, T. and H. Fujita, On the non-stationary Navier-Stokes system, Rend. Sem. Mat. Univ. Padova **32** (1962), 243–260.

[21] Kato, T., Strong $L^p$-solutions of the Navier-Stokes equation in $R^m$, with applications to weak solutions, Math. Z. **187** (4) (1984), 471–480.

[22] Kato, T., Strong solutions of the Navier-Stokes equations in Morrey spaces, Bol. Soc. Brasil. Math. **22** (1992), 127–155.

[23] Kato, T. and G. Ponce, The Navier-Stokes equation with weak initial data, Int. Math. Res. Notes **10** (1994), 435–444.

[24] Kozono, H. and M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, Comm. Partial Differential Equations **19** (1994), 959–1014.

[25] Khai, D. Q. and N. M. Tri, Solutions in mixed-norm Sobolev-Lorentz spaces to the initial value problem for the Navier-Stokes equations, Journal of Mathematical Analysis and Applications **417** (2014), 819–833.

[26] Khai, D. Q. and N. M. Tri, Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces, Preprint arXiv:1601.01441.

[27] Khai, D. Q. and N. M. Tri, Well-posedness for the Navier-Stokes equations with data f in homogeneous Sobolev-Lorentz spaces, Preprint arXiv: 1601.01742.

[28] Koch, H. and D. Tataru, Well-posedness for the Navier-Stokes equations, Adv. Math. **157** (2001), 22–35.

[29] Lemarie-Rieusset, P. G., *Recent Developments in the Navier-Stokes Problem*, Chapman and Hall/CRC Research Notes in Mathematics, vol. 431, Chapman and Hall/CRC, Boca Raton, FL, 2002.

[30] Miao, C. and B. Yuan, On the well-posedness of the Cauchy problem for an MHD system in Besov spaces, Math. Methods Appl. Sci. **32** (2009), 53–76.

[31] Planchon, F., *Solutions Globales et Comportement Asymptotique pour les Equations de Navier-Stokes*, Doctoral Thesis, Ecole Polytechnique, France (1996).

[32] Peetre, J., *New Thoughts on Besov Spaces*, Duke Univ. Math. Series, 1976.

[33] Taylor, M. E., Analysis on Morrey spaces and applications to Navier-Stokes equations and other evolution equations, Comm. Partial Differential Equations **17** (1992), 1407–1456.

[34] Weissler, F. B., The Navier-Stokes initial value problem in $L^p$, Arch. Rat. Mech. Anal. **74** (1981), 219–230.

(Received October 27, 2014)
(Revised October 23, 2015)

Institute of Mathematics
Vietnam Academy of Science and Technology
18 Hoang Quoc Viet, 10307 Cau Giay
Hanoi, Vietnam
E-mail: triminh@math.ac.vn
Khaitoantin@gmail.com