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On a Tower of Good Affinoids in $X_0(p^n)$ and the Inertia Action on the Reduction

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Abstract. Coleman and McMurdy calculate the stable reduction of $X_0(p^3)$ for any prime number $p \geq 13$, on the basis of rigid geometry in [CM]. Further, in [CM2], they compute also the inertia action on the stable reduction of $X_0(p^3)$. In [T], we have determined the stable model of $X_0(p^4)$ for any prime $p \geq 13$. In this paper, we calculate the reductions of some "good" affinoids in $X_0(p^n)$ and determine the inertia action on them. As a result, we study the middle cohomology of the reductions in terms of the type theory for $GL_2(\mathbb{Q}_p)$ given in [BH].

1. Introduction

Let K be a non-archimedean local field, and let h be a non-negative integer. As in [Ca], the non-abelian Lubin-Tate theory asserts that the local Langlands correspondence and the local Jacquet-Langlands correspondence for $GL_h(K)$ are realized in the cohomology of the Lubin-Tate space. Harris-Taylor and Boyer prove this by using global automorphic representations in [HT] and [Bo] in the cases where K has mixed and equal characteristics respectively. The correspondence given by them seems not so explicit. In a series of papers [BH1], [BH2] and [BH3], without geometry, Bushnell-Henniart study supercuspidal representations of $GL_h(K)$ in a purely representation-theoretic manner on the basis of [BK], and give an explicit description of the local Langlands correspondence for essentially tame representations. See [He, §6] for more details on explicit local Langlands correspondence. To know a purely local and geometric proof of non-abelian Lubin-Tate theory, it needs to understand purely local geometric properties of Lubin-Tate spaces. In [We], when the residual characteristic of K is odd and $h = 2$, Weinstein classifies types of irreducible components in the stable

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reduction of the Lubin-Tate curve with full-level structure up to purely inseparable map. To prove this, he regards the projective limit of Lubin-Tate spaces as a perfectoid space, which is called the Lubin-Tate perfectoid space. Then, he constructs a family of affinoids in the Lubin-Tate perfectoid curve and determines the reductions of them. As a result, with the help of the non-abelian Lubin-Tate theory, the CM theory and the type theory due to Bushnell and Henniart in [BH], he deduces that any irreducible component in the stable reduction of the Lubin-Tate curve with finite full-level structure admits a purely inseparable map from one of the smooth compactifications of the reductions of the affinoids in the Lubin-Tate perfectoid curves. In this paper, without ambiguity of the purely inseparability and without depending on the above theories, we explicitly determine the reductions of some affinoids in the modular curve $X_0(p^n)$ for any prime $p \geq 13$. Since some of the smooth compactifications of the reductions have positive genera, we conclude that the reductions actually appear as Zariski open subsets of irreducible components in the stable reduction of $X_0(p^n)$ by [IT4, Proposition 7.11] (cf. Corollary 3.15). Such verifications are useful for a complete understanding of a concrete configuration of the stable reduction of $X_0(p^n)$ for each n. Note that the Lubin-Tate curve in the case where $K = \mathbb{Q}_p$ and $h = 2$ is isomorphic to the generic fiber of the formal completion of the Katz-Mazur model of a modular curve at a supersingular point with j-invariant $\neq 0, 1728$. See [KM] for the Katz-Mazur model. Except for some examples, a concrete understanding of configurations of stable reductions of Lubin-Tate curves or of modular curves with fixed finite level structures is not known. Our ultimate aim is to understand local geometry of Lubin-Tate space, to give a geometric realization of Galois representations and the type theory and, as a result, an explicit and geometric understanding of the local Langlands correspondence in a purely local manner (cf. [Ha, Questions 8,9 in §3]). We believe that the study in this direction will give a new insight to explicit local Langlands correspondence. Actually, in the higher dimensional case, we will give a geometric realization for epipelagic representations, for which the explicit local Langlands correspondence is studied in [BH4] up to unramified twists, and remove the ambiguity of unramified twists in [IT5].

Let p be a prime number. For a non-negative integer n, let $X_0(p^n)$ be the modular curve over $\mathbb{Q}$, which is the moduli space whose valued point corresponds to an isomorphism class of a pair (E, C), where E is a generalized elliptic curve and C is its cyclic subgroup of order p^n. We consider the stable reduction of $X_0(p^n)$ at p. Assume that $p \geq 13$. In [CM], Coleman and McMurdy calculate the stable reduction of $X_0(p^3)$ on the basis of rigid geometry. As a result, they find that several copies of the Artin-Schreier curve with affine model $a^p - a = s^2$ appear as irreducible components in the stable reduction of $X_0(p^3)$. Moreover, they determine the inertia action on the stable reduction of $X_0(p^3)$ in [CM2]. In [T], we computed the stable reduction of $X_0(p^4)$ using techniques in [CM]. As a result, in [T], we proved that some copies of the Hermitian curve appear as irreducible components in the stable reduction of $X_0(p^4)$. The main part in loc. cit. is in proving that singular residue classes in some affinoid of $X_0(p^4)$ are basic wide open spaces whose underlying affinoids reduce to the affine curves defined by $a^p - a = t^{p+1}$. In this paper, we will partially generalize the results to general level in some sense. In the following, we explain the contents of the generalization.

By $p \geq 13$ and Howe's result in [CM, Theorem B.1], there exists a supersingular elliptic curve over $\mathbb{F}_p$ such that its j-invariant is neither 0 nor 1728. We fix such an elliptic curve A. We regard $X_0(p^n)$ as the rigid analytic curve over $\mathbb{Q}_p$ and focus on the tubular neighborhood in $X_0(p^n)$ of A, which we denote by $W_A(p^n)$. We define several affinoids in $W_A(p^n)$. The space $W_A(p)$ is known to be isomorphic to an annulus $A(p^{-1}, 1)$. We fix an isomorphism $W_A(p) \simeq A(p^{-1}, 1)$ appropriately. We consider two special circles $\mathbf{TS}_A = C[p^{-\frac{p}{p+1}}] \subset W_A(p)$ and $\mathbf{SD}_A = C[p^{-1/2}] \subset W_A(p)$. Let $\pi_f,\ \pi_\nu \colon X_0(p^n) \to X_0(p^{n-1})$ be natural level-lowering finite morphisms (cf. Definition 2.3). For $a, b \in \mathbb{Z}_{\geq 0}$, we put $\pi_{a,b} = \pi_\nu^a \circ \pi_f^b$. Let $n \geq 2$ be an integer. Then we define

$$\mathbf{Y}^A_{a,b} = \pi^{-1}_{a,b-1}(\mathbf{TS}_A) \subset W_A(p^n) \quad \text{with } a + b = n \geq 2,\ a, b \geq 1,$$

$$\mathbf{Z}^A_{c,d} = \pi^{-1}_{c,d}(\mathbf{SD}_A) \subset W_A(p^n) \quad \text{with } c + d = n - 1 \geq 2,\ c, d \geq 1.$$

In this paper, we will compute the reductions of $\mathbf{Y}^A_{n,1}$, $\mathbf{Z}^A_{n,1}$ for $n \geq 1$ and $\mathbf{Y}^A_{n,2}$ for $n \geq 2$ and analyze their singular residue classes. To calculate the reductions of them, we use some machinery constructed in [T, Corollary 2.19]. Further, we explicitly describe the inertial action on the reductions and study the middle cohomology of the reductions.

For an irreducible affine curve over an algebraic closure of a finite field, its genus means the genus of the smooth compactification of its normalization. Let $\mathbb{C}_p$ be the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let K be a complete subfield of $\mathbb{C}_p$, and let $\mathbb{F}_K$ be the residue field of K. For an affinoid space $\mathbf{W}$ over K, we write $\overline{\mathbf{W}}$ for its canonical reduction. Let $\mathrm{Red}\colon \mathbf{W}(\mathbb{C}_p) \to \overline{\mathbf{W}}(\overline{\mathbb{F}}_p)$ be the reduction map. For a point $P \in \overline{\mathbf{W}}(\mathbb{F}_K)$, let $R_{\mathbf{W}}(P)$ be the rigid analytic space over K such that $R_{\mathbf{W}}(P)(\mathbb{C}_p) = \mathrm{Red}^{-1}(P)$, which we call the residue class in $\mathbf{W}$ at P. In particular, if P is a singular point on $\overline{\mathbf{W}}$, we call $R_{\mathbf{W}}(P)$ the singular residue class at P. For any $n \geq 1$, the reduction $\overline{\mathbf{Y}}^A_{n,1}$ is defined by $st(s-t)^{p-1} = 1$ with genus $(p-1)/2$. This type of curve does not fit into the classification of Weinstein, because we consider stable reduction of $X_0(p^n)$. This curve is a quotient of the Deligne-Lusztig curve for $SL_2(\mathbb{F}_p)$, which is defined by $x^q y - xy^q = 1$. The curve defined by $x^q y - xy^q = 1$ is called also the Drinfeld curve. For any $n \geq 1$, the reduction $\overline{\mathbf{Z}}^A_{n,1}$ is defined by

$$Z^p + X^{p+1} + X^{-(p+1)} = 0 \ \text{ in } \mathbb{A}^2_{\overline{\mathbb{F}}_p}, \tag{1.1}$$

which has genus 0. This curve has the $2(p+1)$ singular points defined by $X = \zeta$ with $\zeta^{2(p+1)} = 1$. Each singular residue class S in $\mathbf{Z}^A_{n,1}$ is a basic wide open whose underlying affinoid $\mathbf{X}_S$ reduces to the affine curve defined by $a^p - a = s^2$ with genus $(p-1)/2$. The complement $S \setminus \mathbf{X}_S$ is an annulus of width $(4p^n)^{-1}$. For $n \geq 2$, the reduction $\overline{\mathbf{Y}}^A_{n,2}$ is defined by

$$xy(x-y)^{p-1} = 1, \quad Z^p + 1 + x^{-(p+1)} + y^{-(p+1)} = 0 \ \text{ in } \mathbb{A}^3_{\overline{\mathbb{F}}_p}, \tag{1.2}$$

which has genus $(p-1)/2$. The curve (1.2) has the $p+1$ singular points which are defined by $(x, y) = (-\zeta, \zeta)$ with $\zeta^{p+1} = -1$. Each singular residue class T in $\mathbf{Y}^A_{n,2}$ is a basic wide open whose underlying affinoid reduces to the affine curve defined by $a^p - a = t^{p+1}$ with genus $p(p-1)/2$. We can show that the complement $T \setminus \mathbf{X}_T$ is an annulus of width $(p^n(p+1))^{-1}$. When $n = 2$, these things are proved in [T]. However, there is a gap in the proof of [T, Corollary 4.8], and this will be fixed in the proof of Proposition 3.8.1. We write $\mathcal{T}_n$ for the set of the singular residue classes in $\mathbf{Y}^A_{n,2}$. As a result, we prove that, for each $n \geq 2$, the map $\pi_\nu\colon \mathbf{Y}^A_{n+1,2} \to \mathbf{Y}^A_{n,2}$ induces a purely inseparable map $\bar{\pi}_\nu\colon \overline{\mathbf{Y}}^A_{n+1,2} \to \overline{\mathbf{Y}}^A_{n,2}$ and $\pi_\nu(\mathcal{T}_{n+1}) = \mathcal{T}_n$. The restriction of π_ν induces $\pi_\nu\colon \coprod_{T' \in \mathcal{T}_{n+1}} \mathbf{X}_{T'} \to \coprod_{T \in \mathcal{T}_n} \mathbf{X}_T$. We show that this map induces a purely inseparable map $\bar{\pi}_\nu \colon \coprod_{T' \in \mathcal{T}_{n+1}} \overline{\mathbf{X}}_{T'} \to \coprod_{T \in \mathcal{T}_n} \overline{\mathbf{X}}_T$. Similar things are proved for $\mathbf{Z}^A_{n,1}$ with $n \geq 1$.

In [CM2, §6.3], the inertia action on the stable reduction of $X_0(p^3)$ is described using Fake CM, Weil pairing etc. In this paper, we give explicit descriptions of the inertia actions on the reductions of the affinoids explained above in terms of the Lubin-Tate theory. In particular, we make more explicit the inertia action on the components with affine model $a^p - a = s^2$ in the stable reduction of $X_0(p^3)$ described in [CM2, Corollary 6.11]. As application, we explicitly understand the structure of the middle cohomology of the irreducible components as representations of the inertia subgroup. We can conclude that the restrictions to the inertia subgroup of all two-dimensional Galois representations of exponential conductor ≤ 4 and with trivial determinant character appear in the middle cohomology of them. Using the description and the type theory in [BH], we can also describe the cohomology in terms of the language of the local Langlands correspondence. See Corollaries 4.9, 4.14 and 4.22 for precise statements. Such descriptions in finite levels in a purely local manner are not known except for [Yo]. Unfortunately, by using these descriptions, we cannot construct the local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ for representations of exponential Artin conductor 4. A complete treatment in this direction for representations of exponential Artin conductor three is given in [IT4]. The results in this paper will be used in a subsequent paper in which we determine the stable reduction of $X_0(p^5)$.

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Notation. We fix some p-adic notations. We let $\mathbb{C}_p$ be the completion of a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Let K be a complete subfield of $\mathbb{C}_p$. Let $\mathcal{O}_K$ denote the ring of integers of K, and let $\mathfrak{p}_K$ denote the maximal ideal of $\mathcal{O}_K$. Set $\mathbb{F}_K = \mathcal{O}_K/\mathfrak{p}_K$. We simply write $\mathbb{F}$ for $\mathbb{F}_{\mathbb{C}_p}$. For an element $a \in \mathcal{O}_K$, we write $\bar{a} \in \mathbb{F}_K$ for the reduction of a. Let $v(\cdot)$ denote the valuation of K such that $v(p) = 1$, and let $|\cdot|$ denote the absolute value given by $|x| = p^{-v(x)}$ for $x \in K^*$ and $|0| = 0$. For a positive integer $n \geq 1$, we set $U_K^n = 1 + \mathfrak{p}_K^n$. This is an open compact subgroup of K^*.

For a non-negative integer n, let $\mathbb{Q}_{p^n}$ denote the unramified extension of $\mathbb{Q}_p$ of degree n in $\overline{\mathbb{Q}}_p$. We write $\mathbb{Z}_{p^n}$ for $\mathcal{O}_{\mathbb{Q}_{p^n}}$. Let $\mathbb{F}_{p^n}$ be the extension of $\mathbb{F}_p$ of degree n. Then, we have $\mathbb{F}_{p^n} = \mathbb{F}_{\mathbb{Q}_{p^n}}$ in the above notation.

Put $\mathcal{R} = p^{\mathbb{Q}}$. Let K be a complete subfield of $\mathbb{C}_p$. For $r \in \mathcal{R}$, we let $B_K[r]$ and $B_K(r)$ denote the closed and open disks over K of radius r around 0, i.e. the rigid spaces over K whose $\mathbb{C}_p$-valued points are $\{x \in \mathbb{C}_p \mid |x| \leq r\}$ and $\{x \in \mathbb{C}_p \mid |x| < r\}$ respectively. If $r, s \in \mathcal{R}$ and $r \leq s$, let $A_K[r,s]$ and $A_K(r,s)$ be the rigid spaces over K whose $\mathbb{C}_p$-valued points are $\{x \in \mathbb{C}_p \mid r \leq |x| \leq s\}$ and $\{x \in \mathbb{C}_p \mid r < |x| < s\}$, which we call a closed annulus and an open annulus respectively. By the width of such an annulus, we mean $\log_p(s/r)$. A closed annulus of width 0 is called a circle, which we denote the circle $A_K[s,s]$ also by $C_K[s]$.

Let K be a complete subfield of $\mathbb{C}_p$ and A a reduced K-affinoid algebra. Let $|\cdot|_{\mathrm{sup}}$ be the supremum norm on A (cf. [BGR, §6.2]). We set

$$A^\circ = \{f \in A \mid |f|_{\mathrm{sup}} \leq 1\},$$
$$A^{\circ\circ} = \{f \in A \mid |f|_{\mathrm{sup}} < 1\}$$

and $\overline{A} = A^\circ / A^{\circ\circ}$. The ring A° is the set of all topologically bounded elements in A, and $A^{\circ\circ}$ is the set of all topologically nilpotent elements in A. For the affinoid space $\mathbf{X} = \operatorname{Sp} A$, the canonical reduction $\overline{\mathbf{X}}$ means $\operatorname{Spec} \overline{A}$ (cf. [BGR, §6.3]). The reduction of $\mathbf{X}$ always means the canonical reduction of $\mathbf{X}$.

2. Preliminaries

In this section, we collect some known facts and recall some machinery in Proposition 2.5 needed to compute the reductions of affinoids in later sections.

2.1. Kronecker's polynomial

As in [dSh, §0] or [T, §2.1], we recall Kronecker's polynomial

$$F_p(j, X) = (X - j(p\tau)) \prod_{0 \leq a \leq p-1} \left(X - j\left(\frac{\tau + a}{p}\right)\right) \in \mathbb{Z}[j, X].$$

Kronecker proved that

the equation $F_p(j, X) = 0$ gives a plane model of $X_0(p)$ over $\mathbb{Q}$,

$F_p(j, X) = F_p(X, j), \quad F_p(j, X) \equiv (j^p - X)(j - X^p) \mod p\mathbb{Z}[j, X].$

We write $F_p(S, T) = (S^p - T)(S - T^p) + pf(S, T)$, where $f(S, T) \in \mathbb{Z}[S, T]$ is a symmetric polynomial.

Let $p \geq 13$ be a prime number. Let A be a supersingular elliptic curve over $\mathbb{F}_p$ with $j(A) \neq 0, 1728$. Let $\beta_0 \in \mathbb{Z}_p^*$ lift $j(A) \in \mathbb{F}_p$. We put $F_p^{\beta_0}(X, Y) = F_p(X + \beta_0, Y + \beta_0)$.

Lemma 2.1 ([T, Lemma 2.1]). 1. *We have*

$$F_p^{\beta_0}(X, Y) = (X^p - Y)(X - Y^p) \mod p\mathbb{Z}_p[X, Y].$$

2. *We set* $f_1(X, Y) = \left(F_p^{\beta_0}(X, Y) - (X^p - Y)(X - Y^p)\right)/p \in \mathbb{Z}_p[X, Y]$. *Then* $f_1(X, Y)$ *is symmetric and* $f_1(0, 0)$ *is a unit of* $\mathbb{Z}_p$.

Let c_0 be the leading coefficient of f_1 and set $g(X, Y) = f_1(X, Y) - c_0$. We define $g(X)$ and $h(X)$ by

$$g(X, Y) \equiv Xg(X) + h(X)Y \mod (Y^2).$$

Let c_1 be the leading coefficient of $h(X)$, which is also the one of $g(X)$ by Lemma 2.1.2.

Lemma 2.2 ([T, Lemma 2.2]). *We consider the equation* $F_p^{\beta_0}(X, Y) = 0$. *Assume that* $0 < v(X), v(Y) < 1$ *and* $v(X) < v(Y^p)$. *Then we have*

$$Y = X^p + \frac{pc_0}{X} + \frac{pg(X, Y)}{X} + \sum_{n=1}^{\infty} \left(\frac{Y^p}{X}\right)^n \mathfrak{H}(X, Y), \tag{2.1}$$

where we put $\mathfrak{H}(X, Y) = (pc_0/X) + (pg(X, Y)/X)$.

2.2. Circles in $X_0(p)$

We briefly recall supersingular annuli from [CM, §3.1] or [T, §2.2]. Only in this subsection, we do not assume that $p \geq 13$. We think of $X_0(p^n)$ as the rigid analytic curve over $\mathbb{Q}_p$ whose points over $\mathbb{C}_p$ are in a one-to-one correspondence with isomorphism classes of pairs, (E, C), where $E/\mathbb{C}_p$ is a generalized elliptic curve and C is a cyclic subgroup of order p^n. We implicitly make use of this correspondence, when we speak loosely of "the point (E, C)."

DEFINITION 2.3. Let $C[p^i]$ denote the kernel of multiplication by p^i in C. Let

$$\pi_f, \pi_\nu \colon \coprod_{n \geq 1} X_0(p^n) \to \coprod_{n \geq 0} X_0(p^n)$$

be the level-lowering maps given by $\pi_f(E, C) = (E, pC)$ and $\pi_\nu(E, C) = (E/C[p], C/C[p])$ respectively.

Let $a, b \in \mathbb{Z}_{\geq 0}$. Then by setting $\pi_{a,b} = \pi_f^b \circ \pi_\nu^a$, we obtain maps

$$\pi_{a,b} \colon \coprod_{n \geq a+b} X_0(p^n) \to \coprod_{n \geq 0} X_0(p^n).$$

Let $w_n \colon X_0(p^n) \to X_0(p^n)$ be the automorphism defined by $(E, C) \mapsto (E/C, E[p^n]/C)$, which is called the Atkin-Lehner involution. We define

$$w \colon \coprod_{n \geq 0} X_0(p^n) \to \coprod_{n \geq 0} X_0(p^n)$$

by $w_n = w|_{X_0(p^n)}$. The Atkin-Lehner involution is compatible with the level-lowering maps in the sense that $\pi_f \circ w = w \circ \pi_\nu$ or equivalently, $w \circ \pi_f = \pi_\nu \circ w$, since w is an involution.

DEFINITION 2.4 ([CM, Definition 3.3]). For a fixed elliptic curve A over a finite field, let $W_A(p^n)$ represent the rigid subspace of $X_0(p^n)$ whose points over $\mathbb{C}_p$ are represented by pairs (E, C) with $\overline{E} \simeq A$.

The $W_A(1)$ is just a residue disk of the j-line. If E is an elliptic curve over $\mathbb{C}_p$, we let $h(E)$ denote the minimum of 1 and the valuation of a lifting of the Hasse invariant of the reduction of a non-singular model of E mod p, if it exists, and 0 otherwise. When A is a supersingular elliptic curve, it is well-known that $W_A(p)$ is isomorphic, over $\mathbb{Q}_{p^2}$, to an open annulus of width $i(A) = |\mathrm{Aut}(A)|/2$. This means that one can choose a parameter x_A on $W_A(p)$ over $\mathbb{Q}_{p^2}$ which identifies it with the open annulus $A_{\mathbb{Q}_{p^2}}(p^{-i(A)}, 1)$. In fact, we can and will always do this in such a way that $v(x_A(E, C)) = i(A)h(E)$ when C is a canonical subgroup of order p in E and otherwise $i(A)(1 - h(E/C))$ (cf. [Bu, Theorem 3.3 and §4]). See also [Ka] for canonical subgroups.

In [CM, §3.1], Coleman-McMurdy considered the following concentric circles in $B_{\mathbb{Q}_{p^2}}(1)$:

$$\mathbf{SD}_A = C_{\mathbb{Q}_{p^2}}\left[p^{-\frac{i(A)}{2}}\right] \text{ which they call the "self-dual circle" or the "Atkin-Lehner circle,"}$$

$$\mathbf{TS}_A = C_{\mathbb{Q}_{p^2}}\left[p^{-\frac{pi(A)}{p+1}}\right] \text{ which they call the "too supersingular circle."}$$

We fix an isomorphism $W_A(1) \simeq B_{\mathbb{Q}_{p^2}}(1)$ as in de Shalit's theorem in [CM, Theorem 3.5] or [T, Theorem 2.9]. Under the above identification $W_A(1) \simeq B_{\mathbb{Q}_{p^2}}(1)$, for $r \in (0,1) \cap \mathbb{Q}$, let $\mathbf{C}_r^{A,0}$ and $\mathbf{C}_{\geq r}^{A,0}$ denote the circle $C_{\mathbb{Q}_{p^2}}[p^{-r}]$ and the closed ball $B_{\mathbb{Q}_{p^2}}[p^{-r}]$ respectively.

