A Uniform Boundedness Result for Solutions to the Liouville Type Equation with Boundary Singularity

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Abstract. We give blow-up behavior of a sequence of solutions of a Liouville-type problem with a singular weight and Dirichlet boundary conditions. As an application we derive a compactness criterion in the same spirit of the well known Brezis-Merle’s result.

1. Introduction and Main Results

We set \( \Delta = \partial_{11} + \partial_{22} \) on open set \( \Omega \) of \( \mathbb{R}^2 \) with a smooth boundary. We consider the following equation:

\[
(P) \begin{cases} 
-\Delta u = |x|^{-2\alpha} V e^u & \text{in } \Omega \subset \mathbb{R}^2, \\
u = 0 & \text{in } \partial \Omega.
\end{cases}
\]

Here we assume that

\[ \alpha \in (0, 1/2), \ 0 \in \partial \Omega. \]

The above equation was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1-15], where one can find some existence and compactness results.

Among other results, we can see in [10] the following important Theorem. (Since \( 0 \in \partial \Omega \), all the conditions of this theorem are satisfied in our setting).

THEOREM (Brezis-Merle [10]). If \( (u_i) \) is a sequence of solutions of problem \( (P) \) with \( (V_i) \) satisfying \( 0 < a \leq V_i \leq b < +\infty \), then, for any compact subset \( K \) of \( \Omega \), it holds

\[ \sup_K u_i \leq c \]
with $c$ depending on $a, b, \alpha, K, \Omega$.

If we assume that $V$ is more regular, we can have another type of estimates called a sup $+$ inf type inequalities. It was proved by Shafrir see [15], that, if $(u_i)_i$ is a sequence of functions solutions of the previous equation without assumption on the boundary with $V_i$ satisfying $0 < a \leq V_i \leq b < +\infty$, then it holds

$$C \left( \frac{a}{b} \right) \sup_{K} u_i + \inf_{\Omega} u_i \leq c,$$

where $c$ is a constant depending on $a, b, K, \Omega$.

Now, if we assume that $(V_i)_i$ is uniformly Lipschitzian with its Lipschitz constant $A$ then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$; see [9].

We find in [4-7], estimates of type sup $+$ inf for Liouville type equation with singular weight.

In this paper we give a blow-up analysis for sequences of solutions of a Liouville-type problem with singular weight.

We have the following problem for the Liouville equation with singular weight (as in [10, Problem 1]).

**Problem.** Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$ with $0 \leq V_i \leq b$ for some positive constant $b$. Also, we consider a sequence of solutions $(u_i)$ of $(P)$ relative to $(V_i)$ such that

$$\int_{\Omega} |x|^{-2\alpha} e^{u_i} dx \leq C.$$

Is it possible to have

$$\|u_i\|_{L^\infty} \leq C = C(b, \alpha, \Omega, C)?$$

In this paper we derive a uniform boundedness result for the solutions to an elliptic equation with exponential nonlinearity when the prescribed curvature is uniformly Lipschitzian. For a regular case $\alpha = 0$ one can find in [3] a result close to the result of the present paper.

For the blow-up analysis, the following condition is sufficient.

$$0 \leq V_i \leq b.$$ 

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary.
But for the uniform boundedness result, we assume that 
\[ \| \nabla V_i \|_{L^\infty} \leq A. \]

We have

**Theorem 1.1.** Assume \( \alpha \in (0, 1/2) \) and \( \max_\Omega u_i \to +\infty \), where \((u_i)\) are solutions of the problem \((P)\) with

\[
0 \leq V_i \leq b, \quad \text{and} \quad \int_\Omega |x|^{-2\alpha} e^{u_i} dx \leq C, \text{ for all } i \in \mathbb{N}.
\]

Then, after passing to a subsequence, there are a function \( u \), a number \( N \in \mathbb{N} \) and \( N \) points \( x_1 = 0, x_2, \ldots, x_N \in \partial \Omega - \{0\} \), such that

\[
\int_{\partial \Omega} \partial_\nu u_i \phi \to \int_{\partial \Omega} \partial_\nu u \phi + \sum_{j=1}^N \alpha_j \phi(x_j), \quad \alpha_1 \geq 4\pi(1-\alpha), \alpha_j \geq 4\pi.
\]

for any \( \phi \in C^0(\partial \Omega) \), and

\[
u_i \to u \text{ in } C^1_{\text{loc}}(\bar{\Omega} - \{x_1, \ldots, x_N\}).
\]

or, \( x_1, x_2, \ldots, x_N \in \partial \Omega - \{0\} \), and

\[
\int_{\partial \Omega} \partial_\nu u_i \phi \to \int_{\partial \Omega} \partial_\nu u \phi + \sum_{j=1}^N \alpha_j \phi(x_j), \text{ with } \alpha_j \geq 4\pi.
\]

for any \( \phi \in C^0(\partial \Omega) \), and

\[
u_i \to u \text{ in } C^1_{\text{loc}}(\bar{\Omega} - \{x_1, \ldots, x_N\}).
\]

In the following theorem, we have a compactness result which concerns the problem \((P)\).

**Theorem 1.2.** Assume that \((u_i)\) are solutions of \((P)\) relative to \((V_i)\) with the following conditions:

\[
\alpha \in (0, 1/2), \quad 0 \in \partial \Omega,
\]

\[
0 \leq V_i \leq b, \quad \| \nabla V_i \|_{L^\infty} \leq A, \quad \text{and} \quad \int_\Omega |x|^{-2\alpha} e^{u_i} \leq C.
\]

We have

\[
\| u_i \|_{L^\infty} \leq c(b, \alpha, A, C, \Omega).
\]
2. Proof of the Theorems

Proof of Theorem 1.1.

Since \( \int_{\Omega} |x|^{-2\alpha} e^{u_i} \leq C \), we have, by the Brezis-Merle result see [10],
\( e^{ku_i} \in L^1(\Omega) \), \( k > 2 \) and because \( \alpha \in (0, 1/2) \) the elliptic estimates imply that
\[
    u_i \in W^{2,p}(\Omega) \cap C^{1,\epsilon}(\bar{\Omega}).
\]

We denote by \( \partial_{\nu} u_i \) the inner normal derivative of \( u_i \). By the maximum principle, \( \partial_{\nu} u_i \geq 0 \).

By the Stokes formula, we obtain
\[
    \int_{\partial\Omega} \partial_{\nu} u_i d\sigma \leq C.
\]

Thus, (using the weak convergence in the space of Radon measures), we have the existence of a positive Radon measure \( \mu \) such that
\[
    \int_{\partial\Omega} (\partial_{\nu} u_i) \phi d\sigma \to \mu(\phi), \quad \forall \phi \in C^0(\partial\Omega).
\]

Let us consider a point \( x_0 \in \partial\Omega \). We say that \( x_0 \) is regular if, \( x_0 \neq 0 \) and \( \mu(\{x_0\}) < 4\pi(1 - \alpha) \). A point \( x_0 \in \partial\Omega \) is a nonregular point, if the previous conditions are not satisfied.

For a regular point \( x_0 \in \partial\Omega \), we may assume that the following curve, \( B(x_0, \epsilon) \cap \partial\Omega := I_\epsilon \) is an interval. (In this case, it is simpler to construct the following test function \( \eta_\epsilon \).

\textit{Case 1.} \( \mu(\{0\}) \geq 4\pi(1 - \alpha) \).

This means that 0 is a nonregular point for the measure \( \mu \).

Let us consider the following function

\[
    \eta_\epsilon \equiv 1, \quad \text{on} \quad I_\epsilon, \quad 0 < \epsilon < \delta/2, \\
    \eta_\epsilon \equiv 0, \quad \text{outside} \quad I_{2\epsilon}, \\
    0 \leq \eta_\epsilon \leq 1, \\
    ||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \frac{C_0(\Omega, x_0)}{\epsilon}.
\]
We take a \( \tilde{\eta}_\epsilon \) such that
\[
\begin{aligned}
-\Delta \tilde{\eta}_\epsilon &= 0 \quad \text{in } \Omega, \\
\tilde{\eta}_\epsilon &= \eta_\epsilon \quad \text{on } \partial \Omega.
\end{aligned}
\]

We use the following estimate, see [8, 11],
\[
\|\nabla u_i\|_{L^q} \leq C_q, \quad \forall \ i \text{ and } 1 < q < 2.
\]

We deduce from the last estimate that \((u_i)\) converge weakly in \( W^{1,q}_0(\Omega) \), almost everywhere to a function \( u \geq 0 \) and \( \int_\Omega e^u < +\infty \) (by Fatou lemma). Also, \( V_i \) converge \( * \)-weakly in \( L^\infty \) to a nonnegative function \( V \). The function \( u \) is in \( W^{1,q}_0(\Omega) \) solution of:
\[
\begin{aligned}
-\Delta u &= |x|^{-2\alpha}V e^u \in L^1(\Omega) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

As in the corollary 1 of Brezis-Merle result, see [10], we have \( e^{ku} \in L^1(\Omega), k > 2 \). We have \( \alpha \in (0, 1/2) \), by the elliptic estimates, \( u \in W^{2,p}(\Omega) \cap C^{1,\epsilon}_{\text{loc}}(\bar{\Omega}) \).

We can write
\[
-\Delta((u_i - u)\tilde{\eta}_\epsilon) = |x|^{-2\alpha}(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2\nabla (u_i - u) \cdot \nabla \tilde{\eta}_\epsilon.
\]

We use the interior estimate of Brezis-Merle, see [10],

**Step 1.** Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between \( \tilde{\eta}_\epsilon \) and \( u \) to obtain
\[
\int_\Omega |x|^{-2\alpha}V e^u \tilde{\eta}_\epsilon dx = \int_{\partial \Omega} (\partial_\nu u)\eta_\epsilon \leq 4\epsilon\|\partial_\nu u\|_{L^\infty} = C\epsilon.
\]

We have
\[
\begin{aligned}
-\Delta u_i &= |x|^{-2\alpha}V_i e^{u_i} \quad \text{in } \Omega, \\
u_i &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

We use the Green formula between \( u_i \) and \( \tilde{\eta}_\epsilon \) to have
\[
\int_\Omega |x|^{-2\alpha}V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial \Omega} (\partial_\nu u_i)\eta_\epsilon d\sigma \rightarrow \mu(\eta_\epsilon)
\]
\[
\leq \mu(I_{2\epsilon}) \leq 4\pi - \epsilon_0, \quad \epsilon_0 > 0.
\]
From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,
\[ (4) \quad \int_{\Omega} |x|^{-2\alpha} |(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon. \]

**Step 2.** Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^3\}$ and $\Omega_{\epsilon, 3} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^3\}$, $\epsilon > 0$. Then, for $\epsilon$ small enough, $\Sigma_\epsilon$ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon, 3}$ is $k_2 \epsilon^3 \leq \mu_L(\Omega - \Omega_{\epsilon, 3}) \leq k_1 \epsilon^3$.

**Remark.** For the unit ball $\bar{B}(0, 1)$, our new manifold is $\bar{B}(0, 1 - \epsilon^3)$.

We write
\[ (5) \quad \int_{\Omega} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx = \int_{\Omega_{\epsilon, 3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx 
+ \int_{\Omega - \Omega_{\epsilon, 3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx. \]

**Step 2.1.** Estimate of $\int_{\Omega - \Omega_{\epsilon, 3}} |\nabla(u_i - u) \cdot \nabla\tilde{\eta}_\epsilon| dx$.

First, we know from the elliptic estimates that $||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1/\epsilon^2$, $C_1$ depends on $\Omega$.

We know that $|\nabla u_i|$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle theorem), then, $h = |\nabla u|$ a.e. Let $q'$ be the conjugate of $q$.

We have for any $f \in L^{q'}(\Omega)$
\[ \int_{\Omega} |\nabla u_i| fdx \rightarrow \int_{\Omega} |\nabla u| fdx. \]

If we take $f = 1_{\Omega - \Omega_{\epsilon, 3}}$, we have

for $\epsilon > 0 \exists i_1 = i_1(\epsilon) \in \mathbb{N}, \ i \geq i_1, \ \int_{\Omega - \Omega_{\epsilon, 3}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon, 3}} |\nabla u| + \epsilon^3$. 
Then, for $i \geq i_1(\epsilon)$

$$
\int_{\Omega - \Omega_{\epsilon^3}} |\nabla u_i| \leq mes(\Omega - \Omega_{\epsilon^3})||\nabla u||_{L^\infty} + \epsilon^3 = C\epsilon.
$$

Thus, we obtain

$$
(6) \quad \int_{\Omega - \Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 1).
$$

The constant $C_1$ does not depend on $\epsilon$ but on $\Omega$.

**Step 2.2. Estimate of $\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx$.**

We know that $\Omega_{\epsilon^3} \subset \subset \Omega$, and (because of Brezis-Merle’s interior estimates) $u_i \rightarrow u$ in $C^1(\Omega_{\epsilon^3})$. We have

$$
||\nabla(u_i - u)||_{L^\infty(\Omega_{\epsilon^3})} \leq \epsilon^3, \text{ for } i \geq i_3 = i_3(\epsilon).
$$

We write

$$
\int_{\Omega_{\epsilon^3}} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq ||\nabla(u_i - u)||_{L^\infty(\Omega_{\epsilon^3})}||\nabla \tilde{\eta}_\epsilon||_{L^\infty} \leq C_1\epsilon \text{ for } i \geq i_3.
$$

For $\epsilon > 0$ and for $i \in \mathbb{N}$, $i \geq \max\{i_1, i_2, i_3\}$,

$$
(7) \quad \int_{\Omega} |\nabla(u_i - u) \cdot \nabla \tilde{\eta}_\epsilon| dx \leq \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2).
$$

From (4) and (7) we have, for $\epsilon > 0$ there is $i_3 = i_3(\epsilon) \in \mathbb{N}, i_3 = \max\{i_0, i_1, i_2\}$ such that

$$
(8) \quad \int_{\Omega} |\Delta[(u_i - u)\tilde{\eta}_\epsilon]| dx \leq 4\pi - \epsilon_0 + \epsilon C_1(2k_1||\nabla u||_{L^\infty} + 2 + C).
$$

We choose $\epsilon > 0$ small enough to have a good estimate of (1).

Indeed, we have

$$
\begin{aligned}
\left\{ \\
-\Delta[(u_i - u)\tilde{\eta}_\epsilon] = g_{i,\epsilon} \text{ in } \Omega, \\
(u_i - u)\tilde{\eta}_\epsilon = 0 \text{ on } \partial\Omega
\end{aligned}
$$

with $||g_{i,\epsilon}||_{L^1(\Omega)} \leq 4\pi - \epsilon_0/2.$
We can use Theorem 1 of [10] to conclude that there are \( q \geq \tilde{q} > 1 \) such that
\[
\int_{V_\epsilon(x_0)} e^{\tilde{q}|u_i-u|} dx \leq \int_{\Omega} e^{q|u_i-u|\tilde{\eta}} dx \leq C(\epsilon, \Omega),
\]
where \( V_\epsilon(x_0) \) is a neighborhood of \( x_0 \) in \( \bar{\Omega} \). Here we used the fact that in a neighborhood of \( x_0 \), we have for some \( C > 0, 1 - C\epsilon \leq \tilde{\eta}_\epsilon \leq 1 \), by the elliptic estimates.

Thus, for each \( x_0 \in \partial\Omega - \{\bar{x}_1, \ldots, \bar{x}_m\} \) there is \( \epsilon_{x_0} > 0, q_{x_0} > 1 \) such that
\[
\int_{B(x_0, \epsilon_{x_0})} e^{q_{x_0}u_{i}} dx \leq C, \quad \forall \ i.
\]
Now, we consider a cutoff function \( \eta \in C^\infty(\mathbb{R}^2) \) such that
\[
\eta \equiv 1 \text{ on } B(x_0, \epsilon_{x_0}/2) \text{ and } \eta \equiv 0 \text{ on } \mathbb{R}^2 - B(x_0, 2\epsilon_{x_0}/3).
\]
We write
\[
\Delta (u_i\eta) = |x|^{-2\alpha} V_i e^{u_i\eta} - 2\nabla u_i \cdot \nabla \eta + u_i \Delta \eta.
\]
By the elliptic estimates, \((u_i\eta)_i\) is uniformly bounded in \( W^{2,q_1}(\Omega) \) and also, in \( C^1(\bar{\Omega}) \).

Finally, we have, for some \( \epsilon > 0 \) small enough,
\[
\|u_i\|_{C^{1,q}[B(x_0, \epsilon)]} \leq c_3 \quad \forall \ i.
\]
We have proved that, there is a finite number of points \( \bar{x}_1, \ldots, \bar{x}_m \) such that the sequence \((u_i)_i\) is locally uniformly bounded in \( \bar{\Omega} - \{\bar{x}_1, \ldots, \bar{x}_m\} \).

**Case 2.** \( \mu(\{0\}) < 4\pi(1 - \alpha) \).

This means that 0 is a regular point for the measure \( \mu \).

Let us consider \( B_\epsilon(0) \), a ball of center 0 and radius \( \epsilon > 0 \). As in the previous case, we use the uniform estimate in \( W^{1,q}_0(\Omega), (1 \leq q < 2) \) and Brezis-Merle’s method, see [10], to have
\[
e^{u_i} \in L^{(1-\epsilon')/(1-\alpha-\epsilon')}(B_\epsilon(0)),
\]
with a uniform bound.

Thus, by the Hölder inequality we have
\[
u_i \in L^\infty(B_\epsilon(0)).
\]
If we take $\mu(\{0\}) < 4\pi$, by the Brezis-Merle estimate we have $e^{u_i} \in L^r(B_\epsilon(0))$ with $r > 1$, but this $r$ may not be large enough to ensure $u_i \in L^\infty(B_\epsilon(0))$, because we have the term $|x|^{-2\alpha}$ in the equation.

Then, by the elliptic estimates, for $\alpha \in (0,1)$

$$u_i \in W^{2,1+\epsilon'}(B_\epsilon(0)) \cap C^{0,\epsilon'}(B_\epsilon(0)) \cap C^{2,\epsilon'}_{loc}(\Omega - \{0, x_1, x_2, \ldots, x_N\}),$$

and, for $\alpha \in (0,1/2)$, we have

$$u_i \in W^{2,1+\epsilon'}(B_\epsilon(0)) \cap C^{1,\epsilon'}(B_\epsilon(0)) \cap C^{2,\epsilon'}_{loc}(\Omega - \{0, x_1, x_2, \ldots, x_N\}).$$

And thus, we have

$$\partial_j u_i \rightarrow \partial_j u + \sum_{j=1}^N \alpha_j \delta_{x_j},$$

$$\alpha_j \geq 4\pi$$

weakly in the sense of measures on $\partial \Omega$.

As explained in the first step, if we consider a neighborhood of a regular point $x_0 \neq 0$, we are far from the singularity and the scheme of the first step work in this case; we have the uniform convergence of $\partial \nu u_i$ around $x_0$.

In the case $\alpha \in (0,1/2)$, the elliptic estimates gives us the $C^1$ convergence of $u_i$. $\square$

**Proof of Theorem 1.2.** Without loss of generality, we may assume that $0$ is a blow-up point (if $0$ is not a blow-up point we are in the regular case). Also, by a conformal transformation, we can assume that $\Omega = B_1^+$, the half ball, and $\partial^+ B_1^+$ is the exterior part, a part which does not contain $0$ and on which $u_i$ converges in the $C^1$ norm to $u$. Let us consider $B_\epsilon^+$ the half ball with radius $\epsilon > 0$.

In order to apply the Pohozaev identity, we need a good function $u_i$. The fact that $\alpha \in (0,1/2)$ implies that

$$u_i \in W^{2,p} \cap C^1(\bar{\Omega}).$$

Thus

$$\partial_j u_i \in W^{1,p} \cap C^0(\bar{\Omega}).$$

Thus

$$\partial_j u_i, \partial_k u_i \in W^{1,p} \cap C^0(\bar{\Omega}).$$
Thus we can do integration by parts.

The Pohozaev identity gives

\begin{align}
(13) \quad & 2(1 - \alpha) \int_{B_\epsilon^+} |x|^{-2\alpha} V_i e^{u_i} dx + \int_{B_\epsilon^+} x \cdot \nabla V_i |x|^{-2\alpha} e^{u_i} dx \\
&= \int_{\partial^+ B_\epsilon^+} g(\nabla u_i) d\sigma + \int_{\partial B_\epsilon^+} x \cdot \nu V_i e^{u_i},
\end{align}

\begin{align}
(14) \quad & 2(1 - \alpha) \int_{B_\epsilon^+} |x|^{-2\alpha} V e^u dx + \int_{B_\epsilon^+} x \cdot \nabla |x|^{-2\alpha} e^u dx \\
&= \int_{\partial^+ B_\epsilon^+} g(\nabla u) d\sigma + \int_{\partial B_\epsilon^+} x \cdot \nu V e^u
\end{align}

Here \( g(\nabla u_i) \) means a quantity which depends on \( \nabla u_i \) and for which we have a uniform convergence to \( g(\nabla u) \). (On \( \partial^+ B_\epsilon^+ \)). In fact we have:

\[ g(\nabla u_i) = (\nu \cdot \nabla u_i)(x \cdot \nabla u_i) - x \cdot \nu |\nabla u_i|^2 / 2. \]

We use again the fact that \( u_i = u = 0 \) on \( \{x_1 = 0\} \) to obtain

\begin{align}
& 2(1 - \alpha) \int_{B_\epsilon^+} |x|^{-2\alpha} V_i e^{u_i} dx - 2(1 - \alpha) \int_{B_\epsilon^+} |x|^{-2\alpha} V e^u dx \\
&+ \int_{B_\epsilon^+} x \cdot \nabla V_i |x|^{-2\alpha} e^{u_i} dx - \int_{B_\epsilon^+} x \cdot \nabla |x|^{-2\alpha} e^u dx \\
&= \int_{\partial^+ B_\epsilon^+} g(\nabla u_i) - g(\nabla u) d\sigma + o(1) = o(1).
\end{align}

First, we tend \( i \) to infinity and then \( \epsilon \) tend to 0. We obtain

\begin{align}
(15) \quad & \lim_{\epsilon \to 0} \lim_{i \to +\infty} 2(1 - \alpha) \int_{B_\epsilon^+} |x|^{-2\alpha} V_i e^{u_i} dx = 0.
\end{align}

However,

\[ \int_{B_\epsilon^+} |x|^{-2\alpha} V_i e^{u_i} dx = \int_{\partial B_\epsilon^+} \partial_{x_i} u_i + o(\epsilon) + o(1) \to \alpha_1 > 0, \]

which is a contradiction. □
References


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