J. Math. Sci. Univ. Tokyo
23 (2016), 487–497.

A Uniform Boundedness Result for Solutions to the Liouville Type Equation with Boundary Singularity

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Abstract. We give blow-up behavior of a sequence of solutions of a Liouville-type problem with a singular weight and Dirichlet boundary conditions. As an application we derive a compactness criterion in the same spirit of the well known Brezis-Merle's result.

1. Introduction and Main Results

We set $\Delta = \partial_{11} + \partial_{22}$ on open set Ω of $\mathbb{R}^2$ with a smooth boundary. We consider the following equation:

$$(P) \begin{cases} -\Delta u = |x|^{-2\alpha} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{in } \partial\Omega. \end{cases}$$

Here we assume that

$$\alpha \in (0, 1/2),\ 0 \in \partial\Omega.$$

The above equation was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1-15], where one can find some existence and compactness results.

Among other results, we can see in [10] the following important Theorem. (Since $0 \in \partial\Omega$, all the conditions of this theorem are satisfied in our setting).

THEOREM (Brezis-Merle [10]). *If (u_i) is a sequence of solutions of problem (P) with (V_i) satisfying $0 < a \le V_i \le b < +\infty$, then, for any compact subset K of Ω, it holds*

$$\sup_K u_i \le c$$

2010 *Mathematics Subject Classification*. 35J60, 35B44, 35B45.
Key words: Blow-up, boundary, singularity, a priori estimate, Lipschitz condition.

with c depending on a, b, α, K, Ω.

If we assume that V is more regular, we can have another type of estimates called a $\sup + \inf$ type inequalities. It was proved by Shafrir see [15], that, if $(u_i)_i$ is a sequence of functions solutions of the previous equation without assumption on the boundary with V_i satisfying $0 < a \leq V_i \leq b < +\infty$, then it holds

$$C\left(\frac{a}{b}\right) \sup_K u_i + \inf_\Omega u_i \leq c,$$

where c is a constant depending on a, b, K, Ω.

Now, if we assume that $(V_i)_i$ is uniformly Lipschitzian with its Lipschitz constant A then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$; see [9].

We find in [4-7], estimates of type $\sup + \inf$ for Liouville type equation with singular weight.

In this paper we give a blow-up analysis for sequences of solutions of a Liouville-type problem with singular weight.

We have the following problem for the Liouville equation with singular weight (as in [10, Problem 1]).

Problem. Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i \leq b$ for some positive constant b. Also, we consider a sequence of solutions (u_i) of (P) relative to (V_i) such that

$$\int_\Omega |x|^{-2\alpha} e^{u_i} dx \leq C.$$

Is it possible to have

$$||u_i||_{L^\infty} \leq C = C(b, \alpha, \Omega, C)?$$

In this paper we derive a uniform boundedness result for the solutions to an elliptic equation with exponential nonlinearity when the prescribed curvature is uniformly Lipschitzian. For a regular case $\alpha = 0$ one can find in [3] a result close to the result of the present paper.

For the blow-up analysis, the following condition is sufficient.

$$0 \leq V_i \leq b.$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary.

But for the uniform boundedness result, we assume that

$$||\nabla V_i||_{L^\infty} \leq A.$$

We have

THEOREM 1.1. *Assume* $\alpha \in (0, 1/2)$ *and* $\max_\Omega u_i \to +\infty$, *where* (u_i) *are solutions of the problem* (P) *with*

$$0 \leq V_i \leq b, \text{ and } \int_\Omega |x|^{-2\alpha} e^{u_i} dx \leq C, \textit{ for all } i \in \mathbb{N}.$$

