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Wave Equations in a Complex Domain

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Abstract. We consider Cauchy problems for 2-dimensional wave equations in a complex domain. We assume that the initial values have singularities along the union of two hypersurfaces which are normally crossing at the origin. We show that the solutions have singularities not only along the corresponding characteristic hypersurfaces but also on the light cone issuing from the origin.

1. Introduction

Let $x = (x_0, x') = (x_0, x_1, x_2) \in \mathbf{C}^3$. We consider a 2-dimensional wave operator of the following form:

$$Pu = \Box u(x) + P_1(x_0)\partial_0 u(x) + P_0(x_0)u(x). \tag{1}$$

Here we have denoted $\Box = \partial_0^2 - \partial_1^2 - \partial_2^2, \partial_k = \partial/\partial x_k$, and we assume that $P_j(x_0)$ is holomorphic at the origin. We consider the following Cauchy problem:

$$Pu = 0,\ u(0, x') = u_0(x'),\ \partial_1 u(0, x') = u_1(x'). \tag{2}$$

We assume that the initial values $u_0(x')$ and $u_1(x')$ have some singularities in the initial hypersurface $\{x_0 = 0\}$, and study how the singularities of the solution propagate in the complex domain.

Remark. If the initial values are holomorphic on $\{x' \in \mathbf{C}^2;\ x_1 \neq 0\}$, it is known that the singularities of the solution propagate along the characteristic hypersurfaces $\{x \in \mathbf{C}^3;\ x_1 \pm x_0 = 0\}$ (See [1, 2, 8]). If the initial values have singularities on a swallow's tail $X \subset \{x_0 = 0\}$ a similar result is also known (See [3, 4, 5]). However, if the singularity set of the initial values has a different form, the singularities of the solution may

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propagate in a different way. We treat this last case, for which few results are known (See [6, 7]).

In this article, we assume that $u_0(x')$ and $u_1(x')$ are single-valued holomorphic functions on $\{x' \in \mathbf{C}^2;\ |x'| < R,\ x_1 \neq 0,\ x_2 \neq 0\}$ for some $R > 0$. Therefore the initial values have singularities along $X_1 \cup X_2$, where $X_j = \{x' \in \mathbf{C}^2;\ x_j \neq 0\}$. Precisely speaking, we assume that $x_1^{n_0} x_2^{n_0} u_j(x')$ are holomorphic near the origin for some positive $n_0 \in \mathbf{Z}$. We shall show the following result.

THEOREM 1. *Let* $Y = Y_0 \cup Y_{1,+} \cup Y_{1,-} \cup Y_{2,+} \cup Y_{2,-}$, *where*

$$Y_0 = \{x \in \mathbf{C}^3;\ -x_0^2 + x_1^2 + x_2^2 = 0\}$$

is the light cone issuing from the origin, and

$$Y_{j,\pm} = \{x \in \mathbf{C}^3;\ x_j \pm x_0 = 0\}$$

is a characteristic hypersurface issuing form X_j *and propagating backwards or forwards. The solution is holomorphic on the universal covering space* $\mathcal{R}(\omega \setminus Y)$ *of* $\omega \setminus Y$, *where* ω *is a small neighborhood of the origin.*

Plan of the paper. In section 2 we shall construct a formal solution of the Cauchy problem using a kind of asymptotic expansion. In section 3, we shall calculate the singularities of this expansion. It will turn out that this expansion contains a superfluous singularities on $\{x_1^2 + x_2^2 = 0\}$. We shall show that we can remove these superfluous singularities from the solution. In section 4, we shall complete the proof of Theorem 1.

2. Construction of the Solution

We first consider the problem in the following special case.

PROPOSITION 1. *If*

$$P_1(x_0) = 0,\ u_0 = 0,\ u_1 = 1/(x_1 x_2) \tag{3}$$

in (1) *and* (2), *then Theorem 1 is true.*

We shall prove this proposition in section 4. In section 2 and section 3, we admit this result, and we restrictively consider an operator P satisfying condition (3).

At first, we define

$$N(j) = \begin{cases} 0, & \text{if } j = 0, \\ \dfrac{1}{1} + \dfrac{1}{2} + \cdots + \dfrac{1}{j}, & \text{if } j \geq 1. \end{cases}$$

for $j \in \mathbf{Z}_+ = \{0, 1, 2, ...\}$. Let $\varphi(x, \theta) = x_0 + x_1 \cos\theta + x_2 \sin\theta$ for $\theta \in \mathbf{C}$. If $j \in \mathbf{Z}$, we define

$$\varphi_j(x, \theta) = \begin{cases} (-1)^{j-1}(j-1)!\varphi(x, \theta)^{-j}, & \text{if } j \geq 1, \\ \dfrac{\varphi(x, \theta)^{-j}}{(-j)!}(\log \varphi(x, \theta) - N(-j)), & \text{if } j \leq 0. \end{cases}$$

Finally, we define $\Phi_j(x) = \int_0^{\pi/2} \varphi_j(x, \theta) d\theta$. Then we have the following result.

LEMMA 1. (i) *We have* $\partial_0 \Phi_j = \Phi_{j+1}$ *and* $\Box \Phi_j = 0$.
(ii) *If* $\mathrm{Re}\, x_1 < 0$, $\mathrm{Re}\, x_2 < 0$, *and* $j \geq 1$, *then we have*

$$\Phi_j(0, x') = -\int_0^\infty \int_0^\infty e^{x_1 \xi_1 + x_2 \xi_2} |\xi|^{j-2} d\xi_1 d\xi_2.$$

In particular, we have $\Phi_2(0, x') = -1/(x_1 x_2)$.

PROOF. Statement (i) is trivial. Let us prove (ii). We have

$$\begin{aligned} \Phi_j(0, x') &= (-1)^{j-1}(j-1)! \int_0^{\pi/2} (x_1 \cos\theta + x_2 \sin\theta)^{-j} d\theta \\ &= -\int_0^{\pi/2} \int_0^\infty e^{r(x_1 \cos\theta + x_2 \sin\theta)} r^{j-1} dr d\theta \\ &= -\int_0^\infty \int_0^\infty e^{x_1 \xi_1 + x_2 \xi_2} |\xi|^{j-2} d\xi_1 d\xi_2. \ \Box \end{aligned}$$

Let $\omega \subset \mathrm{C}^3$ be a small neighborhood of the origin. We denote the set of holomorphic functions on ω by $\mathcal{O}(\omega)$. We seek for a solution $u(x)$ of (2) in the following form:

$$\begin{aligned} u(x) \quad &= u^+(x) + u^-(x), \\ u^\pm(x) &= \sum_{-\infty<j\le 1} u_j^\pm(x_0)\Phi_j(\pm x_0, x'). \end{aligned}$$

Here we consider unknown functions $u_j^\pm(x) \in \mathcal{O}(\omega)$. We substitute this expression in (2). Then using (i) of Lemma 1 and condition (3), we obtain

$$\begin{aligned} Pu = &\sum_{-\infty<j\le 1} \{Pu_j^+(x_0)\cdot\Phi_j(x_0,x') + 2\partial_0 u_j^+(x_0)\cdot\Phi_{j+1}(x_0,x')\} \\ &+ \sum_{-\infty<j\le 1} \{Pu_j^-(x_0)\cdot\Phi_j(-x_0,x') - 2\partial_0 u_j^-(x_0)\cdot\Phi_{j+1}(-x_0,x')\}. \end{aligned}$$

Therefore we need to solve

$$\partial_0 u_j^\pm = \frac{\mp 1}{2} P u_{j+1}^\pm \tag{4}$$

for $j \le 1$, where we have defined $u_2^\pm = 0$. On the other hand, we have

$$\begin{aligned} u(0,x') \quad &= \sum_{-\infty<j\le 1} (u_j^+(0) + u_j^-(0))\Phi_j(0,x') = 0, \\ \partial_0 u(0,x') &= \sum_{-\infty<j\le 1} \Big(\partial_0 u_j^+(0) + \partial_0 u_j^-(0)\Big)\Phi_j(0,x') \\ &\quad + \sum_{-\infty<j\le 1} \Big(u_j^+(0) - u_j^-(0)\Big)\Phi_{j+1}(0,x') = \frac{1}{x_1 x_2}. \end{aligned}$$

Therefore we need to solve

$$\begin{aligned} &u_j^+(0) + u_j^-(0) = 0, && j \le 1, \\ &u_j^+(0) - u_j^-(0) = -1, && j = 1, \\ &u_j^+(0) - u_j^-(0) = -\partial_0 u_{j+1}^+(0) - \partial_0 u_{j+1}^-(0), && j \le 0. \end{aligned}$$

In other words, we need solve (4) under the following condition:

$$u_j^\pm(0) = \begin{cases} \mp\dfrac{1}{2}, & j = 1, \\ \mp\dfrac{1}{2}\Big(\partial_0 u_{j+1}^+(0) + \partial_0 u_{j+1}^-(0)\Big), & j \le 0. \end{cases} \tag{5}$$

Then we have the following results.

Lemma 2. (i) *We have* $u_j^+(0) + u_j^-(0) = 0$ *for* $j \le 1$.
(ii) *We have* $u_j^+(0) - u_j^-(0) = -\partial_0 u_{j+1}^+(0) - \partial_0 u_{j+1}^-(0)$ *for* $j \le 0$.
(iii) *If* j *is even, then we have* $u_j^+(x_0) = u_j^-(x_0)$.
(iv) *If* j *is odd, then we have* $u_j^+(x_0) = -u_j^-(x_0)$.
(v) *We have* $u_1^\pm = \mp 1/2,\ u_0^\pm = \int_0^{x_0} P_0(t)dt/4$.

Proof. Statements (i), (ii) and (v) are direct results of (4) and (5). We can prove (iii) and (iv) by induction on j. □

Lemma 3. *Let* $C > 0$ *be large, and let* $R > 0$ *be small* (*compared with* $1/C$). *Then we have* $|u_j^\pm| \le (1-j)!C^{2-j}$ *for* $j \le 1$ *on* $\omega(R) = \{x \in \mathbf{C}^3;\ |x| < R\}$.

Proof. We define $u_j^\pm(x_0) = \sum_{k\ge 0} u_{jk}^\pm x_0^k$, where the coefficients $u_{jk}^\pm$ are determined in the following way. For $j = 1$, we define $u_{1k}^\pm = \mp\delta_{0k}/2$. Let $j_0 \le 0$. We assume that we have determined $u_{jk}^\pm$ for $j_0 + 1 \le j \le 1$. For $(j,k) = (j_0, 0)$ we define $u_{j0}^\pm = \mp(u_{j+1,1}^+ + u_{j+1,1}^-)/2$. Let $k_0 \ge 1$. We assume that we have determined $u_{jk}^\pm$ also for $j = j_0,\ 0 \le k \le k_0 - 1$. For $(j,k) = (j_0, k_0)$ we define

$$u_{jk}^\pm = \frac{\mp 1}{2k}(k(k+1)u_{j+1,k+1}^\pm + \sum_{k'+k''=k-1} P_{0,k'} u_{j+1,k''}^\pm).$$

By induction we can easily prove

$$|u_{jk}^\pm| \le \frac{(1-j+k)!}{k!} C^{3-2j+k},$$

if $C > 0$ is large. Therefore we have

$$|u^\pm| \le \sum_{k\ge 0} |x_0|^k |u_{jk}^\pm| \le \sum_{k\ge 0} 2^{1-j}(1-j)!C^{3-2j}(2C|x_0|)^k \le (2C)^{3-2j}(1-j)!.$$

Changing $R > 0$ and $C > 0$, we obtain the lemma. □

3. Singularities of the Solution

Let $D(x) = -x_0^2 + x_1^2 + x_2^2$. In this section we prove the following statements:

(a) If $j \geq 1$, then $\Phi_j(x)$ is holomorphic on $\omega_1 = \{x_1 + x_0 \neq 0,\ x_2 + x_0 \neq 0,\ D \neq 0\}$.

(b) If $j \leq 0$, then $\Phi_j(x)$ is holomorphic on $\omega_2 = \{x_1 + x_0 \neq 0,\ x_2 + x_0 \neq 0,\ D \neq 0,\ x_1^2 + x_2^2 \neq 0\}$.

In other words, our asymptotic expansion contains superfluous singularities on $\{x_1^2 + x_2^2 = 0\}$, which we need to remove from the solution later.

We first show statement (a). It suffices to prove this statement for $j = 1$, because differentiating $\Phi_1(x)$ by x_0 for $(j-1)$-times, we obtain the other cases. We have

$$\Phi_1(x) = \int_0^{2\pi} \frac{d\theta}{\varphi(x,\theta)} = \int_0^1 \frac{2ds}{(x_0 - x_1)s^2 + 2x_2 s + x_0 + x_1},$$

where $s = \tan(\theta/2)$, as usual. Let us define $\alpha_\pm(x) = (-x_2 \pm \sqrt{D(x)})/(x_0 - x_1)$. It follows that

$$\begin{aligned}
\Phi_1(x) &= \frac{1}{x_0 - x_1} \int_0^1 \frac{2ds}{(s - \alpha_+(x))(s - \alpha_-(x))} \\
&= \frac{2}{(x_0 - x_1)(\alpha_+(x) - \alpha_-(x))} \int_0^1 \Big(\frac{1}{s - \alpha_+(x)} - \frac{1}{s - \alpha_-(x)} \Big) ds \\
&= \frac{1}{\sqrt{D(x)}} \log \frac{\alpha_-(x) - \alpha_+(x)\alpha_-(x)}{\alpha_+(x) - \alpha_+(x)\alpha_-(x)}.
\end{aligned}$$

It follows that

$$\Phi_1(x) = \frac{1}{\sqrt{D(x)}} \log \frac{x_0 + x_1 + x_2 + \sqrt{D(x)}}{x_0 + x_1 + x_2 - \sqrt{D(x)}}.$$

Here we have

$$\begin{aligned}
(6) \qquad & x_0 + x_1 + x_2 + \sqrt{D} = 0 \ \text{ or } \ x_0 + x_1 + x_2 - \sqrt{D} = 0 \\
& \iff (x_1 + x_0)(x_2 + x_0) = 0.
\end{aligned}$$

This means that $\Phi_1(x)$ is holomorphic on ω_1.

Remark. One may think that $\{D(x)=0\}$ is a removable singularity of $\Phi_1(x)$, but it is not true. We can prove this in the following way. Let $\omega_0=\{x_1+x_0\neq 0,\ x_2+x_0\neq 0\}$, and let $\pi:\mathcal{R}(\omega_0)\ni\tilde{x}\longmapsto x\in\omega_0$ be the natural projection of the universal covering space. Assume that $\tilde{x}^0\in\mathcal{R}(\omega_0)$ satisfies $D(x^0)=0$. We remark $x_0^0+x_1^0+x_2^0\neq 0$. In fact, if this is not true, then we have $x_1^2+x_2^2=x_0^2=(x_1+x_2)^2$ at $x=x^0$, and we have $x_1^0x_2^0=0$. This, together with $x_0^0+x_1^0+x_2^0=0$ means either $x_1^0+x_0^0=0$ or $x_2^0+x_0^0=0$, which is a contradiction. In a neighborhood of $\tilde{x}^0$, we have

$$\begin{aligned}\Phi_1(\tilde{x}) &= \frac{1}{\sqrt{D}}\log\Big(\frac{1+\sqrt{D}/(x_0+x_1+x_2)}{1-\sqrt{D}/(x_0+x_1+x_2)}\Big)\\ &= \frac{1}{\sqrt{D}}\Big(\log 1+2\sum_{k\geq 1}\frac{1}{2k+1}\Big(\frac{\sqrt{D}}{x_0+x_1+x_2}\Big)^{2k+1}\Big).\end{aligned}$$

Moving $\tilde{x}\in\mathcal{R}(\omega_0)$, we can choose an arbitrary branch of $\log 1$ in this expression, therefore $\Phi_1(\tilde{x})$ is not holomorphic at $\tilde{x}=\tilde{x}^0$ in case of $\log 1\neq 0$.

We next show statement (b) above. For the moment we assume $j\leq -1$. Let

$$y=(y_0,y_1,y_2)=\Big(x_0,\frac{x_1+ix_2}{2},\frac{x_1-ix_2}{2}\Big).$$

We always denote $\partial_k=\partial/\partial x_k$. Then we have

$$\begin{aligned}\Phi_j(x) &= \frac{1}{(-j)!}\int_0^{\pi/2}\varphi(x,\theta)^{-j}(\log\varphi(x,\theta)-N(-j))d\theta\\ &= \int_0^1\psi_j(x,s)(\log\varphi(x,\theta)-N(-j))ds,\end{aligned}$$

where we have defined

$$\psi_j(x,s)=\frac{1}{(-j)!}\varphi(x,\theta)^{-j}\frac{2}{1+s^2}.$$

Since $\varphi(x,\theta)=x_0+\dfrac{(1-s^2)x_1}{1+s^2}+\dfrac{2sx_2}{1+s^2}=y_0-\dfrac{s+i}{s-i}y_1-\dfrac{s-i}{s+i}y_2$, we obtain

$$\psi_j(x,s)=\sum_{k+l+m=-j}\frac{2}{k!\ l!\ m!}y_0^k(-y_1)^l(-y_2)^m(s+i)^{l-m-1}(s-i)^{-l+m-1}$$

$$= \sum_{\substack{k+l+m=-j\\ l\geq m+1}} \frac{2}{k!\ l!\ m!} y_0^k(-y_1)^l(-y_2)^m \frac{(s+i)^{l-m-1}}{(s-i)^{l-m+1}}$$
$$+ \sum_{\substack{k+l+m=-j\\ l\geq m+1}} \frac{2}{k!\ l!\ m!} y_0^k(-y_1)^m(-y_2)^l \frac{(s-i)^{l-m-1}}{(s+i)^{l-m+1}}$$
$$+ \sum_{k+2l=-j} \frac{2}{k!\ l!\ l!}\ y_0^k(-y_1)^l(-y_2)^l\ (s+i)^{-1}(s-i)^{-1}.$$

Therefore we have

$$\begin{aligned}\psi_j(x,s) = \quad & \sum_{(7)} \frac{2}{k!\ l!\ m!} \binom{l-m-1}{l'} y_0^k(-y_1)^l(-y_2)^m \frac{(2i)^{l'}}{(s-i)^{l'+2}} \\ + \quad & \sum_{(7)} \frac{2}{k!\ l!\ m!} \binom{l-m-1}{l'} y_0^k(-y_1)^m(-y_2)^l \frac{(-2i)^{l'}}{(s+i)^{l'+2}} \\ + \quad & \sum_{k+2l=-j} \frac{1}{k!\ l!\ l!} \cdot \frac{y_0^k(-y_1)^l(-y_2)^l}{i} \left(\frac{1}{s-i} - \frac{1}{s+i}\right),\end{aligned}$$

where

$$k+l+m=-j,\ 0\leq l'\leq l-m-1. \tag{7}$$

It follows that

$$\psi_j(x,s) = \sum_{\substack{0\leq n\leq -j\\ q\in\{-1,+1\}}} A_{jnq}(x)(s+iq)^{-n-1},$$

where

$$A_{j,0,q}(x) = \sum_{k+2l=-j} \frac{iq}{k!\ l!\ l!}\ y_0^k y_1^l y_2^l \tag{8}$$
$$A_{j,n,+1}(x) = \sum_{(11)} \frac{2(-2i)^{n-1}}{k!\ l!\ m!} \binom{l-m-1}{n-1} y_0^k(-y_1)^m(-y_2)^l, \tag{9}$$
$$A_{j,n,-1}(x) = \sum_{(11)} \frac{2(2i)^{n-1}}{k!\ l!\ m!} \binom{l-m-1}{n-1} y_0^k(-y_1)^l(-y_2)^m \tag{10}$$

for $n \geq 1$. Here we have defined

$$k + l + m = -j, \ n \leq l - m \tag{11}$$

for $n \geq 1$. By simple calculation, we can easily see the following results (We omit the proof).

LEMMA 4. (i) *If $j \leq 0$ is odd, we have $A_{j0q}(-x_0, x') = -A_{j0q}(x)$, and thus $A_{j0q}(0, x') = 0$.*
(ii) *If $j \leq 0$ is even, we have $A_{j0q}(-x_0, x') = A_{j0q}(x)$.*
(iii) *If $j \leq -1$, we have $\partial_0 A_{j0q} = A_{j+1,0,q}$.*
(iv) *We have $\Box_x A_{j0q} = 0$.*
(v) *We have $A_{00q} = iq$.*

LEMMA 5. *Both $A_{j,n,+1}(x)$ and $y_2^{-n} A_{j,n,+1}(x)$ are entire functions and we have*

$$|A_{j,n,+1}(x)| \leq \frac{(2|y_2|)^n (3|y|)^{-j-n}}{(-j-n)!n!} \leq \frac{(5|y|)^{-j}}{(-j)!}.$$

Both $A_{j,n,-1}(x)$ and $y_1^{-n} A_{j,n,-1}(x)$ are entire functions and we have

$$|A_{j,n0,-1}(x)| \leq \frac{(2|y_1|)^n (3|y|)^{-j-n}}{(-j-n)!n!} \leq \frac{(5|y|)^{-j}}{(-j)!}.$$

We define

$$\Phi_{jnq}(x) = A_{jnq}(x) \int_0^1 (s + iq)^{-n-1} (\log \varphi(x, \theta) - N(-j)) ds. \tag{12}$$

The property of this function is quite different between the case $n \geq 1$ and the case $n = 0$. To see this, we first remark the following fact.

LEMMA 6. *Let $f_\pm(x) = \log \dfrac{\alpha_\pm - 1}{\alpha_\pm}$. Then we have*

$$f_\pm(x) = \log \frac{-x_0 + x_1 - x_2 \pm \sqrt{D(x)}}{-x_2 \pm \sqrt{D(x)}},$$

and $f_\pm(x)$ is holomorphic on $\mathcal{R}(\omega_1)$.

PROOF. We need to prove that $f_\pm(x)$ is holomorphic at an arbitrary point $\tilde{x}^0 \in \mathcal{R}(\omega_1)$. We first consider the case $x_1^0 = x_0^0$. Then we have $x_2^0 \neq 0$, and we assume that we are considering a branch $\sqrt{D(x^0)} = \sqrt{(x_2^0)^2} = x_2^0$. In a neighborhood of $\tilde{x}^0$, we have

$$
\begin{aligned}
f_+(x) &= \log \frac{-x_0 + x_1 - x_2 + x_2\sqrt{1-(x_0^2 - x_1^2)x_2^{-2}}}{-x_2 + x_2\sqrt{1-(x_0^2-x_1^2)x_2^{-2}}} \\
&= \log \frac{-x_0 + x_1 - \frac{1}{2}(x_0^2 - x_1^2)x_2^{-1} - \sum_{k\geq 2} \frac{(2k-3)!!}{2k!!}\cdot\frac{(x_0-x_1)^k(x_0+x_1)^k}{x_2^{2k-1}}}{-\frac{1}{2}(x_0^2-x_1^2)x_2^{-1} - \sum_{k\geq 2}\frac{(2k-3)!!}{2k!!}\cdot\frac{(x_0-x_1)^k(x_0+x_1)^k}{x_2^{2k-1}}} \\
&= \log \frac{2x_2 + x_0 + x_1 + 2\sum_{k\geq 2}\frac{(2k-3)!!}{2k!!}\cdot\frac{(x_0-x_1)^{k-1}(x_0+x_1)^k}{x_2^{2k-2}}}{x_0 + x_1 + 2\sum_{k\geq 2}\frac{(2k-3)!!}{2k!!}\cdot\frac{(x_0-x_1)^{k-1}(x_0+x_1)^k}{x_2^{2k-2}}}.
\end{aligned}
$$

Here we have $2x_2^0 + x_0^0 + x_1^0 = 2x_2^0 + 2x_0^0 \neq 0$, $x_1^0 + x_0^0 \neq 0$, and thus $f_+(x)$ is holomorphic in a neighborhood of $\tilde{x}^0$. On the other hand, for this branch neither $-x_0 + x_1 - x_2 - \sqrt{D(x)}$ nor $-x_2 - \sqrt{D(x)}$ vanishes at $\tilde{x}^0$. Therefore

$$
f_-(x) = \log \frac{-x_0 + x_1 - x_2 - \sqrt{D(x)}}{-x_2 - \sqrt{D(x)}}
$$

is holomorphic at $\tilde{x}^0$.

We next consider the case $x_1^0 \neq x_0^0$. We have

$$
\begin{aligned}
&-x_0 + x_1 - x_2 \pm \sqrt{D(x)} = 0 && \Longrightarrow (x_2 + x_0)(x_1 - x_0) = 0, \\
&-x_2 \pm \sqrt{D(x)} = 0 && \Longrightarrow (x_1 + x_0)(x_1 - x_0) = 0.
\end{aligned}
$$

From our assumption it follows that neither $-x_0 + x_1 - x_2 \pm \sqrt{D(x)}$ nor $-x_2 \pm \sqrt{D(x)}$ vanishes at $\tilde{x} = \tilde{x}^0$. Therefore $f_\pm(x)$ is holomorphic at $\tilde{x}^0$. □

LEMMA 7. *Let $R > 0$ be sufficiently small. We define $\omega_1(R) = \{x \in \mathbf{C}^3;\ |x| < R,\ x_1 + x_0 \neq 0,\ x_2 + x_0 \neq 0,\ D(x) \neq 0\}$. If $n \geq 1$, then $\Phi_{jnq}(x)$ is holomorphic on $\mathcal{R}(\omega_1(R))$ and we have*

$$
|\Phi_{jnq}(x)| \leq \frac{20(10|x|)^{-j}}{(-j)!}\Big(\sum_{l\in\{-1,+1\}} |f_l(x)| + |\log(x_1+x_0)| + |\log(x_2+x_0)| + 1\Big)
$$

on $\mathcal{R}(\omega_1(R))$.

PROOF. From (12) we have $\Phi_{jnq}(x) = \sum_{1 \le p \le 5} \Phi_{jnqp}(x)$, where

$$\begin{aligned}
\Phi_{jnq1}(x) &= \frac{-A_{jnq}(x)}{n} \Big[(s+iq)^{-n}(\log\varphi(x,\theta) - N(-j))\Big]_{s=0}^{s=1} \\
\Phi_{jnq2}(x) &= \frac{A_{jnq}(x)}{n} \int_0^1 \frac{1}{(s+iq)^n} \cdot \frac{1}{s-\alpha_+(x)} ds, \\
\Phi_{jnq3}(x) &= \frac{A_{jnq}(x)}{n} \int_0^1 \frac{1}{(s+iq)^n} \cdot \frac{1}{s-\alpha_-(x)} ds, \\
\Phi_{jnq4}(x) &= -\frac{A_{jnq}(x)}{n} \int_0^1 \frac{1}{(s+iq)^n} \cdot \frac{1}{s+i} ds, \\
\Phi_{jnq5}(x) &= -\frac{A_{jnq}(x)}{n} \int_0^1 \frac{1}{(s+iq)^n} \cdot \frac{1}{s-i} ds.
\end{aligned}$$

We need to prove

$$(13)\quad |\Phi_{jnqk}(x)| \le \frac{4(10|x|)^{-j}}{(-j)!} \Big(\sum_{l \in \{-1,+1\}} |f_l(x)| + |\log(x_1+x_0)| + |\log(x_2+x_0)| + 1 \Big)$$

for $1 \le k \le 5$.

We have $[\varphi(x,\theta)]_{s=0} = x_1 + x_0$, and $[\varphi(x,\theta)]_{s=1} = x_2 + x_0$. It follows that

$$\Phi_{jnq1}(x) = \frac{-A_{jnq}(x)}{n} \Big(\frac{\log(x_2+x_0) - N(-j)}{(1+iq)^n} - \frac{\log(x_1+x_0) - N(-j)}{(iq)^n} \Big).$$

We have $N(-j) \le -j \le 2^{-j}$, and from Lemma 5 we obtain

$$\begin{aligned}
|\Phi_{jnq1}(x)| &\le \frac{(5|y|)^{-j}}{(-j)!} \cdot (|\log(x_2+x_0)| + |\log(x_1+x_0)| + 2^{-j+1}) \\
&\le \frac{2(10|x|)^{-j}}{(-j)!} \cdot (|\log(x_2+x_0)| + |\log(x_1+x_0)| + 1).
\end{aligned}$$

Therefore (13) is true for $k = 1$.

We next consider $\Phi_{jnq2}(x)$. If $a \ne b$, then we have

$$\frac{1}{(s-a)(s-b)^n} = \frac{(-1)^{n-1}}{(a-b)^n} \cdot \frac{1}{s-a} - \sum_{n'+n''=n-1} \frac{1}{(a-b)^{n'+1}} \cdot \frac{1}{(s-b)^{n''+1}}.$$

It follows that

$$\Phi_{jnq2}(x) = \frac{1}{n} \cdot \frac{A_{jnq}(x)}{(\alpha_+(x)+iq)^n} \int_0^1 \frac{(-1)^{n-1}}{s-\alpha_+(x)} ds - \frac{1}{n} \sum_{n'+n''=n-1} \frac{A_{jnq}(x)}{(\alpha_+(x)+iq)^{n'+1}} \int_0^1 \frac{1}{(s+iq)^{n''+1}} ds$$

and thus

$$\Phi_{jnq2}(x) = \frac{1}{n} \cdot \frac{A_{jnq}(x)}{(\alpha_+(x)+iq)^n} \Big((-1)^{n-1} \log \frac{\alpha_+(x)-1}{\alpha_+(x)} + \log \frac{iq}{1+iq} \Big) + \frac{1}{n} \sum_{\substack{n'+n''=n-1 \\ n''\neq 0}} \frac{A_{jnq}(x)}{n''(\alpha_+(x)+iq)^{n'+1}} \Big(\frac{1}{(1+iq)^{n''}} - \frac{1}{(iq)^{n''}} \Big). \tag{14}$$

We need to prove that $\Phi_{jnq2}(x)$ is holomorphic at an arbitrary point $\tilde{x}^0 \in \mathcal{R}(\omega_1)$. From Lemma 6, $\log((\alpha_+(x)-1)/(\alpha_+(x))$ is holomorphic at $\tilde{x}^0 \in \mathcal{R}(\omega_1)$. Therefore $\Phi_{jnq2}(x)$ is holomorphic at $\tilde{x}^0$ if $\alpha_+(\tilde{x}^0)+iq \neq 0$. Let us consider the case $\alpha_+(\tilde{x}^0)+iq = 0$. We have

$$\frac{1}{\alpha_+(x)+iq} = \frac{(x_0-x_1)(\alpha_-(x)+iq)}{2(x_1-iqx_2)}. \tag{15}$$

If $q=+1$, then $x_1 - iqx_2 = 2y_2$. From (9) it follows that $(x_1-iqx_2)^{-n}A_{j,n,+1}$ is holomorphic at $\tilde{x} = \tilde{x}^0$. This means that $\Phi_{j,n,+1,2}(x)$ is holomorphic on $\mathcal{R}(\omega_1)$. We next give an estimate for $\Phi_{jnq2}(x)$ for $q=1$. From (15) we have $|(x_1-iqx_2)/(\alpha_+(x)+iq)| \leq |iq(x_0-x_1) - x_2 - \sqrt{D(x)}| \leq 6|x|$. Let $0 \leq n' \leq n-1$. From Lemma 5 and (15) we have

$$\begin{aligned} \Big| \frac{A_{j,n,+1}(x)}{(\alpha_+(x)+i)^{n'+1}} \Big| &\leq \frac{|x_1-iqx_2|^n (3|y|)^{-j-n}}{|\alpha_+(x)+i|^{n'+1}(-j-n)!n!} \\ &\leq \Big(\frac{|x_1-iqx_2|}{|\alpha_+(x)+i|} \Big)^{n'+1} \frac{|x_1-iqx_2|^{n-n'-1}(3|x|)^{-j-n}}{(-j-n)!n!} \\ &\leq \frac{(10|x|)^{-j}}{(-j)!}. \end{aligned}$$

for $0 \leq n' \leq n-1$. From (14) it follows that

$$|\Phi_{jnq2}(x)| \leq \frac{(10|x|)^{-j}}{(-j)!} \Big(\Big| \log \frac{(\alpha_+(x)-1)}{\alpha_+(x)} \Big| + \Big| \log \frac{i}{i+1} \Big| \Big)$$

$$
+ \sum_{\substack{n'+n''=n-1 \\ n''\neq 0}} \frac{2}{n} \cdot \frac{(10|x|)^{-j}}{(-j)!}
$$

$$
\leq \frac{4(10|x|)^{-j}}{(-j)!} \sum_{l\in\{-1,+1\}} (|f_l(x)|+1).
$$

Therefore (13) is true for $(q,k)=(1,2)$. The same result is true for $\Phi_{jnqk}(x)$ for $q \in \{+1,-1\}$ and $k \in \{2,3\}$. Finally, $\Phi_{jnq4}(x)$ and $\Phi_{jnq5}(x)$ are holomorphic functions at the origin which satisfy the same estimate. □

REMARK. By Lemma 7, $\Phi_{jnq}(x)$ is holomorphic on $\mathcal{R}(\omega_1(R))$ if $n \geq 1$. To the contrary, $\Phi_{jnq}(x)$ is not holomorphic on $\mathcal{R}(\omega_1(R))$ if $n=0$. Let us prove that $\Phi_{j0q}(x)$ is singular at $\tilde{x}^0 \in \mathcal{R}(\omega_1(R))$ if $x_1^0 - iqx_2^0 = 0$. It suffices to prove this for $j=0$, because we have $\Phi_{00q}(x) = \partial_0^{-j}\Phi_{j0q}$. Furthermore, it suffices to prove that $\partial_2\Phi_{00q}$ is singular at $\tilde{x}^0$. From (iv) of Lemma 4 and (12), we have

$$
\begin{aligned}
\partial_2\Phi_{00q}(x) &= \partial_2\Big(\int_0^1 \frac{iq}{s+iq}\log\varphi(x,\theta)ds\Big) \\
&= \partial_2\Big(\int_0^1 \frac{iq}{s+iq}\log\Big(\frac{(x_0-x_1)s^2+2x_2s+(x_0+x_1)}{s^2+1}\Big)ds\Big) \\
&= \int_0^1 \frac{iq}{s+iq}\cdot\frac{2s}{(x_0-x_1)s^2+2x_2s+(x_0+x_1)}ds \\
&= \int_0^1 \frac{iq}{s+iq}\cdot\frac{2s}{(x_0-x_1)(s-\alpha_+(x))(s-\alpha_-(x))}ds \\
&= \frac{iq\alpha_+}{(\alpha_+(x)+iq)\sqrt{D(x)}}\int_0^1\Big(\frac{1}{s-\alpha_+(x)}-\frac{1}{s+iq}\Big)ds \\
&\quad - \frac{iq\alpha_-}{(\alpha_-(x)+iq)\sqrt{D(x)}}\int_0^1\Big(\frac{1}{s-\alpha_-(x)}-\frac{1}{s+iq}\Big)ds \\
&= \frac{iq\alpha_+}{(\alpha_+(x)+iq)\sqrt{D(x)}}\log\frac{(1-\alpha_+(x))iq}{-\alpha_+(x)(1+iq)} \\
&\quad - \frac{iq\alpha_-}{(\alpha_-(x)+iq)\sqrt{D(x)}}\log\frac{(1-\alpha_-(x))iq}{-\alpha_-(x)(1+iq)}.
\end{aligned}
$$

By Lemma 6, the logarithmic functions are holomorphic on $\mathcal{R}(\omega_1)$. At $x=x^0$, either $\alpha_+(x)+iq$ or $\alpha_-(x)+iq$ vanishes, and $\partial_2\Phi_{00q}(x)$ may have a singularity at this point, according to the branch of the logarithmic function.

We next show that nevertheless the solution of (2) is holomorphic on $\mathcal{R}(\omega_1(R))$. For this purpose, we define

$$g_q(x) = \sum_{j\le 0}(u_j^+(x_0)A_{j0q}(x) + u_j^-(x_0)A_{j0q}(-x_0, x')).$$

Then we have the following result.

Lemma 8. *We have* $g_q(x) = 0$.

Proof. From (i) of Lemma 2, we have

$$g_q(0, x') = \sum_{j\le 0}(u_j^+(0) + u_j(0))A_{j0q}(0, x') = 0. \tag{16}$$

From (ii) of Lemma 2, we have

$$\begin{aligned}
\partial_0 g_q(0, x') &= \sum_{j\le 0}(\partial_0 u_j^+(0) + \partial_0 u_j^-(0))A_{j0q}(0, x') \\
&\quad + \sum_{j\le 0}(u_j^+(0) - u_j^-(0))\partial_0 A_{j0q}(0, x') \\
&= \sum_{j\le 0}(\partial_0 u_j^+(0) + \partial_0 u_j^-(0))A_{j0q}(0, x') \\
&\quad - \sum_{j\le 0}(\partial_0 u_{j+1}^+(0) + \partial_0 u_{j+1}^-(0))\partial_0 A_{j0q}(0, x') \\
&= \sum_{j\le 1}(\partial_0 u_j^+(0) + \partial_0 u_j^-(0))(A_{j0q}(0, x') - \partial_0 A_{j-1,0,q}(0, x')) \\
&\quad + (\partial_0 u_0^+(0) + \partial_0 u_0(0))\partial_0 A_{00q}(0, x').
\end{aligned}$$

From (iii) and (v) of Lemma 4 we have $A_{j0q} - \partial_0 A_{j-1,0,q} = 0$ and $\partial_0 A_{00q} = 0$. It follows that

$$\partial_0 g_q(0, x') = 0. \tag{17}$$

We have

$$Pg_q(x) = \sum_{j\le 0}(Pu_j^+(x_0)\cdot A_{j0q}(x) + Pu_j^-(x_0)\cdot A_{j0q}(-x_0, x'))$$

$$+ 2\sum_{j\le 0}\Big(\partial_0 u_j^+(x_0)\cdot\partial_0 A_{j0q}(x) - \partial_0 u_j^-(x_0)\cdot\partial_0 A_{j0q}(-x_0,x')\Big).$$

From (iii) and (v) of Lemma 4, we have

$$\begin{aligned}Pg_q(x) &= \sum_{j\le 0}(Pu_j^+(x_0)\cdot A_{j0q}(x) + Pu_j^-(x_0)\cdot A_{j0q}(-x_0,x'))\\ &\quad + 2\sum_{j\le -1}\Big(\partial_0 u_j^+(x_0)\cdot A_{j+1,0,q}(x) - \partial_0 u_j^-(x_0)\cdot A_{j+1,0,q}(-x_0,x')\Big).\end{aligned}$$

From (4) we have

$$Pg_q = 0. \tag{18}$$

From (16)–(18)we obtain the lemma. □

Corollary. *We have* $\sum_{j\le 0} u_j^{\pm}(x_0)A_{j0q}(\pm x_0, x') = 0.$

Proof. Let j be odd. By Lemma 2, we have $u_j^+(x_0) = -u_j^-(x_0)$, and by Lemma 4, we have $A_{j0q}(-x_0,x') = -A_{j0q}(x)$. It follows that

$$u_j^+(x_0)A_{j0q}(x) = u_j^-(x_0)A_{j0q}(-x_0,x'). \tag{19}$$

We can similarly prove (19) if j is even. From Lemma 8 we have

$$\sum_{j\le 0} u_j^+(x_0)A_{j0q}(x) = \sum_{j\le 0} u_j^-(x_0)A_{j0q}(-x_0,x') = 0. \ \square$$

Proof of Proposition 1. If $j\le 0$, we have

$$\begin{aligned}\Phi_j(x) &= \sum_{\substack{0\le n\le -j\\ q\in\{-1,+1\}}}\Phi_{jnq}(x)\\ &= \sum_{\substack{1\le n\le -j\\ q\in\{-1,+1\}}}\Phi_{jnq}(x)\\ &\quad + \sum_{q\in\{-1,+1\}} A_{j0q}(x)\int_0^1 (s+iq)^{-1}(\log\varphi - N(-j))ds.\end{aligned}$$

We define

$$\Phi'_j(x) = \Phi_j(x) - \sum_{q\in\{-1,+1\}} A_{j0q}(x)\int_0^1 (s+iq)^{-1}\log\varphi ds$$
$$= \sum_{\substack{1\le n\le -j\\ q\in\{-1,+1\}}} \Phi_{jnq}(x) - \sum_{q\in\{-1,+1\}} N(-j)A_{j0q}(x)\log(1-iq).$$

If $j \geq 1$, we define $\Phi'_j(x) = \Phi_j(x)$. From Lemma 8, we have

$$u(x) = \sum_{\substack{-\infty<j\le 1\\ q\in\{+1,-1\}}} u_j^q(x_0)\Phi_j(qx_0, x') = \sum_{\substack{-\infty<j\le 1\\ q\in\{+1,-1\}}} u_j^q(x_0)\Phi'_j(qx_0, x')$$

By Lemma 5 and Lemma 7, this function is holomorphic on $\mathcal{R}(\omega_1(R))$. □

4. Proof of the Theorem

From now on, we do not assume (3), and give the proof of Theorem 1 in several steps.

Lemma 9. *If $P_1(x_0) = 0$ and $u_0 = 0$ in (1), then Theorem 1 is true.*

Proof. For some positive integer n_0 we have $u_1(x') = \sum_{j,k\geq -n_0} \cdot u_{1jk}x_1^j x_2^k$, and we can rewrite as $u_1 = u_1^0 + u_1^1 + u_1^2 + u_1^3$, where

$$u_1^0(x') = \sum_{j,k\geq 0} u_{1jk}x_1^j x_2^k,$$
$$u_1^1(x') = \sum_{\substack{0>j\geq -n_0\\ k\geq 0}} u_{1jk}x_1^j x_2^k,$$
$$u_1^2(x') = \sum_{\substack{j\geq 0\\ 0>k\geq -n_0}} u_{1jk}x_1^j x_2^k,$$
$$u_1^3(x') = \sum_{\substack{0>j\geq -n_0\\ 0>k\geq -n_0}} u_{1jk}x_1^j x_2^k.$$

We can apply the Cauchy-Kowalewski theorem to the case $(u_0, u_1) = (0, u_1^0)$, and the corresponding solution is holomorphic at the origin. We can apply the usual result for singular Cauchy problems to the case $(u_0, u_1) = (0, u_1^1)$, and the corresponding solution is holomorohic outside of $\{x_1 - x_0 = 0\} \cup \{x_1 + x_0 = 0\}$. The solution corresponding to the case of $(u_0, u_1) = (0, u_1^2)$ is holomorohic outside of $\{x_2 - x_0 = 0\} \cup \{x_2 + x_0 = 0\}$ (See [1] for these two cases). Finally, we consider the case of $(u_0, u_1) = (0, u_1^3)$. For this case, we may assume $u_1 = x_1^j x_2^k$ for some negative integers j and k. Then we can reduce the problem to the case $u_1 = 1/(x_1 x_2)$ and can apply Proposition 1. □

PROPOSITION 2. *If $u_0 = 0$, then Theorem 1 is true.*

PROOF. Let $P(x_0, D)$ be a general operator of the form (1). If

$$u(x) = \exp\Big(-\frac{1}{2}\int_0^{x_0} P_1(t)dt\Big)v(x),$$

then we have $Pu = \exp(-\int_0^{x_0} P_1(t)dt/2)Qv$, where

$$Qv = \Box v + \Big(P_0(x_0) - \frac{1}{2}\partial_0 P_1(x_0) - \frac{1}{4}P_1(x_0)^2\Big)v.$$

We can apply Lemma 9 to Q, and the corresponding solution $v(x)$ is holomophic on $\mathcal{R}(\omega \setminus Y)$. □

LEMMA 10. *If $P_0(x_0) = \partial_0 P_1(x_0)$ and $u_1 = 0$, then Theorem 1 is true.*

PROOF. In this case, we need to solve

$$Pu = \Box u + \partial_0(P_1 u) = 0,\ u(0, x') = u_0,\ \partial_0 u(0, x') = 0. \tag{20}$$

Let $Q = \Box + P_1(x_0)\partial_0$. By Proposition 2, there exist $v, w \in \mathcal{O}(\mathcal{R}(\omega_1(R)))$ satisfying

$$\begin{aligned} &Qv = 0,\ v(0, x') = 0,\ \partial_0 v(0, x') = u_0, \\ &Pw = 0,\ w(0, x') = 0,\ \partial_0 w(0, x') = P_1(0)u_0(x'). \end{aligned}$$

Then $u = w + \partial_0 v$ satisfies (20). □

PROPOSITION 3. *If $u_1 = 0$, then Theorem 1 is true.*

Proof. Let $P(x_0, D)$ be a general operator of the form (1). Let $a(x_0)$ be the solution of

$$\begin{cases} \dfrac{d^2a}{dx_0^2} - \dfrac{2}{a}\Big(\dfrac{da}{dx_0}\Big)^2 - P_1\dfrac{da}{dx_0} + \Big(\dfrac{dP_1}{dx_0} - P_0\Big)a = 0, \\ a(0) = 1,\ \dfrac{da}{dx_0}(0) = 0. \end{cases} \tag{21}$$

Both a and $1/a$ are holomorphic at the origin. We define $u = av$. Then we have $Pu = aQv$, where

$$Q = \Box + Q_1\partial_0 + Q_0,$$
$$Q_1 = \frac{2}{a}\cdot\frac{da}{dx_0} + P_1,\ Q_0 = \frac{1}{a}\Big(\frac{d^2a}{dx_0^2} + P_1\frac{da}{dx_0}\Big) + P_0.$$

From (21), we have $Q_0(x_0) = \partial_0 Q_1(x_0)$. We can apply Lemma 10 to Q, and we have a solution $v \in \mathcal{O}(\mathcal{R}(\omega_1(R)))$ of

$$Qv = 0,\ v(0, x') = u_0(x'),\ \partial_0 v(0, x') = 0.$$

Then, $u = av$ is the solution to the present problem. $\Box$

Theorem 1 is a direct consequence of Propostion 2 and Proposition 4.

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