2.3. Affinoids in $W_A(p^n)$

From now until the end of the paper, we assume that $p \geq 13$ and fix a supersingular elliptic curve $A/\mathbb{F}_p$ with $j(A) \neq 0, 1728$. Let $n \in \mathbb{Z}_{\geq 2}$. We define

$$\begin{aligned} &\mathbf{Y}_{a,b}^A = \pi_{a,b-1}^{-1}(\mathbf{TS}_A) \subset W_A(p^n) \text{ for } a+b=n,\ a,b \geq 1, \\ (2.2)\quad & \\ &\mathbf{Z}_{c,d}^A = \pi_{c,d}^{-1}(\mathbf{SD}_A) \subset W_A(p^n) \text{ for } c+d=n-1 \geq 2,\ c,d \geq 1. \end{aligned}$$

In the following proposition, we consider embeddings of $\mathbf{Y}_{a,b}^A$ and $\mathbf{Z}_{c,d}^A$ into products of subspaces of $W_A(1)$. Let $\beta_0 \in \mathbb{Z}_p^*$ be a lifting of $j(A)$.

Proposition 2.5 ([T, Corollary 2.19]). *Let π_n^0 be the map $\prod_{0 \leq i \leq n} \pi_{i,n-i} \colon W_A(p^n) \hookrightarrow W_A(1)^{\times(n+1)}$.*
1. *The affinoid $\mathbf{Y}_{a,b}^A$ with $a+b=n \geq 2$ is isomorphic to the following space by the map π_n^0:*

$$\left\{ (\{X_i\}_{0 \leq i \leq n}) \in \left(\prod_{0 \leq i \leq a-1} \mathbf{C}^{A,0}_{\frac{1}{p^{a-i-1}(p+1)}} \right) \right.$$

$$\times \mathbf{C}^{A,0}_{\geq \frac{p}{p+1}} \times \left(\prod_{a+1 \leq i \leq n} \mathbf{C}^{A,0}_{\frac{1}{p^{i-a-1}(p+1)}} \right) \Bigg| $$

$$F_p^{\beta_0}(X_i, X_{i+1}) = 0 \ \ \textit{for}\ 0 \leq i \leq n, \quad X_{a-1} \neq X_{a+1} \Bigg\}.$$

2. *The affinoid* $\mathbf{Z}^A_{c,d}$ *with* $c+d = n-1 \geq 2$ *is isomorphic to the following space by the map* π^0_n*:*

$$\Bigg\{ (\{X_i\}_{0 \leq i \leq n}) \in \left(\prod_{0 \leq i \leq c} \mathbf{C}^{A,0}_{\frac{1}{2p^{c-i}}} \right) \times \left(\prod_{c+1 \leq i \leq n} \mathbf{C}^{A,0}_{\frac{1}{2p^{i-c-1}}} \right) \Bigg|$$

$$F_p^{\beta_0}(X_i, X_{i+1}) = 0 \ \textit{for}\ 0 \leq i \leq n \Bigg\}.$$

3. Reductions of Affinoids

In [We, Theorem 1.0.1], Weinstein proves the following:

THEOREM 3.1. *Let K be a non-archimedean local field. Let q be the cardinality of the residue field $\mathbb{F}_K$. We simply write $\mathfrak{p}$ for $\mathfrak{p}_K$. Let $\mathbf{X}(\mathfrak{p}^n)$ denote the Lubin-Tate curve with Drinfeld level $\mathfrak{p}^n$-structure. Assume that q is odd. Then, every irreducible component in the stable reduction of $\mathbf{X}(\mathfrak{p}^n)$ admits a purely inseparable map to one of the following four projective smooth curves over $\mathbb{F}$:*

1. *The projective line $\mathbb{P}^1$,*
2. *The curve with affine model $x^q y - x y^q = 1$,*
3. *The curve with affine model $a^q + a = t^{q+1}$,*
4. *The curve with affine model $a^q - a = s^2$.*

Roughly speaking, if we replace K by $\mathbb{Q}_p$, and the second curve by the curve with affine model $xy(x-y)^{p-1} = 1$, similar things are expected to hold for irreducible components in the stable reduction of $X_0(p^n)$ except for ordinary components. In this section, without ambiguity of purely inseparability, we compute the reductions of some of the affinoids defined in (2.2) by using Proposition 2.5. As a result of these computations, we deduce some informations on irreducible components in the stable reduction of $X_0(p^n)$ in Corollary 3.15.

3.1. Reduction of $\mathbf{Y}^A_{n,1}$

In this subsection, we compute the reduction of $\mathbf{Y}^A_{n,1}$ for $n \geq 1$. The reduction of $\mathbf{Y}^A_{1,1}$ is essentially calculated in [E, §2.1.3] and [E2, Theorem 2.1.1].

By Proposition 2.5.1, we have

$$\mathbf{Y}^A_{n,1} \simeq \left\{ (\{X_i\}_{0\leq i\leq n+1}) \in \left(\prod_{0\leq i\leq n-1} \mathbf{C}^{A,0}_{1/(p^{n-i-1}(p+1))} \right) \times \mathbf{C}^{A,0}_{\geq p/(p+1)} \times \mathbf{C}^{A,0}_{1/(p+1)} \;\middle|\; F_p^{\beta_0}(X_i, X_{i+1}) = 0 \ (0 \leq i \leq n),\ X_{n-1} \neq X_{n+1} \right\}.$$

We simply write κ for pc_0. We choose elements $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{C}_p$ such that $\alpha_n^{p^{n-1}(p+1)} = \kappa$ and $\alpha_n^p = \alpha_{n-1}$ for $n \geq 2$. On $\mathbf{Y}^A_{n,1}$, we set

$$\begin{aligned} X_0 = \alpha_n z, \quad X_i &= \alpha_n^{p^i} x_i \quad (1 \leq i \leq n-2), \\ X_{n-1} = \alpha_n^{p^{n-1}} x, \quad X_n &= \alpha_n^{p^n} u, \quad X_{n+1} = \alpha_n^{p^{n-1}} y, \end{aligned} \tag{3.1}$$

where we have $v(z), v(x_i), v(x), v(y) = 0$ for any i and $v(u) \geq 0$. For $\alpha \in \mathbb{Q}_{>0}$, if $v(f-g) > \alpha$, we write $f \equiv g \mod \alpha+$. Then, as in [T, Corollary 4.5], by $F_p^{\beta_0}(X_i, X_{i+1}) = 0$ for $i = n-1, n$ and $X_{n-1} \neq X_{n+1}$, we obtain

$$xy(x-y)^{p-1} \equiv 1 \mod 0+ \quad \text{on } \mathbf{Y}^A_{n,1}. \tag{3.2}$$

Lemma 3.2. 1. *For $n \geq 1$, the reduction of $\mathbf{Y}^A_{n,1}$ is defined by*

$$z^{p^{n-1}} y \left(z^{p^{n-1}} - y \right)^{p-1} = 1. \tag{3.3}$$

This is a smooth affine curve with genus $(p-1)/2$.
2. *The curve defined by (3.3) is isomorphic to the curve defined by*

$st(s-t)^{p-1} = 1$.
3. *The map* π_ν *induces a purely inseparable map* $\overline{\pi}_\nu \colon \overline{\mathbf{Y}}^A_{n+1,1} \to \overline{\mathbf{Y}}^A_{n,1}$; $(z, y) \mapsto (z^p, y)$.

PROOF. By $F_p^{\beta_0}(X_i, X_{i+1}) = 0$ for $0 \leq i \leq n-2$ on $\mathbf{Y}^A_{n,1}$, we obtain $z^{p^{n-1}} \equiv x \mod 0+$ by (2.1). Hence, the first assertion follows from (3.2). The second assertion follows from [T, Lemma 4.8]. The map $\pi_\nu \colon \mathbf{Y}^A_{n+1,1} \to \mathbf{Y}^A_{n,1}$ is given by $(\{X_i\}_{0 \leq i \leq n+2}) \mapsto (\{X_i\}_{1 \leq i \leq n+2})$. The third assertion follows from the proof of 1. □

3.2. Reduction of $\mathbf{Y}^A_{n,2}$

In this subsection, we compute the reduction of $\mathbf{Y}^A_{n,2} \subset W_A(p^{n+2})$ for any $n \geq 2$. In [T, §4.3], when $n = 2$, we have calculated the reduction of $\mathbf{Y}^A_{2,2}$. To deduce defining equations of $\overline{\mathbf{Y}}^A_{n,2}$ in Corollary 3.4, we give defining congruences in appropriate moduli of the affinoid $\mathbf{Y}^A_{n,2}$ in Proposition 3.3.

In the following, every formal group is always assumed to be one-dimensional. Let $\mathscr{G}$ be the formal $\mathbb{Z}_{p^2}$-module over $\mathbb{Z}_{p^2}$ whose κ-multiplication has the following form:

$$[\kappa]_{\mathscr{G}}(X) = X^{p^2} - \kappa X.$$

See [Iw, Chapter IV] for more details on formal groups. Let $\varpi'_2 \in \mathbb{C}_p$ be an element such that $[\kappa^2]_{\mathscr{G}}(\varpi'_2) = 0$ and $[\kappa]_{\mathscr{G}}(\varpi'_2) \neq 0$. We set $\beta = [\kappa]_{\mathscr{G}}(\varpi'_2)$ and $\theta_1 = \varpi'_2/\beta$. Then we have

$$\beta^{p^2-1} = \kappa, \quad \theta_1^{p^2} - \theta_1 = \kappa^{-1}. \tag{3.4}$$

We put $K'_2 = \mathbb{Q}_{p^2}(\beta, \theta_1) = \mathbb{Q}_{p^2}(\varpi'_2)$. Note that we have $v(\beta\theta_1) = (p^2(p^2-1))^{-1}$. By multiplying the second equality in (3.4) by β^{p^2}, taking the (p^2-1)-th power of it and dividing it by $(\beta\theta_1)^{p^2(p^2-1)}$, we obtain

$$\left\{1 - \left(\kappa/(\beta\theta_1)^{p^2-1}\right)\right\}^{p^2-1} = \kappa/(\beta\theta_1)^{p^2(p^2-1)}.$$

This induces $\kappa/(\beta\theta_1)^{p^2(p^2-1)} \equiv 1 + (\kappa/(\beta\theta_1)^{p^2-1}) \mod 1+$. We take elements $\{\alpha_n\}_{n=1}^{\infty} \subset \mathbb{C}_p$ such that $\alpha_n^{p^{n-1}} = (\beta\theta_1)^{p-1}$ and $\alpha_n^p = \alpha_{n-1}$ for $n \geq 2$. We have $v(\alpha_{n-1}) = (p^n(p+1))^{-1}$ and

$$\kappa/\alpha_{n-1}^{p^n(p+1)} \equiv 1 + \left(\kappa/\alpha_{n-1}^{p^{n-2}(p+1)}\right) \mod 1+. \tag{3.5}$$

We set $\gamma_{n-1} = \alpha_{n-1}^{p^2-1}$.

By Proposition 2.5.1, we have

$$\mathbf{Y}_{n,2}^{A} \simeq \left\{ (\{X_i\}_{0\le i\le n+2}) \in \left(\prod_{0\le i\le n-1} \mathbf{C}^{A,0}_{1/(p^{n-i-1}(p+1))} \right) \right.$$

$$\times\, \mathbf{C}^{A,0}_{\ge p/(p+1)} \times \mathbf{C}^{A,0}_{1/(p+1)} \times \mathbf{C}^{A,0}_{1/(p(p+1))} \,\Bigg|$$

$$\left. F_p^{\beta_0}(X_i, X_{i+1}) = 0 \ (0 \le i \le n+1), \quad X_{n-1} \neq X_{n+1} \right\}. \tag{3.6}$$

Let $g(T)$ be the polynomial in §2.1. For a while, we assume that $n = 2$. We set

$$X_0 = \alpha_1^p x, \quad X_4 = \alpha_1^p y \quad \text{with } v(x) = 0 \text{ and } v(y) = 0 \tag{3.7}$$

on $\mathbf{Y}_{2,2}^{A}$.

Let $f(S,T)$ and $\mathfrak{H}(X,Y)$ be as in §2.1. We set

$$H(x,y) = c_0^{-1}\left(\mathfrak{H}(x^p, y^p) - f(x,y)\{xy(x-y)^{p-2}\}^p\right).$$

Let H' be as in [T, (4.19)]. Then, we have $H(x,y) \equiv H' \mod 0+$ by [T, (4.15), (4.16) and Lemma 4.12].

We set $\phi_2 = g(\alpha_1^p y) - g(\alpha_1^p x)$ and

$$xy(x-y)^{p-1} = 1 + \gamma_1 Z + \alpha_1^p xy\phi_2(x-y)^{-1}. \tag{3.8}$$

Note that ϕ_2 is divisible by α_1^p. We have $v(Z) \ge 0$ and its consequence $\left(xy(x-y)^{p-1}\right)^p \equiv 1 \mod p^{-1}+$ on $\mathbf{Y}_{2,2}^{A}$. This can be shown as follows. By plugging in [T, (4.15), (4.16), (4.18)] into the congruence in [T, Lemma 4.12] and using

$$\frac{g(\alpha_1^{p^2}u) - g(\alpha_1^{p^2}v)}{u-v}uv$$
$$\equiv \left(\frac{g(\alpha_1^p x) - g(\alpha_1^p y)}{x-y}xy\right)^p \mod \left(\frac{p-1}{p} - \frac{1}{p+1}\right)+,$$

we obtain this.

By [T, (4.19)] and $H(x,y) \equiv H' \mod 0+$, we have

$$
\begin{aligned}
&Z^p - \gamma_1 + \{xy(x-y)^{p-1}\}^{-1} + \alpha_1^p \phi_2 (x-y)^{-p} \\
&\quad + \alpha_1^p (y^p g(\alpha_1^p x) + x^p g(\alpha_1^p y))(x-y)^{p(p-1)} + x^{-(p+1)} + y^{-(p+1)} \\
&\equiv c_1 \alpha_1^p (xy)^{-p} (x+y)^p - \alpha_1^{p(p+1)} H(x,y) \mod p^{-1}+ .
\end{aligned} \tag{3.9}
$$

The term $\{xy(x-y)^{p-1}\}^{-1} + \alpha_1^p \phi_2 (x-y)^{-p}$ in the left hand side of (3.9) equals

$$
\left(1 + \alpha_1^p xy\phi_2 (x-y)^{-1}\right) / \left(xy(x-y)^{p-1}\right).
$$

Further, by (3.8), we acquire

$$
\begin{aligned}
\frac{1 + \alpha_1^p xy\phi_2 (x-y)^{-1}}{xy(x-y)^{p-1}} &= \frac{1 + \alpha_1^p xy\phi_2 (x-y)^{-1}}{1 + \alpha_1^p xy\phi_2 (x-y)^{-1} + \gamma_1 Z} \\
&\equiv 1 - \gamma_1 Z \mod p^{-1}+,
\end{aligned}
$$

because we have $v(\gamma_1 \alpha_1^{2p}) = (p-1)p^{-2} + 2(p(p+1))^{-1} > p^{-1}$ and $v(Z) \geq 0$. Hence, by taking (x,y,Z) as (u_2, v_2, Z_2), the congruence (3.9) can be written to the form (3.11) for $n = 2$ below.

PROPOSITION 3.3. *Let $n \geq 2$ be an integer. We consider the isomorphism (3.6) and set $X_0 = \alpha_{n-1}^p u_n$ with $v(u_n) = 0$ on $\mathbf{Y}_{n,2}^A$. Then, there exist rigid analytic functions v_n and Z_n such that, on the affinoid $\mathbf{Y}_{n,2}^A$, we have $v(v_n) \geq 0$, $v(Z_n) \geq 0$,*

$$
\begin{aligned}
u_n v_n (u_n - v_n)^{p-1} \equiv 1 &+ \gamma_{n-1} Z_n \\
&+ \alpha_{n-1}^p u_n v_n \phi_n (u_n - v_n)^{-1} \mod p^{-(n-1)}+,
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
&(Z_n + 1)^p - \gamma_{n-1}(Z_n + 1) \\
&\quad + \alpha_{n-1}^p (v_n^p g(\alpha_{n-1}^p u_n) + u_n^p g(\alpha_{n-1}^p v_n))(u_n - v_n)^{p(p-1)} \\
&\quad + u_n^{-(p+1)} + v_n^{-(p+1)} \\
&\equiv c_1 \alpha_{n-1}^p (u_n v_n)^{-p} (u_n + v_n)^p \\
&\quad - \alpha_{n-1}^{p(p+1)} H(u_n, v_n) \mod p^{-(n-1)}+,
\end{aligned} \tag{3.11}
$$

where we set $\phi_n = g(\alpha_{n-1}^p v_n) - g(\alpha_{n-1}^p u_n)$. *The functions* $\{u_n, v_n, Z_n\}_{n\geq 2}$ *satisfy*

$$(3.12)\qquad \begin{aligned} &u_{n+1}^p \equiv u_n, \quad v_{n+1}^p \equiv v_n \mod p^{-(n-1)}+, \\ &Z_{n+1}^p \equiv Z_n \mod p^{-n}+ \quad \text{on } \mathbf{Y}_{n+1,2}^A. \end{aligned}$$

Proof. We prove the first assertion by induction on n. We have already proved the assertion for $n = 2$. By assuming the assertion in the case n, we prove the assertion in the case $n+1$. We simply write H, Z, u, w and v for $H(u_n, v_n)$, Z_n, u_n, u_{n+1} and v_n. We consider every congruence below on $\mathbf{Y}_{n+1,2}^A$. We recall the identification of $\mathbf{Y}_{n+1,2}^A$ in (3.6). We set $X_0 = \alpha_n^p w$ and $X_1 = \alpha_n^{p^2} u$ with $v(w) = 0$ and $v(u) = 0$. Then we have

$$(3.13)\qquad w^p \equiv u \mod p^{-(n-1)}+$$

by (2.1). We set

$$(3.14)\qquad \begin{aligned} h' = -v^{-1}\Big\{&Z - \gamma_n + 1 + w^{-(p+1)} - c_1\alpha_n^p (uv)^{-1}(u+v) \\ &+ \alpha_n^p v g(\alpha_n^p w)(u-v)^{p-1}\Big\}^{-1} \end{aligned}$$

and put $W = Z - \alpha_n^{p(p+1)} H$. By the congruence (3.10) deduced from the induction hypothesis, we have $uv(u-v)^{p-1} \equiv 1 \mod (p^{n-1}(p+1))^{-1}+$. Therefore, we have $\alpha_n^{p^2}\{uv(u-v)^{p-1}\}^p g(\alpha_n^{p^2} v) \equiv \alpha_n^{p^2} g(\alpha_n^{p^2} v) \mod p^{-(n-1)}+$. Hence, by multiplying (3.11) by v^p and using (3.14), we acquire

$$(3.15)\qquad -h'^{-p} + v^{-1}\left(1 + \alpha_n^{p^2} v g(\alpha_n^{p^2} v)\right) \equiv \gamma_n^p v^p W \mod p^{-(n-1)}+.$$

We put

$$(3.16)\qquad v' = v\left(1 + \alpha_n^{p^2} v g(\alpha_n^{p^2} v)\right)^{-1}.$$

We take the power series $g_1(X) \in \mathbb{Z}_p[[X]]$ such that

$$X' = X(1 + Xg(X))^{-1} \iff X = X'(1 + X' g_1(X'))^{-1}.$$

By (3.16), we have

$$v = v'\left(1+\alpha_n^{p^2} v' g_1(\alpha_n^{p^2} v')\right)^{-1}. \tag{3.17}$$

Note that $g_1(\alpha_n^{p^2} v')$ is a rigid analytic function on $\mathbf{Y}^A_{n+1,2}$.

By (3.15), (3.16) and $v(v')=0$, we have $v(h')=0$ and

$$v' \equiv h'^p - \gamma_n^p (vh'^2)^p W \equiv h'^p - \gamma_n^p h'^{p(p+2)} W \mod p^{-(n-1)}+. \tag{3.18}$$

We put

$$h = h'\left(1+\alpha_n^p h' g_1(\alpha_n^p h')\right)^{-1}. \tag{3.19}$$

Then we have

$$h' = h\left(1+\alpha_n^p h g(\alpha_n^p h)\right)^{-1}. \tag{3.20}$$

We write v with respect to h. By substituting (3.18) to (3.17), and using (3.19) and $v(\alpha_n^{2p^2}\gamma_n^p) > p^{-(n-1)}$, we acquire

$$\begin{aligned} v &\equiv \frac{h'^p - \gamma_n^p h'^{p(p+2)} W}{1+\alpha_n^{p^2}\left(h'^p - \gamma_n^p h'^{p(p+2)} W\right) g_1(\alpha_n^{p^2} h'^p)} \\ &\equiv h^p - \gamma_n^p \frac{h'^{p(p+2)}}{\left\{1+\alpha_n^{p^2} h'^p g_1(\alpha_n^{p^2} h'^p)\right\}^2} W \mod p^{-(n-1)}+. \end{aligned}$$

Since we have $\gamma_n^p h'^{p(p+2)}\left\{1+\alpha_n^{p^2} h'^p g_1(\alpha_n^{p^2} h'^p)\right\}^{-2} \equiv \gamma_n^p h^{p(p+2)}$ mod $p^{-(n-1)}+$ by the definition of h in (3.19), we obtain

$$v \equiv h^p - \gamma_n^p h^{p(p+2)} W \mod p^{-(n-1)}+. \tag{3.21}$$

We set

$$\mathscr{Z}' = wh(w-h)^{p-1}, \quad f = w^2 h^{p+2}(w-h)^{p-2}.$$

By substituting (3.13) and (3.21) to $uv(u-v)^{p-1}$, we acquire

$$uv(u-v)^{p-1} \equiv \mathscr{Z}'^p - \gamma_n^p f^p W \mod p^{-(n-1)}+. \tag{3.22}$$

We introduce a new parameter Z' by

$$\mathscr{Z}' = 1 + \gamma_n Z' + \alpha_n^p wh \left(g(\alpha_n^p h) - g(\alpha_n^p w)\right)(w-h)^{-1}. \tag{3.23}$$

Substituting this and (3.22) to the left hand side of the congruence (3.10) in the case n, and dividing it by γ_n^p, we obtain

$$Z'^p \equiv Z + f^p W = Z + f^p \left(Z - \alpha_n^{p(p+1)} H\right) \mod p^{-n}+. \tag{3.24}$$

By this and the induction hypothesis, we obtain $v(Z') \geq 0$. We set $H' = H(w,h)$. Then we have $\alpha_n^{p(p+1)} H'^p \equiv \alpha_n^{p(p+1)} H \mod p^{-n}+$ by plugging in (3.13) and (3.21) into $\alpha_n^{p(p+1)} H$. In characteristic p, there is no common root of two equations $wh(w-h)^{p-1} = 1$ and $1 + f = 0$. Hence, by $v(w) = 0$ and $v(h) = 0$, we have $v(1+f) = 0$. We set

$$Z'' = (1+f)^{-1}\left(Z' + \alpha_n^{p+1} f H'\right). \tag{3.25}$$

Note that $v(Z'') \geq 0$ by $v(Z') \geq 0$ and $v(1+f) = 0$. By (3.24) and (3.25), we acquire

$$Z''^p \equiv Z \mod p^{-n}+. \tag{3.26}$$

We put $W_1 = Z'' - \alpha_n^{p+1} H'$ and

$$h'' = h - \gamma_n h^{p+2} W_1. \tag{3.27}$$

We show that the parameter h'' plays a role of the parameter v_{n+1} in the congruences (3.10) and (3.11) for the case $n+1$. By (3.21) and the definition of h'' in (3.27), we acquire

$$h''^p \equiv v \mod p^{-(n-1)}+. \tag{3.28}$$

By (3.14), we obtain

$$\begin{aligned} &Z - \gamma_n + 1 + w^{-(p+1)} + (h'v)^{-1} + \alpha_n^p v g(\alpha_n^p w)(u-v)^{p-1} \\ &\quad = c_1 \alpha_n^p (uv)^{-1}(u+v). \end{aligned} \tag{3.29}$$

In the following, we rewrite the term $(h'v)^{-1}$ in the left hand side of the equality (3.29) under the variables (h'', w, Z''). By (3.20) and (3.28), we obtain

$$(h'v)^{-1} \equiv (hh''^p)^{-1}\left(1 + \alpha_n^p h g(\alpha_n^p h)\right) \mod p^{-n}+. \tag{3.30}$$

Hence, by the definition of h'' in (3.27), we obtain

$$
\begin{aligned}
(3.31)\qquad 1+\alpha_n^p hg(\alpha_n^p h) \equiv 1 &+ \alpha_n^p h'' g(\alpha_n^p h'') \\
&+ g(\alpha_n^p h'')\alpha_n^p \gamma_n h''^{p+2} W_1 \mod p^{-n}+,
\end{aligned}
$$

$$
(3.32)\qquad (hh''^p)^{-1} \equiv h''^{-(p+1)} - \gamma_n W_1 \mod p^{-n}+ .
$$

By considering (3.31) $\times$ (3.32) and using (3.30), we acquire

$$
(3.33)\qquad (h'v)^{-1} \equiv h''^{-(p+1)}\left(1+\alpha_n^p h'' g(\alpha_n^p h'')\right) - \gamma_n W_1 \mod p^{-n}+ .
$$

Hence, by (3.26), (3.29) and (3.33), we obtain

$$
\begin{aligned}
(3.34)\qquad &(Z''+1)^p - \gamma_n(Z''+1) + w^{-(p+1)} + h''^{-(p+1)} \\
&+ \alpha_n^p h''^{-p} g(\alpha_n^p h'') + \alpha_n^p v g(\alpha_n^p w)(u-v)^{p-1} \\
&\equiv c_1 \alpha_n^p (uv)^{-1}(u+v) - \alpha_n^{p(p+1)} H' \mod p^{-n}+ .
\end{aligned}
$$

By (3.23), we have

$$
(3.35)\qquad wh(w-h)^{p-1} = 1 + \gamma_n Z' + \alpha_n^p wh\left(g(\alpha_n^p h) - g(\alpha_n^p w)\right)(w-h)^{-1}.
$$

We rewrite this equality under the variables (Z'', h''). Substituting $h = h'' + \gamma_n h''^{p+2} W_1$ to $wh(w-h)^{p-1}$, we acquire

$$
(3.36)\qquad wh(w-h)^{p-1} \equiv wh''(w-h'')^{p-1} + \gamma_n f W_1 \mod p^{-n}+ .
$$

Hence, by (3.35), (3.36) and $Z'' = Z' - fW_1$, we obtain

$$
\begin{aligned}
(3.37)\qquad wh''(w-h'')^{p-1} = 1 &+ \gamma_n Z'' + \alpha_n^p wh''\left(g(\alpha_n^p h'') - g(\alpha_n^p w)\right) \\
&\times (w-h'')^{-1} \mod p^{-n}+ .
\end{aligned}
$$

Since we have $w^p h''^p (w-h'')^{p(p-1)} \equiv 1 \mod p^{-n}+$ by (3.37), on the term in the left hand side of (3.34), we obtain $\alpha_n^p h''^{-p} g(\alpha_n^p h'') \equiv \alpha_n^p w^p g(\alpha_n^p h'')(w-h'')^{p(p-1)} \mod p^{-n}+$. By taking (Z'', w, h'') as $(Z_{n+1}, u_{n+1}, v_{n+1})$, the required assertion in the case $n+1$ follows from (3.13), (3.28), (3.34) and (3.37). The second assertion (3.12) follows from (3.13) and (3.26). $\square$

Corollary 3.4. *Let $n \geq 2$ be an integer.*

1. *Over $K_2'(\alpha_{n-1})$, the reduction $\overline{\mathbf{Y}}^A_{n,2}$ is defined by*

$$xy(x-y)^{p-1} = 1, \quad Z^p + 1 + x^{-(p+1)} + y^{-(p+1)} = 0.$$

2. *The map π_ν induces a purely inseparable map $\bar{\pi}_\nu \colon \overline{\mathbf{Y}}^A_{n+1,2} \to \overline{\mathbf{Y}}^A_{n,2}$; $(Z, x, y) \mapsto (Z^p, x^p, y^p)$.*

Proof. We obtain the first assertion by considering (3.10) and (3.11) mod 0+, and the second assertion by considering (3.12) mod 0+. □

Remark 3.5. By the isomorphism $w_{n+2} \colon \mathbf{Y}^A_{n,2} \xrightarrow{\sim} \mathbf{Y}^A_{2,n}$, if we replace π_ν in Corollary 3.4.2 by π_f, the similar things as Corollary 3.4 hold for $\overline{\mathbf{Y}}^A_{2,n}$.

3.3. Singular residue classes in $\mathbf{Y}^A_{n,2}$

In this subsection, we analyze the singular residue classes in $\mathbf{Y}^A_{n,2}$ for $n \geq 2$. As a result, we show that each singular residue class in $\mathbf{Y}^A_{n,2}$ is a basic wide open whose underlying affinoid reduces to the affine curve defined by $a^p - a = t^{p+1}$. When $n = 2$, the analysis of the singular residue classes in $\mathbf{Y}^A_{2,2}$ given in [T, §4.4] is incomplete. There is a gap in arguments in [T, Corollary 4.18], and the gap will be fixed in this subsection.

Let K be a non-archimedean local field and A a K-affinoid algebra. For a finite extension L over K, we write A_L for the base change $A \widehat{\otimes}_K L$. Now, we introduce an elementary lemma in rigid geometry.

Lemma 3.6. *Let K be a non-archimedean local field. Let $f \colon Y \to X$ be a morphism between reduced rigid analytic varieties over K. Let $\{U_i\}_{i \in I}$ be an admissible affinoid covering of X. For any $i \in I$, assume that the inverse $V_i = f^{-1}(U_i)$ is an affinoid, and over some finite extension L_i over K, the morphism f induces an isomorphism between reductions:*

$$\overline{f_{L_i}} = \overline{f \times_K L_i} \colon \overline{V_i \times_K L_i} \xrightarrow{\sim} \overline{U_i \times_K L_i}. \tag{3.38}$$

We write $U_i = \operatorname{Sp} A_i$ and $V_i = \operatorname{Sp} B_i$. Furthermore, we assume that

$$(A_{i,L_i})^{\circ\circ} = \mathfrak{p}_{L_i}(A_{i,L_i})^\circ, \quad (B_{i,L_i})^{\circ\circ} = \mathfrak{p}_{L_i}(B_{i,L_i})^\circ. \tag{3.39}$$

Then, $f \colon Y \to X$ is an isomorphism.

Proof. Since $\{U_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ are admissible affinoid coverings of X and Y respectively, to prove that $f\colon Y\to X$ is an isomorphism, it suffices to show that, for any $i\in I$, the restriction $f\colon V_i\to U_i$ is an isomorphism. Now, we fix $i\in I$. Let $f^*\colon A_i\to B_i$ be the morphism of K-affinoid algebras corresponding to $f\colon V_i\to U_i$. By faithfully flat descent, to prove that f^* is an isomorphism, it suffices to show that the base change of f^* from K to L_i:

$$f^*_{L_i}\colon A_{i,L_i}\to B_{i,L_i}$$

is an isomorphism. The image of $(A_{i,L_i})^\circ$ by $f^*_{L_i}$ is contained in $(B_{i,L_i})^\circ$. Hence, we have the restriction map

$$f^*_{\mathcal{O}_{L_i}}=f^*_{L_i}|_{(A_{i,L_i})^\circ}\colon (A_{i,L_i})^\circ\to(B_{i,L_i})^\circ.$$

Note that $f^*_{\mathcal{O}_{L_i}}\otimes_{\mathcal{O}_{L_i}}L_i=f^*_{L_i}$. By the assumptions (3.38) and (3.39), the morphism $f^*_{\mathcal{O}_{L_i}}$ induces the isomorphism

$$\overline{f^*_{\mathcal{O}_{L_i}}}\colon \overline{A_{i,L_i}}=(A_{i,L_i})^\circ/\mathfrak{p}_{L_i}(A_{i,L_i})^\circ\xrightarrow{\sim}\overline{B_{i,L_i}}=(B_{i,L_i})^\circ/\mathfrak{p}_{L_i}(B_{i,L_i})^\circ. \tag{3.40}$$

Since $(A_{i,L_i})^\circ$ and $(B_{i,L_i})^\circ$ are reduced, they are separated (cf. [BGR, Proposition 4(iii) in §6.2.1]). Clearly, they are $\mathfrak{p}_{L_i}$-torsion free. They are $\mathfrak{p}_{L_i}$-adic complete (cf. [BGR, Theorem 1 in §6.2.4]). By these properties, the isomorphism (3.40) implies that $f^*_{\mathcal{O}_{L_i}}$ is an isomorphism. Therefore, the map $f^*_{L_i}=f^*_{\mathcal{O}_{L_i}}\otimes_{\mathcal{O}_{L_i}}L_i\colon A_{i,L_i}\to B_{i,L_i}$ is also an isomorphism. Hence, we have known that $f\colon V_i\to U_i$ is an isomorphism. □

We go back to the original situation. We keep the same notation as in the previous subsection. We change variables

$$u_n=\frac{r_n+1}{2s_n},\quad v_n=\frac{r_n-1}{2s_n}, \tag{3.41}$$

similarly as in [CM, the proof of Proposition 5.2] and [T, §4.4]. We simply write Z, r, s and ϕ for Z_n, r_n, s_n and $\alpha^p_{n-1}\phi_n$ respectively. The congruence (3.10) has the following form:

$$\frac{r^2-1}{4s^{p+1}}(1-s^p\phi)\equiv 1+\gamma_{n-1}Z \mod p^{-(n-1)}+ \tag{3.42}$$

(cf. [T, (4.20)]). Let $\mathcal{T}_n = \{\zeta \in \mathbb{F}_{p^2} \mid 4\zeta^{p+1}+1=0\}$. The set $\mathcal{T}_n$ naturally corresponds to the set of the singular residue classes in $\mathbf{Y}^A_{n,2}$. Let $\zeta \in \mathcal{T}_n$. We write $T_{n,\zeta} \subset \mathbf{Y}^A_{n,2}$ for the singular residue class at the point on $\overline{\mathbf{Y}}^A_{n,2}$, which is defined by $(r,s)=(0,\zeta)$. On $T_{n,\zeta}$, we have $\alpha_{n-1}^{p(p+1)}H(u,v) \equiv \alpha_{n-1}^{p(p+1)}d_n \mod p^{-(n-1)}+$ with some constant $d_n \in \mathbb{Z}_{p^2}$. Let $s_{0,\zeta} \in \mu_{p^2-1}(\mathbb{Z}_{p^2})$ be the element such that $\bar{s}_{0,\zeta}=\zeta$. We simply write s_0 for $s_{0,\zeta}$. We simply write K for the local field $K_2'(\alpha_{n-1})$ in §3.2.

LEMMA 3.7. 1. *Let c_2 be the leading coefficient of the polynomial $(g(X)-c_1)/X$. We set $s=s_0+\mathfrak{s}$ on $T_{n,\zeta}$. Then, on $T_{n,\zeta}$, we have*

$$\mathfrak{s} \equiv (4s_0^p)^{-1}(r^2-\alpha_{n-1}^{2p}c_2s_0^{p-1})+t(\mathfrak{s},r) \mod 2pv(\alpha_{n-1})+ \tag{3.43}$$

with some $t(\mathfrak{s},r) \in \mathcal{O}_K[[\mathfrak{s}]][r]$ satisfying $t(\mathfrak{s},r) \in (\alpha_{n-1}^{2p}r, \alpha_{n-1}^{2p}\mathfrak{s}, \mathfrak{s}^p)$.
2. *On $T_{n,\zeta}$, we have*

$$\begin{aligned}(Z-1)^p+\gamma_{n-1}(Z-1)-2r^{p+1}+\mathcal{F}(\mathfrak{s},r,Z)+\alpha_{n-1}^{p(p+1)}d_n \\ \equiv 0 \mod p^{-(n-1)}+,\end{aligned} \tag{3.44}$$

with some $\mathcal{F}(\mathfrak{s},r,Z) \in \mathcal{O}_K[[\mathfrak{s},r]][Z]$ contained in the ideal $(\alpha_{n-1}^{2p}r^p,\ \alpha_{n-1}^{p}\mathfrak{s}^p,\ r^{2p},\ \alpha_{n-1}^{p}r^{p+1})$.

PROOF. In this proof, we simply write α, γ, u and v for α_{n-1}^p, γ_{n-1}, u_n and v_n respectively. In the following, we consider everything on $T_{n,\zeta}$.

We prove the first assertion. By (3.42) and $v(\gamma) > 2v(\alpha)$, we have

$$4s^{p+1} \equiv r^2-1-r^2s^p\phi+s^p\phi \mod 2v(\alpha)+. \tag{3.45}$$

For $a \in \mathbb{Q}_{\geq 0}$, we write $f \equiv g \mod a$ if $v(f-g) \geq a$. By (3.45) and $4s_0^{p+1} \equiv -1 \mod 1$, we obtain

$$4(s^{p+1}-s_0^{p+1}) \equiv r^2-r^2s^p\phi+s^p\phi \mod 2v(\alpha)+.$$

By this and $s=s_0+\mathfrak{s}$, we have

$$4s_0^p\mathfrak{s} \equiv r^2-r^2s^p\phi+s^p\phi+4s_0^pt_0(\mathfrak{s}) \mod 2v(\alpha)+, \tag{3.46}$$

where we set $t_0(\mathfrak{s}) = -(s_0^{-1}\mathfrak{s})^p(s_0+\mathfrak{s})$. Recall that $g(X) \in \mathbb{Z}_p[X]$. Then, we have

$$
\begin{aligned}
g(\alpha u) &= g\left(\alpha\frac{r+1}{2(s_0+\mathfrak{s})}\right) = g\left(\alpha\frac{r+1}{2s_0}\sum_{i=0}^{\infty}\left(-\frac{\mathfrak{s}}{s_0}\right)^i\right) \in \mathcal{O}_K[[\mathfrak{s}]][r], \\
g(\alpha v) &= g\left(\alpha\frac{r-1}{2(s_0+\mathfrak{s})}\right) = g\left(\alpha\frac{r-1}{2s_0}\sum_{i=0}^{\infty}\left(-\frac{\mathfrak{s}}{s_0}\right)^i\right) \in \mathcal{O}_K[[\mathfrak{s}]][r].
\end{aligned} \tag{3.47}
$$

Therefore, we obtain

$$
\phi = \alpha(g(\alpha v) - g(\alpha u)) \in \alpha^2\mathcal{O}_K[[\mathfrak{s}]][r]. \tag{3.48}
$$

Furthermore, we can write

$$
\phi \equiv -\alpha^2 c_2 s_0^{-1} + \phi_1 \mod 2v(\alpha)+ \tag{3.49}
$$

with some $\phi_1 \in (\alpha^2\mathfrak{s})\mathcal{O}_K[[\mathfrak{s}]][r]$. Hence, on the right hand side of (3.46), we obtain

$$
r^2 - r^2 s^p\phi + s^p\phi + 4s_0^p t_0(\mathfrak{s}) \equiv r^2 - \alpha^2 c_2 s_0^{p-1} + 4s_0^p t(\mathfrak{s}, r) \mod 2v(\alpha)+
$$

with some $t(\mathfrak{s}, r) \in (\alpha^2 r, \alpha^2\mathfrak{s}, \mathfrak{s}^p) \subset \mathcal{O}_K[[\mathfrak{s}]][r]$. The claim follows from this and $(4s_0^p)^{-1} \times (3.46)$.

We prove the second assertion. We compute the terms in (3.11) one by one under the variables (r, s, Z). We set

$$
f(r) = \frac{2(1+r^{p+1})}{(r^2-1)^p}.
$$

On the terms on the left hand side of (3.11), by simple computations, we have

$$
\begin{aligned}
&\alpha(v^p g(\alpha u) + u^p g(\alpha v))(u-v)^{p(p-1)} \\
&\qquad\qquad \equiv \frac{\alpha r^p}{2s^{p^2}}(g(\alpha u) + g(\alpha v)) + \frac{\phi}{2s^{p^2}} \mod 1, \\
&u^{-(p+1)} + v^{-(p+1)} \equiv 4s^{p+1}\frac{2(1+r^{p+1})}{(r^2-1)^{p+1}} \\
&\qquad\qquad \equiv f(r)\left(\frac{1-s^p\phi}{1+\gamma Z}\right) \mod p^{-(n-1)}+
\end{aligned} \tag{3.50}
$$

by (3.41) and (3.42) (cf. [T, Lemma 4.15 (3),(4)]). We simply write R for $\mathcal{O}_K[[\mathfrak{s}, r]]$. By $s = s_0 + \mathfrak{s}$ and (3.47), we have

$$\frac{\alpha r^p}{2s^{p^2}}(g(\alpha u) + g(\alpha v)) \equiv \left(\frac{c_1\alpha}{s_0^{p^2}}\right) r^p + \mathfrak{F}_1(\mathfrak{s}, r) \mod p^{-(n-1)}+ \tag{3.51}$$

with some $\mathfrak{F}_1(\mathfrak{s}, r) \in (\alpha\mathfrak{s}^p, \alpha^2 r^p) \subset R$. We have

$$\begin{aligned} c_1\alpha\left(\frac{u+v}{uv}\right)^p &= c_1\alpha\left(\frac{4sr}{r^2-1}\right)^p \\ &\equiv -4c_1\alpha(sr)^p\sum_{i=0}^{\infty} r^{2pi} \\ &\equiv -4c_1\alpha(s_0 r)^p + \mathfrak{F}_2(\mathfrak{s}, r) \mod p^{-(n-1)}+ \end{aligned} \tag{3.52}$$

with some $\mathfrak{F}_2(\mathfrak{s}, r) \in (\alpha r^{p+1}, \alpha\mathfrak{s}^p) \subset R$. By (3.51), (3.52) and the definition of s_0, we acquire

$$\begin{aligned} &\frac{\alpha r^p}{2s^{p^2}}(g(\alpha u) + g(\alpha v)) - c_1\alpha\left(\frac{u+v}{uv}\right)^p \\ &\equiv \left(\frac{4s_0^{p(p+1)}+1}{s_0^{p^2}}\right) c_1\alpha r^p + \mathfrak{F}_1(\mathfrak{s}, r) - \mathfrak{F}_2(\mathfrak{s}, r) \\ &\equiv \mathfrak{F}_1(\mathfrak{s}, r) - \mathfrak{F}_2(\mathfrak{s}, r) \mod p^{-(n-1)}+ . \end{aligned} \tag{3.53}$$

Note that $f(r) \equiv -2(1+r^{p+1})\sum_{i=0}^{\infty} r^{2pi} \mod 1$. Therefore, by using (3.48) and $v(\alpha^2\gamma) > p^{-(n-1)}$, we obtain

$$\begin{aligned} \left(-\frac{f(r)s^p}{1+\gamma Z} + \frac{1}{2s^{p^2}}\right)\phi &\equiv \left(-f(r)s^p + \frac{1}{2s^{p^2}}\right)\phi \\ &\equiv \left(2s^p + \frac{1}{2s^{p^2}}\right)\phi + \mathfrak{G}_1(\mathfrak{s}, r) \\ &\equiv \left(\frac{4s_0^{p(p+1)}+1}{2s_0^{p^2}}\right)\phi + \mathfrak{G}_2(\mathfrak{s}, r) \\ &\equiv \mathfrak{G}_2(\mathfrak{s}, r) \mod p^{-(n-1)}+ \end{aligned} \tag{3.54}$$

with some $\mathfrak{G}_1(\mathfrak{s}, r) \in (\alpha^2 r^p) \subset R$ and $\mathfrak{G}_2(\mathfrak{s}, r) \in (\alpha^2 r^p, \alpha\mathfrak{s}^p) \subset R$.

By $2v(\gamma) > p^{-(n-1)}$ and $v(\gamma) > 2v(\alpha)$, we have

$$(3.55)\quad \begin{aligned} &(Z+1)^p - \gamma(Z+1) + \frac{f(r)}{1+\gamma Z} \\ &\quad \equiv (Z+1)^p - \gamma(Z+1) + f(r)(1-\gamma Z) \\ &\quad \equiv (Z-1)^p + \gamma(Z-1) - 2r^{p+1} + \mathfrak{G}(r,Z) \mod p^{-(n-1)}+, \end{aligned}$$

with some $\mathfrak{G}(r,Z) \in (r^{2p}, \alpha^2 r^p) \subset \mathcal{O}_K[[r]][Z]$. We put

$$\begin{aligned} \mathcal{F}(\mathfrak{s},r,Z) &= \mathfrak{F}_1(\mathfrak{s},r) - \mathfrak{F}_2(\mathfrak{s},r) + \mathfrak{G}_2(\mathfrak{s},r) + \mathfrak{G}(r,Z) \\ &\in (\alpha^2 r^p, \alpha\mathfrak{s}^p, r^{2p}, \alpha r^{p+1}) \subset \mathcal{O}_K[[\mathfrak{s},r]][Z]. \end{aligned}$$

Then, the required assertion follows from (3.11), (3.50), (3.53), (3.54), (3.55) and $\alpha^{p+1}H(u,v) \equiv \alpha^{p+1}d_n \mod p^{-(n-1)}+$. $\square$

Let $\mathbb{Q}_p^{\mathrm{ur}}$ be the maximal unramified extension of $\mathbb{Q}_p$ in $\mathbb{C}_p$. We choose elements b_n and ζ_0' in $\mathbb{Q}_p^{\mathrm{ur}}$ such that $b_n^p + b_n = -d_n$ and $\zeta_0'^{p^2-1} = -1$ respectively. We put $\zeta_0 = -2\zeta_0'^{p+1}$ and $M_n = \mathbb{Q}_{p^2}(b_n, \zeta_0', \alpha_{n-1})$. Further, we set

$$(3.56)\qquad \beta_n = \zeta_0 \alpha_{n-1}^{p+1} \in \mathcal{O}_{M_n}, \quad \gamma_{0,n}' = 1 + \alpha_{n-1}^{p+1} b_n \in \mathcal{O}_{M_n}^*.$$

Note that $v(\beta_n) = p^{-n}$. We have

$$(3.57)\quad \begin{aligned} &\beta_n^{p-1} \equiv -\gamma_{n-1}, \quad (\gamma_{0,n}' - 1)^p + \gamma_{n-1}(\gamma_{0,n}' - 1) + \alpha_{n-1}^{p(p+1)} d_n \\ &\quad \equiv 0 \mod p^{-(n-1)} + . \end{aligned}$$

Let $\mathbf{X}_{n,\zeta} \subset T_{n,\zeta}$ be the affinoid which is defined by $v(r) \geq (p^{n-1}(p+1))^{-1}$. By (3.44) and the second congruence in (3.57), we obtain

$$(3.58)\quad (Z - \gamma_{0,n}')^p + \gamma_{n-1}(Z - \gamma_{0,n}') \equiv 2r^{p+1} \mod p^{-(n-1)} + \quad \text{on } \mathbf{X}_{n,\zeta}.$$

On $\mathbf{X}_{n,\zeta}$, by (3.43) and $v(\mathfrak{s}) > 0$, we have $v(\mathfrak{s}) \geq 2(p^{n-1}(p+1))^{-1}$. By (3.58), we have $v(Z - \gamma_{0,n}') \geq p^{-n}$ on $\mathbf{X}_{n,\zeta}$. On $\mathbf{X}_{n,\zeta}$, we put

$$(3.59)\quad \begin{gathered} r = \alpha_{n-1}^p \zeta_0' t, \quad \mathfrak{s} = \alpha_{n-1}^{2p} s', \quad Z = \gamma_{0,n}' + \beta_n a \\ \text{with } v(t), v(s'), v(a) \geq 0. \end{gathered}$$

By substituting (3.59) to (3.43) and dividing it by α_{n-1}^{2p}, we obtain

$$s' \equiv (4s_0^p)^{-1}\left((\zeta_0' t)^2 - c_2 s_0^{p-1}\right) \mod 0+ \quad \text{on } \mathbf{X}_{n,\zeta}. \tag{3.60}$$

There is an error in the congruence $s_1 \equiv (t^2 - c_2 s_0^{p-1})/s_0^p \mod 0+$, which is stated in [T, Lemma 4.16]. It should be corrected as $s_1 \equiv (t^2 - c_2 s_0^{p-1})/(4s_0^p)$ mod 0+.

We have $\beta_n^p \equiv -\zeta_0 \alpha_{n-1}^{p(p+1)} \mod pv(\beta_n)+$ by $\zeta_0^p \equiv -\zeta_0 \mod 0+$ and (3.56). Hence, by substituting (3.59) to (3.58), we acquire

$$\beta_n^p \left(a^p - a - t^{p+1}\right) \equiv 0 \mod p^{-(n-1)} + \quad \text{on } \mathbf{X}_{n,\zeta}. \tag{3.61}$$

PROPOSITION 3.8. *Let $n \geq 2$ be an integer.*
1. *Over M_n, the affinoid $\mathbf{X}_{n,\zeta}$ reduces to the affine curve defined by $a^p - a = t^{p+1}$. The complement $T_{n,\zeta} \setminus \mathbf{X}_{n,\zeta}$ is an annulus with width $(p^n(p+1))^{-1}$.*
2. *The map π_ν induces a purely inseparable map $\bar{\pi}_\nu \colon \overline{\mathbf{X}}_{n+1,\zeta} \to \overline{\mathbf{X}}_{n,\zeta^p}$; $(a,t) \mapsto (a^p, t^p)$.*

PROOF. The first assertion in 1 follows from (3.60) and (3.61).

We prove the second assertion in 1. In the following, we consider on $T_{n,\zeta} \setminus \mathbf{X}_{n,\zeta}$. We have $0 < v(r) < (p^{n-1}(p+1))^{-1}$ by the definitions of $\mathbf{X}_{n,\zeta}$ and $T_{n,\zeta}$. We set

$$s = s_0 + \mathfrak{s}, \quad Z = \gamma_{0,n}' + z \quad \text{with } v(\mathfrak{s}), v(z) > 0.$$

Note that s_0 is a unit. By (3.43), we have $v(\mathfrak{s}) = 2v(r)$. By this, $v(r) < v(\alpha_{n-1}^p)$ and (3.44), we have

$$z^p - \gamma_{n-1} z - 2r^{p+1} \equiv 0 \mod (p+1)v(r) + .$$

By considering the Newton polygon of this polynomial, we acquire $pv(z) = (p+1)v(r)$. Now, we set $\mathfrak{z} = z/(2r)$. Then, we have $v(\mathfrak{z}) = v(r)/p$ and $0 < v(\mathfrak{z}) < (p^n(p+1))^{-1}$. We consider a morphism between rigid analytic curves over M_n

$$\pi \colon T_{n,\zeta} \setminus \mathbf{X}_{n,\zeta} \to \mathcal{A} = A_{M_n}(p^{-\frac{1}{p^n(p+1)}}, 1); \ (\mathfrak{s}, \mathfrak{z}, r) \mapsto \mathfrak{z}. \tag{3.62}$$

We will show that this is an isomorphism by using Lemma 3.6. We note that

$$\{A_{M_n}[p^{-\rho_2}, p^{-\rho_1}]\}_{0<\rho_1<\rho_2<(p^n(p+1))^{-1},\ \rho_1,\rho_2\in\mathbb{Q}}$$

is an admissible affinoid covering of $\mathcal{A}$. Now, we fix such two rational numbers $\rho_1 < \rho_2$. We simply write $\mathbf{A}$ for $A_{M_n}[p^{-\rho_2}, p^{-\rho_1}]$. To apply Lemma 3.6, we should check (3.38) and (3.39) via $\pi^{-1}(\mathbf{A}) \to \mathbf{A}$. The inverse image $\pi^{-1}(\mathbf{A})$ is the affinoid which is defined by

$$\rho_1 \le v(\mathfrak{z}) \le \rho_2, \quad v(r) = pv(\mathfrak{z}), \quad v(\mathfrak{s}) = 2pv(\mathfrak{z}). \tag{3.63}$$

We will compute the reductions of $\mathbf{A} = \operatorname{Sp} A_1$ and $\pi^{-1}(\mathbf{A}) = \operatorname{Sp} A_2$, and understand the induced map $\overline{\pi^{-1}(\mathbf{A})} \to \overline{\mathbf{A}}$. First, we recall the reduction of $\mathbf{A}$. For each $i \in \{1,2\}$, we write $\rho_i = \frac{m_1^{(i)}}{m_2^{(i)}}$ with positive integers $m_1^{(i)}, m_2^{(i)}$ such that $(m_1^{(i)}, m_2^{(i)}) = 1$, and take an element $c_i \in \mathcal{O}_{\mathbb{C}_p}$ such that $c_i^{m_2^{(i)}} = p^{m_1^{(i)}}$. We have $v(c_i) = \rho_i$. We simply write M for $M_n(c_1, c_2)$. On $\mathbf{A}$, we put

$$\mathfrak{z} = c_1\mathfrak{t}_1, \quad d = c_2/c_1. \tag{3.64}$$

Then, we have $0 \le v(\mathfrak{t}_1) \le v(d)$ on $\mathbf{A}$, and isomorphisms

$$A_{1,M} \simeq M\langle \mathfrak{t}_1, \mathfrak{t}_2\rangle/(\mathfrak{t}_1\mathfrak{t}_2 - d), \quad \overline{\mathbf{A}} \simeq \operatorname{Spec} \mathbb{F}_M[\mathfrak{t}_1, \mathfrak{t}_2]/(\mathfrak{t}_1\mathfrak{t}_2) \simeq \mathbb{A}^1_{\mathbb{F}_M} \cup \mathbb{A}^1_{\mathbb{F}_M},$$

where the two affine lines intersect at the origins. Note that

$$(A_{1,M})^{\circ\circ} = \mathfrak{p}_M (A_{1,M})^{\circ}. \tag{3.65}$$

Secondly, we compute the reduction of $\pi^{-1}(\mathbf{A})$. In the following, we consider on $\pi^{-1}(\mathbf{A})$. We set

$$\begin{aligned} &r = c_1^p r_1, \quad \mathfrak{s} = c_1^{2p}\mathfrak{s}_1 \\ &\qquad \text{with } 0 \le v(r_1) \le pv(d), \quad 0 \le v(\mathfrak{s}_1) \le 2pv(d) \end{aligned} \tag{3.66}$$

and

$$T = M\langle r_1, r_2, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}_1, \mathfrak{t}_2\rangle,$$

$$B = T/(r_1r_2 - d^p,\ \mathfrak{s}_1\mathfrak{s}_2 - d^{2p},\ \mathfrak{t}_1\mathfrak{t}_2 - d).$$

By (3.63), we can consider natural surjections

$$\varrho\colon T \xrightarrow{\varrho_1} B \xrightarrow{\varrho_2} A_{2,M}. \tag{3.67}$$

We write ϱ for the composite $\varrho_2 \circ \varrho_1$. We consider (3.43) $\bmod 2pv(\alpha_{n-1})$. By $c_1^{-2p} \times$ (3.43), we have

$$\mathfrak{s}_1 \equiv (4s_0^p)^{-1}r_1^2 + (t(c_1^{2p}\mathfrak{s}_1)/c_1^{2p}) \mod 2pv(\alpha_{n-1}/c_1).$$

Hence, by using $t(\mathfrak{s}) \in \mathfrak{s}^p\mathcal{O}_K[\mathfrak{s}]$, we obtain

$$\mathfrak{s}_1 \equiv (4s_0^p)^{-1}r_1^2\,(1+\mathfrak{f}(r_1)) \mod 2pv(\alpha_{n-1}/c_1) \tag{3.68}$$

with some $\mathfrak{f}(r_1) \in \mathfrak{p}_M\mathcal{O}_M\langle r_1\rangle$. By this, we have

$$\mathfrak{s}_2 = \frac{d^{2p}}{\mathfrak{s}_1} \equiv 4s_0^p r_2^2\,(1+\mathfrak{f}(r_1))^{-1} \mod 2pv(\alpha_{n-1}/c_1). \tag{3.69}$$

We write A_ϱ for the subring $\varrho_2(B^\circ) = \varrho(T^\circ) \subset A_{2,M}$. In the sequel, under the notation of (3.44), we show that

$$\mathcal{F}(\mathfrak{s}, r, Z)/r^{p+1} \in \mathfrak{p}_M A_\varrho. \tag{3.70}$$

By (3.68), we have

$$\mathfrak{s}_1 = (4s_0^p)^{-1}r_1^2\,(1+\mathfrak{f}(r_1)) + \left(\frac{\alpha_{n-1}}{c_1}\right)^{2p}\mathfrak{g}$$

with some $\mathfrak{g} \in A_\varrho$. Hence, by (3.66) and $r_1r_2 = d^p$, we have

$$\begin{aligned}\frac{\alpha_{n-1}^p\mathfrak{s}^p}{r^{p+1}} &= (\alpha_{n-1}c_1^{p-1})^p\frac{\mathfrak{s}_1^p}{r_1^{p+1}}\\ &= \left(\frac{\alpha_{n-1}c_1^{p-1}}{4s_0^p}\right)^p r_1^{p-1}\,(1+\mathfrak{f}(r_1))^p + \alpha_{n-1}^{p^2}\left(\frac{\alpha_{n-1}}{c_2}\right)^{p(p+1)} r_2^{p+1}\mathfrak{h}\end{aligned}$$

with some $\mathfrak{h} \in A_\varrho$. Note that $v(\alpha_{n-1}/c_2) > 0$. Therefore, we obtain

$$\frac{\alpha_{n-1}^p\mathfrak{s}^p}{r^{p+1}} \in \mathfrak{p}_M A_\varrho. \tag{3.71}$$

Similarly, by (3.66) and $r_1 r_2 = d^p$, we have

$$\frac{\alpha_{n-1}^{2p}}{r} = \alpha_{n-1}^{p}\left(\frac{\alpha_{n-1}}{c_2}\right)^{p} r_2 \in \mathfrak{p}_M A_\varrho. \tag{3.72}$$

Then, (3.70) follows from (3.71) and (3.72).

By using $z = 2r\mathfrak{z}$, (3.64), (3.66), $r_1 r_2 = d^p$, $v(\gamma_{n-1}) = (p^2-1)v(\alpha_{n-1})$ and $v(c_1) < v(c_2) < v(\alpha_{n-1})$, we obtain

$$\begin{aligned} \frac{\gamma_{n-1} z}{(2rc_1)^p} &= \frac{\gamma_{n-1}\cdot 2rc_1\mathfrak{t}_1}{2^p c_1^p r^p} \\ &= \left(\frac{\gamma_{n-1}}{2^{p-1}c_1^{p-1}c_2^{p(p-1)}}\right) r_2^{p-1}\mathfrak{t}_1 \equiv 0 \mod v(\alpha_{n-1}/c_1). \end{aligned} \tag{3.73}$$

Hence, by considering $(2rc_1)^{-p} \times$ (3.44), and using (3.70) and (3.73), we acquire

$$r_1(1+\mathfrak{g}_1) \equiv \mathfrak{t}_1^p \mod v(\alpha_{n-1}/c_1) \tag{3.74}$$

with some $\mathfrak{g}_1 \in \mathfrak{p}_M A_\varrho$. By this, we have

$$r_2 = \frac{d^p}{r_1} \equiv \mathfrak{t}_2^p(1+\mathfrak{g}_1) \mod v(\alpha_{n-1}/c_1). \tag{3.75}$$

By (3.68), (3.69), (3.74) and (3.75), we know that $\operatorname{Spec}(A_\varrho \otimes_{\mathcal{O}_M} \mathbb{F}_M)$ is isomorphic to

$$\begin{aligned} &\operatorname{Spec}\mathbb{F}_M[r_1, r_2, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}_1, \mathfrak{t}_2]/(r_1 - \mathfrak{t}_1^p,\ r_2 - \mathfrak{t}_2^p, \\ &\qquad \mathfrak{s}_1 - (4\overline{s_0}^p)^{-1} r_1^2,\ \mathfrak{s}_2 - 4\overline{s_0}^p r_2^2,\ \mathfrak{t}_1\mathfrak{t}_2) \\ &\xrightarrow{\sim} \operatorname{Spec}\mathbb{F}_M[\mathfrak{t}_1, \mathfrak{t}_2]/(\mathfrak{t}_1\mathfrak{t}_2) \simeq \mathbb{A}^1_{\mathbb{F}_M} \cup \mathbb{A}^1_{\mathbb{F}_M}. \end{aligned} \tag{3.76}$$

Since this is a reduced scheme, we conclude that ϱ is distinguished by [BLR, Proposition 1.1]. Namely, the residue norm $|\cdot|_\varrho$ equals the supremum norm $|\cdot|_{\sup}$ on $A_{2,M}$. By [BGR, Proposition 3 (i) in §6.4.3], we obtain

$$A_\varrho = (A_{2,M})^\circ \supset (A_{2,M})^{\circ\circ} = \mathfrak{p}_M (A_{2,M})^\circ = \mathfrak{p}_M A_\varrho. \tag{3.77}$$

Therefore, the reduction $\overline{\pi^{-1}(\mathbf{A})} = \operatorname{Spec}(A_{2,M})^\circ/(A_{2,M})^{\circ\circ} = \operatorname{Spec}(A_\varrho \otimes_{\mathcal{O}_M} \mathbb{F}_M)$ is isomorphic to the scheme (3.76). Hence, π induces an isomorphism

$$\overline{\pi} \colon \overline{\pi^{-1}(\mathbf{A})} \xrightarrow{\sim} \overline{\mathbf{A}};\ (r_1, r_2, \mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{t}_1, \mathfrak{t}_2) \mapsto (\mathfrak{t}_1, \mathfrak{t}_2). \tag{3.78}$$

By (3.65), (3.77) and (3.78), the required assertion follows from Lemma 3.6.

The second assertion follows from (3.12) and (3.59). $\square$

REMARK 3.9. We can show that π in (3.62) is an isomorphism in the proof of Proposition 3.8.1 by applying [IT3, Lemma 2.1]. To apply this lemma, we show that, for each circle $C \subset \mathcal{A}$, the inverse image $\pi^{-1}(C)$ is an affinoid, and the induced morphism

$$\overline{\pi}\colon \overline{\pi^{-1}(C)} \to \overline{C}$$

is an isomorphism over some finite extension of M_n by using Lemma 3.7. In the proof of [IT3, Lemma 2.1], we depend on theory of adic spaces. Here, to avoid adic formalism, we have given a proof of Proposition 3.8.1 by using Lemma 3.6.

3.4. Reduction of $\mathbf{Z}_{n,1}^A$

In this subsection, we compute the reduction of $\mathbf{Z}_{n,1}^A \subset W_A(p^{n+2})$ for $n \geq 1$. The reduction of $\mathbf{Z}_{1,1}^A$ is already computed in [CM, §7].

By Proposition 2.5.2, we have

$$\mathbf{Z}_{n,1}^A \simeq \left\{ (\{X_i\}_{0\leq i\leq n+2}) \in \left(\prod_{0\leq i\leq n} \mathbf{C}^{A,0}_{\frac{1}{2p^{n-i}}} \right) \times \mathbf{C}^{A,0}_{\frac{1}{2}} \times \mathbf{C}^{A,0}_{\frac{1}{2p}} \;\middle|\; F_p^{\beta_0}(X_i, X_{i+1}) = 0 \ (0 \leq i \leq n+1) \right\}. \tag{3.79}$$

We choose a square root $\sqrt{\kappa}$ of κ. We set $L = \mathbb{Q}_{p^2}(\sqrt{\kappa})$. We consider the formal $\mathcal{O}_L$-module $\mathscr{F}$ over $\mathcal{O}_L$ whose $\sqrt{\kappa}$-multiplication is given by

$$[\sqrt{\kappa}\,]_{\mathscr{F}}(X) = X^{p^2} - \sqrt{\kappa}X.$$

Let $\varpi_2 \in \mathcal{O}_{\mathbb{C}_p}$ be an element such that $[\kappa]_{\mathscr{F}}(\varpi_2) = 0$ and $[\sqrt{\kappa}\,]_{\mathscr{F}}(\varpi_2) \neq 0$. We put $\varpi_1 = [\sqrt{\kappa}\,]_{\mathscr{F}}(\varpi_2)$ and $K_2 = L(\varpi_2)$. Further, we set

$$\beta = \varpi_1^2, \quad \theta = \varpi_2/\varpi_1. \tag{3.80}$$

Then, we easily check that

$$\beta^{\frac{p^2-1}{2}} = \sqrt{\kappa}, \quad \theta^{p^2} - \theta = (\sqrt{\kappa})^{-1}. \tag{3.81}$$

Note that we have $K_2 = L(\varpi_1, \theta)$. We set $K_3 = L(\beta, \theta) \subset K_2$. Let $\{\alpha_n\}_{n=1}^{\infty} \subset \mathcal{O}_{\mathbb{C}_p}$ be elements such that $\alpha_n^{p^{n-1}} = \theta^{-1}$ and $\alpha_n^p = \alpha_{n-1}$ for $n \geq 2$. We set $\gamma_n = \alpha_n^{p-1}$. Then, we have $v(\alpha_n) = (2p^{n+1})^{-1}$ and $v(\gamma_n) = (p-1)(2p^{n+1})^{-1}$. By multiplying the second equality in (3.81) by $\sqrt{\kappa}\theta^{-p^2}$ and taking the second power of it, we acquire

$$\kappa\left(1 - 2\alpha_n^{p^{n-1}(p^2-1)}\right) \equiv \alpha_n^{2p^{n+1}} \mod (3/2)+. \tag{3.82}$$

For a while, we assume that $n = 1$. We set

$$\begin{aligned} &X_0 = \alpha_1^p x_1, \quad X_3 = \alpha_1^p y_1 \quad \text{with } v(x_1) = 0,\ v(y_1) = 0, \\ &x_1 y_1 = 1 + \gamma_1 Z_1. \end{aligned} \tag{3.83}$$

By [T, (3.5)], we acquire

$$Z_1^p + x_1^{p+1}(1 - \gamma_1 Z_1) + x_1^{-(p+1)} \equiv 2\gamma_1 \mod (2p)^{-1}+ \ \text{ on } \mathbf{Z}_{1,1}^A. \tag{3.84}$$

In particular, we have $v(Z_1) \geq 0$ on $\mathbf{Z}_{1,1}^A$.

Proposition 3.10. *On $\mathbf{Z}_{n,1}^A$, we set $X_0 = \alpha_n^p x_n$ with $v(x_n) = 0$. Then, there exists a rigid analytic function Z_n on $\mathbf{Z}_{n,1}^A$ satisfying $v(Z_n) \geq 0$,*

$$Z_n^p + x_n^{p+1}(1 - \gamma_n Z_n) + x_n^{-(p+1)} \equiv 2\gamma_n \mod (2p^n)^{-1}+ \ \text{ on } \mathbf{Z}_{n,1}^A. \tag{3.85}$$

Further, the functions $\{Z_n, x_n\}_{n\geq 1}$ satisfy

$$\begin{aligned} &x_{n+1}^p \equiv x_n \mod (2p^n)^{-1}+, \\ &Z_{n+1}^p \equiv Z_n \mod (2p^{n+1})^{-1}+ \ \text{ on } \mathbf{Z}_{n+1,1}^A. \end{aligned} \tag{3.86}$$

Proof. The second assertion is proved in the following proof of the first assertion. We prove the first assertion by induction on n. When $n = 1$, this follows from (3.84).

Assuming the assertion in the case n, we prove the assertion in the case $n+1$. In the following, we consider everything on $\mathbf{Z}_{n+1,1}^A$. We set $X_0 = \alpha_{n+1}^p x_{n+1}$ and $X_1 = \alpha_n^p x_n$ with $v(x_{n+1}) = 0$ and $v(x_n) = 0$. Then, by $F_p^{\beta_0}(X_0, X_1) = 0$ and (2.1), we acquire $x_{n+1}^p \equiv x_n \mod (2p^n)^{-1}+$. We introduce a new parameter Z_{n+1} by

$$Z_n + x_{n+1}^{p+1}(1 - \gamma_{n+1} Z_{n+1}) + x_{n+1}^{-(p+1)} = 2\gamma_{n+1}. \tag{3.87}$$

Substituting this to the congruence (3.85) obtained by the induction hypothesis, and dividing it by $\gamma_n x_{n+1}^{p(p+1)}$, we have $Z_{n+1}^p \equiv Z_n \mod (2p^{n+1})^{-1}+$. Substituting this to (3.87), we obtain the required congruence in the case $n+1$. Hence, we have proved the required assertion. □

Corollary 3.11. *Let $n \geq 1$ be an integer.*
1. *The reduction of the affinoid $\mathbf{Z}_{n,1}^A$ is defined by $Z^p + x^{p+1} + x^{-(p+1)} = 0$. The genus of the curve is* 0.
2. *The map π_ν induces a purely inseparable map $\bar{\pi}_\nu \colon \overline{\mathbf{Z}}_{n+1,1}^A \to \overline{\mathbf{Z}}_{n,1}^A$; $(Z, x) \mapsto (Z^p, x^p)$.*

Proof. We obtain the required assertions 1 and 2 by considering (3.85) and (3.86) $\mod 0+$ respectively. □

Remark 3.12. Corollary 3.11 for $n = 1, 2$ is proved in [CM, Proposition 8.2] and [T, Proposition 3.1 and Lemma 4.1].

Remark 3.13. As in Remark 3.5, the same things as Corollary 3.11 can be proved for $\mathbf{Z}_{1,n}^A$.

3.5. Singular residue classes in $\mathbf{Z}_{n,1}^A$

We show that each singular residue class in $\mathbf{Z}_{n,1}^A$ is a basic wide open whose underlying affinoid reduces to the curve defined by $a^p - a = s^2$. When $n = 1$, there is a gap in the proof of [T, Corollary 3.6], and the gap is fixed in this subsection in the same way as the proof of Proposition 3.8.1.

Let $S_{n,\zeta_1} \subset \mathbf{Z}_{n,1}^A$ denote the singular residue class at the point $(Z, x) = (0, x_n)$ on $\overline{\mathbf{Z}}_{n,1}^A$, where $x_n = \zeta_1 \in \mu_{2(p+1)}(\mathbb{F}_{p^2})$. Let $\zeta_1 \in \mu_{2(p+1)}(\mathbb{F}_{p^2})$, and let $\widetilde{\zeta}_1 \in \mu_{2(p+1)}(\mathbb{Z}_{p^2})$ be the element such that $\overline{\widetilde{\zeta}_1} = \zeta_1$. We set

$$x_{0,\zeta_1,n} = \widetilde{\zeta}_1 \left(1 - \gamma_n \widetilde{\zeta}_1^{p+1}\right), \quad \gamma_{0,\zeta_1,n} = -2\widetilde{\zeta}_1^{p+1}. \tag{3.88}$$

For simplicity, we write γ_0 and x_0 for $\gamma_{0,\zeta_1,n}$ and $x_{0,\zeta_1,n}$ respectively. There exists an element $\zeta' \in \mu_{4(p-1)}(\mathbb{Q}_{p^{12}})$ such that $\zeta'^{2(p-1)} = \widetilde{\zeta}_1^{p+1} \in \{\pm 1\}$. We set $F_n = K_3(\alpha_n) \cdot \mathbb{Q}_{p^{12}}$ and

$$\alpha'_n = -\zeta'^2 \alpha_n, \quad \beta_n = \zeta' \widetilde{\zeta}_1 \beta^{\frac{p(p^2-1)}{4}} \alpha_n^{-\frac{p(p^{n+1}-1)}{2}} \in \mathcal{O}_{F_n}. \tag{3.89}$$

We have $v(\alpha'_n) = (2p^{n+1})^{-1}$ and $v(\beta_n) = (4p^n)^{-1}$. By (3.88) and (3.89), we have

$$\begin{aligned} &\gamma_0^p + x_0^{p+1}(1-\gamma_n\gamma_0) + x_0^{-(p+1)} \equiv 2\gamma_n \mod (2p^n)^{-1}+, \\ &{\alpha'_n}^p \equiv \gamma_n x_0^{p+1}\alpha'_n \equiv -\beta_n^2 x_0^{-(p+3)} \mod (2p^n)^{-1}+\,. \end{aligned} \tag{3.90}$$

By (3.85) and the first congruence in (3.90), we obtain

$$\begin{aligned} &(Z_n-\gamma_0)^p - \gamma_n x_0^{p+1}(Z_n-\gamma_0) \\ &\qquad + (x_n^{p+1} - x_0^{p+1})(1-\gamma_n Z_n) + x_n^{-(p+1)} - x_0^{-(p+1)} \\ &\quad \equiv 0 \mod (2p^n)^{-1}+ \quad \text{on } \mathbf{Z}^A_{n,1}. \end{aligned} \tag{3.91}$$

Let $\mathbf{X}_{n,\zeta_1} \subset S_{n,\zeta_1}$ be the affinoid defined by $v(x_n - x_0) \geq (4p^n)^{-1}$. On $\mathbf{X}_{n,\zeta_1}$, we have

$$\begin{aligned} &(x_n^{p+1} - x_0^{p+1})(1-\gamma_n Z_n) + x_n^{-(p+1)} - x_0^{-(p+1)} \\ &\quad \equiv -\gamma_n x_0^p (Z_n-\gamma_0)(x_n-x_0) + x_0^{-(p+3)}(x_n-x_0)^2 \end{aligned}$$

mod $(2p^n)^{-1}+$ (cf. [T, Corollary 3.6]). Hence, by (3.91), we acquire $v(Z_n - \gamma_0) \geq (2p^{n+1})^{-1}$ and

$$\begin{aligned} &(Z_n-\gamma_0)^p - \gamma_n x_0^{p+1}(Z_n-\gamma_0) + x_0^{-(p+3)}(x_n-x_0)^2 \\ &\quad \equiv 0 \mod (2p^n)^{-1}+ \end{aligned} \tag{3.92}$$

on $\mathbf{X}_{n,\zeta_1}$. On $\mathbf{X}_{n,\zeta_1}$, we put

$$x_n = x_0 + \beta_n s, \quad Z_n = \gamma_0 + \alpha'_n a \quad \text{with } v(s),\ v(a) \geq 0. \tag{3.93}$$

By substituting (3.93) to (3.92) and using the second congruence in (3.90), we acquire

$${\alpha'_n}^p \left(a^p - a - s^2\right) \equiv 0 \mod (2p^n)^{-1}+ \ \text{ on } \mathbf{X}_{n,\zeta_1}. \tag{3.94}$$

PROPOSITION 3.14. *Let $n \geq 1$ be an integer.*

1. *Over F_n, the affinoid $\mathbf{X}_{n,\zeta_1}$ reduces to the affine curve defined by $a^p - a = s^2$. The complement $S_{n,\zeta_1} \setminus \mathbf{X}_{n,\zeta_1}$ is an annulus with width $(4p^{n+1})^{-1}$.*

2. *The map* π_ν *induces a purely inseparable map* $\bar{\pi}_\nu\colon \overline{\mathbf{X}}_{n+1,\zeta_1} \to \overline{\mathbf{X}}_{n,\zeta_1^p}$; $(a,s) \mapsto (a^p, s^p)$.

PROOF. We obtain the first assertion in 1 by dividing (3.94) by ${\alpha'_n}^p$. We prove the second assertion in 1. In the following, we consider everything on $S_{n,\zeta_1} \setminus \mathbf{X}_{n,\zeta_1}$. By definition, we have $0 < v(x_n - x_0) < (4p^n)^{-1}$. We set

$$Z_n = \gamma_0 + z, \quad x_n = x_0 + x \quad \text{with } v(x), v(z) > 0.$$

We have

$$\begin{aligned} x_n^{-(p+1)} - x_0^{-(p+1)} &= \frac{1}{x_0^{p+1}} \left(\left(\sum_{i=0}^{\infty} \left(-\frac{x}{x_0} \right)^i \right)^{p+1} - 1 \right) \\ &\equiv -\frac{x}{x_0^{p+2}} + \frac{x^2}{x_0^{p+3}} + x^3 \mathfrak{F}(x) \mod 1 \end{aligned}$$

with some $\mathfrak{F}(x) \in \mathcal{O}_{F_n}[[x]]$. Hence, by (3.91), we acquire

$$\begin{aligned} & z^p - \gamma_n x_n^{p+1} z + (1 - \gamma_n \gamma_0)(x_0^p x + x_0 x^p + x^{p+1}) \\ & \quad - \frac{x}{x_0^{p+2}} + \frac{x^2}{x_0^{p+3}} + x^3 \mathfrak{F}(x) \equiv 0 \mod (2p^n)^{-1}. \end{aligned} \tag{3.95}$$

By $v(\gamma_n) > v(x) > 0$ and the above congruence, we have

$$z^p - \gamma_n x_0^{p+1} z + \frac{x}{x_0^{p+2}} (x_0^{2(p+1)} - 1) + \frac{x^2}{x_0^{p+3}} \equiv 0 \mod 2v(x)+.$$

Since $v(x_0^{2(p+1)} - 1) = v(\gamma_n)$, the third term vanishes and we obtain

$$z^p - \gamma_n x_0^{p+1} z + \frac{x^2}{x_0^{p+3}} \equiv 0 \mod 2v(x)+.$$

By considering the Newton polygon of this polynomial, we obtain $pv(z) = 2v(x)$ and $0 < v(z) < (2p^{n+1})^{-1}$. Hence, we have

$$z^p + \frac{x^2}{x_0^{p+3}} \equiv 0 \mod 2v(x)+. \tag{3.96}$$

By setting $\mathfrak{x} = x/z^{\frac{p-1}{2}}$ and dividing (3.96) by z^{p-1}, we have $z \equiv -(\mathfrak{x}^2/x_0^{p+3})$ mod $v(z)+$. Therefore, we have $0 < v(\mathfrak{x}) < (4p^{n+1})^{-1}$. We consider a morphism between rigid analytic curves over F_n

$$\pi \colon S_{n,\zeta_1} \setminus \mathbf{X}_{n,\zeta_1} \to \mathcal{A} = A_{F_n}(p^{-\frac{1}{4p^{n+1}}}, 1);\ (z, \mathfrak{x}) \mapsto \mathfrak{x}.$$

Then, in the same way as the proof of Proposition 3.8.1, by using (3.95), for each closed annulus $\mathbf{A} \subset \mathcal{A}$, we can check (3.38) and (3.39) with respect to $\pi^{-1}(\mathbf{A}) \to \mathbf{A}$. Hence, the required assertion follows from Lemma 3.6. We omit the details.

The second assertion follows from (3.86) and (3.93). □

3.6. Conclusion

As a result of the computations of the reductions in the previous subsections, we state a conclusion in this section that the reductions are related to irreducible components in the stable reduction of $X_0(p^n)$.

COROLLARY 3.15. *The reductions of the affinoids in Corollary 3.4.1, Propositions 3.8.1 and 3.14.1 are isomorphic to Zariski open subsets of irreducible components in the stable reduction of* $X_0(p^{n+2})$.

PROOF. All the smooth compactifications of the reductions have positive genera. Hence, the required assertion follows from [IT4, Proposition 7.11]. □

4. Inertial Action and the Middle Cohomology

In this section, we will describe the inertia action on the reductions which are computed in the previous section, and analyze the middle cohomology of the reductions as representations of the inertia subgroup through the type theory in [BH]. Throughout this section, let K be a non-archimedean local field in $\overline{\mathbb{Q}}_p$.

4.1. Preliminary

We recall the action of inertia on the reduction of a reduced affinoid from [CM2, §6]. Let K^{ur} be the maximal unramified extension of K in $\overline{\mathbb{Q}}_p$. If $\mathbf{Y}$ is a reduced affinoid over K, there is a homomorphism

$$w_{\mathbf{Y}} \colon I_K = \mathrm{Aut}_{\mathrm{cont}}(\mathbb{C}_p/K^{\mathrm{ur}}) \to \mathrm{Aut}(\overline{\mathbf{Y}}) \tag{4.1}$$

such that

$$\overline{\sigma(P)} = w_{\mathbf{Y}}(\sigma)(\overline{P}) \tag{4.2}$$

for each $P \in \mathbf{Y}(\mathbb{C}_p)$ and $\sigma \in I_K$. Let $\mathbf{Y}_{\mathbb{C}_p}$ denote the base change of $\mathbf{Y}$ to $\mathbb{C}_p$. Let $A(\mathbf{Y}_{\mathbb{C}_p})$ denote $\mathcal{O}_{\mathbf{Y}_{\mathbb{C}_p}}(\mathbf{Y}_{\mathbb{C}_p})$. Then, we have $\mathbf{Y}_{\mathbb{C}_p} = \operatorname{Sp} A(\mathbf{Y}_{\mathbb{C}_p})$. The inertia subgroup I_K preserves $A(\mathbf{Y}_{\mathbb{C}_p})^\circ$ and $A(\mathbf{Y}_{\mathbb{C}_p})^{\circ\circ}$. Since we have

$$\overline{\mathbf{Y}} = \operatorname{Spec}\left(A(\mathbf{Y}_{\mathbb{C}_p})^\circ / A(\mathbf{Y}_{\mathbb{C}_p})^{\circ\circ}\right),$$

the existence of the homomorphism (4.1) follows.

Moreover, inertia action satisfies some compatibility with respect to morphisms in the following sense.

Lemma 4.1 ([CM2, Lemma 6.1]). *If $f \colon \mathbf{X} \to \mathbf{Y}$ is a morphism between reduced affinoids over K, then $w_{\mathbf{Y}}(\sigma) \circ \bar{f} = \bar{f} \circ w_{\mathbf{X}}(\sigma)$ for any $\sigma \in I_K$.*

4.2. Inertia action on the reductions in §3.4 and §3.5

We compute the inertia action on the reduction of affinoids whose reductions have been computed in §3.4 and §3.5.

Proposition 4.2. *Let the notation be as in (3.84). Let $\sigma \in I_{\mathbb{Q}_p}$ and $P \in \mathbf{Z}^A_{1,1}(\mathbb{C}_p)$. We write $\sigma(\alpha_1) = \xi_\sigma \alpha_1$. Then, we have*

$$\begin{aligned} x_1(\sigma(P)) &= \xi_\sigma^p \sigma(x_1(P)), \\ Z_1(\sigma(P)) &= \xi_\sigma^{3p-1}\sigma(Z_1(P)) + \gamma_1^{-1}(\xi_\sigma^{2p} - 1). \end{aligned} \tag{4.3}$$

Proof. For any $0 \le i \le 3$, we have $\sigma(X_i(P)) = X_i(\sigma(P))$. By $X_0 = \alpha_1^p x_1$ and $X_3 = \alpha_1^p y_1$ in (3.83), we acquire $x_1(\sigma(P)) = \xi_\sigma^p \sigma(x_1(P))$ and $y_1(\sigma(P)) = \xi_\sigma^p \sigma(y_1(P))$. This proves the first equality in (4.3). By $\gamma_1 = \alpha_1^{p-1}$, we have $\sigma(\gamma_1) = \xi_\sigma^{p-1}\gamma_1$. Therefore, by using $x_1 y_1 = 1 + \gamma_1 Z_1$ in (3.83), we obtain

$$\begin{aligned} Z_1(\sigma(P)) &= \gamma_1^{-1}(x_1(\sigma(P))y_1(\sigma(P)) - 1) = \gamma_1^{-1}(\xi_\sigma^{2p}\sigma(x_1(P))\sigma(y_1(P)) - 1) \\ &= \gamma_1^{-1}(\xi_\sigma^{2p}\sigma(\gamma_1)\sigma(Z_1(P)) + \xi_\sigma^{2p} - 1) \\ &= \xi_\sigma^{3p-1}\sigma(Z_1(P)) + \gamma_1^{-1}(\xi_\sigma^{2p} - 1). \end{aligned}$$

This implies the second equality in (4.3). □

Let m be a positive integer prime to p. We choose a uniformizer $\varpi \in K$ and its m-th root $\varpi^{1/m}$. Let $t_{K,m}\colon I_K \twoheadrightarrow \mu_m(\mathbb{F})$ be the character defined by $\sigma \mapsto \overline{\sigma(\varpi^{1/m})/\varpi^{1/m}}$ for $\sigma \in I_K$, which is independent of choices of ϖ and its m-th root. Furthermore, we have $t_{K,m}(\sigma) = \overline{\sigma(x)/x}$ for any $x \in \mathbb{C}_p$ whose valuation is $1/m$. We call $t_{K,m}$ the tame character of order m. We have $t_{K,mn}^m = t_{K,n}$ for any $n, m \geq 1$. For $K = \mathbb{Q}_p$, we simply write t_m for $t_{\mathbb{Q}_p,m}$.

We will describe the inertia action on the reduction $\coprod_{\zeta_1 \in \mu_{2(p+1)}(\mathbb{F})} \overline{\mathbf{X}}_{1,\zeta_1}$. First, we consider the case $n = 1$.

Proposition 4.3. *Let β and θ be as in (3.80). Let $\sigma \in I_{\mathbb{Q}_p}$. We set*

$$\iota_\sigma = \sigma(\sqrt{\kappa})/\sqrt{\kappa} \in \{\pm 1\}, \quad \theta_\sigma = \sigma(\theta) - \iota_\sigma \theta.$$

Then, we have $\bar{\theta}_\sigma \in \mathbb{F}_{p^2}$. Moreover, we set $b_{\sigma,\zeta_1} = 2\mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}\left(\bar{\theta}_\sigma/\zeta'^{2p}\right)$, where ζ' associated to ζ_1 is chosen as in §3.5. The element σ acts on $\coprod_{\zeta_1 \in \mu_{2(p+1)}(\mathbb{F})} \overline{\mathbf{X}}_{1,\zeta_1}$ by

$$\sigma\colon \overline{\mathbf{X}}_{1,\zeta_1} \to \overline{\mathbf{X}}_{1,t_2(\sigma)\zeta_1};\ (a, s) \mapsto \left(t_2(\sigma)(a + b_{\sigma,\zeta_1}), t_4(\sigma)^{p+2}s\right).$$

Proof. By applying σ to the second equality in (3.81) and using $\iota_\sigma \in \{\pm 1\}$, we have

$$\sigma(\theta)^{p^2} - \sigma(\theta) = \iota_\sigma(\sqrt{\kappa})^{-1} = \iota_\sigma(\theta^{p^2} - \theta).$$

Hence, we obtain $\theta_\sigma^{p^2} \equiv \theta_\sigma \mod 0+$ by $v(\theta) = -(2p^2)^{-1}$. Therefore, we obtain $\theta_\sigma \in \mathcal{O}_{\mathbb{C}_p}$ and $\bar{\theta}_\sigma \in \mathbb{F}_{p^2}$. We set $\iota'_\sigma = \sigma(\beta)/\beta$. By the first equality in (3.81), we have ${\iota'_\sigma}^{(p^2-1)/2} \equiv \iota_\sigma \mod 0+$. By $\alpha_1 = \theta^{-1}$ and (3.82) for $n = 1$, we have $(\sigma(\alpha_1)/\alpha_1)^{2p^2} \equiv 1 \mod 0+$ and hence $(\sigma(\alpha_1)/\alpha_1)^2 \equiv 1 \mod 0+$. Note that $(p^2-1)/2$ is even. By (3.89) and $\zeta', \widetilde{\zeta_1} \in \mathbb{Q}_p^{\mathrm{ur}}$, we have $\sigma(\beta_1)/\beta_1 \equiv {\iota'_\sigma}^{p(p^2-1)/4} \mod 0+$. Let $P \in \mathbf{X}_{1,\zeta_1}(\mathbb{C}_p)$. Recall that $\xi_\sigma \equiv \iota_\sigma \mod 0+$ and $\sigma(\gamma_0) = \gamma_0$ by $\sigma \in I_{\mathbb{Q}_p}$ and (3.88). Hence, by (3.93), (4.3) and $\sigma \in I_{\mathbb{Q}_p}$, we have

$$s(\sigma(P)) \equiv \xi_\sigma {\iota'_\sigma}^{p(p^2-1)/4} \sigma(s(P)) \equiv \iota_\sigma {\iota'_\sigma}^{p(p^2-1)/4} s(P) \mod 0+,$$

$$\begin{aligned} a(\sigma(P)) &= \xi_\sigma^{3p-1}(\sigma(\alpha'_1)/\alpha'_1)\sigma(a(P)) \\ &\quad + (\gamma_1\alpha'_1)^{-1}\left(\xi_\sigma^{2p} - 1\right) + \gamma_0 {\alpha'_1}^{-1}(\xi_\sigma^{3p-1} - 1). \end{aligned} \tag{4.4}$$

By $\alpha_1 = \theta^{-1}$, we have

$$\theta_\sigma = \sigma(\alpha_1)^{-1} - \iota_\sigma \alpha_1^{-1} = \sigma(\alpha_1)^{-1}(1 - \iota_\sigma \xi_\sigma) = (\xi_\sigma \alpha_1)^{-1}(1 - \iota_\sigma \xi_\sigma).$$

Hence, by $\iota_\sigma^2 = 1$, we obtain

$$\xi_\sigma = \iota_\sigma(1 - \xi_\sigma \alpha_1 \theta_\sigma). \tag{4.5}$$

By considering the $(2p)$-th power of this equality and using $v(\alpha_1) = (2p^2)^{-1}$ and $v(\theta_\sigma) \geq 0$, we obtain

$$\xi_\sigma^{2p} \equiv 1 - 2(\xi_\sigma \alpha_1 \theta_\sigma)^p \mod p^{-1}.$$

Hence, by the definition of α_1' in (3.89), $\gamma_1 = \alpha_1^{p-1}$ and $\zeta'^{4(p-1)} = 1$, we have

$$(\gamma_1 \alpha_1')^{-1} \left(\xi_\sigma^{2p} - 1\right) \equiv 2(\xi_\sigma \theta_\sigma)^p / \zeta'^2 \equiv 2\iota_\sigma(\theta_\sigma / \zeta'^{2p})^p \mod 0+. \tag{4.6}$$

Similarly, by considering the $(3p-1)$-th power of (4.5) and using (3.88) and $\widetilde{\zeta}_1^{p+1} = \zeta'^{2(p-1)}$, we have

$$\gamma_0 {\alpha_1'}^{-1}(\xi_\sigma^{3p-1} - 1) \equiv 2(\widetilde{\zeta}_1^{p+1}\zeta'^2)^{-1}\iota_\sigma \theta_\sigma \equiv 2\iota_\sigma(\theta_\sigma / \zeta'^{2p}) \mod 0+. \tag{4.7}$$

By $\sigma \in I_{\mathbb{Q}_p}$, we have $\sigma(\alpha_1')/\alpha_1' = \sigma(\alpha_1)/\alpha_1 \equiv \iota_\sigma \mod 0+$. We have $t_2(\sigma) = \overline{\iota_\sigma}$ and $\overline{\iota_\sigma}\overline{\iota'_\sigma}^{p(p^2-1)/4} = t_4(\sigma)^{p+2}$. Hence, by this, (4.4), (4.6) and (4.7), the required assertion follows. □

REMARK 4.4. Compare Proposition 4.3 with [CM2, Corollary 6.11].

COROLLARY 4.5. *The inertia subgroup $I_{\mathbb{Q}_p}$ acts on $\coprod_{\zeta_1 \in \mu_{2(p+1)}(\mathbb{F})} \overline{\mathbf{X}}_{n,\zeta_1}$ by*

$$\sigma \colon \overline{\mathbf{X}}_{n,\zeta_1} \to \overline{\mathbf{X}}_{n,t_2(\sigma)\zeta_1};\ (a,s) \mapsto \left(t_2(\sigma)(a + b_{\sigma,\zeta_1^{p^{n-1}}}), t_4(\sigma)^{p^{-(n-2)}+2} s\right)$$

for $\sigma \in I_{\mathbb{Q}_p}$.

PROOF. The required assertion follows from Proposition 3.14.2, Lemma 4.1 and Proposition 4.3. □

Let W_K denote the Weil group of K. Let $\mathbf{a}_K\colon W_K^{\mathrm{ab}} \xrightarrow{\sim} K^*$ be the Artin reciprocity map normalized such that the geometric Frobenius is sent to a prime element. We write $\mathbf{a}_K\colon W_K \to K^*$ for the composite $W_K \twoheadrightarrow W_K^{\mathrm{ab}} \xrightarrow{\mathbf{a}_K} K^*$. Let $\mathbf{a}_K\colon I_K \twoheadrightarrow \mathcal{O}_K^*$ be the restriction of $\mathbf{a}_K\colon W_K \to K^*$ to I_K.

COROLLARY 4.6. *Let $L = \mathbb{Q}_{p^2}(\sqrt{\kappa})$. We fix the isomorphism $(\mathcal{O}_L/(\kappa))^* \simeq \mathbb{F}_{p^2}^* \times \mathbb{F}_{p^2}$; $a + b\sqrt{\kappa} \mapsto (\bar{a}, \bar{a}^{-1}\bar{b})$. We write* $\mathbf{a}$ *for the composite*

$$I_L \xrightarrow{\mathbf{a}_L} \mathcal{O}_L^* \xrightarrow{\text{can.}} (\mathcal{O}_L/(\kappa))^* \simeq \mathbb{F}_{p^2}^* \times \mathbb{F}_{p^2} \xrightarrow{\mathrm{pr}_2} \mathbb{F}_{p^2}.$$

For each $\zeta_1 \in \mu_{2(p+1)}(\mathbb{F})$, the group I_L stabilizes $\overline{\mathbf{X}}_{n,\zeta_1}$ and acts on it by

$$\sigma\colon \overline{\mathbf{X}}_{n,\zeta_1} \to \overline{\mathbf{X}}_{n,\zeta_1};\ (a, s) \mapsto \left(a - 2\mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}\left(\mathbf{a}(\sigma)/\zeta'^{2p^n}\right), t_{L,2}(\sigma)s\right) \quad \text{for } \sigma \in I_L.$$

PROOF. Let $\mathscr{F}'$ be the formal $\mathcal{O}_L$-module over $\mathcal{O}_L$ such that

$$[\sqrt{\kappa}]_{\mathscr{F}'}(X) = X^{p^2} + \sqrt{\kappa}X.$$

Let ϖ_1' and ϖ_2' be elements satisfying

$$\varpi_1' \neq 0, \quad [\sqrt{\kappa}]_{\mathscr{F}'}(\varpi_1') = 0, \quad [\sqrt{\kappa}]_{\mathscr{F}'}(\varpi_2') = \varpi_1'. \tag{4.8}$$

If we set $\theta' = \varpi_2'/\varpi_1'$, by (4.8), we obtain

$$\theta'^{p^2} - \theta' = -(\sqrt{\kappa})^{-1}.$$

Hence, by (3.80) and $v(\theta) = v(\theta') = -(2p^2)^{-1}$, we acquire

$$\theta \equiv -\theta' + c \mod 0+ \tag{4.9}$$

with some $c \in \mu_{p^2-1}(\mathcal{O}_L)$. We write $\mathbf{a}_L(\sigma) = a_\sigma + b_\sigma\sqrt{\kappa} + \sum_{i=2}^{\infty} b_{i,\sigma}(\sqrt{\kappa})^i \in \mathcal{O}_L^*$ with $a_\sigma \in \mu_{p^2-1}(\mathcal{O}_L)$, $b_\sigma, b_{i,\sigma} \in \mu_{p^2-1}(\mathcal{O}_L) \cup \{0\}$. By the Lubin-Tate theory (cf. [Iw, Chapter VI]), we have

$$[a_\sigma]_{\mathscr{F}'}(\varpi_1') = \sigma(\varpi_1'), \quad [a_\sigma + b_\sigma\sqrt{\kappa}]_{\mathscr{F}'}(\varpi_2') = \sigma(\varpi_2').$$

This implies

$$a_\sigma = \frac{\sigma(\varpi_1')}{\varpi_1'}, \quad a_\sigma \varpi_2' + b_\sigma \varpi_1' = \sigma(\varpi_2').$$

Dividing the second equality by $\sigma(\varpi_1')$, we obtain $b_\sigma/a_\sigma = \sigma(\theta') - \theta'$. Hence, by (4.9) and $\sigma \in I_L$, we obtain $\bar{\theta}_\sigma = -\mathbf{a}(\sigma)$. Hence, the required assertion follows from Corollary 4.5. □

For a positive integer m prime to p, let X_m be the affine smooth curve over $\mathbb{F}$ which is defined by $a^p - a = t^m$. Let $\overline{X}_m$ be the smooth compactification of X_m. By the Riemann-Hurwitz formula, the genus of $\overline{X}_m$ equals $\frac{(p-1)(m-1)}{2}$. Let $\mathbb{F}_p \times \mu_m(\mathbb{F})$ act on X_m by $(a, t) \mapsto (a + \zeta, \mu t)$ for $(\zeta, \mu) \in \mathbb{F}_p \times \mu_m(\mathbb{F})$. Let $\ell \neq p$ be a prime number. For an affine variety X over $\mathbb{F}$ and an integer i, we simply write $H^i_{\mathrm{c}}(X)$ for the i-th étale cohomology group with compact support $H^i_{\mathrm{c}}(X, \overline{\mathbb{Q}}_\ell)$. For a finite abelian group A, let $A^\vee$ denote the character group $\mathrm{Hom}_{\mathbb{Z}}(A, \overline{\mathbb{Q}}_\ell^*)$. By [Ka2, Corollary 2.2.(1)], we have an isomorphism

$$H^1_{\mathrm{c}}(X_m) \simeq \bigoplus_{\psi \in \mathbb{F}_p^\vee \setminus \{1\},\ \chi \in \mu_m(\mathbb{F})^\vee \setminus \{1\}} \psi \otimes \chi \tag{4.10}$$

as $\mathbb{F}_p \times \mu_m(\mathbb{F})$-representations.

Let $\mathbf{b}\colon I_{\mathbb{Q}_p(\sqrt{\kappa})} \to \mathbb{F}_p$ be the composite of the map $\mathbf{a}\colon I_{\mathbb{Q}_p(\sqrt{\kappa})} \simeq I_L \to \mathbb{F}_{p^2}$ in Corollary 4.6 and the trace map $\mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}\colon \mathbb{F}_{p^2} \to \mathbb{F}_p$. Let $\zeta \in \mathbb{F}_{p^2}$ be an element such that $\zeta^{p-1} = -1$. Let $\mathrm{Tr}'\colon \mathbb{F}_{p^2} \to \mathbb{F}_p$ be the map defined by $x \mapsto \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x\zeta^{-1})$. Let $\mathbf{b}'$ be the composite of $\mathbf{a}$ and Tr'. We define an equivalence relation $\sim$ on $\mathbb{F}_p^\vee \setminus \{1\}$ by $\psi \sim \psi^{-1}$ for any $\psi \in \mathbb{F}_p^\vee \setminus \{1\}$. For each $\psi \in \mathbb{F}_p^\vee \setminus \{1\}$, we define a two-dimensional irreducible representation of $I_{\mathbb{Q}_p}$ by

$$\tau_\psi^{(\prime)} = \mathrm{Ind}_{I_{\mathbb{Q}_p(\sqrt{\kappa})}}^{I_{\mathbb{Q}_p}} \left((\psi \circ \mathbf{b}^{(\prime)}) \otimes t_{\mathbb{Q}_p(\sqrt{\kappa}),2} \right). \tag{4.11}$$

By the Frobenius reciprocity, the isomorphism class of this representation depends only on the equivalence class $[\psi] \in \left(\mathbb{F}_p^\vee \setminus \{1\}\right)/\sim$ and, for $[\psi] \neq [\psi'] \in \left(\mathbb{F}_p^\vee \setminus \{1\}\right)/\sim$, we have $\tau_\psi \not\simeq \tau_{\psi'}$ as $I_{\mathbb{Q}_p}$-representations.

Corollary 4.7. *We have an isomorphism*

$$\bigoplus_{\zeta_1 \in \mu_{2(p+1)}(\mathbb{F})} H^1_{\mathrm{c}}(\overline{\mathbf{X}}_{n,\zeta_1}) \simeq \bigoplus_{\psi \in (\mathbb{F}_p^\vee \setminus \{1\})/\sim} (\tau_\psi \oplus \tau'_\psi)^{\oplus(p+1)}$$

as $I_{\mathbb{Q}_p}$-representations.

Proof. For simplicity, we write t'_2 for $t_{\mathbb{Q}_p(\sqrt{\kappa}),2}$. By Corollary 4.5, the stabilizer of $\overline{\mathbf{X}}_{n,\zeta_1}$ in $I_{\mathbb{Q}_p}$ equals $I_{\mathbb{Q}_p(\sqrt{\kappa})}$. By Corollary 4.6 and (4.10) for $m=2$, we have an isomorphism

$$H^1_{\mathrm{c}}(\overline{\mathbf{X}}_{n,\zeta_1}) \simeq \begin{cases} \bigoplus_{\psi\in\mathbb{F}_p^\vee\setminus\{1\}}((\psi\circ\mathbf{b})\otimes t'_2) & \text{if } \zeta_1\in\mu_{p+1}(\mathbb{F}), \\ \bigoplus_{\psi\in\mathbb{F}_p^\vee\setminus\{1\}}((\psi\circ\mathbf{b}')\otimes t'_2) & \text{if } \zeta_1\notin\mu_{p+1}(\mathbb{F}) \end{cases} \tag{4.12}$$

as $I_{\mathbb{Q}_p(\sqrt{\kappa})}$-representations. Note that $\zeta'^{2(p-1)}=\zeta_1^{p+1}\in\{\pm1\}$. For $\epsilon\in\mu_4(\mathbb{F})$, let ι_ϵ be the automorphism of X_2 defined by $(a,s)\mapsto(\epsilon^2a,\epsilon s)$. Then, for any $\zeta\in\mathbb{F}_p$, we have an equality $\iota_\epsilon\circ\zeta=\epsilon^2\zeta\circ\iota_\epsilon$ as automorphisms of X_2. By this, Corollary 4.5 and (4.12), we obtain isomorphisms

$$\bigoplus_{\zeta_1\in\mu_{p+1}(\mathbb{F})} H^1_{\mathrm{c}}(\overline{\mathbf{X}}_{n,\zeta_1}) \simeq \bigoplus_{\psi\in(\mathbb{F}_p^\vee\setminus\{1\})/\sim} \tau_\psi^{\oplus(p+1)},$$
$$\bigoplus_{\zeta_1\notin\mu_{p+1}(\mathbb{F})} H^1_{\mathrm{c}}(\overline{\mathbf{X}}_{n,\zeta_1}) \simeq \bigoplus_{\psi\in(\mathbb{F}_p^\vee\setminus\{1\})/\sim} \tau_\psi'^{\,\oplus(p+1)}$$

as $I_{\mathbb{Q}_p}$-representations. □

In the following two paragraphs, we fix some notations and collect some known facts on supercuspidal representations of $GL_2(\mathbb{Q}_p)$ and two-dimensional Galois representations.

For an admissible irreducible representation π of $GL_2(\mathbb{Q}_p)$, let $c(\pi)$ denote its conductor in the sense of [Tu, §3], and let ω_π denote its central character. For an integer $n\geq1$, we set

$$K_0(p^n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in GL_2(\mathbb{Z}_p)\ \middle|\ c\equiv 0 \pmod{p^n}\right\}.$$

Let π be a supercuspidal representation of $GL_2(\mathbb{Q}_p)$ in the sense of [BH, §9.1], and let $\pi^{K_0(p^n)}$ denote the $K_0(p^n)$-fixed part of π. If $\mathrm{c}(\pi)=n$, by [De, Théorèm 2.2.6], we have

$$\dim\pi^{K_0(p^n)}=\begin{cases} 1 & \text{if } \omega_\pi \text{ is trivial,}\\ 0 & \text{otherwise.}\end{cases} \tag{4.13}$$

Let Π_n^0 be the set consisting of all isomorphism classes of supercusidal representations of $GL_2(\mathbb{Q}_p)$ with conductor n and with trivial central character.

Let E be any quadratic extension of $\mathbb{Q}_p$, and let $(E^*)^\vee$ be the set of continuous characters $E^* \to \overline{\mathbb{Q}}_\ell^*$. Let $(E^*)^{\vee,0} \subset (E^*)^\vee$ denote the subset consisting of all characters which do not factor through the Norm map $\mathrm{N}_{E/\mathbb{Q}_p}\colon E^* \to \mathbb{Q}_p^*$. We identify a character $\chi \in (E^*)^\vee$ with the character of W_E via the class field theory. For a character $\chi \in (E^*)^\vee$, let $\mathrm{a}(\chi)$ denote the exponent of the Artin conductor of χ. We simply write $\mathrm{Ind}_{E/\mathbb{Q}_p}\chi$ for the two-dimensional representation $\mathrm{Ind}_{W_E}^{W_{\mathbb{Q}_p}}\chi$. For $\chi \in (E^*)^{\vee,0}$, the representation $\mathrm{Ind}_{E/\mathbb{Q}_p}\chi$ is irreducible (cf. [BH, §34.1]). Let $d_{E/\mathbb{Q}_p}$ be the exponent of the relative discriminant of the extension, and let $f_{E/\mathbb{Q}_p}$ be the residue class degree of the extension. For an irreducible smooth representation σ of $W_{\mathbb{Q}_p}$ of degree 2, let $\mathrm{a}(\sigma)$ denote its Artin conductor exponent. We write $\det\sigma$ for the determinant character of σ. By [Se, Corollary in VI §2], we have

$$\mathrm{a}(\mathrm{Ind}_{E/\mathbb{Q}_p}\chi) = d_{E/\mathbb{Q}_p} + f_{E/\mathbb{Q}_p}\mathrm{a}(\chi).$$

This formula induces

$$\mathrm{a}(\mathrm{Ind}_{E/\mathbb{Q}_p}\chi) = \begin{cases} 2\mathrm{a}(\chi) & \text{if } E \text{ is unramifed,} \\ 1 + \mathrm{a}(\chi) & \text{if } E \text{ is totally ramified.} \end{cases} \tag{4.14}$$

Let $\mathrm{Gal}(E/\mathbb{Q}_p)$ denote the Galois group of the extension.

Let LL and LL_ℓ denote the local Langlands correspondence and the ℓ-adic local Langlands correspondence for $GL_2(\mathbb{Q}_p)$ in the sense of [BH, §34 and §35] respectively. We have

$$\mathrm{LL}(\pi)\big|_{I_{\mathbb{Q}_p}} = \mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}} \tag{4.15}$$

for any supercuspidal representation π by [BH, §35]. Note that, for a supercuspidal representation π, we have

$$\mathrm{c}(\pi) = \mathrm{a}\left(\mathrm{LL}(\pi)\right) \tag{4.16}$$

by [Tu, §3] and

$$\omega_\pi = \det \mathrm{LL}(\pi) \tag{4.17}$$

as characters of $\mathbb{Q}_p^*$ by [BH, Proposition in §33.4]. There are just two non-isomorphic totally ramified extensions of $\mathbb{Q}_p$, for which we write L_1 and L_2. We set

$$\mathcal{X}_i = \left\{ \chi_i \in (L_i^*)^{\vee,0} \mid \mathrm{a}(\chi_i) = 2,\ \chi_i|_{\mathbb{Q}_p^*} = 1 \right\} / \mathrm{Gal}(L_i/\mathbb{Q}_p).$$

For any $[\chi_i] \in \mathcal{X}_i$, the pair $(L_i/\mathbb{Q}_p, \chi_i)$ is a minimal admissible pair in the sense of [BH, §18.2]. We check this. First, note that χ_i does not factor through $\mathrm{Nr}_{L_i/\mathbb{Q}_p}$. Note that the level of χ_i in the sense of [BH, Definition in §1.8] equals one. We have $\mathrm{Nr}_{L_i/\mathbb{Q}_p}(U_{L_i}^1) = \mathrm{Nr}_{L_i/\mathbb{Q}_p}(U_{L_i}^2) = U_{\mathbb{Q}_p}^1$. Hence, if $\chi_i|_{U_{L_i}^1}$ factors through the Norm map $\mathrm{Nr}_{L_i/\mathbb{Q}_p}$, the condition $\chi_i|_{U_{L_i}^2} = 1$ implies $\chi_i|_{U_{L_i}^1} = 1$. However, this is inconsistent with $a(\chi_i) = 2$. Therefore, $\chi|_{U_{L_i}^1}$ does not factor through $\mathrm{Nr}_{L_i/\mathbb{Q}_p}$. This implies that the pair is admissible and minimal.

By [BH, Theorem in §34.1 and Tame Langlands correspondence (1) in p. 219], we have a bijection

$$(4.18) \qquad \mathscr{L} \colon \coprod_{i=1}^{2} \mathcal{X}_i \xrightarrow{\sim} \Pi_3^0;\ [\chi_i] \mapsto \mathrm{LL}^{-1}\left(\mathrm{Ind}_{L_i/\mathbb{Q}_p}\left(\Delta_{\chi_i}^{-1}\chi_i\right)\right),$$

where Δ_{χ_i} is defined in [BH, §34.4]. Note that Δ_{χ_i} is a tamely ramified character of L_i^* of order 4 by [BH, (34.4.2) and Lemma (1) in §34.4]. We have $\det\left(\mathrm{Ind}_{L_i/\mathbb{Q}_p}\left(\Delta_{\chi_i}^{-1}\chi_i\right)\right) = 1$ by the definition of Δ_{χ_i} and [BH, Proposition in §29.2]. Hence, the well-definedness of the map $\mathscr{L}$ follows from (4.14), (4.16) and (4.17).

We explain the surjectivity of (4.18) in more detail. Let $\pi \in \Pi_3^0$. By [BH, Tame Langlands correspondence (1) in p. 219], there exists an irreducible two-dimensional $W_{\mathbb{Q}_p}$-representation τ such that $\mathrm{LL}(\tau) \simeq \pi$. By [BH, Theorem in §34.2], there exists an admissible pair $(E/\mathbb{Q}_p, \xi)$ such that $\mathrm{Ind}_{E/\mathbb{Q}_p}\xi \simeq \tau$. Since $\mathrm{a}(\tau) = 3$ by (4.16), the field E must be totally ramified over $\mathbb{Q}_p$ by (4.14), and $\mathrm{a}(\xi)$ equals two. Hence, we may assume that $E \simeq L_1$. Let $\kappa_{E/\mathbb{Q}_p}$ be the non-trivial character of $\mathbb{Q}_p^*$ factoring through $\mathbb{Q}_p^*/\mathrm{Nr}_{E/\mathbb{Q}_p}(E^*)$. By (4.17) and [BH, Proposition in §29.2], we acquire

$$(4.19) \qquad 1 = \omega_\pi = \det\tau = \det(\mathrm{Ind}_{E/\mathbb{Q}_p}\xi) = \kappa_{E/\mathbb{Q}_p} \otimes (\xi|_{\mathbb{Q}_p^*}).$$

We set $\chi = \Delta_\xi \xi$. Then, by [BH, (34.4.2)] and (4.19), we have $\chi|_{\mathbb{Q}_p^*} = 1$. Since Δ_ξ is tamely ramified, we have $\mathrm{a}(\chi) = \mathrm{a}(\xi) = 2$. By [BH, Proposition-

Definition (2) in §34.4], we acquire $\Delta_\chi = \Delta_\xi$. As a result, we obtain $[\chi] \in \mathscr{X}_1$ and $\mathscr{L}(\chi) = \pi$.

LEMMA 4.8. *Let L be a totally ramified quadratic extension of $\mathbb{Q}_p$. Let $\tau = \mathrm{Ind}_{L/\mathbb{Q}_p}\chi$ be an irreducible representation of degree two with some character $\chi\colon L^* \to \overline{\mathbb{Q}}_\ell^*$. Then, we have*

$$\tau|_{I_{\mathbb{Q}_p}} \simeq \mathrm{Ind}_{I_L}^{I_{\mathbb{Q}_p}}(\chi|_{I_L})$$

as $I_{\mathbb{Q}_p}$-representations.

PROOF. Let $\sigma \in \mathrm{Gal}(L/\mathbb{Q}_p)$ be the non-trivial element. Then, we have

$$\tau|_{W_L} \simeq \chi \oplus \chi^\sigma \tag{4.20}$$

as W_L-representations. Let (e_1, e_2) be the basis of $\tau|_{W_L}$ on which W_L acts through the characters χ and χ^σ respectively. By the irreducibility of τ, the group $W_{\mathbb{Q}_p}$ permutes the subspaces $W_1 = \overline{\mathbb{Q}}_\ell e_1$ and $W_2 = \overline{\mathbb{Q}}_\ell e_2$ in τ. Since L is totally ramified over $\mathbb{Q}_p$, we have the canonical isomorphisms $I_{\mathbb{Q}_p}/I_L \xrightarrow{\sim} W_{\mathbb{Q}_p}/W_L \simeq \{\pm 1\}$. Hence, the action of the inertia subgroup $I_{\mathbb{Q}_p}$ also permutes W_1 and W_2. By (4.20), we have $\tau|_{I_L} \simeq (\chi|_{I_L}) \oplus (\chi^\sigma|_{I_L})$ as I_L-representations. Therefore, the required assertion follows. $\square$

We choose a uniformizer ϖ_{L_i} of L_i such that $\varpi_{L_i}^2 = pu_i$ with $u_i \in \mathbb{Z}_p^*$ and fix the isomorphism $U_{L_i}^1/U_{L_i}^2 \simeq \mathbb{F}_p$; $1 + \varpi_{L_i}x \mapsto \bar{x}$. For $[\chi] \in \mathscr{X}_{L_i}$, by $\chi|_{\mathbb{Q}_p^*} = 1$, we have $\chi(\varpi_{L_i}) \in \{\pm 1\}$. By $\mathrm{a}(\chi) = 2$, we have $\chi|_{U_{L_i}^2} = 1$. Then, we have a bijection

$$\mathscr{X}_i \xrightarrow{\sim} \left(\{\pm 1\} \times (\mathbb{F}_p^\vee \setminus \{1\})\right)/\simeq;\ [\chi] \mapsto [(\chi(\varpi_{L_i}), \chi|_{U_{L_i}^1/U_{L_i}^2})], \tag{4.21}$$

where $\simeq$ is the equivalence relation on $\{\pm 1\} \times (\mathbb{F}_p^\vee \setminus \{1\})$ which is defined by $(\iota, \psi) \simeq (\iota, \psi^{-1})$ for any $(\iota, \psi) \in \{\pm 1\} \times (\mathbb{F}_p^\vee \setminus \{1\})$. By (4.18) and (4.21), we have $|\Pi_3^0| = 2(p-1)$, which is stated also in [Tu, Remark after Theorem 3.9].

Let D denote the quaternion division algebra over $\mathbb{Q}_p$. Let $\mathcal{O}_D$ be the maximal order of D, and $\mathfrak{p}_D$ the unique maximal ideal of $\mathcal{O}_D$. We set $U_D^n = 1 + \mathfrak{p}_D^n$ for any integer $n \geq 1$. Let LJ denote the local Jacquet-Langlands correspondence for $GL_2(\mathbb{Q}_p)$ (cf. [BH, §56]). For $\pi \in \Pi_n$, we set

$$d(\pi) = \left(\dim \pi^{K_0(p^n)} \dim \mathrm{LJ}(\pi)\right)/2 \in \mathbb{Z}.$$

Corollary 4.9. 1. *We have*

$$d(\pi) = \begin{cases} (p+1)/2 & \text{if } \pi \in \Pi_3^0, \\ 0 & \text{if } \pi \in \Pi_3 \setminus \Pi_3^0. \end{cases}$$

2. *Let* $\widetilde{\zeta} \in \mu_{2(p-1)}(\mathbb{Q}_{p^2}) \setminus \mu_{p-1}(\mathbb{Q}_p)$. *We take* $(\mathbb{Q}_p(\sqrt{\kappa}), \mathbb{Q}_p(\widetilde{\zeta}\sqrt{\kappa}))$ *and* $(\sqrt{\kappa}, \widetilde{\zeta}\sqrt{\kappa})$ *as* (L_1, L_2) *and their uniformizers* $(\varpi_{L_1}, \varpi_{L_2})$ *respectively. We set* $\Pi_{3,L_i}^0 = \mathscr{L}(\mathcal{X}_i) \subset \Pi_3^0$. *Let* $\mathscr{L}_i \colon \Pi_{3,L_i}^0 \to (\mathbb{F}_p^\vee \setminus \{1\})/\sim$ *be the composite of the isomorphism* $\mathscr{L} \colon \Pi_{3,L_i}^0 \to \mathcal{X}_i$, *the map (4.21) and the map* $\big(\{\pm 1\} \times (\mathbb{F}_p^\vee \setminus \{1\})\big)/\simeq \to (\mathbb{F}_p^\vee \setminus \{1\})/\sim;\ [(\iota, \psi)] \mapsto [\psi]$. *Then, we have isomorphisms*

$$\tau_{\mathscr{L}_1(\pi)} \simeq \mathrm{LL}_\ell(\pi)|_{I_{\mathbb{Q}_p}} \text{ for } \pi \in \Pi_{3,L_1}^0, \quad \tau'_{\mathscr{L}_2(\pi)} \simeq \mathrm{LL}_\ell(\pi)|_{I_{\mathbb{Q}_p}} \text{ for } \pi \in \Pi_{3,L_2}^0$$

as $I_{\mathbb{Q}_p}$*-representations.*
3. *We have an isomorphism*

$$\bigoplus_{\zeta_1 \in \mu_{2(p+1)}(\mathbb{F})} H_{\mathrm{c}}^1(\overline{\mathbf{X}}_{n,\zeta_1}) \simeq \bigoplus_{\pi \in \Pi_3} \Big(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}}\Big)^{\oplus d(\pi)}$$

as $I_{\mathbb{Q}_p}$*-representations.*

Proof. For $\pi \in \Pi_3^0$, the admissible D^*-representation $\mathrm{LJ}(\pi)$ has the form $\mathrm{Ind}_{L_i^* U_D^1}^{D^*} \chi$ with some embedding $L_i^* \hookrightarrow D^*$ and some character χ by [BH, §56]. Hence, we have $\dim \mathrm{LJ}(\pi) = [D^* : L_i^* U_D^1] = [\mathbb{F}_{p^2} : \mathbb{F}_p] = p+1$ for any $\pi \in \Pi_3^0$ (cf. [Tu, Theorem 3.6]). The first assertion follows from this and (4.13).

We prove the second assertion. Let $[\chi_i] \in \mathcal{X}$. Then, $\Delta_{\chi_i}\big|_{\mathbb{Q}_p^*}$ is a tamely ramified character of order two by [BH, Proposition-Definition (1) in §34.4]. Note that the canonical map $\mathbb{Z}_p^*/U_{\mathbb{Q}_p}^1 \to \mathcal{O}_{L_i}^*/U_{L_i}^1$ gives an isomorphism. Hence, we have

$$\Delta_{\chi_i}|_{I_{L_i}} = t_{L_i,2}. \tag{4.22}$$

For $\pi_i \in \Pi_{3,L_i}^0$, let χ_i be a character of L_i^* such that $[\chi_i]$ corresponds to π_i via (4.18). Let $\psi_i \in \mathbb{F}_p^\vee \setminus \{1\}$ be the character induced by $\chi_i|_{U_{L_i}^1/U_{L_i}^2}$ (cf. (4.21)).

We consider the second isomorphism. Recall that the reciprocity map is compatible with the Norm map. We fix the canonical isomorphism $i\colon I_{L_1} \xrightarrow{\sim} I_L \xleftarrow{\sim} I_{L_2}$. We have a commutative diagram

$$\begin{tikzcd}
I_{L_1} \arrow[r, "\simeq"] \arrow[dr, "i"'] & I_L \arrow[r, "\mathbf{a}_L"] & (\mathcal{O}_L/(\kappa))^* \arrow[r] \arrow[d, "\mathrm{Nr}_{L/L_2}"] & \mathbb{F}_{p^2} \arrow[d, "\mathrm{Tr}'"] \\
& I_{L_2} \arrow[u, "\simeq"] \arrow[r, "\mathbf{a}_{L_2}"] & (\mathcal{O}_{L_2}/(\widetilde{\zeta}^2\kappa))^* \arrow[r] & \mathbb{F}_p,
\end{tikzcd}$$

where the top rightmost horizontal map is given by $a + b\sqrt{\kappa} \mapsto \bar{b}/\bar{a}$, the bottom rightmost horizontal map is given by $a + b\widetilde{\zeta}\sqrt{\kappa} \mapsto \bar{b}/\bar{a}$ and the right vertical map Tr' is given by $x \mapsto \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x/\bar{\widetilde{\zeta}})$. Under the identification $I_{L_2} \simeq I_{L_1}$, this diagram implies $\chi_2|_{I_{L_2}} = \psi_2 \circ \mathbf{b}'$. Hence, by (4.15), Lemma 4.8 and (4.22), we have isomorphisms

$$\begin{aligned}
\mathrm{LL}_\ell(\pi_2)|_{I_{\mathbb{Q}_p}} = \mathrm{LL}(\pi_2)\big|_{I_{\mathbb{Q}_p}} &\simeq \left(\mathrm{Ind}_{L_2/\mathbb{Q}_p}(\Delta_{\chi_2}^{-1}\chi_2)\right)|_{I_{\mathbb{Q}_p}} \\
&\simeq \mathrm{Ind}_{I_{L_1}}^{I_{\mathbb{Q}_p}}((\psi_2 \circ \mathbf{b}') \otimes t_{L_1,2}) = \tau'_{\mathscr{L}_2(\pi_2)}
\end{aligned}$$

as $I_{\mathbb{Q}_p}$-representations. Therefore, the second isomorphism is proved. The first one is proved more easily than the second one.

The third assertion follows from the assertions 1, 2 and Corollary 4.7. □

Remark 4.10. We consider an equivalence relation on Π_3^0 as follows:

$$\pi \sim \pi' \iff \mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}} \simeq \mathrm{LL}_\ell(\pi')\big|_{I_{\mathbb{Q}_p}} \quad \text{as } I_{\mathbb{Q}_p}\text{-representations.}$$

Each equivalence class consists of two isomorphism classes of representations. The wide open rigid analytic curve $W_A(p^n)$ admits a left action of $\mathcal{O}_D^*$ through the Serre-Tate theorem as in [CM, §4B]. By using [CM, Remark 4.7], we know that the affinoid $\coprod_{\zeta_1 \in \mu_{2(p+1)}(\mathbb{F})} \mathbf{X}_{n,\zeta_1}$ is stable under the action of $\mathcal{O}_D^*$. It is expected that we have an isomorphism

$$\bigoplus_{\zeta_1 \in \mu_{2(p+1)}(\mathbb{F})} H_{\mathrm{c}}^1(\overline{\mathbf{X}}_{n,\zeta_1}) \simeq \bigoplus_{\pi \in \Pi_3^0/\sim} \left(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}} \otimes \mathrm{LJ}(\pi)\big|_{\mathcal{O}_D^*}\right)$$

as $I_{\mathbb{Q}_p} \times \mathcal{O}_D^*$-representations.

REMARK 4.11. Assume that $j(A)$ equals 0 or 1728. We set $c(A) = (p+1)/(2i(A)) \in \mathbb{Z}$. The reduction of $\overline{\mathbf{Z}}_{n,1}^{A}$ is defined by

$$Z^p + x^{2c(A)} + x^{-2c(A)} = 0.$$

Hence, the set of all the singular residue classes in $\mathbf{Z}_{n,1}^{A}$ corresponds to $\mu_{4c(A)}(\mathbb{F})$. We set $K(A) = \mu_{i(A)(p-1)}(\mathbb{Z}_{p^2})$. We regard this as a subgroup of D^*, which is not normal. For $\pi \in \Pi_n$, we set

$$d(A,\pi) = (\dim \pi^{K_0(p^n)} \dim \mathrm{LJ}(\pi)^{K(A)})/2 \in \mathbb{Z}.$$

Note that we have $\dim \mathrm{LJ}(\pi)^{K(A)} = (p+1)/i(A)$ for any $\pi \in \Pi_3^0$. Then we have isomorphisms

$$\begin{aligned}\bigoplus_{\zeta_1 \in \mu_{4c(A)}(\mathbb{F})} H_{\mathrm{c}}^1(\overline{\mathbf{X}}_{n,\zeta_1}) &\simeq \bigoplus_{\psi \in (\mathbb{F}_p^\vee \setminus \{1\})/\sim} \left(\tau_\psi \oplus \tau'_\psi\right)^{\oplus 2c(A)} \\ &\simeq \bigoplus_{\pi \in \Pi_3} \left(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}}\right)^{\oplus d(A,\pi)}\end{aligned}$$

as $I_{\mathbb{Q}_p}$-representations.

4.3. Inertia action on $\overline{\mathbf{Y}}_{n,1}^{A}$

Let the notation be as in §3.1. We determine the inertia action on $\overline{\mathbf{Y}}_{n,1}^{A}$.

LEMMA 4.12. *The inertia subgroup $I_{\mathbb{Q}_p}$ acts on the component $\overline{\mathbf{Y}}_{n,1}^{A}$ by*

$$\sigma \colon \overline{\mathbf{Y}}_{n,1}^{A} \to \overline{\mathbf{Y}}_{n,1}^{A};\ (z,y) \mapsto \left(t_{p+1}(\sigma)^{(-1)^{n-1}} z, t_{p+1}(\sigma) y\right) \quad \textit{for } \sigma \in I_{\mathbb{Q}_p}.$$

PROOF. Let $\sigma \in I_{\mathbb{Q}_p}$. For any $P \in \mathbf{Y}_{n,1}^{A}(\mathbb{C}_p)$ and $0 \le i \le n+1$, we have $X_i(\sigma(P)) = \sigma(X_i(P))$. By $X_0 = \alpha z$ and $X_{n+1} = \alpha^{p^{n-1}} y$ in (3.1), we have $z(\sigma(P)) = (\sigma(\alpha)/\alpha)\sigma(z(P))$ and $y(\sigma(P)) = (\sigma(\alpha)/\alpha)^{p^{n-1}}\sigma(y(P))$. Hence, the required assertion follows from $\overline{(\sigma(\alpha)/\alpha)}^{p^{n-1}} = t_{p+1}(\sigma)$ and $t_{p+1}(\sigma)^p = t_{p+1}(\sigma)^{-1}$. $\square$

When $n=1$, Lemma 4.12 is checked in [E, §2.3.3]. For a proper variety X over $\mathbb{F}$ an an integer i, we simply write $H^i(X)$ for the i-th étale cohomology group $H^i(X,\overline{\mathbb{Q}}_\ell)$. Let Y be the smooth projective curve over $\mathbb{F}$ with affine model $xy(x-y)^{p-1}=1$. Let $\mu_{p+1}(\mathbb{F})$ act on Y by $(x,y)\mapsto(\zeta x,\zeta y)$ for $\zeta\in\mu_{p+1}(\mathbb{F})$. Let $I_{\mathbb{Q}_p}$ act on Y by the composite of the $\mu_{p+1}(\mathbb{F})$-action and the tame character $t_{p+1}\colon I_{\mathbb{Q}_p}\to\mu_{p+1}(\mathbb{F})$. Let $\mu_{p+1}(\mathbb{F})^{\vee,0}\subset\mu_{p+1}(\mathbb{F})^{\vee}$ be the subset which consists of all characters not factoring through the $((p+1)/2)$-th power map $\mu_{p+1}(\mathbb{F})\twoheadrightarrow\{\pm1\}$. Note that $|\mu_{p+1}(\mathbb{F})^{\vee,0}|=p-1$. By [E, Proposition 3.2.1] or [IT, §3], we have an isomorphism

$$H^1(Y)\simeq\bigoplus_{\chi\in\mu_{p+1}(\mathbb{F})^{\vee,0}}\chi\circ t_{p+1} \tag{4.23}$$

as $I_{\mathbb{Q}_p}$-representations.

Corollary 4.13. *Let $\overline{\mathbf{Y}}^{A,\mathrm{c}}_{n,1}$ be the smooth compactification of $\overline{\mathbf{Y}}^{A}_{n,1}$. We have an isomorphism*

$$H^1(\overline{\mathbf{Y}}^{A,\mathrm{c}}_{n,1})\simeq\bigoplus_{\chi\in\mu_{p+1}(\mathbb{F})^{\vee,0}}\chi\circ t_{p+1}$$

as $I_{\mathbb{Q}_p}$-representations.

Proof. We consider the purely inseparable map $\overline{\mathbf{Y}}^{A}_{n,1}\to Y$ defined by $(z,y)\mapsto(z^{p^{n-1}},y)$. By Lemma 4.12, this map is $I_{\mathbb{Q}_p}$-equivariant. Hence, the required assertion follows from (4.23). □

We set

$$\mathcal{X}=\left\{\chi\in\left(\mathbb{Q}_{p^2}^*\right)^{\vee,0}\mid \mathrm{a}(\chi)=1,\ \chi|_{\mathbb{Q}_p^*}=1\right\}\Big/\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p).$$

Let $[\chi]\in\mathcal{X}$. Then, the restriction $\chi|_{\mathbb{Z}_{p^2}^*}$ does not factor through $\mathrm{Nr}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}$: if it factors through the Norm map, so does χ itself since $\chi(p)=1$ and $\mathrm{a}(\chi)=1$. Hence, $(\mathbb{Q}_{p^2}/\mathbb{Q}_p,\chi)$ is admissible and minimal by definition.

By [BH, §34.1 and §34.2], we have a bijection

$$\mathcal{X}\xrightarrow{\sim}\Pi_2^0;\ [\chi]\mapsto\mathrm{LL}^{-1}\left(\mathrm{Ind}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}(\Delta_\chi\chi)\right), \tag{4.24}$$

where every Δ_χ is the unramified character of $\mathbb{Q}_{p^2}^*$ of order 2 (cf. [BH, Definition in §34.4]). We fix the isomorphism $\mathbb{Z}_{p^2}^*/U_{\mathbb{Q}_{p^2}}^1 \simeq \mathbb{F}_{p^2}^*;\ x \mapsto \bar{x}$. For any $[\chi] \in \mathcal{X}$, the restriction $\chi|_{\mathbb{Z}_{p^2}^*/U_{\mathbb{Q}_{p^2}}^1}$ induces the character $\bar{\chi} \in \mu_{p+1}(\mathbb{F})^{\vee,0}$ by $\chi|_{\mathbb{Q}_p^*} = 1$ and the fact that $\chi|_{\mathbb{Z}_{p^2}^*}$ does not factor through $\mathrm{Nr}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}$. We define an equivalence relation $\sim$ on $\mu_{p+1}(\mathbb{F})^{\vee,0}$ by $\chi \sim \chi^{-1}$ for any $\chi \in \mu_{p+1}(\mathbb{F})^{\vee,0}$. We have a bijection

$$\mathcal{X} \xrightarrow{\sim} \mu_{p+1}(\mathbb{F})^{\vee,0}/\sim; [\chi] \mapsto [\bar{\chi}]. \tag{4.25}$$

By (4.24) and (4.25), we have $|\Pi_2^0| = (p-1)/2$.

Corollary 4.14. 1. *We have*

$$d(\pi) = \begin{cases} 1 & \text{if } \pi \in \Pi_2^0, \\ 0 & \text{if } \pi \in \Pi_2 \setminus \Pi_2^0. \end{cases}$$

2. *We have an isomorphism*

$$H^1(\overline{\mathbf{Y}}_{n,1}^{A,c}) \simeq \bigoplus_{\pi \in \Pi_2} \left(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}}\right)^{\oplus d(\pi)}$$

as $I_{\mathbb{Q}_p}$-representations.

Proof. We have $\dim \mathrm{LJ}(\pi) = [D^* : \mathbb{Q}_p^* \mathcal{O}_D^*] = 2$ for any $\pi \in \Pi_2^0$ by the construction of LJ given in [BH, §56]. Hence, the first assertion follows from (4.13).

We prove the second assertion. We consider the composite $f \colon \Pi_2^0 \xrightarrow{\sim} \mu_{p+1}(\mathbb{F})^{\vee,0}/\sim$ of the maps (4.24) and (4.25). Let $\pi \in \Pi_2^0$ and $[\chi_0] = f(\pi)$. Then, by (4.15), (4.24) and $I_{\mathbb{Q}_{p^2}} \xrightarrow{\sim} I_{\mathbb{Q}_p}$, we have

$$\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}} \simeq (\chi_0 \circ t_{p+1}) \oplus \left(\chi_0^{-1} \circ t_{p+1}\right)$$

as $I_{\mathbb{Q}_p}$-representations. Hence, the required assertion follows from Corollary 4.13 and the first assertion. □

Remark 4.15. For $\pi \in \Pi_2^0$, the irreducible smooth D^*-representation $\mathrm{LJ}(\pi)$ has the form $\mathrm{Ind}_{\mathbb{Q}_{p^2}^* U_D^1}^{D^*} \chi$ with some character χ by [BH, §56.4]. Note that $\mathbb{Q}_{p^2}^* U_D^1 = \mathbb{Q}_p^* \mathcal{O}_D^*$ and $\mathrm{LJ}(\pi)$ is two-dimensional. Let $\widetilde{\mathrm{LJ}}(\pi)$ denote the character $\chi|_{\mathcal{O}_D^*}$. For $d \in \mathcal{O}_D^*$, let $\bar{d}$ denote its image by the reduction map $\mathcal{O}_D^* \twoheadrightarrow \mathbb{F}_{p^2}^*$. There exists a unique character $\chi_0 \in \mu_{p+1}(\mathbb{F})^{\vee,0}$ such that $\widetilde{\mathrm{LJ}}(\pi)$ equals the character $\mathcal{O}_D^* \to \overline{\mathbb{Q}}_\ell^*$ defined by $d \mapsto \chi_0(\bar{d}^{p-1})$ for $d \in \mathcal{O}_D^*$. It is expected that $\mathcal{O}_D^*$ acts on $\overline{\mathbf{Y}}_{1,1}^A$ by $(x, y) \mapsto (\bar{d}^{p-1}x, \bar{d}^{p-1}y)$ for $d \in \mathcal{O}_D^*$. If this is true, by (4.23), we have an isomorphism

$$H^1(\overline{\mathbf{Y}}_{n,1}^{A,c}) \simeq \bigoplus_{\pi \in \Pi_2^0} \left(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}} \otimes \widetilde{\mathrm{LJ}}(\pi)\right)$$

as $I_{\mathbb{Q}_p} \times \mathcal{O}_D^*$-representations.

Remark 4.16. Assume that $j(A)$ equals 0 or 1728. The reduction $\overline{\mathbf{Y}}_{1,1}^A$ is defined by

$$r^2 = 4s^{2c(A)} + 1$$

with genus $c(A) - 1$. This curve admits an action of $\mu_{2c(A)}(\mathbb{F})$ by $(r, s) \mapsto (r, \zeta s)$ for $\zeta \in \mu_{2c(A)}(\mathbb{F})$. Let $\mu_{2c(A)}(\mathbb{F})^{\vee,0} \subset \mu_{2c(A)}(\mathbb{F})^\vee$ be the subset of the characters not factoring through the $c(A)$-th power map $\mu_{2c(A)}(\mathbb{F}) \to \{\pm 1\}$. Let $\Pi_2^{A,0}$ be the subset of Π_2^0 consisting of the representations which correspond to characters whose restrictions to $K(A)$ are trivial under (4.24). Note that $\left|\Pi_2^{A,0}\right| = c(A) - 1$. For $\pi \in \Pi_2^0$, we have

$$\dim \mathrm{LJ}(\pi)^{K(A)} = \begin{cases} \dim \mathrm{LJ}(\pi) & \text{if } \pi \in \Pi_2^{A,0}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, by (4.23), we obtain isomorphisms

$$H^1(\overline{\mathbf{Y}}_{n,1}^{A,c}) \simeq \bigoplus_{\chi \in \mu_{2c(A)}(\mathbb{F})^{\vee,0}} (\chi \circ t_{2c(A)}) \simeq \bigoplus_{\pi \in \Pi_2} \left(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}}\right)^{\oplus d(A,\pi)}$$

as $I_{\mathbb{Q}_p}$-representations.

4.4. Inertia action on the reductions in §3.2 and §3.3

We compute the inertia action on the reduction of $\mathbf{Y}_{n,2}^A$ for $n \geq 2$.

Lemma 4.17. *Let the notation be as in (3.4) and (3.56). Let $\sigma \in I_{\mathbb{Q}_p}$. We write $\sigma(\alpha_1) = \xi_\sigma \alpha_1$. Then, we have*

$$\begin{aligned} \beta_2^{-1}(1-\xi_\sigma^{p+1}) &\equiv \theta_\sigma/\zeta_0 \mod 0+, \\ (\beta_2\gamma_1)^{-1}(\xi_\sigma^{p(p+1)}-1) &\equiv (\theta_\sigma/\zeta_0)^p \mod 0+. \end{aligned} \tag{4.26}$$

In particular, we have $v(\xi_\sigma^{p+1}-1) \geq p^{-2}$.

Proof. As in (3.4), we have

$$\beta^{p^2-1} = \kappa, \quad \alpha_1 = (\beta\theta_1)^{p-1}, \quad \gamma_1 = \alpha_1^{p^2-1}. \tag{4.27}$$

By this, we have $\sigma(\beta)/\beta \in \mu_{p^2-1}(\mathbb{Q}_{p^2})$. By using this, (3.56) and (4.27), we have

$$\begin{aligned} \beta_2^{-1}(1-\xi_\sigma^{p+1}) &= \frac{1}{\zeta_0\alpha_1^{p+1}}\left(1-\left(\frac{\sigma(\beta\theta_1)}{\beta\theta_1}\right)^{p^2-1}\right) \\ &= \frac{1}{\zeta_0\alpha_1^{p+1}}\left(1-\left(\frac{\sigma(\theta_1)}{\theta_1}\right)^{p^2-1}\right) \\ &= \frac{1}{\zeta_0\alpha_1^{p+1}}\left(1-\left(1+\frac{\theta_\sigma}{\theta_1}\right)^{p^2-1}\right) \\ &\equiv \frac{\theta_\sigma}{\zeta_0\alpha_1^{p+1}\theta_1} = \frac{\theta_\sigma}{\zeta_0\beta^{p^2-1}\theta_1^{p^2}} = \frac{\theta_\sigma}{\zeta_0\kappa\theta_1^{p^2}} \equiv \frac{\theta_\sigma}{\zeta_0} \mod 0+, \end{aligned}$$

where we use $v(\theta_1^{-1}) = p^{-2}$ and $v(\alpha_1) = (p^2(p+1))^{-1}$ at the fourth congruence, and use (3.56) at the last congruence. Similarly, by using (4.27) and $\zeta_0^{p-1} \equiv -1 \mod 0+$, we have

$$\begin{aligned} (\beta_2\gamma_1)^{-1}(\xi_\sigma^{p(p+1)}-1) &= \frac{1}{\zeta_0\alpha_1^{p(p+1)}}\left(\left(\frac{\sigma(\theta_1)}{\theta_1}\right)^{p(p^2-1)}-1\right) \\ &= \frac{1}{\zeta_0\alpha_1^{p(p+1)}}\left(\left(1+\frac{\theta_\sigma}{\theta_1}\right)^{p(p^2-1)}-1\right) \\ &\equiv -\frac{\theta_\sigma^p}{\zeta_0\alpha_1^{p(p+1)}\theta_1^p} = -\frac{\theta_\sigma^p}{\zeta_0(\kappa\theta_1^{p^2})^p} \equiv \left(\frac{\theta_\sigma}{\zeta_0}\right)^p \mod 0+. \end{aligned}$$

Hence, the required assertion follows. □

PROPOSITION 4.18. *Let the notation be as in (3.8). Let $\sigma \in I_{\mathbb{Q}_p}$ and $P \in \mathbf{Y}^A_{2,2}(\mathbb{C}_p)$. Then, we have*

$$x(\sigma(P)) = \xi_\sigma^p \sigma(x(P)), \quad y(\sigma(P)) = \xi_\sigma^p \sigma(y(P)),$$

$$Z(\sigma(P)) \equiv \xi_\sigma^{(p+1)(2p-1)}\sigma(Z(P)) + \gamma_1^{-1}\left(\xi_\sigma^{p(p+1)} - 1\right) \mod p^{-2}+.$$

PROOF. Note that

$$X_i(\sigma(P)) = \sigma(X_i(P)) \text{ for } i = 0, 4. \tag{4.28}$$

We set $\alpha = \alpha_1^p$. By $X_0 = \alpha x$ and $X_4 = \alpha y$ in (3.7), we acquire $x(\sigma(P)) = \xi_\sigma^p \sigma(x(P))$ and $y(\sigma(P)) = \xi_\sigma^p \sigma(y(P))$. We consider (3.8). We have

$$(xy(x-y)^{p-1})(\sigma(P)) = \xi_\sigma^{p(p+1)}\sigma((xy(x-y)^{p-1})(P)). \tag{4.29}$$

We set $\mathfrak{s}(x,y) = \alpha xy\phi_2(x-y)^{-1}$. Since we have

$$\mathfrak{s}(x,y) = X_0X_4(g(X_4) - g(X_0))(X_0 - X_4)^{-1},$$

we acquire $\mathfrak{s}(x(\sigma(P)), y(\sigma(P))) = \sigma(\mathfrak{s}(x(P), y(P)))$ by (4.28) and $g(X) \in \mathbb{Z}_p[X]$. Hence, by (3.8), (4.29) and $\sigma(\gamma_1) = \xi_\sigma^{p^2-1}\gamma_1$, we have

$$\begin{aligned} Z(\sigma(P)) &= \gamma_1^{-1}\left((xy(x-y)^{p-1})(\sigma(P)) - 1 - \mathfrak{s}(x(\sigma(P)), y(\sigma(P)))\right) \\ &= \gamma_1^{-1}\Big(\xi_\sigma^{p(p+1)}\sigma\left(1 + \gamma_1 Z(P) + \mathfrak{s}(x(P), y(P))\right) \\ &\qquad\qquad - 1 - \sigma(\mathfrak{s}(x(P), y(P)))\Big) \\ &= \xi_\sigma^{(p+1)(2p-1)}\sigma(Z(P)) + \gamma_1^{-1}(\xi_\sigma^{p(p+1)} - 1)(1 + \sigma(\mathfrak{s}(x(P), y(P)))). \end{aligned}$$

Hence, the required assertion follows from Lemma 4.17 and $v(\mathfrak{s}(x,y)) > 0$. □

We describe the inertia action on $\coprod_{\zeta \in \mathcal{T}_n} \overline{\mathbf{X}}_{n,\zeta}$. First, we consider the case $n = 2$. By $x = (r+1)/(2s)$ and $y = (r-1)/(2s)$ in (3.41), we have $r = (x+y)/(x-y)$ and $s = 1/(x-y)$. Let $\zeta \in \mathcal{T}_2$ and $P \in \mathbf{X}_{2,\zeta}(\mathbb{C}_p)$. By Proposition 4.18, we obtain

$$r(\sigma(P)) = \sigma(r(P)), \quad s(\sigma(P)) = \xi_\sigma^{-p}\sigma(s(P)). \tag{4.30}$$

Proposition 4.19. *Let* $\sigma \in I_{\mathbb{Q}_p}$. *We fix the isomorphism* $(\mathbb{Z}_{p^2}/(\kappa^2))^* \simeq \mathbb{F}_{p^2}^* \times \mathbb{F}_{p^2};\ a+b\kappa \mapsto (\bar{a}, \bar{a}^{-1}\bar{b})$. *We write* $\mathbf{a}$ *for the composite*

$$I_{\mathbb{Q}_p} \xleftarrow{\sim} I_{\mathbb{Q}_{p^2}} \xrightarrow{\mathbf{a}_{\mathbb{Q}_{p^2}}} \mathbb{Z}_{p^2}^* \xrightarrow{\text{can.}} (\mathbb{Z}_{p^2}/(\kappa^2))^* \simeq \mathbb{F}_{p^2}^* \times \mathbb{F}_{p^2} \xrightarrow{\mathrm{pr}_2} \mathbb{F}_{p^2}.$$

Let ζ_0 *be as in §3.3. We set* $b(\sigma) = \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}\left(\mathbf{a}(\sigma)/\bar{\zeta}_0\right)$. *Then, the element* σ *acts on* $\coprod_{\zeta \in \mathcal{T}_2} \overline{\mathbf{X}}_{2,\zeta}$ *by*

$$\sigma\colon \overline{\mathbf{X}}_{2,\zeta} \to \overline{\mathbf{X}}_{2,t_{p+1}(\sigma)\zeta};\ (a,t) \mapsto \left(a - b(\sigma), t_{p+1}(\sigma)^{-1}t\right).$$

Proof. Let ζ_0' be as in §3.3. By $\zeta_0' \in \mathbb{Q}_p^{\mathrm{ur}}$ and $\sigma \in I_{\mathbb{Q}_p}$, we have $\sigma(\zeta_0') = \zeta_0'$. As in (3.56), we have $\beta_2 = \zeta_0\alpha_1^{p+1}$ and $\gamma_{0,2}' = 1 + \alpha_1^{p+1}b_2$. Then, we have $\sigma(\gamma_{0,2}') = 1 + \xi_\sigma^{p+1}(\gamma_{0,2}' - 1)$. By (3.59) for $n = 2$ and (4.30), we acquire

$$
\begin{aligned}
& t(\sigma(P)) = \xi_\sigma^p \sigma(t(P)), \\
& a(\sigma(P)) = \frac{\sigma(\beta_2)}{\beta_2}\sigma(a(P)) + \gamma_{0,2}'\beta_2^{-1}(\xi_\sigma^{2p(p+1)} - 1) \\
& \qquad + \xi_\sigma^{(p+1)(2p-1)}\beta_2^{-1}(1 - \xi_\sigma^{p+1}) + (\beta_2\gamma_1)^{-1}(\xi_\sigma^{p(p+1)} - 1) \\
& \qquad \equiv \sigma(a(P)) + \beta_2^{-1}(1 - \xi_\sigma^{p+1}) \\
& \qquad + (\beta_2\gamma_1)^{-1}(\xi_\sigma^{p(p+1)} - 1) \mod 0+,
\end{aligned} \tag{4.31}
$$

where we use Lemma 4.17 at the last congruence. Let θ_1 be as in (3.4). We set $\theta_\sigma = \sigma(\theta_1) - \theta_1$. By (3.4), we have $\sigma(\theta_1)^{p^2} - \sigma(\theta_1) = \theta_1^{p^2} - \theta_1$. By $v(\theta_1) = v(\theta_1') = -p^{-2}$, we have $\theta_\sigma \in \mathcal{O}_{\mathbb{C}_p}$ and hence $\overline{\theta}_\sigma \in \mathbb{F}_{p^2}$. By the property of tame character and $\bar{\xi}_\sigma \in \mu_{p^2-1}(\mathbb{F}_{p^2})$, we have

$$t_{p+1}(\sigma) = \overline{\sigma(\alpha_1^{p^2})/\alpha_1^{p^2}} = \bar{\xi}_\sigma^{p^2} = \bar{\xi}_\sigma. \tag{4.32}$$

Let $\mathscr{G}'$ be the formal $\mathbb{Z}_{p^2}$-module over $\mathbb{Z}_{p^2}$ such that $[\kappa]_{\mathscr{G}'}(X) = X^{p^2} + \kappa X$. We take non-zero elements ϖ_1' and ϖ_2' such that

$$[\kappa]_{\mathscr{G}'}(\varpi_1') = 0, \quad [\kappa]_{\mathscr{G}'}(\varpi_2') = \varpi_1'. \tag{4.33}$$

We put $\theta'_1 = \varpi'_2/\varpi'_1$. By (4.33), we have

$$\varpi_1'^{\,p^2-1} = -\kappa, \quad \theta_1'^{\,p^2} - \theta'_1 = -\kappa^{-1}.$$

Hence, by $v(\theta_1) = -p^{-2}$ and (3.4), we obtain $\theta_1 \equiv -\theta'_1 + c \mod 0+$ with some $c \in \mu_{p^2-1}(\mathbb{Q}_{p^2})$. We set $\theta'_\sigma = \sigma(\theta'_1) - \theta'_1$. Then, by $\sigma \in I_{\mathbb{Q}_p}$, we have

$$\theta_\sigma \equiv -\theta'_\sigma \mod 0+\,. \tag{4.34}$$

We set $\mathbf{a}_{\mathbb{Q}_{p^2}}(\sigma) = a_\sigma + b_\sigma\kappa + \sum_{i=2}^{\infty} b_{i,\sigma}\kappa^i \in \mathbb{Z}_{p^2}^*$ with $a_\sigma \in \mu_{p^2-1}(\mathbb{Q}_{p^2})$ and $b_\sigma, b_{i,\sigma} \in \mu_{p^2-1}(\mathbb{Q}_{p^2}) \cup \{0\}$. By the Lubin-Tate theory, we have

$$[a_\sigma]_{\mathscr{G}'}(\varpi'_1) = \sigma(\varpi'_1), \quad [a_\sigma + b_\sigma\kappa]_{\mathscr{G}'}(\varpi'_2) = \sigma(\varpi'_2).$$

This implies

$$a_\sigma = \sigma(\varpi'_1)/\varpi'_1, \quad a_\sigma\varpi'_2 + b_\sigma\varpi'_1 = \varpi'_1.$$

Dividing the second equality by $\sigma(\varpi'_1)$, we obtain $b_\sigma/a_\sigma = \theta'_\sigma$. Hence, by (4.34), we obtain $\bar{\theta}_\sigma = -\mathbf{a}(\sigma)$. Hence, the required assertion follows from (4.26), (4.31) and (4.32). □

Corollary 4.20. *The inertia subgroup $I_{\mathbb{Q}_p}$ acts on $\coprod_{\zeta \in \mathcal{T}_n} \overline{\mathbf{X}}_{n,\zeta}$ by*

$$\sigma\colon \overline{\mathbf{X}}_{n,\zeta} \to \overline{\mathbf{X}}_{n,t_{p+1}(\sigma)^{(-1)^n}\zeta};\ (a,t) \mapsto \Big(a + b(\sigma), t_{p+1}(\sigma)^{(-1)^{n-1}}t\Big)$$

for $\sigma \in I_{\mathbb{Q}_p}$.

Proof. The required assertion follows from Proposition 3.8.2, Lemma 4.1 and Proposition 4.19. □

Corollary 4.21. *We have an isomorphism*

$$\bigoplus_{\zeta \in \mathcal{T}_n} H^1_{\mathrm{c}}(\overline{\mathbf{X}}_{n,\zeta}) \simeq \bigoplus_{\psi \in \mathbb{F}_p^\vee \backslash \{1\},\ \chi \in \mu_{p+1}(\mathbb{F})^\vee} ((\chi \circ t_{p+1}) \otimes (\psi \circ b))^{\oplus p}$$

as $I_{\mathbb{Q}_p}$-representations.

Proof. We note that the induced action of $I_{\mathbb{Q}_p}$ on $\mathcal{T}_n$ factors through the abelian quotient $t_{p+1}\colon I_{\mathbb{Q}_p} \twoheadrightarrow \mu_{p+1}(\mathbb{F})$ by Corollary 4.20. By (4.10) and Corollary 4.20, we obtain

$$\begin{aligned}\bigoplus_{\zeta\in\mathcal{T}_n} H^1_{\mathrm{c}}(\overline{\mathbf{X}}_{n,\zeta}) &\simeq \bigoplus_{\chi'\in\mu_{p+1}(\mathbb{F})^\vee}\ \bigoplus_{\psi\in\mathbb{F}_p^\vee\setminus\{1\},\ \chi\in\mu_{p+1}(\mathbb{F})^\vee\setminus\{1\}} \\ &\qquad \times ((\chi'\circ t_{p+1})\otimes(\chi\circ t_{p+1})\otimes(\psi\circ b)) \\ &\simeq \bigoplus_{\psi\in\mathbb{F}_p^\vee\setminus\{1\},\ \chi\in\mu_{p+1}(\mathbb{F})^\vee} ((\chi\circ t_{p+1})\otimes(\psi\circ b))^{\oplus p}\end{aligned}$$

as $I_{\mathbb{Q}_p}$-representations. $\square$

We set

$$\begin{aligned}\mathcal{X} = \Big\{\chi \in \left(\mathbb{Q}_{p^2}^*\right)^{\vee,0} \mid \mathrm{a}(\chi)=2,\ \chi|_{\mathbb{Q}_p^*}=1,\ \chi|_{U^1_{\mathbb{Q}_{p^2}}} \\ \text{not factoring through } \mathrm{Nr}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}\Big\}\Big/\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p).\end{aligned}$$

For $[\chi]\in\mathcal{X}$, the pair $(\mathbb{Q}_{p^2}/\mathbb{Q}_p,\chi)$ is minimal admissible pair by definition. By [BH, §34.1 and §34.2], we have a bijection

$$\mathcal{X} \xrightarrow{\sim} \Pi_4^0;\ [\chi]\mapsto \mathrm{LL}^{-1}\left(\mathrm{Ind}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}(\Delta_\chi\chi)\right), \tag{4.35}$$

where Δ_χ is the unramified character of $\mathbb{Q}_{p^2}^*$ of order 2 (cf. [BH, §34.4]). No element of Π_4^0 does not come from an admissible pair $(E/\mathbb{Q}_p,\chi)$ with $E/\mathbb{Q}_p$ totally ramified. Assume that such pair exists. Since the central character of π is trivial, we obtain $\chi|_{U^1_{\mathbb{Q}_p}}=1$ by [BH, Proposition in §29.4]. By (4.14), we have $\mathrm{a}(\chi)=3$. By the canonical isomorphism $U^1_{\mathbb{Q}_p}/U^2_{\mathbb{Q}_p}\xrightarrow{\sim}U^2_E/U^3_E$, we must have $\chi|_{U^2_E}=1$. But, by $\mathrm{a}(\chi)=3$, this is a contradiction.

We fix the isomorphism $\mathbb{Z}_{p^2}^*/U^2_{\mathbb{Q}_{p^2}}\xrightarrow{\sim}\mathbb{F}_{p^2}^*\times\mathbb{F}_{p^2};\ a+bp\mapsto(\bar{a},\bar{a}^{-1}\bar{b})$. For any $[\chi]\in\mathcal{X}$, the restriction $\chi|_{\mathbb{Z}_{p^2}^*/U^2_{\mathbb{Q}_{p^2}}}$ induces the character of $\mathbb{F}_{p^2}^*\times\mathbb{F}_{p^2}$, for which we write $\bar{\chi}$. By $\chi|_{\mathbb{Q}_p^*}=1$ and the condition that $\chi|_{U^1_{\mathbb{Q}_{p^2}}}$ does not factor through the Norm map $\mathrm{Nr}_{\mathbb{Q}_{p^2}/\mathbb{Q}_p}\colon\mathbb{Q}_{p^2}^*\to\mathbb{Q}_p^*$, the character $\bar{\chi}$ induces the element of the following set:

$$\begin{aligned}\mathcal{Y}=\{(\chi',\psi)\in\mu_{p+1}(\mathbb{F})^\vee\times\mathbb{F}_{p^2}^\vee \mid \psi \text{ dose not factor through} \\ \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}\colon\mathbb{F}_{p^2}\to\mathbb{F}_p,\ \psi|_{\mathbb{F}_p}=1\}/\simeq,\end{aligned}$$

where $\simeq$ is the equivalence relation defined by $(\chi', \psi) \simeq ({\chi'}^{-1}, \psi \circ (x \mapsto x^p))$. Hence, we obtain the map $\mathscr{L} \colon \mathcal{X} \to \mathcal{Y}$. This is bijective. Hence, we have $|\Pi_4^0| = (p^2-1)/2$.

COROLLARY 4.22. 1. *We have*

$$d(\pi) = \begin{cases} p & \text{if } \pi \in \Pi_4^0, \\ 0 & \text{if } \pi \in \Pi_4 \setminus \Pi_4^0. \end{cases}$$

2. *We have an isomorphism*

$$\bigoplus_{\zeta \in \mathcal{T}_n} H_{\mathrm{c}}^1(\overline{\mathbf{X}}_{n,\zeta}) \simeq \bigoplus_{\pi \in \Pi_4} \left(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}} \right)^{\oplus d(\pi)}$$

as $I_{\mathbb{Q}_p}$-representations.

PROOF. Let $\pi \in \Pi_4^0$. By [BH, §56], we have an isomorphism $\mathrm{LJ}(\pi) \simeq \mathrm{Ind}_{\mathbb{Q}_{p^2}^* U_D^1}^{D^*} \rho$ as D^*-representations, where ρ is some irreducible representation of $\mathbb{Q}_{p^2}^* U_D^1$ of dimension p. Hence, we have $\dim \mathrm{LJ}(\pi) = 2p$ for any $\pi \in \Pi_4^0$. Hence, the first assertion follows from (4.13).

We prove the second assertion. We set $[(\chi, \psi)] = \mathscr{L}(\pi)$. Then, by (4.15), (4.35) and $I_{\mathbb{Q}_{p^2}} \xrightarrow{\sim} I_{\mathbb{Q}_p}$, we have isomorphisms

$$\begin{aligned} \mathrm{LL}_\ell(\pi)|_{I_{\mathbb{Q}_p}} = \mathrm{LL}(\pi)|_{I_{\mathbb{Q}_p}} \simeq ((\chi \circ t_{p+1}) \\ \otimes (\psi \circ \mathbf{a})) \oplus ((\chi^{-1} \circ t_{p+1}) \otimes (\psi \circ (x \mapsto x^p) \circ \mathbf{a})) \end{aligned} \tag{4.36}$$

as $I_{\mathbb{Q}_p}$-representations. We set $\mathrm{tr}' \colon \mathbb{F}_{p^2} \to \mathbb{F}_p;\ x \mapsto \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p}(x/\bar{\zeta}_0)$. Since $\bar{\zeta}_0^{p-1} = -1$, we have $\mathrm{tr}'(x) = 0$ for any $x \in \mathbb{F}_p$. Since p is odd, we have

$$\begin{aligned} &\{\psi \circ \mathrm{tr}' \in \mathbb{F}_{p^2}^\vee \mid \psi \in \mathbb{F}_p^\vee \setminus \{1\}\} \\ &= \{\psi \in \mathbb{F}_{p^2}^\vee \mid \psi \text{ does not factor through} \\ &\qquad \mathrm{Tr}_{\mathbb{F}_{p^2}/\mathbb{F}_p} \colon \mathbb{F}_{p^2} \to \mathbb{F}_p,\ \psi|_{\mathbb{F}_p} = 1\} \end{aligned} \tag{4.37}$$

Note that

$$b = \mathrm{tr}' \circ \mathbf{a}. \tag{4.38}$$

Hence, the required assertion follows from the first assertion, Corollary 4.21, (4.36), (4.37), (4.38) and the bijection $\mathcal{X} \simeq \mathcal{Y}$. $\square$

Remark 4.23. For $\pi \in \Pi_4^0$, let $\widetilde{\mathrm{LJ}}(\pi)$ denote the unique irreducible representation of $\mathcal{O}_D^*$ of dimension p which satisfies an isomorphism $\widetilde{\mathrm{LJ}}(\pi)^{\oplus 2} \simeq \mathrm{LJ}(\pi)\big|_{\mathcal{O}_D^*}$ as $\mathcal{O}_D^*$-representations. It is expected that $\coprod_{\zeta \in \mathcal{T}_n} \overline{\mathbf{X}}_{n,\zeta}$ is stable under the action of $\mathcal{O}_D^*$, and we have an isomorphism

$$\bigoplus_{\zeta \in \mathcal{T}_n} H_{\mathrm{c}}^1(\overline{\mathbf{X}}_{n,\zeta}) \simeq \bigoplus_{\pi \in \Pi_4^0} \Big(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}} \otimes \widetilde{\mathrm{LJ}}(\pi)\Big) \tag{4.39}$$

as $I_{\mathbb{Q}_p} \times \mathcal{O}_D^*$-representations.

Remark 4.24. We consider the equalities which are obtained by taking the dimensions of the both sides of the isomorphisms in Corollaries 4.9, 4.14 and 4.22. A reason why these equalities hold can be explained by using [IT2, Proposition 4.3]. However, we do not explain this in detail, because we have to recall the whole shape of the stable reduction or the stable covering of the wide open rigid curve $W_A(p^n)$ for $2 \leq n \leq 4$ and need some facts in representation theory.

Remark 4.25. Assume that $j(A) \in \{0, 1728\}$. Let $\mathcal{T}_n^A = \big\{\zeta \in \mathbb{F}_{p^2} \mid 4\zeta^{2c(A)} + 1 = 0\big\}$. Let $\Pi_4^{A,0}$ be the subset of Π_4^0 consisting of representations which correspond to characters whose restrictions to $K(A)$ are trivial under (4.35). For $\pi \in \Pi_4^0$, we have

$$\dim \mathrm{LJ}(\pi)^{K(A)} = \begin{cases} \dim \mathrm{LJ}(\pi) & \text{if } \pi \in \Pi_4^{A,0}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we acquire isomorphisms

$$\begin{aligned} \bigoplus_{\zeta \in \mathcal{T}_n^A} H_{\mathrm{c}}^1(\overline{\mathbf{X}}_{n,\zeta}) &\simeq \bigoplus_{\psi \in \mathbb{F}_p^\vee \backslash \{1\},\ \chi \in \mu_{2c(A)}(\mathbb{F})^\vee} \Big(\big(\chi \circ t_{2c(A)}\big) \otimes (\psi \circ b)\Big)^{\oplus p} \\ &\simeq \bigoplus_{\pi \in \Pi_4} \Big(\mathrm{LL}_\ell(\pi)\big|_{I_{\mathbb{Q}_p}}\Big)^{\oplus d(A,\pi)} \end{aligned}$$

as $I_{\mathbb{Q}_p}$-representations.

REMARK 4.26. For a prime $5 \leq p \leq 13$, it is expected that the same isomorphisms as the ones in Remarks 4.11, 4.16 and 4.25 hold. Actually, for $p = 5, 7$, we can check them by using computations given in [T2]. For $p = 3$, the situation is considerably different, because of $|\mathrm{Aut}(A)| = 12$ for the supersingular elliptic curve in characteristic 3. See [Mc, Theorem 3.11 and §4.1] on the stable reduction of $X_0(3^4)$ and the inertia action on it. Since Galois representations in the case $p = 2$ become complicated, the stable reduction of $X_0(p^n)$ is much more difficult to understand.

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