Then, after passing to a subsequence, there are a function u, *a number* $N \in \mathbb{N}$ *and* N *points* $x_1 = 0, x_2, \dots, x_N \in \partial\Omega - \{0\}$, *such that*

$$\int_{\partial\Omega} \partial_\nu u_i \phi \to \int_{\partial\Omega} \partial_\nu u \phi + \sum_{j=1}^N \alpha_j \phi(x_j), \ \alpha_1 \geq 4\pi(1-\alpha), \alpha_j \geq 4\pi.$$

for any $\phi \in C^0(\partial\Omega)$, *and*

$$u_i \to u \text{ in } C^1_{loc}(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

or, $x_1, x_2, \dots, x_N \in \partial\Omega - \{0\}$, *and*

$$\int_{\partial\Omega} \partial_\nu u_i \phi \to \int_{\partial\Omega} \partial_\nu u \phi + \sum_{j=1}^N \alpha_j \phi(x_j), \text{ with } \alpha_j \geq 4\pi.$$

for any $\phi \in C^0(\partial\Omega)$, *and*

$$u_i \to u \text{ in } C^1_{loc}(\bar{\Omega} - \{x_1, \dots, x_N\}).$$

In the following theorem, we have a compactness result which concerns the problem (P).

THEOREM 1.2. *Assume that* (u_i) *are solutions of* (P) *relative to* (V_i) *with the following conditions:*

$$\alpha \in (0, 1/2), \ 0 \in \partial\Omega,$$

$$0 \leq V_i \leq b, \ ||\nabla V_i||_{L^\infty} \leq A, \text{ and } \int_\Omega |x|^{-2\alpha} e^{u_i} \leq C.$$

We have

$$||u_i||_{L^\infty} \leq c(b, \alpha, A, C, \Omega).$$

2. Proof of the Theorems

PROOF OF THEOREM 1.1.

Since $\int_\Omega |x|^{-2\alpha} e^{u_i} \leq C$, we have, by the Brezis-Merle result see [10], $e^{ku_i} \in L^1(\Omega), k > 2$ and because $\alpha \in (0, 1/2)$ the elliptic estimates imply that

$$u_i \in W^{2,p}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).$$

We denote by $\partial_\nu u_i$ the inner normal derivative of u_i. By the maximum principle, $\partial_\nu u_i \geq 0$.

By the Stokes formula, we obtain

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C.$$

Thus, (using the weak convergence in the space of Radon measures), we have the existence of a positive Radon measure μ such that

$$\int_{\partial\Omega} (\partial_\nu u_i)\phi d\sigma \to \mu(\phi), \ \forall \ \phi \in C^0(\partial\Omega).$$

Let us consider a point $x_0 \in \partial\Omega$. We say that x_0 is regular if, $x_0 \neq 0$ and $\mu(\{x_0\}) < 4\pi$, or $x_0 = 0$ and $\mu(\{0\}) < 4\pi(1-\alpha)$. A point $x_0 \in \partial\Omega$ is a nonregular point, if the previous conditions are not satisfied.

For a regular point $x_0 \in \partial\Omega$, we may assume that the following curve, $B(x_0, \epsilon) \cap \partial\Omega := I_\epsilon$ is an interval. (In this case, it is simpler to construct the following test function η_ϵ).

Case 1. $\mu(\{0\}) \geq 4\pi(1-\alpha)$.

This means that 0 is a nonregular point for the measure μ.

Let us consider the following function

$$\begin{cases} \eta_\epsilon \equiv 1, \ \text{on} \ I_\epsilon, \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \ \text{outside} \ I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ ||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \dfrac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde\eta_\epsilon$ such that

$$\begin{cases} -\Delta\tilde\eta_\epsilon = 0 & \text{in } \Omega, \\ \tilde\eta_\epsilon = \eta_\epsilon & \text{on } \partial\Omega. \end{cases}$$

We use the following estimate, see [8, 11],

$$||\nabla u_i||_{L^q} \leq C_q, \ \forall \ i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_\Omega e^u < +\infty$ (by Fatou lemma). Also, V_i converge $*$-weakly in L^∞ to a nonnegative function V. The function u is in $W_0^{1,q}(\Omega)$ solution of :

$$\begin{cases} -\Delta u = |x|^{-2\alpha} V e^u \in L^1(\Omega) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

As in the corollary 1 of Brezis-Merle result, see [10], we have $e^{ku} \in L^1(\Omega), k > 2$. We have $\alpha \in (0, 1/2)$, by the elliptic estimates, $u \in W^{2,p}(\Omega) \cap C^{1,\epsilon}(\bar\Omega)$.

