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Derivatives of Secondary Classes and 2-Normal Bundles of Foliations

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Abstract. Derivatives of secondary characteristic classes for foliations are discussed. It will be shown that one can construct the derivatives in a parallel way to the standard construction of secondary characteristic classes, namely, by using connections and applying the Chern-Weil theory. Some relationship of connections in the construction and transverse TW-connections, which is significant in the study of deformations of the Godbillon-Vey class and the Bott class, are also discussed.

Introduction

It is known that some secondary characteristic classes for foliations admit continuous deformations. That is, the classes vary continuously according to deformations of foliations. If the families are differentiable, we can consider the derivatives of characteristic classes with respect to deformation parameters. Such derivatives are studied by Heitsch et. al. [12], [13], [14], see also [9], [4]. Secondary classes and derivatives of them are constructed in terms of connections and deformations of them so that independence of cohomology classes of choices is to be shown. It is done for secondary classes usually by using the Chern-Simons forms. On the other hand, it usually relies on combinatorial arguments for derivatives of them. In this paper, we give a framework by which derivatives of secondary classes are treated as secondary classes for foliations and deformations. In particular, there will appear a kind of truncated Weil algebras such as WO_q and WU_q. If we restrict ourselves to the Godbillon-Vey class and the Bott class, then

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it is known that deformations are related with transverse projective structures and that transverse projective TW-connections are relevant [5]. We will discuss how connections associated with deformations of foliations and transverse TW-connections are related. Roughly speaking, there is a certain extension of the tangent bundle of ambient manifolds which has some parameters. There is a connection such that the constant term of it with respect to the parameters is a deformation of foliations, and the linear term of it is a deformation of transverse projective structures.

1. Preliminaries

Assumption. *We assume the following throughout the paper. We denote by M a manifold equipped with a foliation $\mathcal{F}$. The foliation $\mathcal{F}$ is assumed to be transversely holomorphic unless otherwise mentioned. The arguments for real (smooth) foliations are almost parallel and easier. The term 'smooth' stands for the term 'of class C^∞', even in the transversely holomorphic case. We work in the smooth category unless otherwise mentioned.*

Let p and q be the dimension and the complex codimension of $\mathcal{F}$, respectively. Then, a foliation chart is given by a triple $(U, V \times B, \varphi)$, where $\varphi\colon U \to V \times B \subset \mathbb{R}^p \times \mathbb{C}^q$. We usually let (x, y) be the natural coordinates on $V \times B$. For simplicity we identify U and $V \times B$, and regard (x, y) as coordinates on U.

Notation. We will frequently compare coefficients of tensors, connections, etc. in what follows. Once a chart is chosen and coefficients are defined, the symbol '$\widehat{\ }$' is used to express another chart and the coefficients on it. For example, if (U, φ) is a chart and if $a_1, \dots, a_q$ are coefficients of a tensor on (U, φ), then $(\widehat{U}, \widehat{\varphi})$ represents a chart such that $U \cap \widehat{U} \neq \varnothing$ and $\widehat{a}_1, \dots, \widehat{a}_q$ represent the coefficients on $(\widehat{U}, \widehat{\varphi})$. The coefficients are often considered as entries of matrices, and the multiplication rule of matrices is applied. For example, if $\omega^1, \dots, \omega^q$ are coefficients of a $\mathbb{C}^q$-valued 1-form and if a^i_j, where $1 \leq i, j \leq q$, are coefficients of a $\mathfrak{gl}_q\mathbb{C}$-valued 2-form, then we set $\omega = (\omega^i) = {}^t(\omega^1 \ \cdots \ \omega^q)$, $a = (a^i_j)$ and define $a \wedge \omega$ to be a $\mathbb{C}^q$-valued 3-form of which the i-th entry is given by $\sum_j a^i_j \wedge \omega^j = a^i_j \wedge \omega^j$. Note that we make use of the Einstein convention. Finally, when coefficients of tensors, etc., are expressed, the Roman indices will begin from one, while the Greek indices will begin from zero.

Notation 1.1. Let U be an open subset of M and E a vector bundle over M. We denote by $\varGamma_U(E)$ the module of the smooth sections to E over U, even in the transversely holomorphic case. If $U = M$, then we denote $\varGamma_M(E)$ also by $\varGamma(E)$.

DEFINITION 1.2. Let U and $\widehat{U}$ be foliation charts and φ the transition function from U to $\widehat{U}$. Then, under the identifications of $U \cong V \times B$ and $\widehat{U} \cong \widehat{V} \times \widehat{B}$, φ is of the form (ψ, γ). We refer γ as the *transverse component* of φ.

DEFINITION 1.3. If $\mathcal{F}$ is a real foliation, then we set $E(\mathcal{F}) = T\mathcal{F}$, namely, the subbundle of TM which consists of vectors tangent to leaves. If $\mathcal{F}$ is transversely holomorphic, then we denote $TM \otimes \mathbb{C}$ by TM by abuse of notations, and define $E(\mathcal{F})$ to be the complex subbundle of TM locally spanned by $E(\mathcal{F})$ and $\dfrac{\partial}{\partial \overline{y}^i}$, where $1 \leq i \leq q$. In the both cases, we set $Q(\mathcal{F}) = TM/E(\mathcal{F})$. We call $Q(\mathcal{F})$ the *normal bundle* in the real case, and the *complex normal bundle* in the transversely holomorphic case. We denote by π the projection from TM to $Q(\mathcal{F})$, and by p the one from $Q(\mathcal{F})$ to M. We locally set $e_i = \pi \left(\dfrac{\partial}{\partial y^i} \right)$, and choose $(e_1, \ldots, e_q)$ as a local trivialization of $Q(\mathcal{F})$ unless otherwise mentioned.

2. 2-Normal Bundles of Foliations

The 2-tangent bundle of a manifold M is by definition of the tangent bundle of the tangent bundle TM [22]. We will first introduce 2-normal bundles of foliations as an analogy.

Notation 2.1. We denote by $T\mathrm{GL}_q\mathbb{C}$ the tangent group of $\mathrm{GL}_q\mathbb{C}$. That is, $T\mathrm{GL}_q\mathbb{C}$ is the tangent bundle of $\mathrm{GL}_q\mathbb{C}$ equipped with the following multiplication. Let I_q be the unit element in $\mathrm{GL}_q\mathbb{C}$. We identify $T_{I_q}\mathrm{GL}_q\mathbb{C}$ with the Lie algebra of the left invariant vector fields on $\mathrm{GL}_q\mathbb{C}$, and denote it by $\mathfrak{gl}_q\mathbb{C}$. Then, we have a natural identification $T\mathrm{GL}_q\mathbb{C} = \mathrm{GL}_q\mathbb{C} \times \mathfrak{gl}_q\mathbb{C}$ as manifolds. If $(A, B), (A', B') \in T\mathrm{GL}_q\mathbb{C} = \mathrm{GL}_q\mathbb{C} \times \mathfrak{gl}_q\mathbb{C}$, then $(A, B)(A', B') = (AA', (A')^{-1}BA' + B')$. Therefore, $T\mathrm{GL}_q\mathbb{C} = \mathrm{GL}_q\mathbb{C} \ltimes \mathfrak{gl}_q\mathbb{C}$ as Lie groups.

A matrix representation of $T\mathrm{GL}_q\mathbb{C}$ is given by

$$(A,B) \mapsto \begin{pmatrix} A & O \\ AB & A \end{pmatrix} \in \mathrm{GL}_{2q}\mathbb{C}.$$

This representation is indeed given by the natural action of $T\mathrm{GL}_q\mathbb{C}$ on $T\mathbb{C}^q$ [22]. If we denote by $\mathfrak{tgl}_q\mathbb{C}$ the Lie algebra of $T\mathrm{GL}_q\mathbb{C}$, then the induced representation is also given by $(X,Y) \mapsto \begin{pmatrix} X & O \\ Y & X \end{pmatrix}$.

Let $Q^{(2)}(\mathcal{F})$ be the complex vector bundle of rank $2q$ over $Q(\mathcal{F})$ defined as follows. Let $l = l(t)$ be a curve in $Q(\mathcal{F})$. We can locally represent $l(t)$ as $l(t) = (z(t), v(t)) \in U \cong V \times B$. Let $l(0) = (z,v)$ and $\dfrac{dl}{dt}(0) = (\dot{z}, \dot{v})$. If $U, \widehat{U}$ are foliation charts and if φ is the transition function from U to $\widehat{U}$, then φ induces a transition function from $p^{-1}(U)$ to $p^{-1}(\widehat{U})$ which we denote by $\widetilde{\varphi}$. If we denote by γ the transverse component of φ, then we have $\widetilde{\varphi} = (\varphi, D\gamma)$, namely, $\widetilde{\varphi} \circ l(t) = (\varphi(z(t)), D\gamma_{y(t)} v(t))$, where $z(t) = (x(t), y(t)) \in U \cap \widehat{U} \subset U$, and

$$\frac{d(\widetilde{\varphi} \circ l)}{dt}(0) = \left(D\varphi_z\, \dot{z}, H\gamma^i_{jk,y}\, \dot{y}^j v^k + D\gamma_y\, \dot{v} \right),$$

where $z = (x,y)$, $\dot{z} = (\dot{x}, \dot{y})$ and $H\gamma^i_{jk,y} = \dfrac{\partial^2 \gamma^i}{\partial y^j \partial y^k}(y)$. We often denote $H\gamma^i_{jk,y}$ by $H\gamma^i_{jk}$. Note that $H\gamma^i_{jk} = H\gamma^i_{kj}$ and that $\dot{y} = \pi_*(\dot{z})$ holds in $Q(\mathcal{F})$. We set $Q^{(2)}(\mathcal{F}) = \{(z, v; \dot{y}, \dot{v})\}$ and define $p^{(2)}_Q, \nu\colon Q^{(2)}(\mathcal{F}) \to Q(\mathcal{F})$ by $p^{(2)}_Q(z, v; \dot{y}, \dot{v}) = (z; v)$ and $\nu(z, v; \dot{y}, \dot{v}) = (z; \dot{y})$, respectively. It is easy to see that $p^{(2)}_Q$ and ν are globally well-defined.

Definition 2.2. We call $p^{(2)}_Q\colon Q^{(2)}(\mathcal{F}) \to Q(\mathcal{F})$ the *2-normal bundle* of $\mathcal{F}$.

The following diagram commutes:

$$\begin{array}{ccc} Q^{(2)}(\mathcal{F}) & \xrightarrow{\ \nu\ } & Q(\mathcal{F}) \\ {\scriptstyle p^{(2)}_Q}\downarrow & & \downarrow{\scriptstyle p_Q} \\ Q(\mathcal{F}) & \xrightarrow[p_Q]{} & M. \end{array}$$

A local description of $Q^{(2)}(\mathcal{F})$ is given as follows. If $q \in Q(\mathcal{F})$, then q is represented as $\left(z, v^1 e_1 + \cdots + v^q e_q\right)$ on a foliation chart. Let $v = \begin{pmatrix} v^1 \\ \vdots \\ v^q \end{pmatrix}$ and regard (z, v) be coordinates on $\pi^{-1}(U)$. Let $(U, V \times B, \varphi)$ and $(\widehat{U}, \widehat{V} \times \widehat{B}, \widehat{\varphi})$ be foliation charts and $q \in \pi^{-1}(U) \cap \pi^{-1}(\widehat{U})$. If both (z, v) and $(\widehat{z}, \widehat{v})$ represent q, then $(\widehat{z}, \widehat{v}) = (\varphi(z), D\gamma_y v)$, where $z = (x, y)$. We set $\eta_i = \dfrac{\partial}{\partial v^i}$. Then, a local trivialization over $\pi^{-1}(U)$ is given by $(e_1, \ldots, e_q, \eta_1, \ldots, \eta_q)$. If we set $p_Q^{21} = p_Q \circ p_Q^{(2)}$, then the transition function from $(p_Q^{21})^{-1}(U)$ to $(p_Q^{21})^{-1}(\widehat{U})$ is give by

$$\begin{aligned} &(\widehat{e}_1, \ldots, \widehat{e}_q, \widehat{\eta}_1, \ldots, \widehat{\eta}_q)_{(\varphi(z), D\gamma_y v)} \begin{pmatrix} D\gamma_y & O \\ H\gamma_y v & D\gamma_y \end{pmatrix} \\ &= (e_1, \ldots, e_q, \eta_1, \ldots, \eta_q)_{(z,v)}, \end{aligned}$$

where $(H\gamma_y v)^i_j = H\gamma^i_{jk,y} v^k$.

Several foliations are naturally defined on $Q(\mathcal{F})$ and $Q^{(2)}$. Let $U \subset M$ be a foliation chart. If we choose $(e_1, \ldots, e_q, \eta_1, \ldots, \eta_q)$ as a local trivialization, then (z, v) are coordinates on $\pi^{-1}(U)$, and $(z, v; \dot{y}, \dot{v})$ are coordinates on $(p_Q^{21})^{-1}(U)$. We can define foliations of $Q(\mathcal{F})$ by locally setting $\mathcal{F}_Q = \{y, v \text{ are constant}\}$ and $\pi^*\mathcal{F} = \{y \text{ is constant}\}$. Similarly, we can define foliations of $Q^{(2)}(\mathcal{F})$ by locally setting

$$\begin{aligned} \mathcal{F}^{(2)} &= \{y, v, \dot{y}, \dot{v} \text{ are constant}\}, \\ \mathcal{F}^{(2)}_1 &= \{y, v, \dot{y} \text{ are constant}\}, \\ \mathcal{F}^{(2)}_2 &= \{y, v \text{ are constant}\}, \\ \mathcal{F}^{(2)}_3 &= \{y, \dot{y} \text{ are constant}\}, \\ \mathcal{F}^{(2)}_4 &= \{y \text{ is constant}\}. \end{aligned}$$

Note that $\pi^*\mathcal{F}$ is indeed the pull-back of $\mathcal{F}$ by π, and that $\mathcal{F}^{(2)}_2 = \pi^{(2)*}\mathcal{F}_Q$, $\mathcal{F}^{(2)}_4 = p_Q^{21*}\mathcal{F}$ and $\mathcal{F}^{(2)}_1 = \mathcal{F}^{(2)}_2 \cap \mathcal{F}^{(2)}_3$. Note also that instead of dealing with $p_Q^{(2)} \colon Q^{(2)}(\mathcal{F}) \to Q(\mathcal{F})$, we can work on $\nu \colon Q^{(2)}(\mathcal{F}) \to Q(\mathcal{F})$ by exchanging v and $\dot{y}$. This point of view is relevant in §5. Finally we remark that the 2-normal bundle $Q^{(2)}(\mathcal{F})$ is closely related to the 2-jet bundle of $\mathcal{F}$ which are usually denoted by $J^2(\mathcal{F})$.

3. Infinitesimal Deformations and 2-Normal Bundles

We will introduce infinitesimal deformations of foliations after Heitsch [13] (cf. [12], [9], see also [16]).

Definition 3.1. Let U be an open subset of M. A section $X \in \Gamma_U$ of $Q(\mathcal{F})$ is said to be *foliated* if $\mathcal{L}_Y X = 0$ for any section Y of $E(\mathcal{F})$, where $\mathcal{L}_Y$ denotes the Lie derivative with respect to Y. We denote by $\Theta_{\mathcal{F}}$ the sheaf of germs of foliated sections of $Q(\mathcal{F})$.

Let ∇^b be a Bott connection on $Q(\mathcal{F})$ and denote by d_{∇^b} the covariant differentiation associated with ∇^b. It is known that $\{\bigwedge^i E(\mathcal{F})^* \otimes Q(\mathcal{F}), d_{\nabla^b}\}$ is a resolution of $\Theta_{\mathcal{F}}$ [12], [9], [11].

Definition 3.2. An infinitesimal deformation of $\mathcal{F}$ is an element of $H^1(M; \Theta_{\mathcal{F}})$.

If σ is a representative of an infinitesimal deformation of $\mathcal{F}$, then σ can be locally represented as $\sigma = e_i \sigma^i$, where $(e_1, \dots, e_q)$ is the local trivialization of $Q(\mathcal{F})$ as above and σ^i are 1-forms such that

$$\widehat{\sigma}^i = D\gamma^i_j \, \sigma^j.$$

In addition, as we identify fibers of $Q(\mathcal{F})$ with $\mathbb{C}^q$ by the trivialization, there is a $\mathfrak{gl}_q\mathbb{C}$-valued 1-form $(\dot{\omega}^i_j)$ such that

$$d\sigma^i + \omega^i_j \wedge \sigma^j + \dot{\omega}^i_j \wedge \theta^j = 0, \tag{3.3}$$

where $(\theta^1, \dots, \theta^q) = (dy^1, \dots, dy^q)$ is the dual to $(e_1, \dots, e_q)$ and $\omega = (\omega^i_j)$ denotes the connection form of ∇^b with respect to $(e_1, \dots, e_q)$. By using a partition of unity, we may assume that

$$\dot{\omega}^i_j = (D\gamma)^{-1}{}^i_k \, \widehat{\omega}^k_l \, D\gamma^l_j.$$

By abuse of notations, we set $\sigma = (\sigma^i)$. Then, (3.3) is represented as $d\sigma + \omega \wedge \sigma + \dot{\omega} \wedge \theta = 0$.

Let $(z, v; \dot{y}, \dot{v})$ be local coordinates for $Q^{(2)}(\mathcal{F})$. As ω is the connection form of a Bott connection, $\omega^i_j = f^i_{jk} dy^k$ holds for some functions f^i_{jk}. We set

$$\rho^i_j = f^i_{jk} v^k,$$

$$\theta^{(2)} = \begin{pmatrix} \widetilde{\theta} \\ \widetilde{\sigma} \end{pmatrix} = \begin{pmatrix} \theta \\ \sigma - \rho\,\theta \end{pmatrix},$$
$$\omega^{(2)} = \begin{pmatrix} \omega & O \\ \dot{\omega} + d\rho + [\omega, \rho] & \omega \end{pmatrix}$$

on $Q(\mathcal{F})$, where $[\omega, \rho] = \omega\rho - \rho\omega$.

LEMMA 3.4 (cf. [5, Theorem 1.20]).

1) *If we set*

$$D^{(2)}\gamma = D^{(2)}\gamma_{(p,v)} = \begin{pmatrix} D\gamma_y & O \\ H\gamma_y\, v & D\gamma_y \end{pmatrix},$$

where $p = (x, y)$*, then,*

$$\widehat{\theta}^{(2)} = (D^{(2)}\gamma)\theta^{(2)}.$$

2) *We have*

$$d\theta^{(2)} + \omega^{(2)} \wedge \theta^{(2)} = 0. \tag{3.5}$$

3) *The family* $\{\omega^{(2)}\}$ *of* $\mathfrak{tgl}_q\mathbb{C}$*-valued one-forms gives rise to a connection on* $Q^{(2)}(\mathcal{F}) \to Q(\mathcal{F})$ *which we denote by* $\nabla^{(2)}$*. Conversely, if* $\{\widetilde{\omega}\}$ *is a family of local connection forms of a connection on* $Q^{(2)}(\mathcal{F}) \to Q(\mathcal{F})$*, then* $\{\iota^*\widetilde{\omega}\}$*, namely, the restriction of* $\{\widetilde{\omega}\}$ *to* M *determines an infinitesimal deformation of* ω *with respect to* θ*.*

PROOF. We have

$$\omega = (D\gamma)^{-1}(dD\gamma) + (D\gamma)^{-1}\widehat{\omega}(D\gamma),$$
$$\dot{\omega} = (D\gamma)^{-1}\widehat{\dot{\omega}}(D\gamma)$$

and $f^i_{jk} = \dfrac{\partial y^i}{\partial \widehat{y}^l}\dfrac{\partial^2 \gamma^l}{\partial y^j \partial y^k} + \dfrac{\partial y^i}{\partial \widehat{y}^l}\widehat{f}^l_{mn}\dfrac{\partial \widehat{y}^m}{\partial y^j}\dfrac{\partial \widehat{y}^n}{\partial y^k}$. Therefore, we have

$$\begin{aligned} \rho^i_j &= f^i_{jk}v^k \\ &= \left(\frac{\partial y^i}{\partial \widehat{y}^l}\frac{\partial^2 \gamma^l}{\partial y^j \partial y^k} + \frac{\partial y^i}{\partial \widehat{y}^l}\widehat{f}^l_{mn}\frac{\partial \widehat{y}^m}{\partial y^j}\frac{\partial \widehat{y}^n}{\partial y^k}\right)\frac{\partial y^k}{\partial \widehat{y}^r}\widehat{v}^r \\ &= (D\gamma^{-1})^i_l(H\gamma^l_{jk}v^k + \widehat{\rho}^l_m D\gamma^m_j). \end{aligned}$$

Hence we have

$$\begin{aligned} d\rho &= -(D\gamma)^{-1}(dD\gamma)(D\gamma)^{-1}(H\gamma)v + (D\gamma)^{-1}(dH\gamma)v + (D\gamma)^{-1}(H\gamma)dv \\ &\quad - (D\gamma)^{-1}(dD\gamma)(D\gamma)^{-1}\widehat{\rho}(D\gamma) + (D\gamma)^{-1}(d\widehat{\rho})(D\gamma) \\ &\quad + (D\gamma)^{-1}\widehat{\rho}(dD\gamma), \\ \omega\rho &= (D\gamma)^{-1}(dD\gamma)(D\gamma)^{-1}(H\gamma)v + (D\gamma)^{-1}(dD\gamma)(D\gamma)^{-1}\widehat{\rho}(D\gamma) \\ &\quad + (D\gamma)^{-1}\widehat{\omega}(H\gamma)v + (D\gamma)^{-1}\widehat{\omega}\widehat{\rho}(D\gamma), \\ \rho\,\omega &= (D\gamma)^{-1}(H\gamma)v(D\gamma)^{-1}(dD\gamma) + (D\gamma)^{-1}(H\gamma)v(D\gamma)^{-1}\widehat{\omega}(D\gamma) \\ &\quad + (D\gamma)^{-1}\widehat{\rho}(dD\gamma) + (D\gamma)^{-1}\widehat{\rho}\,\widehat{\omega}(D\gamma), \\ \rho\,\theta &= (D\gamma)^{-1}(H\gamma)v(D\gamma)^{-1}\widehat{\theta} + (D\gamma)^{-1}\widehat{\rho}\,\widehat{\theta}. \end{aligned}$$

Therefore,

$$\begin{aligned} \widetilde{\sigma} &= \sigma - \rho\,\theta \\ &= (D\gamma)^{-1}\widehat{\sigma} - (D\gamma^{-1})(H\gamma)v(D\gamma)^{-1}\widehat{\theta} - (D\gamma)^{-1}\widehat{\rho}\,\widehat{\theta} \\ &= (D\gamma)^{-1}\widehat{\widetilde{\sigma}} - (D\gamma)^{-1}(H\gamma)v(D\gamma)^{-1}\widehat{\theta}. \end{aligned}$$

The part 1) follows from the last equalities. If we set

$$(D^{(2)}\gamma)^{-1}d(D^{(2)}\gamma) + (D^{(2)}\gamma)^{-1}\,\widehat{\omega}^{(2)}(D^{(2)}\gamma) = \begin{pmatrix} \lambda^1_1 & \lambda^1_2 \\ \lambda^2_1 & \lambda^2_2 \end{pmatrix},$$

then we have

$$\begin{gathered} \lambda^1_1 = \lambda^2_2 = (D\gamma)^{-1}(dD\gamma) + (D\gamma)^{-1}\widehat{\omega}(D\gamma) = \omega, \\ \lambda^1_2 = 0, \end{gathered}$$

and

$$\left\{ \begin{aligned} \lambda^2_1 &= -(D\gamma)^{-1}(H\gamma)v(D\gamma)^{-1}(dD\gamma) \\ &\qquad + (D\gamma)^{-1}(dH\gamma)v + (D\gamma)^{-1}(H\gamma)dv \\ &\qquad - (D\gamma)^{-1}(H\gamma)v(D\gamma)^{-1}\widehat{\omega}(D\gamma) \\ &\qquad + (D\gamma)^{-1}(\widehat{\dot{\omega}} + d\widehat{\rho} + [\widehat{\omega}, \widehat{\rho}])(D\gamma) \\ &\qquad + (D\gamma)^{-1}\widehat{\omega}(H\gamma)v \\ &= \dot{\omega} + d\rho + [\omega, \rho]. \end{aligned} \right. \tag{3.6}$$

Consequently, $\{\omega^{(2)}\}$ gives rise to a connection on $Q^{(2)}(\mathcal{F})$. This shows the first claim of 3). On the other hand, we have

$$\begin{aligned} d\widetilde{\sigma} &= d\sigma - d(\rho\,\theta) \\ &= -\dot{\omega}\wedge\theta - \omega\wedge\sigma - d\rho\wedge\theta - \rho\, d\theta \\ &= -(\dot{\omega}+d\rho)\wedge\theta - \omega\wedge(\widetilde{\sigma}+\rho\,\theta) + \rho\,\omega\wedge\theta \\ &= -(\dot{\omega}+d\rho+[\omega,\rho])\wedge\theta - \omega\wedge\widetilde{\sigma}. \end{aligned}$$

This shows the part 2). The latter part of 3) follows from the fact that the equality (3.5) reduces to the defining condition of infinitesimal deformations of ω. This completes the proof. □

Remark 3.7. Lemma 3.4 implies that $\{\theta^{(2)}\}$ formally defines a foliation of $Q^{(2)}(\mathcal{F})$. Indeed, if each $\theta^{(2)}$ is a local trivialization of $T^*Q(\mathcal{F})$, then a foliation is defined and $\omega^{(2)}$ defines a Bott connection.

Remark 3.8. Let

$$R^{(2)} = d\omega^{(2)} + \omega^{(2)}\wedge\omega^{(2)}$$

be the curvature form of $\nabla^{(2)}$ with respect to $\theta^{(2)}$. If we denote by $R = d\omega + \omega\wedge\omega$ the curvature form of ∇^b with respect to θ, then, we have

$$R^{(2)} = \begin{pmatrix} R & O \\ [R,\rho] + d\dot{\omega} + [\dot{\omega},\omega] & R \end{pmatrix},$$

where $[R,\rho] = R\rho - \rho R$ and $[\dot{\omega},\omega] = \dot{\omega}\wedge\omega + \omega\wedge\dot{\omega}$. By the equality (3.5), we have

$$R^{(2)}\wedge\theta^{(2)} = 0.$$

There are several choices in defining ρ, $\theta^{(2)}$ and $\omega^{(2)}$. We will study how they affect $\{\omega^{(2)}\}$. We denote by ρ', $\theta^{(2)\prime}$, $\omega^{(2)\prime}$ etc., newly obtained ones. First we fix θ, ω and σ. Suppose that $\{a^i_{jk}\}$ is a family of functions such that

$$\text{(3.9)} \qquad \begin{cases} a^i_{jk} = a^i_{kj}, \\ a^i_{jk} = (D\gamma)^{-1}{}^i_l\,\widehat{a}^l_{mn}(D\gamma)^m_j(D\gamma)^n_k. \end{cases}$$

Then,

$$\dot\omega'^i_j = \dot\omega^i_j + a^i_{jk}\theta^k.$$

Conversely, any infinitesimal deformation of ω with respect to σ is of this form. Accordingly,

$$\omega^{(2)\prime} = \omega^{(2)} + \begin{pmatrix} O & O \\ A\theta & O \end{pmatrix},$$

where $(A\theta)^i_j = a^i_{jk}\theta^k$. We can replace ω in a similar way, namely, let $\{b^i_{jk}\}$ be a family of functions such that

$$\begin{cases} b^i_{jk} = b^i_{kj}, \\ b^i_{jk} = (D\gamma)^{-1}{}^i_l \widehat{b}^l_{mn} (D\gamma)^m_j (D\gamma)^n_k. \end{cases} \tag{3.10}$$

If we set $(B\theta)^i_j = b^i_{jk}\theta^k$, then the connection form of a Bott connection is of the form $\omega + B\theta$ and vice versa. Let $\omega' = \omega + B\theta$ and ∇' be the connection defined by ω'. Then, the 'identity map' from $H^*(\bigwedge^i E(\mathcal{F})^* \otimes Q(\mathcal{F}), d_\nabla)$ to $H^*(\bigwedge^i E(\mathcal{F})^* \otimes Q(\mathcal{F}), d_{\nabla'})$ given by $[\sigma] \mapsto [\sigma]$ gives an isomorphism. Indeed, if σ is an infinitesimal deformation, then the infinitesimal deformation $\dot\omega'$ of ω' with respect to σ is given by

$$\dot\omega' = \dot\omega + B\sigma,$$

where $(B\sigma)^i_j = b^i_{jk}\sigma^k$. Accordingly,

$$\rho' = \rho + Bv,$$
$$\theta^{(2)\prime} = \theta^{(2)} - \begin{pmatrix} 0 \\ B\theta v \end{pmatrix},$$
$$\omega^{(2)\prime} = \omega^{(2)} + \begin{pmatrix} B\theta & O \\ B\sigma & B\theta \end{pmatrix} + \begin{pmatrix} O & O \\ d(Bv) + [B\theta, \rho] + [\omega, Bv] + [B\theta, Bv] & O \end{pmatrix},$$

where $(B\theta v)^i = b^i_{jk}\theta^j v^k$. Next, we modify σ. Let σ' be anther representative of the infinitesimal deformation $[\sigma] \in H^1(M; \Theta_{\mathcal{F}})$. Then,

$$\sigma'^i = \sigma^i + df^i + \omega^i_j f^j + g^i_j \theta^j \tag{3.11}$$

holds for a family of functions $\{f_i\}$ such that $\widehat{f^i} = (D\gamma)^i_j f^j$ and a family of functions $\{g^i_j\}$ such that $\widehat{g}^i_j = (D\gamma)^i_k g^k_l (D\gamma)^{-1}{}^l_j$ (we again make use of a partition of unity in order to assume that $e_i g^i_j \theta^j$ is globally well-defined). An infinitesimal deformation of ω with respect to σ' is given by

$$\dot{\omega}'^i_j = \dot{\omega}^i_j + \Omega^i_{lj} f^l + (dg + [\omega, g])^i_j, \tag{3.12}$$

where $\omega^i_j = \Gamma^i_{jk} dy^k$ and $\Omega^i_{jk} = d\Gamma^i_{jk} + \Gamma^i_{ml}\Gamma^m_{jk} dy^l$ so that $\Omega^i_{jk} \wedge dy^k = (d\omega + \omega \wedge \omega)^i_j$. Therefore, $\theta^{(2)}$ and $\omega^{(2)}$ are replaced by

$$\theta^{(2)\prime} = \theta^{(2)} + \begin{pmatrix} 0 \\ df + \omega f + g\theta \end{pmatrix}$$
$$\omega^{(2)\prime} = \omega^{(2)} + \begin{pmatrix} O & O \\ \Omega f + dg + [\omega, g] & O \end{pmatrix}.$$

Finally, we can replace θ by $\theta' = (D\zeta)\theta$, where ζ is a local biholomorphic diffeomorphism. Then, the connection form of a Bott connection, say ∇', is given by

$$\omega' = -(dD\zeta)(D\zeta)^{-1} + (D\zeta)\omega(D\zeta)^{-1}.$$

In this case, the mapping $D\zeta \colon H^*(\bigwedge^i E(\mathcal{F})^* \otimes Q(\mathcal{F}), d_\nabla) \to H^*(\bigwedge^i E(\mathcal{F})^* \otimes Q(\mathcal{F}), d_{\nabla'})$ given by $[\sigma] \mapsto [(D\zeta)\sigma]$ is an isomorphism. Indeed, we have

$$d((D\zeta)\sigma) + \omega' \wedge ((D\zeta)\sigma) + (D\zeta)\dot{\omega}(D\zeta)^{-1} \wedge ((D\zeta)\theta) = 0.$$

We also see that an infinitesimal deformation of ω' with respect to σ' is given by $(D\zeta)\dot{\omega}(D\zeta)^{-1}$. We have

$$\rho' = -H\zeta v(D\zeta)^{-1} + (D\zeta)\rho(D\zeta)^{-1},$$
$$\theta^{(2)\prime} = (D\zeta)\theta^{(2)} + \begin{pmatrix} 0 \\ (H\zeta v)dy \end{pmatrix},$$
$$\omega^{(2)\prime} = -(dD^{(2)}\zeta)(D^{(2)}\zeta)^{-1} + (D^{(2)}\zeta)\,\omega^{(2)}(D^{(2)}\zeta)^{-1}, \tag{3.13}$$

where

$$\rho'^i_j = -\frac{\partial^2 \widehat{y}^i}{\partial y^l \partial y^m}\frac{\partial y^l}{\partial \widehat{y}^j} v^m + \frac{\partial \widehat{y}^i}{\partial y^l} f^l_{mn} \frac{\partial y^m}{\partial \widehat{y}^j} v^n,$$
$$((H\zeta v)dy)^i = \frac{\partial^2 \widehat{y}^i}{\partial y^l \partial y^m} v^l dy^m.$$

The equality (3.13) is shown by essentially the same calculations to show (3.6).

LEMMA 3.14 (cf. [5, Theorem 1.20]). *Let* $e_j^H = e_j - \eta_i \rho_j^i = e_j - \eta_i f_{jk}^i v^k$. *If we set* $\rho = (\rho_j^i)$ *and* $F = \begin{pmatrix} I_q & O \\ -\rho & I_q \end{pmatrix}$*, then we have the following.*

1) $(\widehat{e}_1^H, \dots, \widehat{e}_q^H, \widehat{\eta}_1, \dots, \widehat{\eta}_q) \begin{pmatrix} D\gamma & O \\ O & D\gamma \end{pmatrix} = (e_1^H, \dots, e_q^H, \eta_1, \dots, \eta_q)$.

2) *The connection form of* $\nabla^{(2)}$ *with respect to* $(e_1^H, \dots, e_q^H, \eta_1, \dots, \eta_q)$ *is given by* $\begin{pmatrix} \omega & O \\ \dot{\omega} & \omega \end{pmatrix}$.

3) $F^{-1}\theta^{(2)} = \begin{pmatrix} \theta \\ \sigma \end{pmatrix}$.

PROOF. If we set $\rho = (\rho_j^i)$ and $F = \begin{pmatrix} I_q & O \\ -\rho & I_q \end{pmatrix}$, then we have

$$
\begin{aligned}
&(\widehat{e}_1^H, \dots, \widehat{e}_q^H, \widehat{\eta}_1, \dots, \widehat{\eta}_q) \begin{pmatrix} D\gamma & O \\ O & D\gamma \end{pmatrix} \\
&= (\widehat{e}_1, \dots, \widehat{e}_q, \widehat{\eta}_1, \dots, \widehat{\eta}_q) \begin{pmatrix} I & O \\ -\widehat{\rho} & I \end{pmatrix} \begin{pmatrix} D\gamma & O \\ O & D\gamma \end{pmatrix} \\
&= (e_1, \dots, e_q, \widehat{\eta}_1, \dots, \widehat{\eta}_q) \\
&\quad \times \begin{pmatrix} (D\gamma)^{-1} & O \\ -(D\gamma)^{-1}(H\gamma v)(D\gamma)^{-1} & (D\gamma)^{-1} \end{pmatrix} \begin{pmatrix} D\gamma & O \\ -\widehat{\rho}(D\gamma) & D\gamma \end{pmatrix} \\
&= (e_1^H, \dots, e_q^H, \widehat{\eta}_1, \dots, \widehat{\eta}_q) \\
&\quad \times \begin{pmatrix} I & O \\ \rho & I \end{pmatrix} \begin{pmatrix} I & O \\ -(D\gamma)^{-1}(H\gamma v) - (D\gamma)^{-1}\widehat{\rho}(D\gamma) & I \end{pmatrix} \\
&= (e_1^H, \dots, e_q^H, \widehat{\eta}_1, \dots, \widehat{\eta}_q) \begin{pmatrix} I & O \\ \rho - (D\gamma)^{-1}(H\gamma v) - (D\gamma)^{-1}\widehat{\rho}(D\gamma) & I \end{pmatrix}.
\end{aligned}
$$

Since $\rho = (D\gamma)^{-1}(dD\gamma)v + (D\gamma)^{-1}\widehat{\rho}(D\gamma)$, we have

$$\rho - (D\gamma)^{-1}(H\gamma v) - (D\gamma)^{-1}\widehat{\rho}(D\gamma) = 0.$$

Next, we have $(e_1^H, \dots, e_q^H, \eta_1, \dots, \eta_q) = (e_1, \dots, e_q, \eta_1, \dots, \eta_q)F$ and

$$F^{-1}dF + F^{-1}\omega^{(2)}F = \begin{pmatrix} O & O \\ -d\rho & O \end{pmatrix} + \begin{pmatrix} \omega & O \\ \rho\,\omega + \dot{\omega} + d\rho + [\omega, \rho] - \omega\rho & \omega \end{pmatrix}$$
$$= \begin{pmatrix} \omega & O \\ \dot{\omega} & \omega \end{pmatrix}. \square$$

The vectors e_i^H are versions of horizontal lifts of e_i in the sense of [22, Chapter 2], see also [5, §1]. Finally we again remark that we can exchange v and $\dot{y}$ in the construction.

4. Deformations of the Godbillon-Vey Class and Relation to Projective Structures

If we discuss only the Godbillon-Vey and Bott classes, the construction can be largely simplified. The following vector bundle is relevant. We set $J\gamma = \det D\gamma$. Note that

$$\operatorname{tr} D\gamma^{-1}{}^i_l H\gamma^l_{jk} = D\gamma^{-1}{}^i_l H\gamma^l_{ik} = \frac{\partial \log J\gamma}{\partial y^k}.$$

We set $\operatorname{tr} D\gamma^{-1}H\gamma = \left(\operatorname{tr} D\gamma^{-1}{}^i_l H\gamma^l_{j1} \ \cdots \ \operatorname{tr} D\gamma^{-1}{}^i_l H\gamma^l_{jq}\right)$.

Definition 4.1 (cf. Definition 2.2). Let $K_{\mathcal{F}}^{(2)-1}$ be a vector bundle over $K_{\mathcal{F}}^{-1}$ defined as follows. Let U be a foliation chart and $e = e_1 \wedge \cdots \wedge e_q$ be a local trivialization of $K_{\mathcal{F}}^{-1}$, where $e_i = \pi\left(\dfrac{\partial}{\partial y^i}\right)$. We set $K_{\mathcal{F}}^{(2)-1} = \{(z, w; \dot{y}, \dot{w})\}$, where

$$(\widehat{z}, \widehat{w}; \widehat{\dot{y}}, \widehat{\dot{w}}) = (\varphi(z), J\gamma_y w; D\gamma_y \dot{y}, J\gamma_y \operatorname{tr}(D\gamma_y^{-1}H\gamma_y\dot{y})w + J\gamma_y\dot{w}).$$

The projection to $K_{\mathcal{F}}^{-1}$, denoted by $\underline{p}^{(2)}$, is defined by $\underline{p}^{(2)}(z, w; \dot{y}, \dot{w}) = (z, w)$. We denote by $\underline{\nu}$ the mapping from $K_{\mathcal{F}}^{(2)-1}$ to $Q(\mathcal{F})$ defined by $\underline{\nu}(z, w; \dot{y}, \dot{w}) = (z, \dot{y})$. Then, $\underline{\nu}\colon K_{\mathcal{F}}^{(2)-1} \to Q(\mathcal{F})$ is a vector bundle.

The following diagram commutes:

$$\begin{array}{ccc} K_{\mathcal{F}}^{(2)-1} & \xrightarrow{\underline{p}^{(2)}} & K_{\mathcal{F}}^{-1} \\ {\scriptstyle \underline{\nu}}\downarrow & & \downarrow{\scriptstyle \underline{p}} \\ Q(\mathcal{F}) & \xrightarrow[p_Q]{} & M, \end{array}$$

where $\underline{p}$ is the natural projection.

Remark 4.2. One might notice that $\underline{p}^{(2)}\colon K_{\mathcal{F}}^{(2)-1} \to K_{\mathcal{F}}^{-1}$ is quite similar to $p_Q \circ p\colon Q(\widetilde{\mathcal{F}}) \to M$, where $p\colon Q(\widetilde{\mathcal{F}}) \to Q(\mathcal{F})$ is a certain vector bundle appeared in [5]. This is later discussed.

First we study $\underline{\nu}\colon K_{\mathcal{F}}^{(2)-1} \to Q(\mathcal{F})$. This bundle is related to $p_Q^{(2)}\colon Q^{(2)}(\mathcal{F}) \to Q(\mathcal{F})$ as follows. If we set

$$L\gamma = \begin{pmatrix} J\gamma & 0 \\ J\gamma\,\mathrm{tr}(D\gamma^{-1}H\gamma\dot{y}) & J\gamma \end{pmatrix},$$

then the transition function is given by $L\gamma$. Let ∇ be a Bott connection on $Q(\mathcal{F})$ and $\{\omega\}$ the family of connection forms as in the previous sections. We represent $\omega^i_j = f^i_{jk}dy^k$ and set $\rho^i_j = f^i_{jk}\dot{y}^k$. Recall that $\widetilde{\sigma} = \sigma - \rho\,\theta$ and $\omega^{(2)} = \begin{pmatrix} \omega & O \\ \dot{\omega} + d\rho + [\omega, \rho] & \omega \end{pmatrix}$. We set $\underline{\omega} = \mathrm{tr}\,\omega$, $\underline{\dot{\omega}} = \mathrm{tr}\,\dot{\omega}$ and $\underline{\rho} = \mathrm{tr}\,\rho$. If we represent $\underline{\omega} = f_k dy^k$, then $f_k = f^i_{ik}$ and $\underline{\rho} = f_i\dot{y}^i$. Let $(e_1, \dots, e_q, \eta_1, \dots, \eta_q)$ be the local trivialization of $Q^{(2)}(\mathcal{F})$ as in §2. We set

$$\begin{aligned} \delta &= \frac{1}{q}(\eta_1 \wedge e_2 \wedge \cdots \wedge e_q + e_1 \wedge \eta_2 \wedge e_3 \wedge \cdots \wedge e_q \\ &\qquad + \cdots + e_1 \wedge \cdots \wedge e_{q-1} \wedge \eta_q), \\ \underline{\theta} &= \theta^1 \wedge \cdots \wedge \theta^q, \\ \underline{\widetilde{\sigma}} &= \widetilde{\sigma}^1 \wedge \theta^2 \wedge \cdots \wedge \theta^q + \theta^1 \wedge \widetilde{\sigma}^2 \wedge \theta^3 \wedge \cdots \wedge \theta^q \\ &\qquad + \theta^1 \wedge \cdots \wedge \theta^{q-1} \wedge \widetilde{\sigma}^q, \\ \underline{\theta}^{(2)} &= \begin{pmatrix} \underline{\theta} \\ \underline{\widetilde{\sigma}} \end{pmatrix}, \\ \underline{\omega}^{(2)} &= \begin{pmatrix} \underline{\omega} & 0 \\ \underline{\dot{\omega}} + d\underline{\rho} & \underline{\omega} \end{pmatrix}. \end{aligned}$$

Let A be the vector space spanned by

$$\eta_i \wedge e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge e_q - (-1)^{i-j} \eta_j \wedge e_1 \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge e_q,$$
$$\eta_i \wedge e_1 \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge e_q,$$
$$\eta_i \wedge \eta_j \wedge e_1 \wedge \cdots \wedge \widehat{e_i} \wedge \cdots \wedge \widehat{e_j} \wedge \cdots \wedge e_q,$$

where $i \neq j$ and '$\widehat{\ }$' means omission. Let V be the vector space spanned by e and δ modulo A. Then, $(e\ \delta)$ gives a local trivialization of $K_{\mathcal{F}}^{(2)-1}$. Indeed, if we denote by $A\gamma = J\gamma D\gamma^{-1}$ the adjugate matrix of $D\gamma$, then we have

$$\begin{aligned} \gamma_* e &= \gamma_*(e_1 \wedge \cdots \wedge e_q) \\ &= (\det D\gamma)\widehat{e}_1 \wedge \cdots \wedge \widehat{e}_q + (\operatorname{tr}(A\gamma H\gamma \dot{y}))\widehat{\delta} \\ &= J\gamma \widehat{e} + J\gamma \operatorname{tr}(D\gamma^{-1} H\gamma \dot{y})\widehat{\delta}, \\ \gamma_* \delta &= J\gamma \widehat{\delta}. \end{aligned}$$

Thus (e, δ) can be identified with $\left(e, \dfrac{\partial}{\partial w}\right)$. We have the following

Lemma 4.3 (cf. Lemma 3.4 and [5, Theorem 1.20]). *We have*

1) $\widehat{\underline{\theta}}^{(2)} = (L\gamma)\underline{\theta}^{(2)}$.

2) $d\underline{\theta}^{(2)} + \underline{\omega}^{(2)} \wedge \underline{\theta}^{(2)} = 0$.

3) $\underline{\omega}^{(2)} = (L\gamma)^{-1}(dL\gamma) + (L\gamma)^{-1}\widehat{\underline{\omega}}^{(2)}(L\gamma)$, *namely,* $\{\underline{\omega}^{(2)}\}$ *gives rise to a connection on* $K^{(2)}(\mathcal{F})^{-1}$ *which we denote by* $\underline{\nabla}^{(2)}$.

Proof. First, the coefficients of $\widetilde{\sigma}^k \wedge \theta^2 \wedge \cdots \wedge \theta^q$ in $\widehat{\underline{\widetilde{\sigma}}}$ are equal to $(D\gamma)^k_l G^l_1$, where G^l_1 denotes the $(l,1)$-cofactor of $D\gamma$. Therefore, we have

$$\widehat{\underline{\widetilde{\sigma}}} = \operatorname{tr}((A\gamma)(H\gamma v))\underline{\theta} + (\det D\gamma)\underline{\widetilde{\sigma}}.$$

The part 1) follows from the above equality. The part 2) follows from the following one, namely,

$$
\begin{aligned}
& d\underline{\widetilde{\sigma}} \\
&= d\sum_{k=1}^{q}\left(\theta^1\wedge\cdots\wedge\widetilde{\sigma}^k\wedge\cdots\wedge\theta^q\right) \\
&= -\sum_{k=1}^{q}\sum_{j<k}(-1)^{j-1}\left(\theta^1\wedge\cdots\wedge\theta^{j-1}\wedge\omega^j_l\wedge\theta^l\wedge\theta^{j+1}\wedge\cdots\wedge\widetilde{\sigma}^k\wedge\cdots\wedge\theta^q\right) \\
&\quad -\sum_{k=1}^{q}\sum_{l=1}^{q}(-1)^{k-1} \\
&\quad \times\left(\theta^1\wedge\cdots\wedge\theta^{k-1}\wedge(\dot{\omega}+d\rho+[\omega,\rho])^k_l\wedge\theta^l\wedge\theta^{k+1}\wedge\cdots\wedge\theta^q\right) \\
&\quad -\sum_{k=1}^{q}\sum_{l=1}^{q}(-1)^{k-1}\left(\theta^1\wedge\cdots\wedge\theta^{k-1}\wedge\omega^k_l\wedge\widetilde{\sigma}^l\wedge\theta^{k+1}\wedge\cdots\wedge\theta^q\right) \\
&\quad -\sum_{k=1}^{q}\sum_{k<j}\sum_{l=1}^{q}(-1)^{j-1} \\
&\quad \times\left(\theta^1\wedge\cdots\wedge\widetilde{\sigma}^k\wedge\cdots\wedge\theta^{j-1}\wedge\omega^j_l\wedge\theta^l\wedge\theta^{j+1}\wedge\cdots\wedge\cdots\wedge\theta^q\right) \\
&= -\sum_{k=1}^{q}\sum_{j<k}(-1)^{j-1}\left(\theta^1\wedge\cdots\wedge\theta^{j-1}\wedge\omega^j_j\wedge\theta^j\wedge\theta^{j+1}\wedge\cdots\wedge\widetilde{\sigma}^k\wedge\cdots\wedge\theta^q\right) \\
&\quad -\sum_{k=1}^{q}\sum_{j<k}(-1)^{j-1} \\
&\quad \times\left(\theta^1\wedge\cdots\wedge\theta^{j-1}\wedge\omega^j_k\wedge\theta^k\wedge\theta^{j+1}\wedge\cdots\wedge\widetilde{\sigma}^k\wedge\cdots\wedge\theta^q\right) \\
&\quad -\sum_{k=1}^{q}(-1)^{k-1} \\
&\quad \times\left(\theta^1\wedge\cdots\wedge\theta^{k-1}\wedge(\dot{\omega}+d\rho+[\omega,\rho])^k_k\wedge\theta^k\wedge\theta^{k+1}\wedge\cdots\wedge\theta^q\right) \\
&\quad -\sum_{k=1}^{q}\sum_{l=1}^{q}(-1)^{k-1}\left(\theta^1\wedge\cdots\wedge\theta^{k-1}\wedge\omega^k_l\wedge\widetilde{\sigma}^l\wedge\theta^{k+1}\wedge\cdots\wedge\theta^q\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{k=1}^{q}\sum_{k<j}(-1)^{j-1}\\
&\times\left(\theta^1\wedge\cdots\wedge\widetilde{\sigma}^k\wedge\cdots\wedge\theta^{j-1}\wedge\omega^j_j\wedge\theta^j\wedge\theta^{j+1}\wedge\cdots\wedge\theta^q\right)\\
&-\sum_{k=1}^{q}\sum_{k<j}(-1)^{j-1}\\
&\times\left(\theta^1\wedge\cdots\wedge\widetilde{\sigma}^k\wedge\cdots\wedge\theta^{j-1}\wedge\omega^j_k\wedge\theta^k\wedge\theta^{j+1}\wedge\cdots\wedge\theta^q\right)\\
=&-\sum_{k=1}^{q}\sum_{j\neq k}\omega^j_j\wedge\left(\theta^1\wedge\cdots\wedge\widetilde{\sigma}^k\wedge\cdots\wedge\theta^q\right)\\
&+\sum_{k=1}^{q}\sum_{j\neq k}(-1)^{j-1}\left(\theta^1\wedge\cdots\wedge\theta^{j-1}\wedge\omega^j_k\wedge\widetilde{\sigma}^k\wedge\theta^{j+1}\wedge\cdots\wedge\theta^q\right)\\
&-(\underline{\dot{\omega}}+d\underline{\rho})\wedge\left(\theta^1\wedge\cdots\wedge\theta^q\right)\\
&-\sum_{k=1}^{q}(-1)^{k-1}\left(\theta^1\wedge\cdots\wedge\theta^{k-1}\wedge\omega^k_k\wedge\widetilde{\sigma}^k\wedge\theta^{k+1}\wedge\cdots\wedge\theta^q\right)\\
&-\sum_{k=1}^{q}\sum_{l\neq k}(-1)^{k-1}\left(\theta^1\wedge\cdots\wedge\theta^{k-1}\wedge\omega^k_l\wedge\widetilde{\sigma}^l\wedge\theta^{k+1}\wedge\cdots\wedge\theta^q\right)\\
=&-\underline{\omega}\wedge\underline{\widetilde{\sigma}}-(\underline{\dot{\omega}}+d\underline{\rho})\wedge\underline{\theta}.
\end{aligned}
$$

Finally, we have

$$
\underline{\omega}=(\det D\gamma)^{-1}(d\det D\gamma)+\underline{\widehat{\omega}},
$$

and

$$
\begin{aligned}
&\underline{\dot{\omega}}+d\underline{\rho}\\
&=\underline{\widehat{\dot{\omega}}}+d\left(\left((\det D\gamma)^{-1}\frac{\partial\det D\gamma}{\partial y^i}+\widehat{f}_j(D\gamma)^j_i\right)(D\gamma)^{-1}{}^i_j\widehat{v}^j\right)\\
&=\underline{\widehat{\dot{\omega}}}+d\underline{\widehat{\rho}}+d\left((\det D\gamma)^{-1}\frac{\partial\det D\gamma}{\partial y^i}v^i\right)\\
&=\underline{\widehat{\dot{\omega}}}+d\underline{\widehat{\rho}}-(\det D\gamma)^{-2}(d\det D\gamma)\frac{\partial\det D\gamma}{\partial y^i}v^i\\
&\quad+(\det D\gamma)^{-1}d\left(\frac{\partial\det D\gamma}{\partial y^i}v^i\right).
\end{aligned}
$$

Therefore the part 3) also holds. □

Let e_j^H be as in Lemma 3.14. Note that

$$\delta = \frac{1}{q}(\eta_1 \wedge e_2^H \wedge \cdots \wedge e_q^H + e_1^H \wedge \eta_2 \wedge e_3^H \wedge \cdots \wedge e_q^H + \cdots + e_1^H \wedge \cdots \wedge e_{q-1}^H \wedge \eta_q)$$

holds in $K^{(2)}(\mathcal{F})^{-1}$. We set $e^H = e_1^H \wedge \cdots \wedge e_q^H$.

Lemma 4.4 (cf. Lemma 3.14 and [5, Theorem 1.20]). *We have the following.*

1) $(e^H, \delta) = (e, \delta) \begin{pmatrix} 1 & 0 \\ -\operatorname{tr} \underline{\rho} & 1 \end{pmatrix}$.

2) $(\widehat{e}^H, \widehat{\delta}) \begin{pmatrix} \det D\gamma & 0 \\ 0 & \det D\gamma \end{pmatrix} = (e^H, \delta)$.

3) *The connection form of* $\underline{\nabla}^{(2)}$ *with respect to* (e^H, δ) *is given by* $\begin{pmatrix} \underline{\omega} & 0 \\ \underline{\dot{\omega}} & \underline{\omega} \end{pmatrix}$.

Note that $d\begin{pmatrix} \underline{\omega} & 0 \\ \underline{\dot{\omega}} & \underline{\omega} \end{pmatrix} + \begin{pmatrix} \underline{\omega} & 0 \\ \underline{\dot{\omega}} & \underline{\omega} \end{pmatrix} \wedge \begin{pmatrix} \underline{\omega} & 0 \\ \underline{\dot{\omega}} & \underline{\omega} \end{pmatrix} = \begin{pmatrix} d\underline{\omega} & 0 \\ d\underline{\dot{\omega}} & d\underline{\omega} \end{pmatrix}$. See also [3]. In view of Lemma 4.4, we introduce the following

Definition 4.5. Let $\underline{\dot{\omega}}$ be a 1-form on $Q(\mathcal{F})$ such that $\underline{\dot{\omega}}(\eta_i) = 0$ for $1 \leq i \leq q$. Such a 1-form is called a *generalized infinitesimal deformation* of $\underline{\omega}$ with respect to θ when it is regarded as a coefficient of connection on $K_{\mathcal{F}}^{(2)-1}$. More concretely, we consider a connection of which the connection matrix with respect to $(e^H\ \delta)$ is given by $\begin{pmatrix} \underline{\omega} & 0 \\ \underline{\dot{\omega}} & \underline{\omega} \end{pmatrix}$.

A generalized infinitesimal deformation is an ordinary one if $d\underline{\dot{\omega}}(\eta_i) = 0$. As we will see, the constant and the linear parts of $\underline{\dot{\omega}}$ with respect to $\dot{y}$ are relevant.

Next, we study $\underline{p}^{(2)}\colon K_{\mathcal{F}}^{(2)-1} \to K_{\mathcal{F}}^{-1}$ (we refer to [5] for details of $\mathcal{E}_{\mathcal{F}}, Q(\widetilde{\mathcal{F}})$ and TW-connections. See also [20]). A local trivialization of $K_{\mathcal{F}}^{(2)-1}$ is given by (e_i, δ) and the transition function is given by

$$(e_i \ \delta) = (\widehat{e}_i \ \widehat{\delta}) \begin{pmatrix} D\gamma & 0 \\ \operatorname{tr} D\gamma^{-1} H\gamma w & J\gamma \end{pmatrix}.$$

We set $k_{\mathcal{F}}^{-1} = \{(z, w) \in K_{\mathcal{F}}^{-1} \mid w \neq 0\}$ and denote by $k_{\mathcal{F}}^{(2)-1}$ the restriction of $K_{\mathcal{F}}^{(2)-1}$ to $k_{\mathcal{F}}^{(2)-1}$. Let $\mathcal{E}_{\mathcal{F}}$ be the $\mathbb{C}^*$-principal bundle associated with $K_{\mathcal{F}}^{-1}$. Note that $k_{\mathcal{F}}^{-1}$ and $\mathcal{E}_{\mathcal{F}}$ are indeed the same. On a foliation chart, we can find local coordinates on $\mathcal{E}_{\mathcal{F}}$ such as $(z, u) = (x, y, u) \in (\mathbb{R}^p \times \mathbb{C}^q) \times \mathbb{C}^*$. By taking the logarithm and changing the order, we may make use of $(x, \log u, y)$ as local coordinates which we denote by (x, y^0, y^i). Let $E(\widetilde{\mathcal{F}})$ be the subbundle of $T\mathcal{E}_{\mathcal{F}}$ locally spanned by $\dfrac{\partial}{\partial x}$ and $\dfrac{\partial}{\partial \bar{y}^\mu}$, $0 \leq \mu \leq q$ (we omit $\dfrac{\partial}{\partial \bar{y}^\mu}$ in the real case), and set $Q(\widetilde{\mathcal{F}}) = T\mathcal{E}_{\mathcal{F}}/E(\widetilde{\mathcal{F}})$. A local trivialization of $Q(\widetilde{\mathcal{F}})$ is given by $\left(\dfrac{\partial}{\partial y^0}, e_i\right)$ and the transition function is given by $\begin{pmatrix} 1 & \operatorname{tr}(D\gamma^{-1} H\gamma) \\ 0 & D\gamma \end{pmatrix}$. In order to compare $k_{\mathcal{F}}^{(2)-1}$ with $Q(\widetilde{\mathcal{F}})$, we change the order again and choose $\left(e_i, \dfrac{\partial}{\partial y^0}\right)$ as a local trivialization. Then, the transition function is given by

$$\widetilde{D}\gamma = \begin{pmatrix} D\gamma & 0 \\ \operatorname{tr}(D\gamma^{-1} H\gamma) & 1 \end{pmatrix}.$$

Note that

$$\frac{\partial}{\partial y^0} = w \frac{\partial}{\partial w}.$$

A transverse TW-connection, say ∇^{TW}, is a linear connection on $Q(\widetilde{\mathcal{F}})$ of which the connection form with respect to $\left(e_i, \dfrac{\partial}{\partial y^0}\right)$ is given by

$$\frac{-1}{q+1} \begin{pmatrix} dy^0 I_q & dy \\ 0 & dy^0 \end{pmatrix} + \begin{pmatrix} \nu & 0 \\ L(q) + \alpha & 0 \end{pmatrix},$$

where $dy = {}^t(dy^1 \ \cdots \ dy^q)$. If we change the local trivialization into a horizontal one, namely, to

$$\left(e_i^h, \frac{\partial}{\partial y^0}\right),$$

where $e_i^h = e_i - f_i \dfrac{\partial}{\partial y^0}$, then the connection form is changed into

$$\frac{-1}{q+1}\begin{pmatrix} d\eta^0 I_q & d\eta \\ 0 & d\eta^0 \end{pmatrix} + \begin{pmatrix} \nu & 0 \\ \alpha & 0 \end{pmatrix},$$

where $\eta^0 = dy^0 + f_i dy^i$ and $\eta^i = dy^i$. We further change the local trivialization into $\left(e_i^h, \dfrac{\partial}{\partial w}\right)$. Note that $e_i^h = e_i - f_i w \dfrac{\partial}{\partial w}$. Then, the connection form is changed into

$$\frac{-1}{q+1}\begin{pmatrix} d\eta^0 I_q & \frac{d\eta}{w} \\ 0 & d\eta^0 \end{pmatrix} + \begin{pmatrix} \nu & 0 \\ w\alpha & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix}.$$

Now let (e, δ) be the trivialization of $K_{\mathcal{F}}^{(2)-1} \to Q(\mathcal{F})$ and recall that (e, δ) can be identified with $\left(e, \dfrac{\partial}{\partial w}\right)$. Therefore (e^H, δ) is identified with $\left(e^H, \dfrac{\partial}{\partial w}\right)$. If we regard $K_{\mathcal{F}}^{(2)-1}$ a fiber bundle over M, then we can consider $\left(e_i^h, e^H, \dfrac{\partial}{\partial w}\right)$ as a local trivialization. We can represent a point on $K_{\mathcal{F}}^{(2)-1}$ by (z, v^i, w, u), where z is the projection of the point to M and (v^i, w, u) are the coefficients with respect to $\left(e_i^h, e^H, \dfrac{\partial}{\partial w}\right)$. If ∇^{TW} is a transverse TW-connection and $\underline{\nabla}^{(2)}$ is a connection as in Definition 4.5, then we can ask if there are some relationship between them. For this purpose, we need the following

Definition 4.6. Let $L_{\mathcal{F}}$ be the line bundle over M locally spanned by $\dfrac{\partial}{\partial y^0}$. By abuse of notations, we denote their pull-backs to $Q(\mathcal{F})$ and $K_{\mathcal{F}}^{-1}$ again by $L_{\mathcal{F}}$. Let $Q(\mathcal{F})^h$ be the subbundle of $K_{\mathcal{F}}^{(2)-1}$ over $Q(\mathcal{F})$ locally spanned by $\left\{e_j^h\right\}$, where $e_j^h = e_j - f_j \dfrac{\partial}{\partial y^0}$. We denote by h_Q the projection from $K_{\mathcal{F}}^{(2)-1}$ to $L_{\mathcal{F}}$ determined by the decomposition $K_{\mathcal{F}}^{(2)-1} = Q(\mathcal{F})^h \oplus L_{\mathcal{F}}$. We denote by $(K_{\mathcal{F}}^{-1})^H$ the subbundle of $K_{\mathcal{F}}^{(2)-1}$ over $K_{\mathcal{F}}^{-1}$ locally spanned by e^H. We denote by h_K the projection from $K_{\mathcal{F}}^{(2)-1}$ to $L_{\mathcal{F}}$ determined by the decomposition $K_{\mathcal{F}}^{(2)-1} = (K_{\mathcal{F}}^{-1})^H \oplus L_{\mathcal{F}}$.

We have the following

THEOREM 4.7. *Suppose that*

$$h_Q\left(\nabla^{\mathrm{TW}}_{X^h}\left(e_i^h v^i + \frac{\partial}{\partial w}u\right)\right) = h_K\left(\underline{\nabla}^{(2)}_{X^H}\left(e^H w + \frac{\partial}{\partial w}u\right)\right)$$

for any $X \in TM$ and (v^i, w, u), where if $X = e_i a^i$, then $X^h = e_i^h a^i$ and $X^H = e_i^H a^i$. Then we have $\alpha_i = \dfrac{\partial}{\partial \dot{y}^i}\underline{\dot{\omega}}$, that is, the linear part of a generalized infinitesimal deformation of $\underline{\omega}$ is equal to the infinitesimal deformation of a TW-connection. In general, the linear part with respect to $\dot{y}^i$ of a generalized infinitesimal deformation of $\underline{\omega}$ is an (original) infinitesimal deformation of $\underline{\dot{\omega}}$ in the sense of [3] (even back to [12]).

PROOF. We have

$$\begin{aligned}
h_Q\left(\nabla^{\mathrm{TW}}_{X^h}\left(e_i^h v^i + \frac{\partial}{\partial w}u\right)\right) &= \frac{\partial}{\partial w}(w\alpha_i(X^h)v^i + f_i a^i u) \\
&= \frac{\partial}{\partial w}(w\alpha_i(X)v^i + f_i a^i u), \\
h_K\left(\underline{\nabla}^{(2)}_{X^H}\left(e^H w + \frac{\partial}{\partial w}u\right)\right) &= \frac{\partial}{\partial w}(\underline{\dot{\omega}}(X^H)w + \omega(X^H)u) \\
&= \frac{\partial}{\partial w}(\underline{\dot{\omega}}(X)w + \omega(X)u).
\end{aligned}$$

Since $f_i a^i = \omega(X)$, the equality in the claim holds if and only if $\alpha_i v^i = \underline{\dot{\omega}}$. $\square$

The linear term of a generalized infinitesimal deformations is a kind of *non-linear connections* in the sense of [22].

In order to deal with deformations other than Godbillon-Vey class and Bott class, we need a generalization of Theorem 4.7, which will be discussed elsewhere.

Remark 4.8. Let $\theta = f_i dy^i$ be the connection form of a Bott connection, say $\mathcal{D}$, on $K_{\mathcal{F}}^{-1}$ with respect to $dy^1 \wedge \cdots \wedge dy^q$, and set $\eta^0 = dy^0 + \theta$, $\eta^i = dy^i$. If we set ω^{TW} to be the connection form of a TW-connection with respect to $(\nabla^b, \mathcal{D})$, then

$$\omega^{\mathrm{TW}} = -\frac{1}{q+1}\begin{pmatrix} \eta^0 I_q & \eta \\ 0 & \eta^0 \end{pmatrix} + \begin{pmatrix} \Gamma & 0 \\ \alpha & 0 \end{pmatrix},$$

where Γ is the connection form of ∇^b with respect to $\left(\dfrac{\partial}{\partial y^1}, \dots, \dfrac{\partial}{\partial y^q}, \dfrac{\partial}{\partial y^0}\right)$ [5] and $\eta = {}^t(\eta^1 \ \cdots \ \eta^q)$. If we set $\theta^{\mathrm{TW}} = {}^t(\eta^1 \ \cdots \ \eta^q \ \eta^0)$, then we have, provided that ∇^b is transversely torsion-free [5],

$$d\theta^{\mathrm{TW}} + \omega^{\mathrm{TW}} \wedge \theta^{\mathrm{TW}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ d\theta + \alpha_i \wedge dy^i \end{pmatrix}.$$

Thus, slightly different from Lemmata 3.4 and 4.3, the torsion of a TW-connection for $(\nabla^b, \mathcal{D})$ (as a linear connection) is related to the curvature of $\mathcal{D}$.

Remark 4.9. The Frobenius theorem and the torsion-freeness a Bott connection on $Q(\mathcal{F})$ are related as follows. Let $(e_1, \dots, e_q)$ be a local trivialization of $Q(\mathcal{F})$ and ${}^t(\theta^1, \dots, \theta^q)$ be the dual. If ∇^b is a Bott connection, then $\nabla^b e_j = \sum_{i=1}^{q} e_i \omega^i_j$, where (ω^i_j) is the connection form of ∇^b with respect to $(e_1, \dots, e_q)$. If $X \in E(\mathcal{F})$, then $\nabla^b_X e_j = \pi[X, \widetilde{e}_j]$. Therefore, $d\theta^i(X, \widetilde{e}_j) = X(\theta^i(\widetilde{e}_j)) - \widetilde{e}_j(\theta^i(X)) - \theta^i([X, \widetilde{e}_j]) = -\theta^i(\nabla^b_X e_j) = -\omega^i_j(X)$. On the other hand, we have $\omega^i_k \wedge \theta^k(X, \widetilde{e}_j) = \delta^j_k \omega^i_j(X)$, where δ^i_j denotes the Kronecker delta. Hence we have $d\theta + \omega \wedge \theta = 0$ on $E(\mathcal{F}) \otimes TM$. Thanks to the Frobenius theorem, we have a $\mathfrak{gl}_q\mathbb{C}$-valued 1-form, say $\tau = (\tau^i_j)$, such that $d\theta + \tau \wedge \theta = 0$. Let $A = (a^i_j)$ be a $\mathrm{GL}_q\mathbb{C}$-valued function such that $\theta = Ady$. Then, $(A^{-1}dA + A^{-1}\tau A) \wedge dy = 0$. Therefore, if we can represent $A^{-1}dA + A^{-1}\tau A = (b^i_j)$ and $b^i_j = b^i_{jk} dy^k$, then $b^i_{jk} = b^i_{kj}$. This is the case if τ is the connection form of a transversely torsion-free Bott connection with respect to $(e_1, \dots, e_q)$.

5. Characteristic Classes

We begin with invariant polynomials on $T\mathrm{GL}_q\mathbb{R}$ and $T\mathrm{GL}_q\mathbb{C}$.

LEMMA 5.1. *Let $I(T\mathrm{GL}_q\mathbb{C})$ be the algebra of invariant polynomials. If $X \in \mathfrak{tgl}_q\mathbb{C}$, then we represent $X \in \mathfrak{tgl}_q\mathbb{C}$ as $X = \begin{pmatrix} X_1 & O \\ X_2 & X_1 \end{pmatrix}$.*

1) *Let $f \in I(\mathrm{GL}_q\mathbb{C})$. If we set $\widetilde{f}(X) = f(X_1)$, then $\widetilde{f} \in I(T\mathrm{GL}_q\mathbb{C})$. We denote $\widetilde{f}$ again by f.*

2) *If we set*

$$\widehat{b}_k(X) = \begin{cases} \operatorname{tr} \sum_{i=0}^{k-1} X_1^i X_2 X_1^{k-i-1}, & k > 0, \\ 0, & k = 0, \end{cases}$$

then, $\widehat{b}_k \in I(T\mathrm{GL}_q\mathbb{C})$.

PROOF. If $g = \begin{pmatrix} A & O \\ B & A \end{pmatrix} \in T\mathrm{GL}_q\mathbb{C}$, then

$$\mathrm{Ad}_g X = \begin{pmatrix} AX_1A^{-1} & O \\ BX_1A^{-1} + AX_2A^{-1} - AX_1A^{-1}BA^{-1} & AX_1A^{-1} \end{pmatrix}.$$

The first claim immediately follows from this. In order to show the second claim, we first remark that $\sum_{i=0}^{k-1} X_1^i X_2 X_1^{k-i-1}$ is the bottom-left component of X^k. In other words, if we represent $X^k = \begin{pmatrix} Y_1 & O \\ Y_2 & Y_1 \end{pmatrix}$, then $\widehat{b}_k(X) = \operatorname{tr} Y_2$. It follows that

$$\begin{aligned} \widehat{b}_k(gXg^{-1}) &= \operatorname{tr}(BY_1A^{-1} + AY_2A^{-1} - AY_1A^{-1}BA^{-1}) \\ &= \operatorname{tr}(BY_1A^{-1}) + \operatorname{tr}(AY_2A^{-1}) - \operatorname{tr}(AY_1A^{-1}BA^{-1}) \\ &= \operatorname{tr}(BY_1A^{-1}) + \operatorname{tr} Y_2 - \operatorname{tr}(BA^{-1}AY_1A^{-1}) \\ &= \operatorname{tr} Y_2. \end{aligned}$$

Hence $\widehat{b}_k \in I(T\mathrm{GL}_q\mathbb{C})$. $\square$

Actually we have $\widehat{b}_k(X) = k \operatorname{tr} X_2 X_1^{k-1}$ but we prefer the expression as above for its naturality seen as above.

The polynomial $\widehat{b}_k$ will correspond to the infinitesimal derivative of the Chern characters. If we work on the Chern classes, a certain variant of $\widehat{b}_k$ is useful. In order to introduce it, we recall the relation between the Chern classes and the Chern characters.

Notation 5.2. Let σ_i, $i = 1, \dots, q$, be the i-th elementary symmetric function in $x_1, \dots, x_q$. Formally we set $\sigma_0(x_1, \dots, x_q) = 1$. We set $\tau_j(x_1, \dots, x_q) = x_1^j + \cdots + x_q^j$ for $j = 1, \dots$ and $\tau_0(x_1, \dots, x_q) = q$.

The relation between σ_i and τ_j is well-known (cf. [15, 14.1 and 19.3]).

PROPOSITION 5.3. *If we set* $f(t) = \sum_{i=0}^{q} \sigma_i(x_1, \dots, x_q)\, t^i$, *then*

$$\sum_{j=1}^{\infty} (-1)^j \tau_j(x_1, \dots, x_q)\, t^j = -t \frac{df}{dt}(t) \Big/ f(t)$$

holds in $\mathbb{R}[[t]]$, *where* t^0 *is regarded as* 1.

PROOF. We have $f(t) = (1 + x_1 t) \cdots (1 + x_q t)$. It follows that

$$\frac{df}{dt}(t) \Big/ f(t) = \frac{d \log f}{dt}(t) = \sum_{j=0}^{\infty} (-1)^j \tau_{j+1}(x_1, \dots, x_q) t^j. \qquad \square$$

COROLLARY 5.4. *There is a polynomial* P_j *in* $y_1, \dots, y_j$ *such that*

$$\tau_j = P_j(\sigma_1, \dots, \sigma_j).$$

Example 5.5. The concrete form of the polynomials P_k for small k are as follows. We regard $y_k = 0$ if $k > q$.

$$\begin{aligned} &P_1(y_1) = y_1, \\ &P_2(y_1, y_2) = y_1^2 - 2y_2, \\ &P_3(y_1, y_2, y_3) = y_1^3 - 3y_1 y_2 + 3y_3, \end{aligned}$$

$$
\begin{aligned}
&P_4(y_1, y_2, y_3, y_4) = y_1^4 - 4y_1^2 y_2 + 2y_2^2 + 4y_1 y_3 - 4y_4,\\
&P_5(y_1, y_2, y_3, y_4, y_5)\\
= &\, y_1^5 - 5y_1^3 y_2 + 5y_1 y_2^2 + 5y_1^2 y_3 - 5y_2 y_3 - 5y_1 y_4 + 5y_5,\\
&P_6(y_1, y_2, y_3, y_4, y_5, y_6)\\
= &\, y_1^6 - 6y_1^4 y_2 + 9y_1^2 y_2^2 + 6y_1^3 y_3 - 12 y_1 y_2 y_3\\
&- 6y_1^2 y_4 + 6y_1 y_5 - 2y_2^3 + 6y_2 y_4 + 3y_3^2 - 6y_6.
\end{aligned}
$$

THEOREM 5.6. *Let c_i be the i-th Chern class and $\widehat{c}_j$ the j-th Chern character. Then, $\widehat{c}_j = P_j(c_1, \dots, c_j)/j!$.*

Conversely, if we set $g(t) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \tau_{j+1}(x_1, \dots, x_q)\, t^{j+1}$, then $f(t) = \exp g(t)$. Therefore, the converse of Theorem 5.6 also holds.

THEOREM 5.7. *Let Q_k be the polynomial in $\tau_1, \dots, \tau_k$ determined by*

$$
\sum_{k=0}^{\infty} Q_k t^k = \exp\left(\sum_{j=0}^{\infty} \frac{(-1)^j}{j+1} \tau_{j+1} t^{j+1} \right).
$$

Then, $c_k = Q_k(\widehat{c}_1, 2\widehat{c}_2, \dots, k!\,\widehat{c}_k)$.

Example 5.8. The concrete form of the polynomials Q_k for $k = 1, 2, 3$ is as follows.

$$
\begin{aligned}
&Q_1(\tau_1) = \tau_1,\ Q_2(\tau_1, \tau_2) = \frac{1}{2}\left(\tau_1^2 - \tau_2\right),\\
&Q_3(\tau_1, \tau_2, \tau_3) = \frac{1}{6}\left(\tau_1^3 - 3\tau_1\tau_2 + \tau_3\right).
\end{aligned}
$$

DEFINITION 5.9. If $I = \{i_1, \dots, i_r\}$, then we set $I_l = I \setminus \{i_l\}$ and $\tau_I = \tau_{i_1} \cdots \tau_{i_r}$, where $\tau_\varnothing = 1$. Let $\tau'_1, \tau'_2, \dots$ be formal variables and set

$$
\delta\tau_I(\tau'; \tau) = \sum_{l=1}^{r} \tau'_{i_l} \tau_{I_l}.
$$

We represent Q_i as $Q_i(\tau_1, \dots, \tau_i) = \sum_{|I|=i} a_I \tau_I$, where $|I| = i_1 + \cdots + i_r$, and set

$$b_i = \sum_{|I|=i} a_I \delta\tau_I(\widehat{b}_1, 2\widehat{b}_2, \dots, i!\,\widehat{b}_i; \widehat{c}_1, 2\widehat{c}_2, \dots, i!\,\widehat{c}_i).$$

Example 5.10.

$$b_1 = \widehat{b}_1,\ b_2 = \widehat{b}_1\widehat{c}_1 - \widehat{b}_2 \text{ and } b_3 = \frac{1}{2}\widehat{b}_1\widehat{c}_1^2 - \widehat{b}_1\widehat{c}_2 - \widehat{b}_2\widehat{c}_1 + \widehat{b}_3.$$

We define a subgroup $T\mathrm{SL}_q\mathbb{C}$ of $T\mathrm{GL}_q\mathbb{C}$ by $T\mathrm{SL}_q\mathbb{C} = \mathrm{SL}_q\mathbb{C} \ltimes \mathfrak{sl}_q\mathbb{C} \subset \mathrm{GL}_q\mathbb{C} \ltimes \mathfrak{gl}_q\mathbb{C}$ and denote by $\mathfrak{tsl}_q\mathbb{C}$ the Lie algebra of $T\mathrm{SL}_q\mathbb{C}$. Then, the following is known.

Theorem 5.11 (Takiff [21], Rais-Tauvel [19, Théorème 4.5]).

$$I(T\mathrm{SL}_q\mathbb{C}) = \mathbb{C}[\widehat{c}_2, \dots, \widehat{c}_q, \widehat{b}_2, \dots, \widehat{b}_q],$$
$$I(T\mathrm{SL}_q\mathbb{R}) = \mathbb{R}[\widehat{c}_2, \dots, \widehat{c}_q, \widehat{b}_2, \dots, \widehat{b}_q].$$

The following is a direct consequence of Theorem 5.11.

Theorem 5.12.

$$I(T\mathrm{GL}_q\mathbb{C}) = \mathbb{C}[\widehat{c}_1, \dots, \widehat{c}_q, \widehat{b}_1, \dots, \widehat{b}_q],$$
$$I(T\mathrm{GL}_q\mathbb{R}) = \mathbb{R}[\widehat{c}_1, \dots, \widehat{c}_q, \widehat{b}_1, \dots, \widehat{b}_q].$$

Proof. First we show the theorem for $T\mathrm{GL}_q\mathbb{C}$. We will show that $I(T\mathrm{GL}_q\mathbb{C}) \subset \mathbb{C}[\widehat{c}_1, \dots, \widehat{c}_q, \widehat{b}_1, \dots, \widehat{b}_q]$ and that $\widehat{c}_i$, $\widehat{b}_j$ are algebraically independent. We have

$$\mathfrak{tgl}_q\mathbb{C} = \{(X, Y) \mid X, Y \in \mathfrak{gl}_q\mathbb{C}\},$$

and

$$\mathrm{Ad}_{(A,B)}(X, Y) = (AXA^{-1}, A(BX - XB + Y)A^{-1}),$$
$$\mathrm{ad}_{(X_1,Y_1)}(X_2, Y_2) = ([X_1, X_2], [Y_1, X_2] + [X_1, Y_2]),$$

where $(A, B) \in T\mathrm{GL}_q\mathbb{C}$ and $(X, Y), (X_1, Y_1), (X_2, Y_2) \in \mathfrak{tgl}_q\mathbb{C}$. Let $F \in I(T\mathrm{GL}_q\mathbb{C})$. We may assume that F is homogeneous of degree d. Let f be the polarization of F, i.e., the symmetric multilinear mapping on $(\mathfrak{tgl}_q\mathbb{C})^d$ such that $F(X) = f(X, \dots, X)$ for $X \in \mathfrak{tgl}_q\mathbb{C}$. Let $I = (I_q, O)$ and $J = (O, I_q)$, where I_q denotes the unit matrix. Then $Z(T\mathrm{GL}_q\mathbb{C}) = \{tI + sJ \mid t, s \in \mathbb{C}\}$. If $X = (Y, Z) \in \mathfrak{tgl}_q\mathbb{C}$, then we can decompose X as $X = X' + tI + sJ$, where $X' \in \mathfrak{tsl}_q\mathbb{C}$, $t = \frac{1}{q}\operatorname{tr} Y = \frac{1}{q}\widehat{c}_1(X)$ and $s = \frac{1}{q}\operatorname{tr} Z = \frac{1}{q}\widehat{b}_1(X)$. Let $K = \{K_1, \dots, K_r\}$, where $0 \le r \le d$ and each K_p is either I or J. We set $f_K(X'_1, \dots, X'_{d-r}) = f(X'_1, \dots, X'_{d-r}, K_1, \dots, K_r)$. If $K = \varnothing$, then we set $f_K = f_\varnothing = f$. Note that the order of K_i's does not affect $f_{K_1,\dots,K_r}$ because f is symmetric. Then, f_K is a symmetric, invariant polynomial on $T\mathrm{SL}_q\mathbb{C}$. Therefore, by Theorem 5.11, there is an element of $I(T\mathrm{SL}_q\mathbb{C})$ of which the polarization is equal to f_K. We have

$$\begin{aligned}
&f(X_1, \dots, X_d)\\
&= f(t_1 I + s_1 J, \dots, t_d I + s_d J)\\
&\qquad + (f(X'_1, t_2 I + s_2 J, \dots, t_d I + s_d J) + \cdots\\
&\qquad\qquad + f(t_1 I + s_1 J, \dots, t_{d-1} I + s_{d-1} J, X'_d))\\
&\qquad + \cdots + f(X'_1, \dots, X'_d)\\
&= t_1 t_2 \cdots t_d f_{I,\dots,I}\\
&\quad + (s_1 t_2 \cdots t_d f_{I,\dots,I,J} + t_1 s_2 t_3 \cdots t_d f_{I,\cdots,I,J} +\\
&\qquad + \cdots + t_1 \cdots t_{d-1} s_d f_{I,\cdots,I,J})\\
&\quad + \cdots + s_1 \cdots s_d f_{J,\cdots,J}\\
&\quad + (t_2 \cdots t_d f_{I,\dots,I}(X'_1) + \cdots + s_2 \cdots s_d f_{J,\dots,J}(X'_1))\\
&\qquad + (t_1 \cdots t_{d-1} f_{I,\dots,I}(X'_d) + \cdots + s_1 \cdots s_{d-1} f_{J,\dots,J}(X'_d))\\
&\quad + \cdots + f_\varnothing(X'_1, \dots, X'_d).
\end{aligned}$$

As the above equality holds identically on $X_1, \dots, X_d$, we see that $f \in \mathbb{C}[\widehat{c}_2, \dots, \widehat{c}_q, \widehat{b}_2, \dots, \widehat{b}_q][\widehat{c}_1, \widehat{b}_1] = \mathbb{C}[\widehat{c}_1, \dots, \widehat{c}_q, \widehat{b}_1, \dots, \widehat{b}_q]$. Finally, $\widehat{c}_i$ and $\widehat{b}_j$ are algebraically independent by Lemme 3.3 of [19]. Since there is an obvious inclusion of $I(T\mathrm{GL}_q\mathbb{R})$ into $I(T\mathrm{GL}_q\mathbb{C})$, the same holds for $I(T\mathrm{GL}_q\mathbb{R})$. □

Definition 5.13. Let $J, K \in \{(j_1, \dots, j_q) \mid j_r \in \mathbb{N}\}$. We set $|J| = j_1 + 2j_2 + \cdots + qj_q$ and $|K|' = k_2 + 2k_3 + \cdots + (q-1)k_q$, where $K = (k_1, \dots, k_q)$.

Then, we set

$$\begin{aligned} I_q &= \{c_J \in \mathbb{C}[c_1, \dots, c_q] \mid |J| > q\}, \\ I_{q,q} &= \{c_J \dot{c}_K \in \mathbb{C}[c_1, \dots, c_q, \dot{c}_1, \dots, \dot{c}_q] \mid |J| + |K|' > q\}. \end{aligned}$$

If $c_K \in I_q$, then we set

$$\delta c_K = k_1 c_1^{k_1-1} \dot{c}_1 c_2 \cdots c_q + k_2 c_1 c_2^{k_2-1} \dot{c}_2 c_3 \cdots c_q + \cdots + k_q c_1 \cdots c_{q-1} c_q^{k_q-1} \dot{c}_q.$$

Finally, let DI_q be the ideal of $\mathbb{C}[c_1, \dots, c_q, \dot{c}_1, \dots, \dot{c}_q]$ generated by $I_{q,q}$ and $\{\delta c_J \mid c_J \in I_q\}$, and set

$$\begin{aligned} \mathbb{C}_q[c_1, \dots, c_q] &= \mathbb{C}[c_1, \dots, c_q]/I_q, \\ \mathbb{C}_{q,q}[c_1, \dots, c_q, \dot{c}_1, \dots, \dot{c}_q] &= \mathbb{C}[c_1, \dots, c_q, \dot{c}_1, \dots, \dot{c}_q]/DI_q. \end{aligned}$$

We define $\mathbb{C}_q[\bar{c}_1, \dots, \bar{c}_q]$ by replacing c_i by $\bar{c}_i$.

We formally set $\bar{\dot{c}} = \dot{\bar{c}}$ and $\bar{\dot{u}} = \dot{\bar{u}}$.

Definition 5.14. We set $\deg c_i = \deg \dot{c}_i = \deg \dot{\bar{c}}_i = 2i$, $\deg h_i = \deg \widetilde{u}_i = \deg \dot{h}_i = \deg \dot{u}_i = \deg \dot{\bar{u}}_i = 2i-1$. Let

$$\begin{aligned} \mathrm{WO}_q &= \textstyle\bigwedge[h_1, \dots, h_{[q]}] \otimes \mathbb{R}_q[c_1, \dots, c_q], \\ \mathrm{WU}_q &= \textstyle\bigwedge[\widetilde{u}_1, \widetilde{u}_2, \dots, \widetilde{u}_q] \otimes \mathbb{C}_q[c_1, \dots, c_q] \otimes \mathbb{C}_q[\bar{c}_1, \dots, \bar{c}_q], \\ \mathrm{DWO}_q &= \textstyle\bigwedge[\dot{h}_1, \dot{h}_2, \dots, \dot{h}_q] \wedge \bigwedge[h_1, h_3, \dots, h_{[q]}] \\ &\qquad \otimes \mathbb{R}_{q,q}[c_1, \dots, c_q, \dot{c}_1, \dots, \dot{c}_q], \\ \mathrm{DWU}_q &= \textstyle\bigwedge[\dot{u}_1, \dot{u}_2, \dots, \dot{u}_q] \wedge \bigwedge[\dot{\bar{u}}_1, \dot{\bar{u}}_2, \dots, \dot{\bar{u}}_q] \wedge \bigwedge[\widetilde{u}_1, \widetilde{u}_2, \dots, \widetilde{u}_q] \\ &\qquad \otimes \mathbb{C}_{q,q}[c_1, \dots, c_q, \dot{c}_1, \dots, \dot{c}_q] \otimes \mathbb{C}_{q,q}[\bar{c}_1, \dots, \bar{c}_q, \dot{\bar{c}}_1, \dots, \dot{\bar{c}}_q], \end{aligned}$$

where $[q]$ denotes the greatest odd integer less than or equal to q, and set

$$\begin{aligned} &dh_i = c_i,\ d\dot{h}_i = \dot{c}_i, \\ &d\widetilde{u}_i = c_i - \bar{c}_i,\ d\dot{u}_i = \dot{c}_i,\ d\dot{\bar{u}}_i = \dot{\bar{c}}_i,\ dc_i = d\bar{c}_i = d\dot{c}_i = d\dot{\bar{c}}_i = 0. \end{aligned}$$

Proposition 5.15. *The natural homomorphisms from $H^*(\mathrm{WO}_q)$ to $H^*(\mathrm{DWO}_q)$ and from $H^*(\mathrm{WU}_q)$ to $H^*(\mathrm{DWU}_q)$ are injective. More precisely, $H^*(\mathrm{WO}_q)$ is isomorphic to $\{f \in H^*(\mathrm{DWO}_q) \mid f$ does not involve $\dot{h}_i$*

or $\dot{c}_j\}$*, and* $H^*(\mathrm{WU}_q)$ *is isomorphic to* $\{f \in H^*(\mathrm{DWU}_q) \,|\, f$ *does not involve* $\dot{u}_i$*,* $\dot{\bar{u}}_j$*,* $\dot{c}_k$ *or* $\dot{\bar{c}}_l\}$*.*

Proof. As the number of 'dots' are well-defined on the cohomology level, we can decompose $H^*(\mathrm{DWO}_q)$ according to that number. It is easy to see that $H^*(\mathrm{WO}_q)$ is the part of $H^*(\mathrm{DWO}_q)$ of which the number is equal to zero. The same arguments work on WU_q and DWU_q. □

Example 5.16. We have

$$H^*(\mathrm{DWO}_1) = \langle 1, h_1c_1, \dot{h}_1c_1, \dot{h}_1h_1c_1\rangle,$$

where the bracket means that the cohomology is generated as a linear space. The class h_1c_1 is the Godbillon-Vey class, $2\dot{h}_1c_1$ is the infinitesimal derivative of the Godbillon-Vey class, and $2\dot{h}_1h_1c_1$ is the Fuks-Lodder-Kotschick class which will be introduced in Example 5.28.

Example 5.17. We can also determine $H^*(\mathrm{DWU}_1)$ by using simple spectral sequences. Let

$$A = \langle f \in \mathrm{DWU}_1 \,|\, f \text{ does not involve } \widetilde{u}_1\rangle.$$

Then, A is closed under d. More concretely, if we set

$$B = \langle c_1\bar{c}_1, c_1\dot{\bar{c}}_1^l, \bar{c}_1\dot{c}_1^k, \dot{c}_1^k\dot{\bar{c}}_1^l \,|\, k, l \in \mathbb{N}\rangle,$$

then, we have

$$A = B \oplus \dot{u}_1B \oplus \dot{\bar{u}}_1B \oplus \dot{u}_1\dot{\bar{u}}_1B,$$

where the product is the wedge product. We have

$$H^*(A) = \langle 1, c_1, \bar{c}_1, c_1\bar{c}_1, \dot{u}_1c_1, \dot{\bar{u}}_1\bar{c}_1, \dot{u}_1c_1\bar{c}_1, \dot{\bar{u}}_1c_1\bar{c}_1, \dot{u}_1\dot{\bar{u}}_1c_1\bar{c}_1\rangle,$$
$$H^*(\mathrm{DWU}_1/A) = \widetilde{u}_1\langle 1, c_1, \bar{c}_1, c_1\bar{c}_1, \dot{u}_1c_1, \dot{\bar{u}}_1\bar{c}_1, \dot{u}_1c_1\bar{c}_1, \dot{\bar{u}}_1c_1\bar{c}_1, \dot{u}_1\dot{\bar{u}}_1c_1\bar{c}_1\rangle.$$

We have an exact sequence of complexes

$$0 \to A \to \mathrm{DW}_1 \to \mathrm{DW}_1/A \to 0$$

and that of cohomologies

$$\cdots \to H^{*-1}(\mathrm{DW}_1/A) \xrightarrow{\partial} H^*(A) \to H^*(\mathrm{DW}_1) \\ \to H^*(\mathrm{DW}_1/A) \xrightarrow{\partial} H^{*+1}(A) \to \cdots .$$

The connecting homomorphism ∂ is indeed given by the differential d so that we have

$$H^r(\mathrm{DWU}_1) = \begin{cases} \langle 1 \rangle, & r=0, \\ \left\langle \frac{c_1+\bar{c}_1}{2} \right\rangle, & r=2, \\ \langle \dot{u}_1 c_1, \dot{\bar{u}}_1 \bar{c}_1, \widetilde{u}_1(c_1+\bar{c}_1) \rangle, & r=3, \\ \langle \widetilde{u}_1 c_1 \bar{c}_1 \rangle, & r=5, \\ \langle \dot{u}_1 \dot{\bar{u}}_1 c_1 \bar{c}_1, \widetilde{u}_1 \dot{u}_1 c_1 \bar{c}_1, \widetilde{u}_1 \dot{\bar{u}}_1 c_1 \bar{c}_1 \rangle, & r=6, \\ \langle \widetilde{u}_1 \dot{u}_1 \dot{\bar{u}}_1 c_1 \bar{c}_1 \rangle, & r=7, \\ 0, & \text{otherwise.} \end{cases}$$

Up to multiplication of constants, $c_1+\bar{c}_1$ is the first Chern class, $\widetilde{u}_1(c_1+\bar{c}_1)$ is the imaginary part of the Bott class, $\widetilde{u}_1 c_1 \bar{c}_1$ is the Godbillon-Vey class, and $\dot{u}_1 c_1, \dot{\bar{u}}_1 \bar{c}_1$ are the infinitesimal derivative of the Bott class and its complex conjugate. The absence of the infinitesimal derivative of the Godbillon-Vey class corresponds to the rigidity of the Godbillon-Vey class for transversely holomorphic foliations [4]. Note also that the infinitesimal derivative of the Chern classes are also absent. This is of course due to the integrality (therefore the rigidity under deformations) of the Chern classes. Finally we remark that there is no direct analogue of the Fuks-Lodder-Kotschick class. See also Remark 5.34. Instead of that, the classes of degree 6 and perhaps the class of degree 7 are variants.

Let $\delta_{\mathbb{R}}\colon \mathrm{WO}_q \to \mathrm{DWO}_q$ and $\delta\colon \mathrm{WU}_q \to \mathrm{DWU}_q$ be the (commutative) derivations which satisfy $\delta_{\mathbb{R}} c_j = \dot{c}_j$ and $\delta_{\mathbb{R}} h_i = \dot{h}_i$, and $\delta c_j = \dot{c}_j$, $\delta \bar{c}_j = \dot{\bar{c}}_j$ and $\delta \widetilde{u}_i = \dot{u}_i - \dot{\bar{u}}_i$, respectively. It is easy to see that $\delta_{\mathbb{R}}$ and δ are well-defined and commute with d. Therefore, we have the following

Proposition 5.18. *There are well-defined derivations* $\delta_{\mathbb{R}}\colon H^*(\mathrm{WO}_q) \to H^*(\mathrm{DWO}_q)$ *and* $\delta\colon H^*(\mathrm{WU}_q) \to H^*(\mathrm{DWU}_q)$ *such that*

$$\delta_{\mathbb{R}} h_i = \dot{h}_i,\ \delta_{\mathbb{R}} c_i = \dot{c}_i, \\ \delta(\widetilde{u}_i) = \dot{u}_i - \dot{\bar{u}}_i,\ \delta(c_i) = \dot{c}_i,\ \delta(\bar{c}_i) = \dot{\bar{c}}_i.$$

Some of results in [13] and [4] can be summarized as follows.

THEOREM 5.19. *There is a well-defined bilinear pairing*

$$D\chi_{\mathbb{R}}\colon H^1(M;\Theta_{\mathcal{F}}) \times H^*(\mathrm{DWO}_q) \to H^*(M;\mathbb{R})$$

for real codimension-q foliations, and

$$D\chi\colon H^1(M;\Theta_{\mathcal{F}}) \times H^*(\mathrm{DWU}_q) \to H^*(M;\mathbb{C})$$

for complex codimension-q transversely holomorphic foliations. If $\sigma \in H^1(M;\Theta_{\mathcal{F}})$ is an infinitesimal deformation and if $\alpha \in H^*(\mathrm{WO}_q)$ (resp. $\beta \in H^*(\mathrm{WU}_q)$), then $D\chi_{\mathbb{R}}(\sigma,\delta(\alpha))$ (resp. $D\chi(\sigma,\delta(\beta))$) is the infinitesimal derivative of α (resp. β) with respect to σ.

Before proving Theorem 5.19, we recall the Chern forms and Chern-Simons forms. Let $\omega^{(2)}$ be the connection form of a connection $\nabla^{(2)}$ on $Q^{(2)}(\mathcal{F})$ obtained from a Bott connection and an infinitesimal deformation of $\mathcal{F}$. Then, the curvature form $R^{(2)}$ of $\omega^{(2)}$ is by definition

$$R^{(2)} = d\omega^{(2)} + \frac{1}{2}[\omega^{(2)},\omega^{(2)}] = d\omega^{(2)} + \omega^{(2)} \wedge \omega^{(2)}.$$

DEFINITION 5.20. We define i-th Chern forms $c_i(R^{(2)})$, $0 \le i \le q$, by the condition

$$\det\left(\lambda I_q - \frac{1}{2\pi\sqrt{-1}}R^{(2)}\right) = \sum_{k=0}^{q} c_k(R^{(2)})\lambda^{q-k}.$$

Note that $c_0(R^{(2)}) = 1$. In the real case, we replace $2\pi\sqrt{-1}$ by 2π. In general, if f is a $T\mathrm{GL}_q\mathbb{C}$-invariant polynomial, then we set $f(R^{(2)}) = f(R^{(2)},\dots,R^{(2)})$, where we make use of the Chern convention.

It is well-known that $f(R^{(2)})$ is closed.

Let $\omega_0^{(2)}$ and $\omega_1^{(2)}$ be connection forms. We set $\omega_t^{(2)} = (1-t)\omega_0^{(2)} + t\omega_1^{(2)}$, and represent $f(R_t^{(2)}) = \alpha + \beta \wedge dt$, where α and β do not involve dt.

DEFINITION 5.21. We define the Chern-Simons form of f by

$$\Delta_f(\omega_0^{(2)},\omega_1^{(2)}) = \int_0^1 \beta dt.$$

It is well-known that $d\Delta_f(\omega_0^{(2)}, \omega_1^{(2)}) = f(R_0^{(2)}) - f(R_1^{(2)})$ (see [7]).

We also need a version of the Bott vanishing theorem.

Lemma 5.22. *Let I be the ideal of $\Omega^*(M)$ locally generated by $dy^1, \dots, dy^q$, where $(y^1, \dots, y^q)$ are local coordinates in the transverse direction. If we denote by I^r the ideal of $\Omega^*(M)$ locally generated by $\{\alpha_1 \wedge \cdots \wedge \alpha_r \mid \alpha_1, \dots, \alpha_r \in I\}$, then $\widehat{b}_k$ evaluated by $R^{(2)}$ belongs to I^{k-1}.*

Proof. If we represent $R^{(2)}$ as $R^{(2)} = \begin{pmatrix} R_1 & O \\ R_2 & R_1 \end{pmatrix}$, then each entry of R_1 belongs to I. On the other hand, $b_k(R^{(2)})$ is a certain sum of entries of $R_1^i R_2 R_1^{k-i-1}$. □

Finally, we make use of the following

Lemma 5.23 (cf. [13, Theorem 2.16], [4, Lemma 4.3.17]). *Let $J = (j_1, \dots, j_q) \in \mathbb{N}^q$. If $|J| > q$, then $\delta c_J(R^{(2)}) = 0$, where δc_J is as in Definition 5.13.*

A proof can be found as a part of the proof of [13, Theorem 2.16] and also in [4, Lemma 4.3.17] so that we omit it.

Proof of Theorem 5.19. Let ∇^b be a Bott connection on $Q(\mathcal{F})$, ∇^h a unitary (resp. metric) connection on $Q(\mathcal{F})$ with respect to a Hermitian (resp. Riemannian) metric h, a family of local trivializations $\{\theta\}$ of $Q^*(\mathcal{F})$ and an infinitesimal deformation $\{\sigma\}$ of $\{\theta\}$ which represents an infinitesimal deformation of $\mathcal{F}$. For simplicity we denote $\{\theta\}$ and $\{\sigma\}$ by θ and σ, respectively. Let ω^b and ω^h be connection forms of ∇^b and ∇^h with respect to the dual of θ, and $\dot{\omega}$ an infinitesimal deformation of ω^b with respect to σ. We set $\omega^{(2)} = \begin{pmatrix} \omega^b & O \\ \dot{\omega} & \omega^b \end{pmatrix}$, $\omega_0^{(2)} = \begin{pmatrix} \omega^b & O \\ O & \omega^b \end{pmatrix}$, $R = d\omega^b + \omega^b \wedge \omega^b$ and $R^{(2)} = d\omega^{(2)} + \omega^{(2)} \wedge \omega^{(2)}$. Let $\widetilde{D}\chi$ be the algebra homomorphism from $(E(\mathcal{F})^* \otimes Q(\mathcal{F})) \times \mathrm{DWU}_q$ to $\Omega^*(M)$ determined by the conditions that

$$\begin{aligned} \widetilde{D}\chi(c_i) &= c_i(R), \\ \widetilde{D}\chi(\bar{c}_i) &= \overline{c_i(R)}, \end{aligned}$$

$$\begin{aligned}
\widetilde{D}\chi(\widetilde{u}_i) &= \Delta_{c_i}(\omega^b,\omega^h) - \overline{\Delta_{c_i}(\omega^b,\omega^h)},\\
\widetilde{D}\chi(\dot{c}_i) &= b_i(R^{(2)}),\\
\widetilde{D}\chi(\dot{\bar{c}}_i) &= \overline{b_i(R^{(2)})},\\
\widetilde{D}\chi(\dot{u}_i) &= \Delta_{b_i}(\omega^{(2)},\omega^{(2)}_0),\\
\widetilde{D}\chi(\dot{\bar{u}}_i) &= \overline{\Delta_{b_i}(\omega^{(2)},\omega^{(2)}_0)}.
\end{aligned}$$

In the real case, we define $\widetilde{D}\chi_{\mathbb{R}}$ by the conditions that

$$\begin{aligned}
\widetilde{D}\chi_{\mathbb{R}}(c_i) &= c_i(R),\\
\widetilde{D}\chi_{\mathbb{R}}(h_i) &= \Delta_{c_i}(\omega^b,\omega^h), \quad \text{where } i \text{ is odd},\\
\widetilde{D}\chi_{\mathbb{R}}(\dot{c}_i) &= b_i(R^{(2)}),\\
\widetilde{D}\chi_{\mathbb{R}}(\dot{h}_i) &= \Delta_{b_i}(\omega^{(2)},\omega^{(2)}_0).
\end{aligned}$$

By the construction and Lemma 5.22, $\widetilde{D}\chi$ and $\widetilde{D}\chi_{\mathbb{R}}$ are well-defined and induce bilinear mappings on the cohomology, which we denote by $D\chi$ and $D\chi_{\mathbb{R}}$, respectively. We denote the product foliation of $M\times[0,1]$ by $\mathcal{F}\times[0,1]$, namely, the leaves of $\mathcal{F}\times[0,1]$ are of the form $L\times[0,1]$, where L is a leaf of $\mathcal{F}$. Let $\{a^i_{jk}\}$ be as (3.9). If we set $\omega^{(2)}_t = \begin{pmatrix} \omega & O \\ \dot{\omega}+tA\theta & \omega \end{pmatrix}$, then $\omega^{(2)}_t$ is the connection form of a connection, say $\nabla^{(2)}_t$, on $Q^{(2)}(\mathcal{F})$ with respect to $\theta^{(2)}$. Note that $\omega^{(2)}$ can be also viewed as the connection form of a connection on $Q^{(2)}(\mathcal{F}\times[0,1])$ with respect to the pull-back of $\theta^{(2)}$ to $M\times[0,1]$. Let φ be a cocycle in $\mathrm{DWU_q}$. If we represent $\varphi(\nabla^b,\nabla^{(2)}_t) = \varphi_1+\varphi_2\wedge dt$, where φ_1 and φ_2 do not involve dt, and if we set $\widetilde{\varphi} = \int\varphi_2$, then we have $d\widetilde{\varphi} = \varphi(\nabla^b,\nabla^{(2)}_0) - \varphi(\nabla^b,\nabla^{(2)}_1)$. If we replace $\omega,\dot{\omega}$ by $\omega+B\theta,\dot{\omega}+B\sigma$, where B is defined by (3.10), then we set $\omega_t = \omega+tB\theta$ and $\dot{\omega}_t = \dot{\omega}+tB\sigma$. Let $\nabla^{b\prime}$ and $\nabla^{(2)\prime}$ be the connections defined by $\omega+B\theta$ and $\dot{\omega}+B\sigma$, then, by repeating almost the same argument as above, we can find a primitive of $\varphi(\nabla^b,\nabla^{(2)}) - \varphi(\nabla^{b\prime},\nabla^{(2)\prime})$. Suppose that σ is replaced by σ' by (3.11). By the same argument in the proof of Lemma 3.14, we see that g does not affect $\varphi(\nabla^b,\nabla^{(2)})$. By considering $\sigma_t = \sigma + d(tf) + \omega\,(tf)$ and by repeating again the same argument as above, we see that $\varphi(\nabla^b,\nabla^{(2)})$ and $\varphi(\nabla^b,\nabla^{(2)\prime})$ are cohomologous. If we replace θ by $(D\zeta)\theta$, then $\varphi(\nabla^b,\nabla^{(2)})$ does not change by (3.13), because invariant polynomial are considered. Finally, if we replace

h by another metric, then we can form a 1-parameter family of metrics and connections, and show by similar arguments as above that the cohomology class remains the same. □

Remark 5.24. A related but different construction can be found in [8]. Characteristic classes for deformations of foliations are also studied in [6] from another viewpoint. We also remark that Theorem 5.19 is shown by more combinatorial arguments in [13] and [4].

Remark 5.25. The differential forms b_1 and c_1 can be obtained from $K^{(2)}(\mathcal{F})^{-1}$ and $\underline{\nabla}^{(2)}$ appeared in §4.

Definition 5.26. The elements in the image of δ in $H^*(\mathrm{DWO_q})$ (resp. $H^*(\mathrm{DWU_q})$) are said to be *infinitesimal derivatives* of secondary classes. If $\sigma \in H^1(M;\Theta_{\mathcal{F}})$, then the image of infinitesimal derivatives under $D\chi_{\mathbb{R}}(\sigma,\delta(\ \cdot\ ))$ (resp. $D\chi(\sigma,\delta(\ \cdot\ ))$) are called the *infinitesimal derivatives* with respect to σ.

Example 5.27. We have $\mathrm{DW}_1 = \mathrm{DWO}_1$. Consequently,

$$H^*(\mathrm{DW}_1) = H^*(\mathrm{DWO}_1) = \langle 1, h_1c_1, \dot{h}_1c_1, \dot{h}_1h_1c_1\rangle.$$

Example 5.28. The class $h_1c_1^q \in H^{2q+1}(\mathrm{DWO}_q)$ is the Godbillon-Vey class. The class $(q+1)\dot{h}_1c_1^q \in H^{2q+1}(\mathrm{DWO}_q)$ is the infinitesimal derivative of the Godbillon-Vey class. Note that $(q+1)\dot{h}_1c_1^q = \delta(h_1c_1^q)$ holds in $H^{2q+1}(\mathrm{DWO}_q)$. There is another class which involves $\dot{h}_1$, h_1 and c_1. Indeed,

$$d(\dot{h}_1h_1c_1^q) = \dot{c}_1h_1c_1^q - \dot{h}_1c_1^{q+1} = 0$$

in DWO_q because $c_1^{q+1}, \dot{c}_1c_1^q \in DI_q$. The class $(q+1)\dot{h}_1h_1c_1^q \in H^{2q+1}(\mathrm{DWO_q})$ is introduced by Fuks [10], Lodder [18] and Kotschick [17], and called the *Fuks-Lodder-Kotschick class* in [4].

In the transversely holomorphic case, the Bott class is defined by $u_1c_1^q$ if the complex normal bundle is trivial. In general, the imaginary part of the Bott class is given by $\sqrt{-1}\widetilde{u}_1(c_1^q + c_1^{q-1}\bar{c}_1 + \cdots + \bar{c}_1^q) \in H^{2q+1}(\mathrm{DWU}_q)$ if we choose $\mathbb{R}$ or $\mathbb{C}$ as coefficients of cohomology. On the other hand, the infinitesimal derivative of the Bott class is defined as an element of

$H^{2q+1}(\mathrm{DWU}_q)$ or $H^{2q+1}(M;\mathbb{C})$ even if the complex normal bundle is non-trivial. Indeed, the infinitesimal derivative of the Bott class is given by $(q+1)\dot{u}_1 c_1^q \in H^{2q+1}(\mathrm{DWU}_q)$.

Therefore, the infinitesimal derivative of the imaginary part of the Bott class will be represented by two cocycles in DWU_q, namely,

$$\begin{aligned}\lambda_1 &= (q+1)\sqrt{-1}(\dot{u}_1 c_1^q - \dot{\bar{u}}_1 \bar{c}_1^q),\\ \lambda_2 &= \sqrt{-1}(\dot{u}_1 - \dot{\bar{u}}_1)(c_1^q + \cdots + \bar{c}_1^q)\\ &\quad + \sqrt{-1}\sum_{k=0}^{q}\left((q-k)\dot{c}_1 c_1^{q-k-1}\bar{c}_1^k + k\dot{\bar{c}}_1 c_1^{q-k}\bar{c}_1^{k-1}\right).\end{aligned}$$

We have the following

Lemma 5.29. *The above cocycles λ_1 and λ_2 are cohomologous.*

Proof. Let

$$\begin{aligned}\mu &= \widetilde{u}_1(q\dot{u}_1 c_1^{q-1} + (q-1)\dot{u}_1 c_1^{q-2}\bar{c}_1 + \dot{\bar{u}}_1 c_1^{q-1} + \cdots + q\dot{\bar{u}}_1 c_1^{q-1})\\ &= \sum_{i=0}^{q}\widetilde{u}_1((q-i)\dot{u}_1 c_1^{q-i-1}\bar{c}^i + i\dot{\bar{u}}_1 c_1^{q-i}\bar{c}_1^{i-1}).\end{aligned}$$

Since

$$\begin{aligned}d\mu = q\dot{u}_1 c_1^q &- \sum_{i=1}^{q}\dot{u}_1 c_1^{q-i}\bar{c}_1^i + \sum_{j=0}^{q-1}\dot{\bar{u}}_1 c_1^{q-j}\bar{c}_1^j - q\dot{\bar{u}}_1\bar{c}_1^q\\ &- \widetilde{u}_1(q\dot{c}_1 c_1^{q-1} + (q-1)\dot{c}_1 c_1^{q-2}\bar{c}_1 + \dot{\bar{c}}_1 c_1^{q-1} + \cdots + q\dot{\bar{c}}_1\bar{c}_1^{q-1}),\end{aligned}$$

we have $\lambda_2 + \sqrt{-1}d\mu = \lambda_1$. □

If we assume that normal bundles of foliations are trivial, then we can modify the construction as follows.

Definition 5.30. We set $\deg c_i = \deg \dot{c}_i = \deg \dot{\bar{c}}_i = 2i$, $\deg h_i = \deg u_i = \deg \bar{u}_i = \deg \dot{h}_i = \deg \dot{u}_i = \deg \dot{\bar{u}}_i = 2i-1$. Let

$$\begin{aligned}\mathrm{W}_q &= \bigwedge[h_1, h_2, \ldots, h_q] \otimes \mathbb{R}_q[c_1, \ldots, c_q],\\ \mathrm{DW}_q &= \bigwedge[\dot{h}_1, \dot{h}_2, \ldots, \dot{h}_q] \wedge \bigwedge[h_1, h_2, \ldots, h_q] \otimes \mathbb{R}_{q,q}[c_1, \ldots, c_q, \dot{c}_1, \ldots, \dot{c}_q].\end{aligned}$$

We set

$$dh_i = c_i,\ d\dot{h}_i = \dot{c}_i.$$

When we consider $\mathrm{W}_q \otimes \mathbb{C}$ and $\mathrm{DW}_q \otimes \mathbb{C}$, we denote by h_i and $\dot{h}_i$ by u_i and $\dot{u}_i$. We set

$$du_i = c_i,\ d\bar{u}_i = \bar{c}_i,\ d\dot{u}_i = \dot{c}_i,\ d\dot{\bar{u}}_i = \dot{\bar{c}}_i,\ dc_i = d\bar{c}_i = d\dot{c}_i = d\dot{\bar{c}}_i = 0.$$

Then, $\mathrm{W}_q \otimes \mathbb{C} = \bigwedge[u_1, u_2, \dots, u_q] \otimes \mathbb{C}_q[c_1, \dots, c_q]$. We set

$$\overline{\mathrm{W}_q \otimes \mathbb{C}} = \bigwedge[\bar{u}_1, \bar{u}_2, \dots, \bar{u}_q] \otimes \mathbb{C}_q[\bar{c}_1, \dots, \bar{c}_q].$$

We define $\overline{\mathrm{DW}_q \otimes \mathbb{C}}$ in an obvious way, and set $\mathrm{W}_q^{\mathbb{C}} = (\mathrm{W}_q \otimes \mathbb{C}) \wedge (\overline{\mathrm{W}_q \otimes \mathbb{C}})$ and $\mathrm{DW}_q^{\mathbb{C}} = (\mathrm{DW}_q \otimes \mathbb{C}) \wedge (\overline{\mathrm{DW}_q \otimes \mathbb{C}})$.

PROPOSITION 5.31. *The natural homomorphisms* $H^*(\mathrm{W}_q) \to H^*(\mathrm{DW}_q)$ *and* $H^*(\mathrm{W}_q \otimes \mathbb{C}) \to H^*(\mathrm{DW}_q \otimes \mathbb{C})$ *are injective.*

The proof is almost identical to that of Proposition 5.15.

Let $\delta_{\mathbb{R}}^F \colon \mathrm{W}_q \to \mathrm{DW}_q$ be the (commutative) derivation which satisfies $\delta_{\mathbb{R}}^F c_j = \dot{c}_j$ and $\delta_{\mathbb{R}}^F h_i = \dot{h}_i$. We denote the complexification of $\delta_{\mathbb{R}}^F$ by δ^F. It is easy to see that $\delta_{\mathbb{R}}^F$ and δ^F are well-defined and commute with d. We have the following

PROPOSITION 5.32. *There are well-defined derivations* $\delta_{\mathbb{R}}^F \colon H^*(\mathrm{W}_q) \to H^*(\mathrm{DW}_q)$ *and* $\delta^F \colon H^*(\mathrm{W}_q \otimes \mathbb{C}) \to H^*(\mathrm{DW}_q \otimes \mathbb{C})$ *such that*

$$\delta_{\mathbb{R}}^F h_i = \dot{h}_i,\ \delta_{\mathbb{R}}^F c_i = \dot{c}_i,$$
$$\delta^F u_i = \dot{u}_i,\ \delta^F c_i = \dot{c}_i.$$

Note that a derivation on $H^*(\mathrm{W}_q^{\mathbb{C}})$ with values in $H^*(\mathrm{DW}_q^{\mathbb{C}})$ is naturally defined.

THEOREM 5.33. *Once a homotopy type of trivialization of the normal bundle of* $\mathcal{F}$ *is fixed, there is a well-defined bilinear pairing*

$$D\chi_{\mathbb{R}}^F \colon H^1(M; \Theta_{\mathcal{F}}) \times H^*(\mathrm{DW}_q) \to H^*(M; \mathbb{R})$$

for real codimension-q foliations , and

$$D\chi^F \colon H^1(M;\Theta_{\mathcal{F}}) \times H^*(\mathrm{DW}_q \otimes \mathbb{C}) \to H^*(M;\mathbb{C})$$

for complex codimension-q transversely holomorphic foliations with trivialized complex normal bundles. If $\sigma \in H^1(M;\Theta_{\mathcal{F}})$ is an infinitesimal deformation and if $\alpha \in H^*(\mathrm{W}_q)$, then $D\chi^F_{\mathbb{R}}(\sigma, \delta^F_{\mathbb{R}}(\alpha))$ or $D\chi^F(\sigma, \delta^F(\beta))$ is the infinitesimal derivative of α with respect to σ.

Proof. The theorem is proven in an almost the same way as that of Theorem 5.19. Let s be a trivialization of the normal bundle in the homotopy type we have chosen. