J. Math. Sci. Univ. Tokyo
23 (2016), 349–380.

Blow-Up of Finite-Difference Solutions to Nonlinear Wave Equations

By Norikazu SAITO and Takiko SASAKI

Abstract. Finite-difference schemes for computing blow-up solutions of one dimensional nonlinear wave equations are presented. By applying time increments control technique, we can introduce a numerical blow-up time which is an approximation of the exact blow-up time of the nonlinear wave equation. After having verified the convergence of our proposed schemes, we prove that solutions of those finite-difference schemes actually blow up in the corresponding numerical blow-up times. Then, we prove that the numerical blow-up time converges to the exact blow-up time as the discretization parameters tend to zero. Several numerical examples that confirm the validity of our theoretical results are also offered.

1. Introduction

The purpose of this paper is to establish numerical methods for computing blow-up solutions of one space dimensional nonlinear wave equations with power nonlinearlities. In order to avoid unessential difficulties about boundary conditions, we concentrate our attention to L-periodic functions of x with $L > 0$. That is, setting $S_L = \mathbb{R}/L\mathbb{Z}$, we consider the following initial value problem for the function $u = u(t,x)$ $(t \geq 0,\ x \in S_L)$,

$$(1.1) \qquad \begin{cases} u_{tt} - u_{xx} = |u|^p, & t > 0,\ x \in S_L, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in S_L. \end{cases}$$

Before stating assumptions on nonlinearlity and initial values, we recall a general result for nonlinear wave equations. Set $Q_{T,L} = [0,T] \times S_L$ for $T > 0$.

2010 *Mathematics Subject Classification*. 35L71, 35B44, 65M06.
Key words: Nonlinear wave equation, blow-up, finite-difference method.

PROPOSITION 1.1. *Let $u_0, u_1 \in C^3(S_L)$ and $f \in C^4(\mathbb{R})$ be given. Then, there exists $T > 0$ and a unique classical solution $u \in C^3(Q_{T,L})$ of*

$$\begin{cases} u_{tt} - u_{xx} = f(u), & (t,x) \in Q_{T,L}, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in S_L. \end{cases} \tag{1.2}$$

Moreover, there exists a positive and continuous function $C_{ml}(\eta)$ of $\eta > 0$ satisfying

$$\left\| \frac{\partial^m}{\partial t^m} \frac{\partial^l}{\partial x^l} u \right\|_{L^\infty(Q_{T,L})} \leq C_{ml} \left(\|u\|_{L^\infty(Q_{T,L})} \right)$$

for non-negative integers m, l such that $m + l \leq 3$. Furthermore, if $f(s) \geq 0$ for $s \geq 0$ and $u_0(x) \geq 0$, $u_1(x) \geq 0$ for $x \in S_L$, then we have $u(t,x) \geq 0$ for $(t,x) \in Q_{T,L}$.

This proposition is proved by the standard argument based on the contraction mapping principle (cf. [6, §12.3]) with the aid of the explicit solution formula given as

$$u(t,x) = \frac{1}{2}[u_0(x-t) + u_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)\, d\xi + \frac{1}{2}\int_0^t \int_{x-s}^{x+s} f(u(s,y))\, dyds.$$

Throughout this paper, we make the following assumptions:

$$f(u) = |u|^p \text{ with } p > 1 \text{ is of class } C^4; \tag{1.3}$$

$$u_0,\ u_1 \in C^3(S_L); \tag{1.4}$$

$$u_0(x) \geq 0, \quad u_1(x) \geq 0, \quad x \in S_L. \tag{1.5}$$

Thanks to Proposition 1.1, the problem (1.1) admits a unique non-negative solution $u \in C^3(Q_{T,L})$, which we will call simply a *solution* hereinafter. We note that the condition (1.3) is equivalently written as

$$p = 2 \text{ or } p \text{ is a real number} \geq 4. \tag{1.6}$$

See also Remark 2.5.

The supremum of T in Proposition 1.1 is called the lifespan of a solution and is denoted by T_∞. If $T_\infty = \infty$, then we say that the solution u of (1.1) exists globally-in-time. On the other hand, if $T_\infty < \infty$, we say that u blows up in finite time and call T_∞ the blow-up time of a solution.

As a readily obtainable consequence of Proposition 1.1, we deduce the following proposition.

PROPOSITION 1.2. *Let u be the solution of* (1.1). *Then, the following* (i) *and* (ii) *are equivalent.*

(i) *u blows up in finite time $T_\infty < \infty$.*

(ii) $\lim_{t\uparrow T_\infty} \|u(t)\|_{L^\infty(S_L)} = \infty$.

Any solution u of (1.1) actually blows up. To verify this fact, the functional

$$K(v) = \frac{1}{L}\int_0^L v(x)\ dx \qquad (v \in C(S_L))$$

plays an important role. Obviously, we have

$$K(v) \le \|v\|_{L^\infty(S_L)} \quad (0 \le v \in C(S_L)). \tag{1.7}$$

PROPOSITION 1.3. *Assume that*

$$\alpha = K(u_0) \ge 0, \quad \beta = K(u_1) > 0. \tag{1.8}$$

Then, there exists $T_\infty \in (0,\infty)$ such that the solution u of (1.1) *blows up in finite time T_∞.*

This proposition is not new; however, we briefly review the proof since we will study a discrete analogue of this result in Section 4. As a matter of fact, the key point of the proof is that the solution u of (1.1) satisfies, whenever it exists,

$$\frac{d}{dt}K(u(t)) \ge \beta + \int_0^t K(u(s))^p\ ds > 0, \tag{1.9}$$

$$\left[\frac{d}{dt}K(u(t))\right]^2 \ge \frac{2}{p+1}K(u(t))^{p+1} + M_1 \ge 0, \tag{1.10}$$

where $M_1 = \beta^2 - \frac{2}{p+1}\alpha^{p+1}$ and $K(u(t)) = K(u(t,\cdot))$.

These inequalities, together with the following elementary proposition, implies that $K(u(t))$ cannot exist beyond T_K, which is defined below. Thus, $u(t,x)$ blows up in finite time $T_\infty \in (0, T_K]$, which completes the proof of Proposition 1.3.

PROPOSITION 1.4. *Let a C^1 function $w = w(t)$ satisfy a differential inequality*

$$\frac{d}{dt}w(t) \geq \sqrt{\frac{2}{p+1}w(t)^{p+1} + M_1} \qquad (t > 0) \tag{1.11}$$

with $w(0) = \alpha \geq 0$. Then, $w(t)$ blows up in finite time $T_K \in (0, T_1)$, where

$$T_1 = \int_\alpha^\infty \left[\beta^2 + \frac{2}{p+1}(s^{p+1} - \alpha^{p+1})\right]^{-\frac{1}{2}} ds < \infty.$$

Inequalities (1.9) and (1.10) are derived in the following manner. First, we derive by using Jensen's inequality

$$\frac{d^2}{dt^2}K(u(t)) \geq K(u(t))^p, \tag{1.12}$$

which gives (1.9). Multiplying the both-sides of (1.12) by $(d/dt)K(u(t))$, we have

$$\frac{d}{dt}K(u(t))\frac{d^2}{dt^2}K(u(t)) \geq \frac{d}{dt}K(u(t))K(u(t))^p.$$

Thus

$$\frac{d}{dt}\left[\frac{1}{2}\left(\frac{d}{dt}K(u(t))\right)^2 - \int_\alpha^{K(u(t))} \xi^p \, d\xi\right] \geq 0.$$

Therefore, we get

$$\left[\frac{d}{dt}K(u(t))\right]^2 \geq \beta^2 + \frac{2}{p+1}\left[K(u(t))^{p+1} - \alpha^{p+1}\right],$$

which implies (1.10).

There are a large number of works devoted to blow-up of positive solutions for nonlinear wave equations. To our best knowledge, the first result was obtained by Kawarada [11]. He studied a nonlinear wave equation

$$u_{tt} - \Delta u = f(u) \quad (x \in \Omega,\ t > 0) \tag{1.13}$$

in a smooth bounded domain Ω in $\mathbb{R}^d$ and proved a positive solution actually blows up in finite time if the initial values are sufficiently large. (He did not consider a positive solution explicitly, but as a readily obtainable corollary of his theorem we could obtain the blow-up of a positive solution.) Those results are referred as "large data blow-up" results. After Kawarada's work, a lost of results have been reported. For example, Glassey's papers [7], [8] are well-known. On the other hand, "small data blow-up" results were presented, for example, F. John ([9]) and T. Kato ([10]). See an excellent survey by S. Alinhac ([2]) for more details on blow-up results for nonlinear hyperbolic equations. In contrast to parabolic equations, it seems that there is a little work devoted to asymptotic profiles and blow-up rates of blow-up solutions for hyperbolic equations. Therefore, numerical methods would be important tools to study blow-up phenomena in hyperbolic equations.

However, the computation of blow-up solutions is a difficult task. We do not state here the detail of those issues; see, for example, [4] and [5]. In order to surmount those obstacles, various techniques for computing blow-up solutions of various nonlinear partial differential equations are developed so far. Among them, variable time-increments Δt_n is of use. The pioneering work is done by Nakagawa [13] in 1976. He considered the explicit Euler/finite difference scheme to a semilinear heat equation $u_t - u_{xx} = u^2$ $(t > 0, 0 < x < 1)$ with $u(t,0) = u(t,1) = 0$. The crucial point of his strategy is that the time increment and the discrete time are given, respectively, as

$$\Delta t_n = \tau \min\left\{1, \frac{1}{\|u_h(t_n)\|_{L^2}}\right\}, \quad t_{n+1} = t_n + \Delta t_n = \sum_{k=0}^{n} \Delta t_k$$

with some $\tau > 0$, where $u_h(t_n)$, h being the size of space grids, denotes the piece-wise constant interpolation function of the finite-difference solution at $t = t_n$ and $\|u_h(t_n)\|_{L^2}$ its $L^2(0,1)$ norm. Then, he succeeded in proving that, for a sufficiently large initial value, the finite-difference solution $u_h(t_n)$

actually blows up in finite time

$$T(\tau, h) = \sum_{n=1}^{\infty} \Delta t_n < \infty$$

and

$$\lim_{\tau,h\to 0} T(\tau, h) = T_\infty, \tag{1.14}$$

where τ denotes the size of a time discretization and T_∞ the blow-up time of the equation under consideration. $T(\tau, h)$ is called the *numerical blow-up time*. Later, Nakagawa's result has been extend to several directions; see, for example, Chen [3], Abia et al. [1], Nakagawa and Ushijima [14] and Cho et al. [4]. However, those papers are concerned only with parabolic equations. On the other hand, it seems that little is known for hyperbolic equations and C. H. Cho's work ([5]) is the first result on the subject. He studied the initial-boundary value problem for a nonlinear wave equation

$$\begin{cases} u_{tt} - u_{xx} = u^2 \qquad (t > 0,\ x \in (0,1)), \\ u = 0 \quad (t \geq 0,\ x = 0, 1), \quad u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \end{cases}$$

and the explicit Euler/finite-difference scheme

$$\begin{cases} \dfrac{1}{\tau_n}\left(\dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} - \dfrac{u_j^n - u_j^{n-1}}{\Delta t_{n-1}}\right) = \dfrac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} + (u_j^n)^2, \\ u_0^n = u_N^n = 0, \quad u_j^0 = u_0(x_j), \quad u_j^1(x_j) = u_0(x_j) + \Delta t_0 u_1(x_j), \end{cases} \tag{1.15}$$

where the time and space variable are discretized as $t_n = \Delta t_0 + \Delta t_1 + \cdots + \Delta t_{n-1}$, $x_j = j/N$ and $N \in \mathbb{N}$, and u_j^n denotes the approximation of $u(t_n, x_j)$. He proposed the following time-increments control strategy

$$\Delta t_n = \tau \min\left\{1, \ \frac{1}{\|u_h(t_n)\|_{L^2}^{1/2}}\right\}, \qquad \tau_n = \frac{\Delta t_n + \Delta t_{n-1}}{2}. \tag{1.16}$$

Then, he succeeded in proving that (1.14) actually holds true under some assumptions. One of the crucial assumptions in his theorem is convergence of the finite-difference solutions, that is,

$$\lim_{h\to 0} \max_{0\leq t_n \leq T} |u_j^n - u(t_n, x_j)| = 0 \tag{1.17}$$

for any $T \in (0, T_\infty)$. The proof of this convergence result is still open at present. As a matter of fact, we need some a priori estimates or stability in a certain norm in order to prove (1.17). However, as Cho mentioned in [5, page 487], it is quite difficult to prove a stability that remains true even when $\Delta t_n \to 0$.

Recently, K. Matsuya reported some interesting results on global existence and blow-up of solutions of a discrete nonlinear wave equation in [12]. However, it seems that his results are not directly related with approximation of partial differential equations.

This paper is motivated by the paper [5] and devoted to a study of the finite-difference method applied to (1.1). Thus, we propose finite-difference schemes and prove convergence results (cf. Theorems 3 and 4) for those schemes even when time-increments approaches to zero. To accomplish this purpose, we rewrite the equation as

$$u_t + u_x = \phi, \quad \phi_t - \phi_x = |u|^p,$$

which is based on the formal factorization $u_{tt} - u_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)u = |u|^p$, and then follow the method of convergence analysis proposed by [15] that is originally developed to study time-discretizations for a system of nonlinear Schrödinger equations. Actually, it suffices to prove local stability results in a certain sense (cf. Theorems 1 and 2) in order to obtain convergence results. Moreover, we show that discrete analogues of (1.9) and (1.10) holds true, and therefore, we can deduce approximation of blow-up time (1.14) (cf. Theorem 5).

This paper is organized as follows. In Section 2, after having stated our finite-difference schemes, we mention stability and convergence results for our schemes (Theorems 1, 2, 3 and 4). Therein, approximation of blow-up time is also mentioned (Theorem 5). Section 3 is devoted to the proofs of Theorems 1, 2, 3 and 4. The proof of Theorem 5 is given in Section 4. We conclude this paper by examining several numerical examples in Section 5.

Notation. For $\boldsymbol{v} = (v_1, \dots, v_J)^{\mathrm{T}} \in \mathbb{R}^J$, we set $\|\boldsymbol{v}\| = \max_{1 \le j \le J} |v_j|$, where $\cdot^{\mathrm{T}}$ indicates the transpose of a matrix. We write $\boldsymbol{v} \ge \boldsymbol{0}$ if and only if $v_i \ge 0$

$(1 \leq i \leq J)$. We use the matrix ∞ norm

$$\|E\| = \max_{\boldsymbol{v}\in\mathbb{R}^J} \frac{\|E\boldsymbol{v}\|}{\|\boldsymbol{v}\|} = \max_{1\leq i\leq J} \sum_{j=1}^{J} |E_{ij}|$$

for a matrix $E = (E_{ij}) \in \mathbb{R}^{J\times J}$. Moreover, we write $E \geq O$ if and only if $E_{i,j} \geq 0$ $(1 \leq i, j \leq J)$. The set of all positive integers is denoted by $\mathbb{N}$.

2. Schemes and Main Results

Introducing a new variable $\phi = u_t + u_x$, we first convert (1.1) into the first order system as follows:

$$\begin{cases} u_t + u_x = \phi & (t,x) \in Q_{T,L}, \\ \phi_t - \phi_x = |u|^p & (t,x) \in Q_{T,L}, \\ u(0,x) = u_0(x), \quad \phi(0,x) = u_1(x) + u_0'(x), & x \in S_L. \end{cases} \tag{2.1}$$

Take a positive integer J and set $x_j = jh$ with $h = L/J$. As a discretization of the time variable, we take positive constants $\Delta t_0, \Delta t_1, \ldots$ and set

$$t_0 = 0, \quad t_n = \sum_{k=0}^{n-1} \Delta t_k = t_{n-1} + \Delta t_{n-1} \quad (n \geq 1).$$

Then, our explicit scheme to find

$$u_j^n \approx u(t_n, x_j), \quad \phi_j^n \approx \phi(t_n, x_j) \qquad (1 \leq j \leq J,\ t \geq 0)$$

reads as

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} + \dfrac{u_j^n - u_{j-1}^n}{h} = \phi_j^n \\ \dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t_n} - \dfrac{\phi_{j+1}^n - \phi_j^n}{h} = |u_j^{n+1}|^p \end{cases} \qquad (1 \leq j \leq J,\ n \geq 0) \tag{2.2}$$

where u_0^n and ϕ_{J+1}^n are set as $u_0^n = u_J^n$ and $\phi_{J+1}^n = \phi_1^n$.

We also consider an implicit scheme for the purpose of comparison. However, we do not prefer fully implicit schemes since we need iterative computations for solving resulting nonlinear system. Instead, we consider a linearly-implicit scheme by introducing dual time grids

$$t_{n+\frac{1}{2}} = \frac{\Delta t_0}{2} + t_n \quad (n \geq 0). \tag{2.3}$$

Then, our implicit scheme to find

$$u_j^n \approx u(t_n, x_j), \quad \phi_j^{n+\frac{1}{2}} \approx \phi(t_{n+\frac{1}{2}}, x_j) \qquad (1 \le j \le J,\ n \ge 0)$$

reads as

$$\text{(2.4)} \quad \begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} + \dfrac{1}{2}\left(\dfrac{u_j^{n+1} - u_{j-1}^{n+1}}{h} + \dfrac{u_j^n - u_{j-1}^n}{h}\right) = \phi_j^{n+\frac{1}{2}}, \\ \dfrac{\phi_j^{n+\frac{3}{2}} - \phi_j^{n+\frac{1}{2}}}{\Delta t_n} - \dfrac{1}{2}\left(\dfrac{\phi_{j+1}^{n+\frac{3}{2}} - \phi_j^{n+\frac{3}{2}}}{h} + \dfrac{\phi_{j+1}^{n+\frac{1}{2}} - \phi_j^{n+\frac{1}{2}}}{h}\right) \\ \qquad\qquad\qquad\qquad = |u_j^{n+1}|^p, \end{cases} \quad (1 \le j \le J,\ n \ge 0),$$

where u_0^n and $\phi_{J+1}^{n+\frac{1}{2}}$ are set as $u_0^n = u_J^n$ and $\phi_{J+1}^{n+\frac{1}{2}} = \phi_1^{n+\frac{1}{2}}$.

REMARK 2.1. It is possible to take

$$t_{\frac{1}{2}} = \frac{\Delta t_0}{2}, \quad t_{n+\frac{1}{2}} = \frac{\Delta t_0}{2} + \sum_{k=1}^{n} \tau_k \quad (n \ge 1)$$

as dual time grids instead of (2.3), where $\tau_k = (\Delta t_{k-1} + \Delta t_k)/2$. With this choice, the implicit scheme is modified as

$$\text{(2.5)} \quad \begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t_n} + \dfrac{1}{2}\left(\dfrac{u_j^{n+1} - u_{j-1}^{n+1}}{h} + \dfrac{u_j^n - u_{j-1}^n}{h}\right) = \phi_j^{n+\frac{1}{2}}, \\ \dfrac{\phi_j^{n+\frac{3}{2}} - \phi_j^{n+\frac{1}{2}}}{\tau_n} - \dfrac{1}{2}\left(\dfrac{\phi_{j+1}^{n+\frac{3}{2}} - \phi_j^{n+\frac{3}{2}}}{h} + \dfrac{\phi_{j+1}^{n+\frac{1}{2}} - \phi_j^{n+\frac{1}{2}}}{h}\right) \\ \qquad\qquad\qquad\qquad = |u_j^{n+1}|^p, \end{cases} \quad (1 \le j \le J,\ n \ge 0).$$

Then, we can deduce all the results presented below with obvious modifications.

For $n \ge 0$, we set

$$\boldsymbol{u}^n = (u_1^n, \dots, u_J^n)^{\mathrm{T}} \in \mathbb{R}^J,$$
$$\boldsymbol{\phi}^n = (\phi_1^n, \dots, \phi_J^n)^{\mathrm{T}} \in \mathbb{R}^J, \qquad \boldsymbol{\phi}^{n+\frac{1}{2}} = (\phi_1^{n+\frac{1}{2}}, \dots, \phi_J^{n+\frac{1}{2}})^{\mathrm{T}} \in \mathbb{R}^J.$$

THEOREM 1 (Local stability of the explicit scheme). *Let $\tau = \gamma h$ with some $\gamma \in (0,1)$ and assume that $\Delta t_n \leq \tau$ for $n \geq 0$. Let $\boldsymbol{a} \geq \mathbf{0}, \boldsymbol{b} \geq \mathbf{0} \in \mathbb{R}^J$. Then, the solution $(\boldsymbol{u}^n, \boldsymbol{\phi}^n)$ of the explicit scheme (2.2) with $\boldsymbol{u}^0 = \boldsymbol{a}$ and $\boldsymbol{\phi}^0 = \boldsymbol{b}$ satisfies $\boldsymbol{u}^n \geq \mathbf{0}$ and $\boldsymbol{\phi}^n \geq \mathbf{0}$ for $n \geq 1$. Furthermore, for any $N \in \mathbb{N}$, there exists a constants $h_{R,N} > 0$ depending only on N and $R = \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$ such that, if $h \in (0, h_{R,N}]$, we have*

$$\sup_{1\leq n\leq N} (\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^n\|) \leq 2R. \tag{2.6}$$

THEOREM 2 (Well-posedness and local stability of the implicit scheme). *Let $\tau = 2\gamma h$ with some $\gamma \in (0,1)$ and assume that $\Delta t_n \leq \tau$ for $n \geq 0$. Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^J$. Then, the implicit scheme (2.4) admits a unique solution $(\boldsymbol{u}^n, \boldsymbol{\phi}^{n+\frac{1}{2}})$ for any $n \geq 1$, where $\boldsymbol{u}^0 = \boldsymbol{a}$ and $\boldsymbol{\phi}^{\frac{1}{2}} = \boldsymbol{b}$. Moreover, if $\boldsymbol{a} \geq \mathbf{0}$ and $\boldsymbol{b} \geq \mathbf{0}$, then we have $\boldsymbol{u}^n \geq \mathbf{0}$ and $\boldsymbol{\phi}^{n+\frac{1}{2}} \geq \mathbf{0}$ for $n \geq 1$. Furthermore, for any $N \in \mathbb{N}$, there exists a constants $h_{R,N} > 0$ depending only on N and $R = \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$ such that, if $h \in (0, h_{R,N}]$, we have*

$$\sup_{1\leq n\leq N} \left(\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\|\right) \leq 2R. \tag{2.7}$$

In order to state convergence results, we introduce $\boldsymbol{e}^n = (e_j^n)$, $\boldsymbol{\varepsilon}^n = (\varepsilon_j^n)$ and $\boldsymbol{\varepsilon}^{n+\frac{1}{2}} = (\varepsilon_j^{n+\frac{1}{2}})$ which are given as

$$e_j^n = u(t_n, x_j) - u_j^n, \quad \varepsilon_j^n = \phi(t_n, x_j) - \phi_j^n, \quad \varepsilon_j^{n+\frac{1}{2}} = \phi(t_{n+\frac{1}{2}}, x_j) - \phi_j^{n+\frac{1}{2}}.$$

Recall that T_∞ denotes the blow-up time of the solution $u(t,x)$ of (1.1).

THEOREM 3 (Convergence of the explicit scheme). *Let $\tau = \gamma h$ with some $\gamma \in (0,1)$ and assume that $\Delta t_n \leq \tau$ for $n \geq 0$. Suppose that $(\boldsymbol{u}^n, \boldsymbol{\phi}^n)$ is the solution of the explicit scheme (2.2) for $n \geq 1$, where $(\boldsymbol{u}^0, \boldsymbol{\phi}^0)$ is defined as*

$$u_j^0 = u_0(x_j), \quad \phi_j^0 = u_1(x_j) + u_0'(x_j) \quad (1 \leq j \leq J). \tag{2.8}$$

Let $T \in (0, T_\infty)$ be arbitrarily. Then, there exists positive constants h_0 and M_0 which depend only on

$$p, \quad T, \quad \gamma, \quad M = \max_{0\leq m+l\leq 3} \left\| \frac{\partial^m}{\partial t^m} \frac{\partial^l}{\partial x^l} u \right\|_{L^\infty(Q_{T,L})} \tag{2.9}$$

such that we have

$$\max_{0\le t_n\le T}\left(\|\boldsymbol{e}^n\|+\|\boldsymbol{\varepsilon}^n\|\right)\le M_0(\tau+h)$$

for any $h\in(0,h_0]$.

THEOREM 4 (Convergence of the implicit scheme). *Let* $\tau=2\gamma h$ *with some* $\gamma\in(0,1)$ *and assume that* $\Delta t_n\le\tau$ *for* $n\ge 0$. *Suppose that* $(\boldsymbol{u}^n,\boldsymbol{\phi}^{n+\frac{1}{2}})$ *is the solution of the implicit scheme* (2.4) *for* $n\ge 1$, *where* $(\boldsymbol{u}^0,\boldsymbol{\phi}^{\frac{1}{2}})$ *is defined as*

$$u_j^0=u_0(x_j),\quad \phi_j^{\frac{1}{2}}=u_1(x_j)+u_0'(x_j)\quad(1\le j\le J). \tag{2.10}$$

Let $T\in(0,T_\infty)$ *be arbitrarily. Then, there exists positive constants* h_0 *and* M_0, *which depend only on* (2.9), *such that we have*

$$\max_{0\le t_{n+1}\le T}\left(\|\boldsymbol{e}^n\|+\|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|\right)\le M_0(\tau+h) \tag{2.11}$$

for any $h\in(0,h_0]$.

REMARK 2.2. If taking constant time-increments $\Delta t_n=\tau$ and suitable initial value $\boldsymbol{\phi}^{\frac{1}{2}}$, we can prove

$$\max_{0\le t_{n+1}\le T}\left(\|\boldsymbol{e}^n\|+\|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|\right)\le M_0(\tau^2+h)$$

instead of (2.11).

By using the solutions of the explicit scheme (2.2) and the implicit scheme (2.4), we can calculate the blow-up time T_∞ of the solution of (1.1). To this purpose, we fix

$$1\le q<\infty,\qquad 0<\gamma<1 \tag{2.12}$$

and choose the time increments $\Delta t_0,\Delta t_1,\ldots$ as

$$\Delta t_n=\tau\cdot\min\left\{1,\ \frac{1}{\|\boldsymbol{u}^n\|^q}\right\}\qquad(n\ge 0), \tag{2.13}$$

where τ is taken as

$$\tau = \begin{cases} \gamma h & \text{for the explicit scheme (2.2)} \\ 2\gamma h & \text{for the implicit scheme (2.4).} \end{cases} \tag{2.14}$$

DEFINITION 1. Let $\boldsymbol{u}^n$ be the solution of the explicit scheme (2.2) or the implicit scheme (2.4) with the time increment control (2.13) and (2.14). Then, we set

$$T(h) = \sum_{n=0}^{\infty} \Delta t_n.$$

If $T(h) < \infty$, we say that $\boldsymbol{u}^n$ blows up in finite time $T(h)$.

REMARK 2.3. The blow-up of $\boldsymbol{u}^n$ implies that $\lim_{t_n \to T(h)} \|\boldsymbol{u}^n\| = \lim_{n\to\infty} \|\boldsymbol{u}^n\| = \infty$.

We are now in a position to state numerical blow-up results.

THEOREM 5 (Approximation of the blow-up time). *Let $\boldsymbol{u}^n$ be the solution of the explicit scheme* (2.2) *or the implicit scheme* (2.4) *with the time increment control* (2.13) *and* (2.14), *where the initial value is defined as* (2.8) *or* (2.10), *respectively. In addition to the basic assumptions* (1.4) *and* (1.5) *on initial values, assume that $u_1(x)$ is so large that*

$$u_1(x) + u_0'(x) \geq 0, \not\equiv 0 \quad (x \in S_L). \tag{2.15}$$

Then, we have the following:

(i) $\boldsymbol{u}^n \geq 0$ *and* $\boldsymbol{\phi}^n \geq \mathbf{0}$ *(or* $\boldsymbol{\phi}^{n+\frac{1}{2}} \geq \mathbf{0}$*) for all* $n \geq 0$.

(ii) *If* (1.8) *holds true,* $\boldsymbol{u}^n$ *blows up in finite time* $T(h)$ *and*

$$T_\infty \leq \liminf_{h\to 0} T(h). \tag{2.16}$$

(iii) *In addition to* (1.8), *we assume that*

$$\lim_{t \to T_\infty} K(u(t)) = \infty, \tag{2.17}$$

then we have

$$T_\infty = \lim_{h \to 0} T(h). \tag{2.18}$$

REMARK 2.4. The assumption (2.17) is somewhat restrictive. Essentially the same assumption is considered in [5]. However, we are unable to remove it at present. To find the sufficient condition for (2.17) to hold is an interesting open question.

REMARK 2.5. All results presented above remain valid for $f(u) = u|u|^2$, since it is a C^4 function on $\mathbb{R}$.

3. Proofs of Theorems 1, 2, 3 and 4

We rewrite the explicit scheme (2.2) and the implicit scheme (2.4), respectively, as

$$\begin{cases} \boldsymbol{u}^{n+1} = M_n \boldsymbol{u}^n + \Delta t_n \boldsymbol{\phi}^n \\ \boldsymbol{\phi}^{n+1} = N_n \boldsymbol{\phi}^n + \Delta t_n \boldsymbol{f}(\boldsymbol{u}^{n+1}) \end{cases} \qquad (n \geq 0), \tag{3.1}$$

and

$$\begin{cases} A_n \boldsymbol{u}^{n+1} = B_n \boldsymbol{u}^n + \Delta t_n \boldsymbol{\phi}^{n+\frac{1}{2}} \\ C_n \boldsymbol{\phi}^{n+\frac{3}{2}} = D_n \boldsymbol{\phi}^{n+\frac{1}{2}} + \Delta t_n \boldsymbol{f}(\boldsymbol{u}^{n+1}) \end{cases} \qquad (n \geq 0), \tag{3.2}$$

where

$$M_n = P(-\gamma_n), \quad N_n = P(-\gamma_n)^{\mathrm{T}},$$
$$A_n = P(\delta_n), \quad B_n = P(-\delta_n), \quad C_n = P(\delta_n)^{\mathrm{T}}, \quad D_n = P(-\delta_n)^{\mathrm{T}},$$
$$\gamma_n = \frac{\Delta t_n}{h}, \quad \delta_n = \frac{\Delta t_n}{2h},$$

$$P(\mu) = \begin{pmatrix} 1+\mu & 0 & \cdots & -\mu \\ -\mu & 1+\mu & 0 & \vdots \\ & \ddots & \ddots & 0 \\ \text{\Large 0} & & -\mu & 1+\mu \end{pmatrix},$$

$$\boldsymbol{f}(\boldsymbol{v}) = (|v_1|^p, \ldots, |v_J|^p)^{\mathrm{T}} \quad \text{for} \quad \boldsymbol{v} = (v_1, \ldots, v_J)^{\mathrm{T}}.$$

LEMMA 3.1.

(i) *$P(\mu)$ is non-singular, $P(\mu)^{-1} \geq O$ and $\|P(\mu)^{-1}\| \leq 1$ if $\mu > 0$.*

(ii) *$P(-\mu) \geq O$ and $\|P(-\mu)\| = 1$ if $0 < \mu \leq 1$.*

PROOF. (i) Let $\mu > 0$. The matrix $P(\mu)$ is expressed as $P(\mu) = (1+\mu)(I-G)$, where

$$G = \frac{\mu}{1+\mu} \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Since $\|G\| = \mu(1+\mu)^{-1} < 1$, the matrix $I - G$ is non-singular, $(I-G)^{-1} = \sum_{l=0}^{\infty} G^l \geq O$ and $\|(I-G)^{-1}\| \leq 1/(1-\|G\|) = 1+\mu$. Hence, $P(\mu)$ is also non-singular, $P(\mu)^{-1} = (1+\mu)^{-1} \sum_{l=0}^{\infty} G^l \geq O$ and $\|P(\mu)^{-1}\| \leq (1+\mu)^{-1}\|(I - G)^{-1}\| = 1$.
(ii) Let $0 < \mu \leq 1$. Then, $P(-\mu) \geq O$ is obvious. We further have

$$\|P(-\mu)\| = \max_{1 \leq i \leq J} \sum_{j=1}^{J} |p_{ij}| = (1-\mu) + \mu = 1,$$

where $P(\mu) = (p_{ij})$, which completes the proof. □

Now, we can state the following proofs.

Proofs of Theorems 1 and 2. According to Lemma 3.1, we have $M_n, N_n, B_n, D_n \geq O$ and $\|M_n\| = \|N_n\| = \|B_n\| = \|D_n\| = 1$. Moreover, A_n, C_n are non-singular, $A_n^{-1}, C_n^{-1} \geq O$ and $\|A_n^{-1}\|, \|C_n^{-1}\| \leq 1$. Therefore, the unique existence and non-negativity of solutions of (2.2) and (2.4) are direct consequences of the expressions (3.1) and (3.2), respectively.

Below we are going to show local stability results (2.6) and (2.7). We only state the proof of (2.7); that of (2.6) could be done in the same way. Recall that we are assuming that $\Delta t_j \leq \tau$ for all j and $\tau = 2\gamma h$ with some $\gamma \in (0, 1)$. Choose $N \in \mathbb{N}$ arbitrarily and fix it.

Now we can prove (2.7) by induction on n. First, note that $\|\boldsymbol{u}^0\| + \|\boldsymbol{\phi}^{\frac{1}{2}}\| = \|\boldsymbol{a}\| + \|\boldsymbol{b}\| = R$. Assume that

$$\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\| \leq 2R \tag{3.3}$$

for $0 \leq n \leq N-1$. Since $\boldsymbol{u}^{n+1}$ and $\boldsymbol{\phi}^{n+\frac{3}{2}}$ are given as

$$\boldsymbol{u}^{n+1} = H_n \cdots H_0 \boldsymbol{a} + \sum_{j=0}^{n} \Delta t_{n-j} H_n \cdots H_{n-j+1} A_{n-j}^{-1} \boldsymbol{\phi}^{n-j+\frac{1}{2}},$$

$$\boldsymbol{\phi}^{n+\frac{3}{2}} = L_n \cdots L_0 \boldsymbol{b} + \sum_{j=0}^{n} \Delta t_{n-j} L_n \cdots L_{n-j+1} C_{n-j}^{-1} \boldsymbol{f}(\boldsymbol{u}^{n-j+1})$$

with $H_n = A_n^{-1} B_n$ and $L_n = C_n^{-1} D_n$, we have

$$\|\boldsymbol{u}^{n+1}\| \leq \|\boldsymbol{a}\| + \tau \sum_{j=0}^{n} \|\boldsymbol{\phi}^{n-j+\frac{1}{2}}\| \leq \|\boldsymbol{a}\| + N\tau(2R),$$

$$\|\boldsymbol{\phi}^{n+\frac{3}{2}}\| \leq \|\boldsymbol{b}\| + \tau \sum_{j=0}^{n} \|\boldsymbol{u}^{n-j+1}\|^p \leq \|\boldsymbol{b}\| + N\tau(2R)^p$$

for $0 \leq n \leq N-1$. Hence,

$$\|\boldsymbol{u}^{n+1}\| + \|\boldsymbol{\phi}^{n+\frac{3}{2}}\| \leq R + N\tau[2R + (2R)^p] \tag{3.4}$$

for $0 \leq n \leq N-1$.

At this stage, we define $\tau_{R,N}$ and $h_{R,N}$ as

$$\tau_{R,N} = \frac{R}{N[2R + (2R)^p]}, \qquad h_{R,N} = \frac{\tau_R}{2\gamma}$$

and suppose $h \in (0, h_{R,N}]$.

Then, by (3.4), we get

$$\|\boldsymbol{u}^{n+1}\| + \|\boldsymbol{\phi}^{n+\frac{3}{2}}\| \leq 2R.$$

This completes the proof of (2.7). □

We proceed to the proof of convergence results. Below, we only state the proof of Theorem 4 since that of Theorem 3 is simpler.

PROOF OF THEOREM 4. Let $\{(\boldsymbol{u}^n, \boldsymbol{\phi}^{n+\frac{1}{2}})\}_{n\geq 1}$ be the solution of the implicit scheme (2.4) with the initial condition (2.10). We note that

$$\|\boldsymbol{u}^0\| + \|\boldsymbol{\phi}^{\frac{1}{2}}\| \leq 3M.$$

Hereinafter, set $M' = 3M$. In view of Theorem 2, there exists constants $h_{M'} > 0$ and $T_{M'} > 0$, which depend only on M' and p, such that, if $h \in (0, h_{M'}]$, we have

$$\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\| \leq 2M' \quad (n \in \Lambda_{M'} = \{n \in \mathbb{N} \mid t_n \leq T_{M'}\}).$$

We set

$$\nu = \sup\{n \in \mathbb{N} \mid \|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\| \leq 3M'\},$$
$$\tilde{\Lambda}_\nu = \{n \in \mathbb{N} \mid t_{n+1} \leq T,\ n \leq \nu\}.$$

The rest of the proof is divided into two steps.

Step 1. First, we show that there exist positive constants h_1 and M_0, which depend only on T and M, such that the estimate (2.11) holds for all $h \in (0, h_1]$ and $n \in \tilde{\Lambda}_\nu$.

We have for $n \in \tilde{\Lambda}_\nu$

$$(3.5)\qquad e_j^n - e_j^{n-1} + \frac{\Delta t_{n-1}}{2}\left(\frac{e_j^n - e_{j-1}^n}{h} + \frac{e_j^{n-1} - e_{j-1}^{n-1}}{h}\right) = \Delta t_{n-1} E_j^{n-\frac{1}{2}},$$

where $E_j^{n-\frac{1}{2}} = \varepsilon_j^{n-\frac{1}{2}} - E_{1j}^{n-\frac{1}{2}} - E_{2j}^{n-\frac{1}{2}}$,

$$\begin{aligned} E_{1j}^{n-\frac{1}{2}} &= u_t(t_{n-\frac{1}{2}}, x_j) - \frac{u(t_n, x_j) - u(t_{n-1}, x_j)}{\Delta t_{n-1}}, \\ E_{2j}^{n-\frac{1}{2}} &= u_x(t_{n-\frac{1}{2}}, x_j) \\ &\quad - \frac{1}{2}\left(\frac{u(t_n, x_j) - u(t_n, x_{j-1})}{h} + \frac{u(t_{n-1}, x_j) - u(t_{n-1}, x_{j-1})}{h}\right). \end{aligned}$$

Since (3.5) is equivalently written as

$$\boldsymbol{e}^n = A_{n-1}^{-1} B_{n-1} \boldsymbol{e}^{n-1} + \Delta t_{n-1} A_{n-1}^{-1} \boldsymbol{E}^{n-\frac{1}{2}},$$

where $\boldsymbol{E}^{n-\frac{1}{2}} = (E_j^{n-\frac{1}{2}})$, we have from Lemma 3.1

$$\begin{aligned} \|\boldsymbol{e}^n\| &\leq \|\boldsymbol{e}^{n-1}\| + \Delta t_{n-1} \|\boldsymbol{E}^{n-\frac{1}{2}}\| \\ &\leq \|\boldsymbol{e}^{n-1}\| + \Delta t_{n-1}(\|\boldsymbol{E}_1^{n-\frac{1}{2}}\| + \|\boldsymbol{E}_2^{n-\frac{1}{2}}\|) + \Delta t_{n-1} \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|. \end{aligned}$$

From the standard error estimates for the difference quotients, we obtain

$$\|\boldsymbol{E}_1^{n-\frac{1}{2}}\| \leq CM\Delta t_{n-1}, \quad \|\boldsymbol{E}_2^{n-\frac{1}{2}}\| \leq CM(\Delta t_{n-1} + h)$$

for $n \in \tilde{\Lambda}_\nu$. Consequently,

$$\|\boldsymbol{e}^n\| \leq \|\boldsymbol{e}^{n-1}\| + CM\Delta t_{n-1}(\Delta t_{n-1} + h) + \Delta t_{n-1} \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\| \tag{3.6}$$

for $n \in \tilde{\Lambda}_\nu$.

Similarly, we have for $n \in \tilde{\Lambda}_\nu$

$$\varepsilon_j^{n+\frac{1}{2}} - \varepsilon_j^{n-\frac{1}{2}} - \frac{\Delta t_{n-1}}{2}\left(\frac{\varepsilon_{j+1}^{n+\frac{1}{2}} - \varepsilon_j^{n+\frac{1}{2}}}{h} + \frac{\varepsilon_{j+1}^{n-\frac{1}{2}} - \varepsilon_j^{n-\frac{1}{2}}}{h}\right) = \Delta t_{n-1} \xi_j^n,$$

or, equivalently,

$$\boldsymbol{\varepsilon}^{n+\frac{1}{2}} = C_{n-1}^{-1} D_{n-1} \boldsymbol{\varepsilon}^{n-\frac{1}{2}} + \Delta t_{n-1} C_{n-1}^{-1} \boldsymbol{\xi}^n,$$

where $\xi_j^n = -\xi_{1j}^n + \xi_{2j}^n + \xi_{3j}^n$, $\boldsymbol{\xi}^n = (\xi_j^n)$ and

$$\begin{aligned}
\xi_{1j}^n &= \phi_t(t_n, x_j) - \frac{\phi(t_{n+\frac{1}{2}}, x_j) - \phi(t_{n-\frac{1}{2}}, x_j)}{\Delta t_{n-1}}, \\
\xi_{2j}^n &= \phi_x(t_n, x_j) \\
&\quad -\frac{1}{2}\left(\frac{\phi(t_{n+\frac{1}{2}}, x_{j+1}) - \phi(t_{n+\frac{1}{2}}, x_j)}{h} + \frac{\phi(t_{n-\frac{1}{2}}, x_{j+1}) - \phi(t_{n-\frac{1}{2}}, x_j)}{h}\right), \\
\xi_{3j}^n &= |u(t_n, x_j)|^p - |u_j^n|^p.
\end{aligned}$$

We know

$$\|\boldsymbol{\xi}_1^n\| \le CM\Delta t_{n-1}, \quad \|\boldsymbol{\xi}_2^n\| \le CM(\Delta t_{n-1} + h)$$

for $n \in \tilde{\Lambda}_\nu$. Since $|u(t_n, x_j)| \le M$ and $|u_j^n| \le 3M'$, we can estimate as

$$\Big||u(t_n, x_j)|^p - |u_j^n|^p\Big| \le C_{2p} M^{p-1} |u(t_n, x_j) - u_j^n|$$

for $n \in \tilde{\Lambda}_\nu$ and $1 \le j \le J$, where C_2p denotes a constant depending only on p. Hence, we deduce

$$\|\boldsymbol{\xi}_3^n\| \le CM^{p-1}\|\boldsymbol{e}^n\|$$

for $n \in \tilde{\Lambda}_\nu$. Thus, we obtain

$$(3.7) \quad \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \le \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\| + CM\Delta t_{n-1}(\Delta t_{n-1} + h) + CM^{p-1}\Delta t_{n-1}\|\boldsymbol{e}^n\|.$$

Summing up (3.6) and (3.7), we deduce

$$\begin{aligned}
(3.8) \quad \|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \le \|\boldsymbol{e}^{n-1}\| + \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\| + CM\Delta t_{n-1}(\Delta t_{n-1} + h) \\
+ CM^{p-1}\Delta t_{n-1}\|\boldsymbol{e}^n\| + \Delta t_{n-1}\|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|.
\end{aligned}$$

Setting $M^* = M + M^{p-1}$, we have from (3.8)

$$\begin{aligned}
&(1 - CM^*\Delta t_{n-1})(\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\|) \\
&\le \|\boldsymbol{e}^{n-1}\| + (1 + \Delta t_{n-1})\|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\| + CM\Delta t_{n-1}(\Delta t_{n-1} + h) \\
&\le (1 + CM^*\Delta t_{n-1})(\|\boldsymbol{e}^{n-1}\| + \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|) + CM^*\Delta t_{n-1}(\Delta t_{n-1} + h).
\end{aligned}$$

At this stage, we define

$$h_1 = \frac{1}{4\gamma CM^*}, \quad \tau_1 = 2\gamma h_1$$

and we assume that $h \in (0, h_1]$. Then, using an elementally inequality $0 \le (1-s)^{-1}(1+s) \le 1+4s$ for $s \in [0, 1/2]$, we have

$$\begin{aligned}
&\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \\
&\le (1 + 4CM^*\Delta t_{n-1})(\|\boldsymbol{e}^{n-1}\| + \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|) + 2CM^*\Delta t_{n-1}(\Delta t_{n-1} + h) \\
&\le e^{4CM^*\Delta t_{n-1}}(\|\boldsymbol{e}^{n-1}\| + \|\boldsymbol{\varepsilon}^{n-\frac{1}{2}}\|) + 2CM^*\Delta t_{n-1}(\Delta t_{n-1} + h).
\end{aligned}$$

Therefore

$$\begin{aligned}
\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| &\le e^{4CM^*t_n}(\|\boldsymbol{e}^0\| + \|\boldsymbol{\varepsilon}^{\frac{1}{2}}\|) + 2CM^* \sum_{j=0}^{n-1} \Delta t_j(\Delta t_j + h) e^{4CM^*t_n} \\
&\le e^{4CM^*T}\|\boldsymbol{\varepsilon}^{\frac{1}{2}}\| + 2CM^*Te^{4CM^*T}(\tau + h).
\end{aligned}$$

On the other hand, we have $\|\boldsymbol{\varepsilon}^{\frac{1}{2}}\| \le (\tau+h)M$, since $\varepsilon_j^{\frac{1}{2}} = \phi(t_{\frac{1}{2}}, x_j) - \phi_j^{\frac{1}{2}} = u_t(t_{\frac{1}{2}}, x_j) + u_x(t_{\frac{1}{2}}, x_j) - u_1(x_j) - u_0'(x_j)$. Therefore, taking

$$M_0 = (Me^{4CM^*T} + 2CM^*Te^{4CM^*T}),$$

we have shown that the desired estimate (2.11) holds for all $h \in (0, h_1]$ and $n \in \tilde{\Lambda}_\nu$.

Step 2. We set

$$h_0 = \min\left\{h_1,\ \frac{M}{2M_0(1+2\gamma)},\ h_{\frac{3}{2M},1}\right\}$$

where $h_{\frac{3}{2}M,1}$ is the constant introduced in Theorem 2 with $R = \frac{3}{2}M$ and $N = 1$. Below we assume $h \in (0, h_0]$.

We prove

$$\max\{n \in \mathbb{N} \mid t_{n+1} \le T\} \le \nu \tag{3.9}$$

by showing a contradiction. Thus, we assume

$$\max\{n \in \mathbb{N} \mid t_{n+1} \le T\} > \nu.$$

Then, we have $\tilde{\Lambda}_\nu = \{1, \dots, \nu\}$ and, since $h_0 \le h_1$ in view of Step 1,

$$\|\boldsymbol{e}^n\| + \|\boldsymbol{\varepsilon}^{n+\frac{1}{2}}\| \le M_0(1 + 2\gamma)h$$

for all $n = 1, \dots, \nu$. Moreover, since $t_{\nu+1} \le T$, it follows from the definition of M that

$$\max_{n=1,\dots,\nu} (\|\boldsymbol{u}(t_n)\| + \|\boldsymbol{\phi}(t_{n+\frac{1}{2}})\|) \le M,$$

where $\boldsymbol{u}(t_n) = (u(t_n, x_j))$ and $\boldsymbol{\phi}(t_{n+\frac{1}{2}}) = (\phi(t_{n+\frac{1}{2}}, x_j))$. Combining those inequalities, we get

$$\|\boldsymbol{u}^n\| + \|\boldsymbol{\phi}^{n+\frac{1}{2}}\| \le M + M_0(1 + 2\gamma)h$$

for all $n = 1, \dots, \nu$. In particular,

$$\|\boldsymbol{u}^\nu\| + \|\boldsymbol{\phi}^{\nu+\frac{1}{2}}\| \le M + M_0 h \le \frac{3}{2}M.$$

Now, we apply Theorem 2 with $\boldsymbol{a} = \boldsymbol{u}^\nu$, $\boldsymbol{b} = \boldsymbol{\phi}^{\nu+\frac{1}{2}}$, $R = \frac{3}{2}M$, and $N = 1$ to obtain

$$\|\boldsymbol{u}^{\nu+1}\| + \|\boldsymbol{\phi}^{\nu+\frac{3}{2}}\| \le 3M.$$

This contradicts the definition of ν. Therefore, (3.9) actually holds true. Hence, by the result of Step 1, we see that the desired estimate (2.11) holds for all $h \in (0, h_0]$ and $n \in \mathbb{N}$ satisfying $t_{n+1} \le T$. This completes the proof of Theorem 4. □

4. Proof of Theorem 5

This section is devoted to the proof of numerical blow-up result, Theorem 5. We shall deal only with the case of the explicit scheme (2.2); the case of the implicit scheme (2.4) is proved in exactly the same way.

Throughout this section, suppose that $(\boldsymbol{u}^n, \boldsymbol{\phi}^n)$ denotes the solution of the explicit scheme (2.2) as in Theorem 5. Further, we suppose that all assumptions of Theorem 5 hold true. In view of (2.15), we may suppose that $\boldsymbol{\phi}^0, \boldsymbol{u}^1 \geq \mathbf{0}, \neq \mathbf{0}$ for a sufficiently small $h > 0$. Consequently, we have $\boldsymbol{u}^n, \boldsymbol{\phi}^n \geq \mathbf{0}, \neq \mathbf{0}$ for $n \geq 1$.

Before stating the proof of Theorem 5, we establish a discrete version of (1.10). To this end, we introduce the functional

$$K_h(\boldsymbol{v}) = \frac{1}{L}\sum_{j=1}^{J} v_j h \qquad (\mathbf{0} \leq \boldsymbol{v} \in \mathbb{R}^J) \tag{4.1}$$

and consider the discrete version $K_h(\boldsymbol{u}^n)$ of $K(u(t))$. We note that $K_h(\boldsymbol{u}^n) \geq 0$ and $K_h(\boldsymbol{\phi}^n) \geq 0$ for $n \geq 0$. In particular,

$$K_h(\boldsymbol{\phi}^0) > 0, \quad \alpha_h = K_h(\boldsymbol{u}^0) \geq 0, \quad \beta_h = K_h(\boldsymbol{u}^1) > 0. \tag{4.2}$$

Lemma 4.1. *$K_h(\boldsymbol{u}^n)$ is a strictly increasing sequence in $n \geq 0$ and it satisfies*

$$\left[\frac{K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)}{\Delta t_n}\right]^2 \geq \frac{1}{p+1} K_h(\boldsymbol{u}^n)^{p+1} + M_{1h} \geq 0 \tag{4.3}$$

for $n \geq 0$, where

$$M_{1h} = \left(\frac{\beta_h - \alpha_h}{\Delta t_0}\right)^2 - \frac{1}{p+1}\alpha_h^{p+1}. \tag{4.4}$$

Proof. We have

$$\begin{aligned} \frac{K_h(\boldsymbol{u}^{n+1}) - K_h(\boldsymbol{u}^n)}{\Delta t_n} &= \frac{1}{L}\sum_{j=1}^{J} \frac{u_j^{n+1} - u_j^n}{\Delta t_n} h \\ &= \frac{1}{L}\sum_{j=1}^{J} \left[-\frac{u_j^n - u_{j-1}^n}{h} + \phi_j^n\right] h = K_h(\boldsymbol{\phi}^n) \end{aligned} \tag{4.5}$$

for $n \geq 0$. In particular, by (4.2)

$$\frac{K_h(\boldsymbol{u}^1) - K_h(\boldsymbol{u}^0)}{\Delta t_0} \geq K_h(\boldsymbol{\phi}^0) > 0 \tag{4.6}$$

By using Jensen's inequality, we have from (4.5)

$$\begin{aligned}\frac{K_h(\boldsymbol{\phi}^{n+1})-K_h(\boldsymbol{\phi}^n)}{\Delta t_n} &= \frac{1}{L}\sum_{j=1}^{J}\left[\frac{\phi_{j+1}^n-\phi_j^n}{h}+\left(u_j^{n+1}\right)^p\right]h \\ &= \frac{1}{L}\sum_{j=1}^{J}\left(u_j^{n+1}\right)^p h \geq K_h(\boldsymbol{u}^{n+1})^p.\end{aligned}$$

Combining these, we obtain

$$\frac{K_h(\boldsymbol{u}^{n+2})-K_h(\boldsymbol{u}^{n+1})}{\Delta t_{n+1}}$$

$$\geq \frac{K_h(\boldsymbol{u}^{n+1})-K_h(\boldsymbol{u}^n)}{\Delta t_n}+\Delta t_n(K_h(\boldsymbol{u}^{n+1}))^p \tag{4.7}$$

$$\geq \frac{K_h(\boldsymbol{u}^1)-K_h(\boldsymbol{u}^0)}{\Delta t_0}+\sum_{k=0}^{n}\Delta t_k(K_h(\boldsymbol{u}^{k+1}))^p > 0 \tag{4.8}$$

for $n \geq 0$. This, together with (4.6), implies that $K_h(\boldsymbol{u}^n)$ is a strictly increasing sequence in $n \geq 0$.

Again, we apply (4.7) to obtain

$$\begin{aligned}&\left[\frac{K_h(\boldsymbol{u}^{n+2})-K_h(\boldsymbol{u}^{n+1})}{\Delta t_{n+1}}\right]^2 \\ &\quad\geq \frac{K_h(\boldsymbol{u}^{n+1})-K_h(\boldsymbol{u}^n)}{\Delta t_n}\left[\frac{K_h(\boldsymbol{u}^{n+1})-K_h(\boldsymbol{u}^n)}{\Delta t_n}+\Delta t_n\left(K_h(\boldsymbol{u}^{n+1})\right)^p\right] \\ &\quad= \left[\frac{K_h(\boldsymbol{u}^{n+1})-K_h(\boldsymbol{u}^n)}{\Delta t_n}\right]^2+\left(K_h(\boldsymbol{u}^{n+1})-K_h(\boldsymbol{u}^n)\right)K_h(\boldsymbol{u}^{n+1})^p.\end{aligned}$$

Hence,

$$\begin{aligned}&\left[\frac{K_h(\boldsymbol{u}^{n+2})-K_h(\boldsymbol{u}^{n+1})}{\Delta t_{n+1}}\right]^2 \\ &\quad\geq \sum_{k=0}^{n}\left(K_h(\boldsymbol{u}^{k+1})-K_h(\boldsymbol{u}^k)\right)K_h(\boldsymbol{u}^{k+1})^p+\left[\frac{K_h(\boldsymbol{u}^1)-K_h(\boldsymbol{u}^0)}{\Delta t_0}\right]^2 \\ &\quad\geq \int_{\alpha_h}^{K_h(\boldsymbol{u}^{k+1})} z^p\,dz+\left(\frac{\beta_h-\alpha_h}{\Delta t_0}\right)^2\end{aligned}$$

$$= \frac{1}{p+1}\left(K_h(\boldsymbol{u}^{n+1})^{p+1}-\alpha_h^{p+1}\right)+\left(\frac{\beta_h-\alpha_h}{\Delta t_0}\right)^2. \tag{4.9}$$

Since $K_h(\boldsymbol{u}^n)$ is non-decreasing in n, the right-hand side of (4.9) is non-negative. This completes the proof of Lemma 4.1. □

REMARK 4.2. Under the assumptions of Theorem 3, we have $M_{1h} \to \beta^2 - \frac{1}{p+1}\alpha^{p+1}$ as $h \to 0$.

REMARK 4.3. In view of (4.6) and (4.8),

$$\frac{K_h(\boldsymbol{u}^{n+2}) - K_h(\boldsymbol{u}^{n+1})}{\Delta t_{n+1}} \geq K_h(\boldsymbol{\phi}^0) \equiv \nu_h,$$

where ν_h is a positive number which is independent of n. This implies that $K_h(\boldsymbol{u}^n)$ is not a bounded sequence in n. In particular, there exists $m \in \mathbb{N}$ such that $K_h(\boldsymbol{u}^m) > 1$.

At this stage, we set

$$G(z) = \sqrt{\frac{1}{p+1} z^{p+1} + M_{1h}}.$$

Note that $G(z)$ is a strictly increasing function in $z \in [\alpha_h, \infty)$.

In view of Lemma 4.1, we can follow exactly the same argument of the proof of [5, Lemma 5.4] and obtain the following lemma.

LEMMA 4.4. *There exists a positive constant C which is independent of h such that*

$$T(h) \leq 2\left(\int_{\alpha_h}^{\infty} \frac{dz}{G(z)} + C\tau\right).$$

In particular, we have $T(h) < \infty$.

Now we can state the following proof.

PROOF OF THEOREM 5. (i) It is a direct consequence of Theorems 1 and 2.
(ii) According to Lemma 4.4, we have $T(h) < \infty$; $\boldsymbol{u}^n$ blows up in finite time. We prove that

$$T_\infty \leq \liminf_{h \to 0} T(h) \equiv T_* \tag{4.10}$$

by showing a contradiction. Thus, we assume that

$$T_* < T_\infty.$$

Then, there exists a subsequence $\{h_i\}_i$ such that $h_i \to 0$ as $i \to \infty$ and that

$$T(h_i) \le T_* + \delta < T_\infty,$$

where $\delta = (T_\infty - T_*)/2$. We have

$$\max_{0 \le t \le T_* + \delta} \|u(t)\|_{L^\infty(S_L)} < \infty. \tag{4.11}$$

On the other hand, the solution $\boldsymbol{u}^n = \boldsymbol{u}^n(h_i)$ of the explicit scheme (2.2) corresponding to the parameter $h = h_i$ satisfies (cf. Remark 2.3)

$$\lim_{n\to\infty} \|\boldsymbol{u}^n(h_i)\| = \lim_{t_n \to T(h_i)} \|\boldsymbol{u}^n(h_i)\| = \infty. \tag{4.12}$$

These (4.11) and (4.12) contradict to Theorem 3. Hence, (4.10) is proved.
(iii) We assume (2.17); thus, $u(t,x)$ and $K(u(t))$ blow up in finite time $t = T_\infty$. We now prove that

$$T^* \equiv \limsup_{h\to 0} T(h) \le T_\infty \tag{4.13}$$

by showing a contradiction. In fact, this, together with (4.10), implies $T_\infty = \lim\limits_{h\to 0} T(h)$, which completes the proof. We assume

$$T_\infty < T^*$$

and set $\epsilon = (T^* - T_\infty)/4$. There exist $R > 0$ and $h_{**} > 0$ such that

$$2\left(\int_R^\infty \frac{dz}{G(z)} + C\gamma h_{**}\right) < \epsilon.$$

Below we fix such R and h_{**}. Further, there exists $t' = t'_R < T_\infty$ such that $K(u(t')) > 2R$. Set

$$T = t' + \frac{T_\infty - t'}{2} = \frac{t' + T_\infty}{2} < T_\infty$$

and let M and M_0 be the positive constants appearing Theorem 3 corresponding to this T. Set

$$h_* = \min\left\{h_{**},\ \frac{T_\infty - t'}{2\gamma},\ \frac{R}{M + M_0(1+\gamma)}\right\}$$

and suppose $h \in (0, h_*]$ below. Then, we have $Mh + M_0(\tau + h) \leq R$ and $\tau \leq T - t'$.

According to Theorem 3, we have

$$\begin{aligned}
&|K(u(t_n)) - K_h(\boldsymbol{u}^n)| \\
&\quad \leq \frac{1}{L}\sum_{j=1}^{J}\int_{x_{j-1}}^{x_j} |u(t_n,x) - u_j^n|\, dx \\
&\quad \leq \frac{1}{L}\sum_{j=1}^{J}\int_{x_{j-1}}^{x_j} (|u(t_n,x) - u(t_n,x_j)| + |u(t_n,x_j) - u_j^n|)\, dx \\
&\quad \leq Mh + M_0(\tau + h) \leq R
\end{aligned}$$

and, therefore,

$$K_h(\boldsymbol{u}^n) \geq K(u(t_n)) - R.$$

There exists $k \in \mathbb{N}$ satisfying $t' \leq t_k < T_\infty$, since $\tau \leq T - t' < T_\infty - t'$. Then,

$$K_h(\boldsymbol{u}^k) \geq K(u(t_k)) - R > R. \tag{4.14}$$

At this stage, we can take a subsequence $\{h_i\}_i$ such that

$$T_\infty + \epsilon < T(h_i)$$

and $h_i \to 0$ as $i \to \infty$. However, in view of Lemma 4.4 and (4.14), we have

$$T(h_i) = t_k + \sum_{n=k}^{\infty} \Delta t_n < T_\infty + 2\left(\int_R^\infty \frac{dz}{G(z)} + C\tau_i\right).$$

Therefore, by the definition of R and h_{**}, we obtain $T(h_i) < T_\infty + \epsilon$, which is a contradiction. Hence, we obtain (4.13). This completes the proof of Theorem 5. □

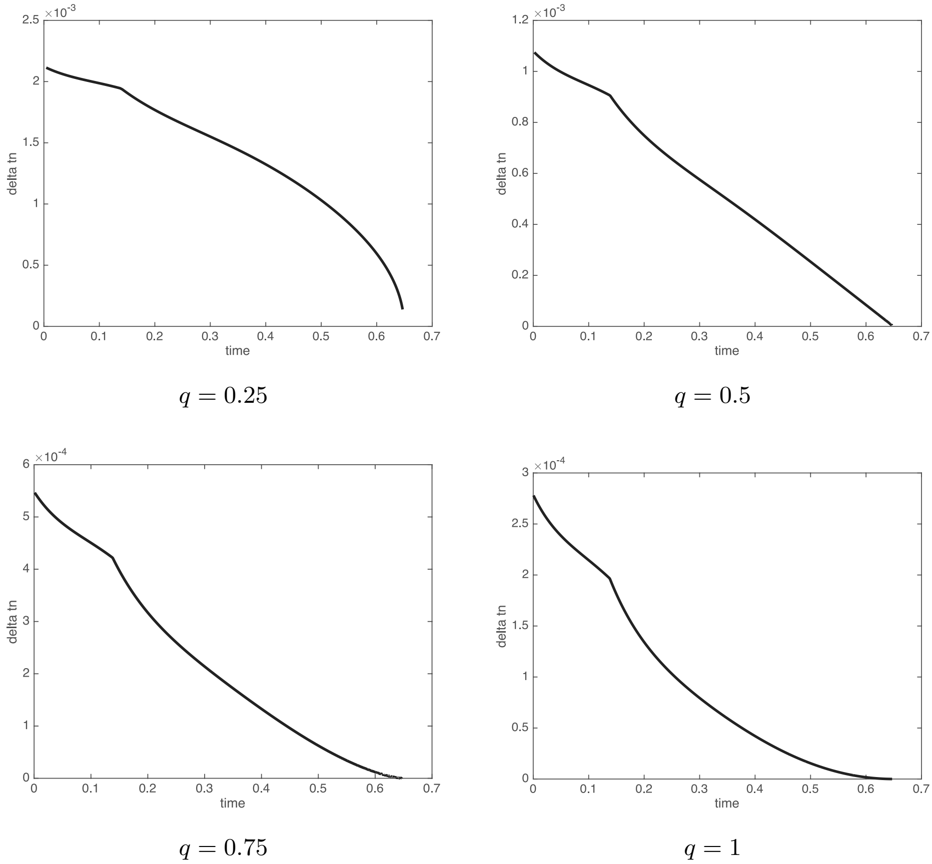


Fig. 1. The history of Δt_n for $p = 2$.

5. Numerical experiments

In this section, we offer some numerical examples and examine the validity of our proposed finite-difference schemes. Suppose $L = 1$ and take

$$u_0(x) = \frac{\lambda}{2}(\sin(4\pi x) + 2), \quad u_1(x) = 2\pi\lambda + \mu$$

as initial values. Then, if $\lambda, \mu > 0$, we have $\alpha = K(u_0) = \lambda > 0$, $\beta = K(u_1) = 2\pi\lambda + \mu > 0$ and $u_0'(x) + u_1(x) \geq \mu > 0$. Below we set $\lambda = 10$ and $\mu = 5$.

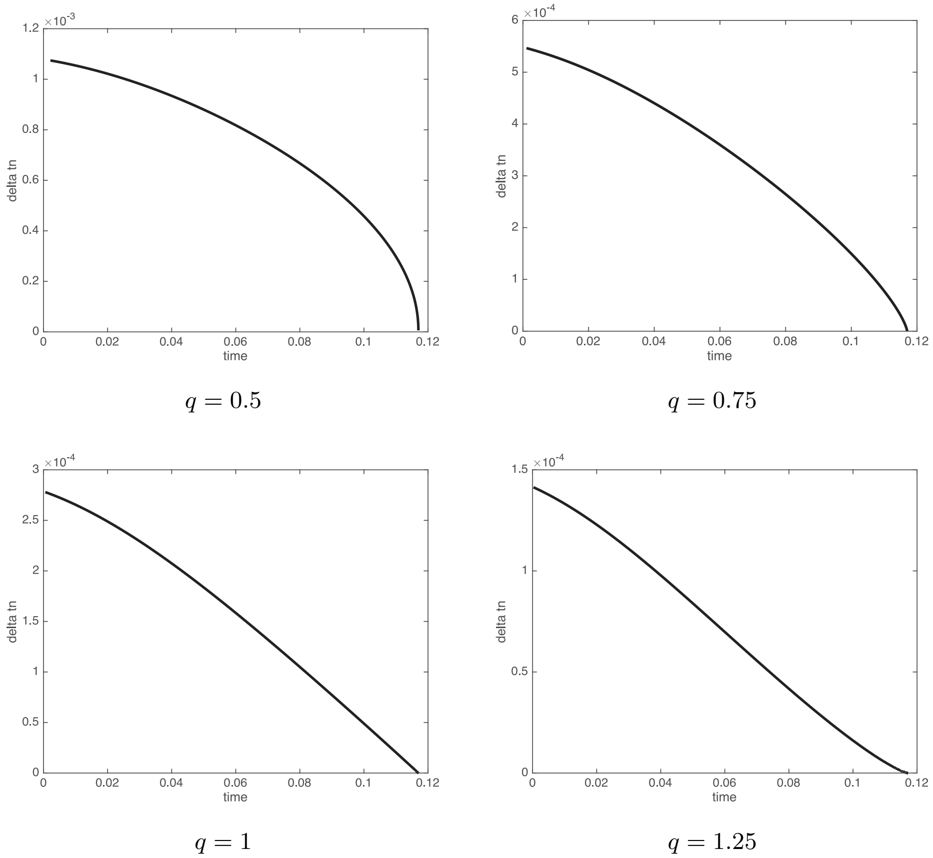
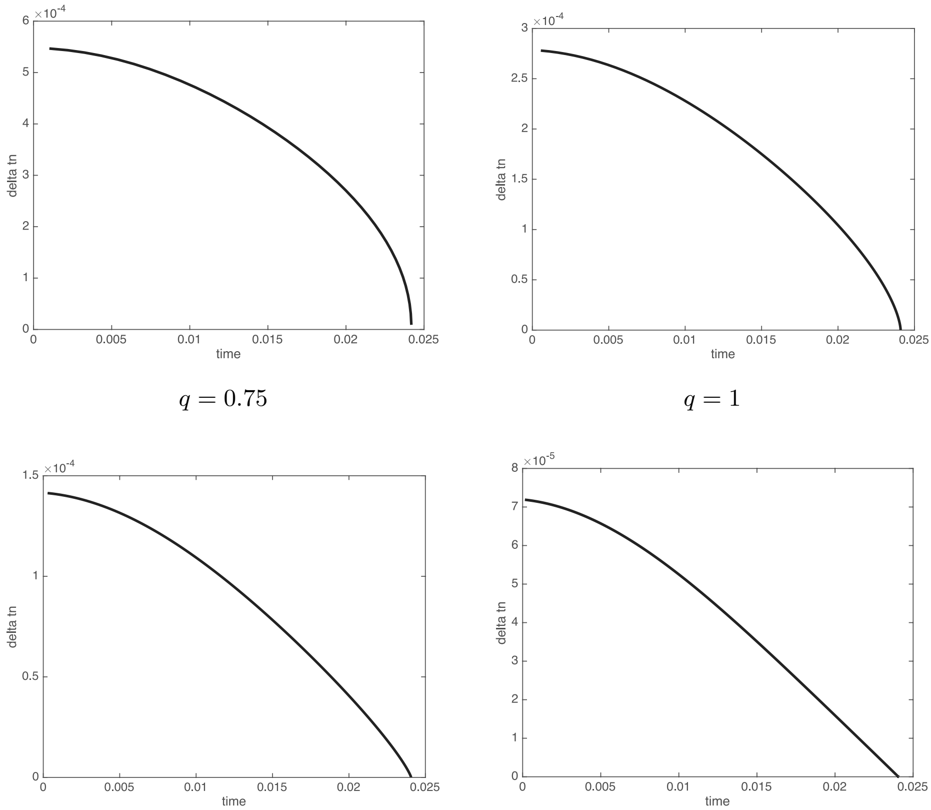


Fig. 2. The history of Δt_n for $p = 3$.

5.1. Choice of q

We first examine the value of q in the definition of Δt_n. We consider the explicit scheme (2.2). In Fig. 1, we plot Δt_n as a function of t_n when $p = 2$. We see that Δt_n deceases as a linear function if $q = 0.5$ whereas it deceases very rapidly if $q = 0.25$ and very slowly if $q = 0.75, 1$. Results for the cases of $p = 3$ and 4 are reported in Fig. 2 and 3, respectively. Here, the case $p = 3$ means the nonlinearlity $f(u) = u|u|^2$; see Remark 2.5. For each p, there is $q = q_*$ such that Δt_n deceases linearly if $q = q_*$ and it deceases very rapidly if $q < q_*$ and very slowly if $q > q_*$.



$q = 0.75$ $q = 1$

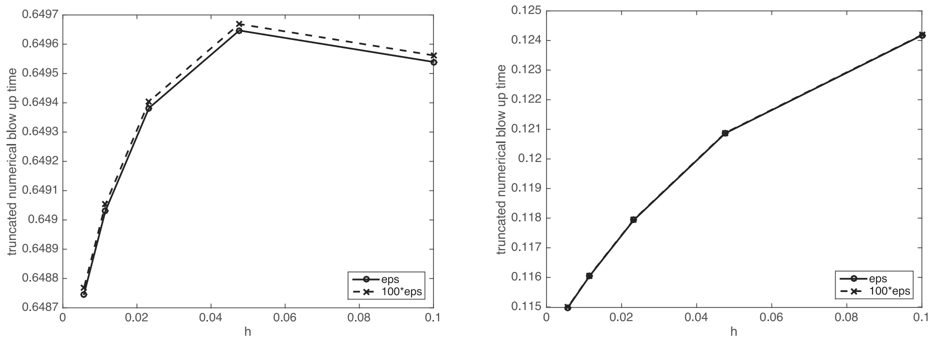
$q = 1.25$ $q = 1.5$

Fig. 3. The history of Δt_n for $p = 4$.

Slowly-deceasing cases are not suitable from the viewpoint of efficiency. On the other hand, we do not prefer rapidly-deceasing cases since it is difficult to capture clearly the variation of a numerical solution near $t = T(h)$ even if Δt_n is quite small.

Consequently, as a better choice, we offer

$$q = \begin{cases} 0.5 & (p = 2) \\ 1 & (p = 3) \\ 1.5 & (p = 4). \end{cases} \tag{5.1}$$



(a) $p = 2$, $\varepsilon = 10^{-12}$ (b) $p = 3$, $\varepsilon = 10^{-7}$

Fig. 4. Truncated numerical blow-up time $T(h;\varepsilon)$ for stopping criteria ε and 100ε.

Below we choose q as (5.1).

5.2. Stopping criterion

The numerical blow-up time is an infinite series defined as

$$T(h) = \sum_{n=0}^{\infty} \Delta t_n.$$

Therefore, in actual computations, we take a sufficiently large n and regard t_n as a reasonable approximation of $T(h)$. For this purpose, we introduce the *truncated numerical blow-up time* $T(h;\varepsilon)$ by setting

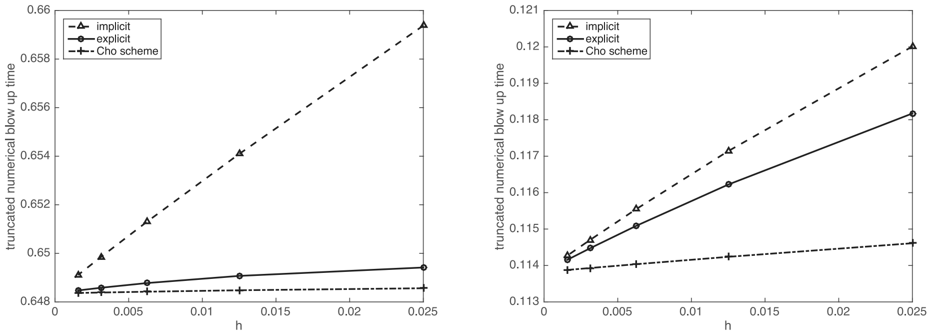
$$T(h;\varepsilon) = \min\left\{t_n \mid \|\boldsymbol{u}^n\| > \varepsilon^{-1}\right\}, \tag{5.2}$$

where $\varepsilon > 0$ is the stopping criterion given below.

We still consider the explicit scheme (2.2) and plot $T(h,\varepsilon)$, $T(h;100\varepsilon)$ for several h in Fig. 4. For suitably small ε and h, $T(h,\varepsilon)$ and $T(h;100\varepsilon)$ are almost equal so that we can take $T(h;\varepsilon)$ as a reasonable approximation of the exact blow-up time.

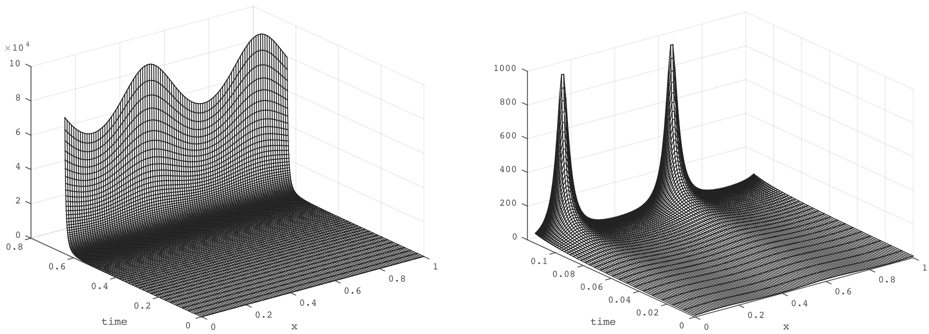
5.3. Comparison of our schemes and Cho's scheme

We compare three finite-difference schemes; the explicit scheme (2.2), the implicit scheme (2.4) and the Cho's scheme (1.17) with obvious modification of the boundary condition.



(a) $p = 2$, $\varepsilon = 10^{-10}$ (b) $p = 3$, $\varepsilon = 10^{-5}$

Fig. 5. Truncated numerical blow-up time $T(h; \varepsilon)$ for three schemes.



(a) $p = 2$ (b) $p = 3$

Fig. 6. Shapes of finite-difference solutions $\boldsymbol{u}^n$ of the explicit scheme (2.2).

Fig. 4, we plot $T(h; \varepsilon)$ for several h by using those three schemes. We see that those $T(h; \varepsilon)$ converge to a certain value, say the exact blow-up time, as $h \to 0$. Thus, we can apply anyone to compute the blow-up solutions. Cho's scheme is better than ours. But, again, it should be kept in mind that our schemes and the numerical blow-up times are guaranteed to converge by the mathematical proof.

Furthermore, we conjecture form those figures that the rate of convergence of $T(h)$ is expressed as

$$|T(h) - T_\infty| \leq Ch = C\tau$$

if τ/h is fixed. We, however, have no mathematical proof; for similar difficulties for parabolic problems, see [4].

We finally give the shapes of solutions $\boldsymbol{u}^n$ of the explicit scheme (2.2) in Fig. 6.

Acknowledgement. We thank C. H. Cho for valuable discussions. This work is supported by JST, CREST, and JSPS KAKENHI Grant Number 23340023.

References

[1] Abia, L. M., López-Marcos, J. C. and J. Martínez, On the blow-up time convergence of semidiscretizations of reaction-diffusion equations, Appl. Numer. Math. **26** (1998), 399–414.

[2] Alinhac, S., *Blowup for Nonlinear Hyperbolic Equations*, Birkhäuser, 1995.

[3] Chen, Y. G., Asymptotic behaviours of blowing-up solutions for finite difference analogue of $u_t = u_{xx} + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **33** (1986), 541–574.

[4] Cho, C. H., Hamada, S. and H. Okamoto, On the finite difference approximation for a parabolic blow-up problems, Japan J. Indust. Appl. Math. **24** (2007), 105–134.

[5] Cho, C. H., A finite difference scheme for blow-up solutions of nonlinear wave equations, Numer. Math. Theory Methods Appl. **3** (2010), 475–498.

[6] Evans, L. C., *Partial Differential Equations*, American Mathematical Society, 1998.

[7] Glassey, R. T., Blow-up theorems for nonlinear wave equations, Math. Z. **132** (1973), 183–203.

[8] Glassey, R. T., Finite-time blow-up for solutions of nonlinear wave equations, Math. Z. **177** (1981), 323–340.

[9] John, F., Blow-up of solutions of nonlinear wave equations in three space dimensions, Manuscripta Math. **28** (1979), 235–268.

[10] Kato, T., Blow-up of solutions of some nonlinear hyperbolic equations, Comm. Pure Appl. Math. **33** (1980), 501–505.

[11] Kawarada, H., On solutions of nonlinear wave equations, J. Phys. Soc. Japan **31** (1971), 280–282.

[12] Matsuya, K., The blow-up theorem of a discrete semilinear wave equation, J. Difference Eq. Appl. **19** (2013), 457–465.

[13] Nakagawa, T., Blowing up of a finite difference solution to $u_t = u_{xx} + u^2$, Appl. Math. Optim. **2** (1976), 337–350.
[14] Nakagawa, T. and T. Ushijima, Finite element analysis of the semi-linear heat equation of blow-up type, Topics Numer. Anal. III, Academic Press, 275–291, 1977.
[15] Sasaki, T., A second-order time-discretization scheme for a system of nonlinear Schrödinger equations, Proc. Japan. Acad. Ser. A Math. Sci. **90** (2014), 15–20.

(Received July 22, 2014)

Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1 Komaba, Meguro
Tokyo 153-8914, Japan
E-mail: norikazu@ms.u-tokyo.ac.jp
tsasaki@ms.u-tokyo.ac.jp