We can write

$$(1) \qquad -\Delta((u_i - u)\tilde\eta_\epsilon) = |x|^{-2\alpha}(V_i e^{u_i} - V e^u)\tilde\eta_\epsilon - 2\nabla(u_i - u) \cdot \nabla\tilde\eta_\epsilon.$$

We use the interior esimate of Brezis-Merle, see [10],

Step 1. Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde\eta_\epsilon$ and u to obtain

$$(2) \qquad \int_\Omega |x|^{-2\alpha} V e^u \tilde\eta_\epsilon dx = \int_{\partial\Omega} (\partial_\nu u)\eta_\epsilon \leq 4\epsilon ||\partial_\nu u||_{L^\infty} = C\epsilon.$$

We have

$$\begin{cases} -\Delta u_i = |x|^{-2\alpha} V_i e^{u_i} & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde\eta_\epsilon$ to have

$$(3) \qquad \int_\Omega |x|^{-2\alpha} V_i e^{u_i} \tilde\eta_\epsilon dx = \int_{\partial\Omega} (\partial_\nu u_i)\eta_\epsilon d\sigma \to \mu(\eta_\epsilon) \leq \mu(I_{2\epsilon}) \leq 4\pi - \epsilon_0, \ \ \epsilon_0 > 0.$$

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$(4) \qquad \int_\Omega |x|^{-2\alpha} |(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon.$$

Step 2. Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon^3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^3}$ is $k_2\epsilon^3 \leq \mu_L(\Omega - \Omega_{\epsilon^3}) \leq k_1\epsilon^3$.

REMARK. For the unit ball $\bar{B}(0,1)$, our new manifold is $\bar{B}(0, 1-\epsilon^3)$.

We write

$$(5) \qquad \int_\Omega |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx + \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx.$$

Step 2.1. Estimate of $\int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx$.

First, we know from the elliptic estimates that $||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1/\epsilon^2$, C_1 depends on Ω.

We know that $(|\nabla u_i|)_i$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q.

We have for any $f \in L^{q'}(\Omega)$

$$\int_\Omega |\nabla u_i| f dx \to \int_\Omega |\nabla u| f dx.$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^3}}$, we have

$$\text{for } \epsilon > 0 \; \exists \; i_1 = i_1(\epsilon) \in \mathbb{N}, \;\; i \geq i_1, \; \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^3}} |\nabla u| + \epsilon^3.$$

Then, for $i \geq i_1(\epsilon)$

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla u_i| \leq mes(\Omega - \Omega_{\epsilon^3})||\nabla u||_{L^\infty} + \epsilon^3 = C\epsilon.$$

Thus, we obtain

$$\int_{\Omega-\Omega_{\epsilon^3}} |\nabla(u_i - u)\cdot\nabla\tilde{\eta}_\epsilon|dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 1). \tag{6}$$

The constant C_1 does not depend on ϵ but on Ω.

Step 2.2. Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u)\cdot\nabla\tilde{\eta}_\epsilon|dx$.

We know that $\Omega_\epsilon \subset\subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \to u$ in $C^1(\Omega_{\epsilon^3})$. We have

$$||\nabla(u_i - u)||_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).$$

We write

$$\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u)\cdot\nabla\tilde{\eta}_\epsilon|dx \leq ||\nabla(u_i - u)||_{L^\infty(\Omega_{\epsilon^3})}||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1\epsilon \text{ for } i \geq i_3.$$

For $\epsilon > 0$ and for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$\int_{\Omega} |\nabla(u_i - u)\cdot\nabla\tilde{\eta}_\epsilon|dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2). \tag{7}$$

From (4) and (7) we have, for $\epsilon > 0$ there is $i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max\{i_0, i_1, i_2\}$ such that

$$\int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]|dx \leq 4\pi - \epsilon_0 + \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2 + C). \tag{8}$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have

$$\begin{cases} -\Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} & \text{in } \Omega, \\ (u_i - u)\tilde{\eta}_\epsilon = 0 & \text{on } \partial\Omega \end{cases}$$

with $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2$.

We can use Theorem 1 of [10] to conclude that there are $q \geq \tilde{q} > 1$ such that

$$\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i-u|} dx \leq \int_\Omega e^{q|u_i-u|\tilde{\eta}_\epsilon} dx \leq C(\epsilon, \Omega),$$

where $V_\epsilon(x_0)$ is a neighberhood of x_0 in $\bar{\Omega}$. Here we used the fact that in a neighborhood of x_0, we have for some $C > 0$, $1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1$, by the elliptic estimates.

Thus, for each $x_0 \in \partial\Omega - \{\bar{x}_1, \dots, \bar{x}_m\}$ there is $\epsilon_{x_0} > 0, q_{x_0} > 1$ such that

$$\int_{B(x_0,\epsilon_{x_0})} e^{q_{x_0} u_i} dx \leq C, \ \ \forall \ i.$$

Now, we consider a cutoff function $\eta \in C^\infty(\mathbb{R}^2)$ such that

$$\eta \equiv 1 \ \text{ on } \ B(x_0, \epsilon_{x_0}/2) \ \text{ and } \ \eta \equiv 0 \ \text{ on } \ \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).$$

We write

$$\Delta(u_i \eta) = |x|^{-2\alpha} V_i e^{u_i} \eta - 2\nabla u_i \cdot \nabla \eta + u_i \Delta \eta.$$

By the elliptic estimates, $(u_i\eta)_i$ is uniformly bounded in $W^{2,q_1}(\Omega)$ and also, in $C^1(\bar{\Omega})$.

Finally, we have, for some $\epsilon > 0$ small enough,

$$||u_i||_{C^{1,\theta}[B(x_0,\epsilon)]} \leq c_3 \ \ \forall \ i. \tag{9}$$

We have proved that, there is a finite number of points $\bar{x}_1, \dots, \bar{x}_m$ such that the squence $(u_i)_i$ is locally uniformly bounded in $\bar{\Omega} - \{\bar{x}_1, \dots, \bar{x}_m\}$.

Case 2. $\mu(\{0\}) < 4\pi(1-\alpha)$.

This means that 0 is a regular point for the measure μ.

Let us consider $B_\epsilon(0)$, a ball of center 0 and radius $\epsilon > 0$. As in the previous case, we use the uniform estimate in $W_0^{1,q}(\Omega), (1 \leq q < 2)$ and Brezis-Merle's method, see [10], to have

$$e^{u_i} \in L^{(1-\epsilon')/(1-\alpha-\epsilon')}(B_\epsilon(0)).$$

with a uniform bound.

Thus, by the Hölder inequality we have

$$u_i \in L^\infty(B_\epsilon(0)).$$

If we take $\mu(\{0\}) < 4\pi$, by the Brezis-Merle estimate we have $e^{u_i} \in L^r(B_\epsilon(0))$ with $r > 1$, but this r may not be large enough to ensure $u_i \in L^\infty(B_\epsilon(0))$, because we have the term $|x|^{-2\alpha}$ in the equation.

Then, by the elliptic estimates, for $\alpha \in (0,1)$

$$(10) \quad u_i \in W^{2,1+\epsilon'}(B_\epsilon(0)) \cap C^{0,\epsilon'}(B_\epsilon(0)) \cap C^{2,\epsilon'}_{loc}(\Omega - \{0, x_1, x_2, \dots, x_N\}),$$

and, for $\alpha \in (0, 1/2)$, we have

$$(11) \quad u_i \in W^{2,1+\epsilon'}(B_\epsilon(0)) \cap C^{1,\epsilon'}(B_\epsilon(0)) \cap C^{2,\epsilon'}_{loc}(\Omega - \{0, x_1, x_2, \dots, x_N\}).$$

And thus, we have

$$(12) \qquad \partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^{N} \alpha_j \delta_{x_j},$$

$$\alpha_j \geq 4\pi \text{ weakly in the sense of measures on } \partial\Omega.$$

As explained in the first step, if we consider a neighborhood of a regular point $x_0 \neq 0$, we are far from the singularity and the scheme of the first step work in this case; we have the uniform convergence of $\partial_\nu u_i$ around x_0. In the case $\alpha \in (0, 1/2)$, the elliptic estimates gives us the C^1 convergence of u_i. □

Proof of Theorem 1.2. Without loss of generality, we may assume that 0 is a blow-up point (if 0 is not a blow-up point we are in the regular case). Also, by a conformal transformation, we can assume that $\Omega = B_1^+$, the half ball, and $\partial^+ B_1^+$ is the exterior part, a part which does not contain 0 and on which u_i converges in the C^1 norm to u. Let us consider B_ϵ^+ the half ball with radius $\epsilon > 0$.

In order to apply the Pohozaev identity, we need a good function u_i.The fact that $\alpha \in (0, 1/2)$ implies that

$$u_i \in W^{2,p} \cap C^1(\bar{\Omega}).$$

Thus

$$\partial_j u_i \in W^{1,p} \cap C^0(\bar{\Omega}).$$

Thus

$$\partial_j u_i . \partial_k u_i \in W^{1,p} \cap C^0(\bar{\Omega}).$$

Thus we can do integration by parts.

The Pohozaev identity gives

$$2(1-\alpha)\int_{B_\epsilon^+}|x|^{-2\alpha}V_ie^{u_i}dx+\int_{B_\epsilon^+}x\cdot\nabla V_i|x|^{-2\alpha}e^{u_i}dx = \int_{\partial^+B_\epsilon^+}g(\nabla u_i)d\sigma+\int_{\partial B_\epsilon^+}x\cdot\nu V_ie^{u_i}, \tag{13}$$

$$2(1-\alpha)\int_{B_\epsilon^+}|x|^{-2\alpha}Ve^{u}dx+\int_{B_\epsilon^+}x\cdot\nabla V|x|^{-2\alpha}e^{u}dx = \int_{\partial^+B_\epsilon^+}g(\nabla u)d\sigma+\int_{\partial B_\epsilon^+}x\cdot\nu Ve^{u} \tag{14}$$

Here $g(\nabla u_i)$ means a quantity which depends on ∇u_i and for which we have a uniform convergence to $g(\nabla u)$. (On $\partial^+B_\epsilon^+$). In fact we have:

$$g(\nabla u_i)=(\nu\cdot\nabla u_i)(x\cdot\nabla u_i)-x\cdot\nu\frac{|\nabla u_i|^2}{2}.$$

We use again the fact that $u_i=u=0$ on $\{x_1=0\}$ to obtain

$$\begin{aligned}2(1-\alpha)\int_{B_\epsilon^+}|x|^{-2\alpha}V_ie^{u_i}dx-2(1-\alpha)\int_{B_\epsilon^+}|x|^{-2\alpha}Ve^{u}dx\\+\int_{B_\epsilon^+}x\cdot\nabla V_i|x|^{-2\alpha}e^{u_i}dx-\int_{B_\epsilon^+}x\cdot\nabla V|x|^{-2\alpha}e^{u}dx\\=\int_{\partial^+B_\epsilon^+}g(\nabla u_i)-g(\nabla u)d\sigma+o(1)=o(1).\end{aligned}$$

First, we tend i to infinity and then ϵ tend to 0. We obtain

$$\lim_{\epsilon\to0}\lim_{i\to+\infty}2(1-\alpha)\int_{B_\epsilon^+}|x|^{-2\alpha}V_ie^{u_i}dx=0. \tag{15}$$

However,

$$\int_{B_\epsilon^+}|x|^{-2\alpha}V_ie^{u_i}dx=\int_{\partial B_\epsilon^+}\partial_\nu u_i+o(\epsilon)+o(1)\to\alpha_1>0,$$

which is a contradiction. □

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(Received March 18, 2014)
(Revised September 3, 2015)

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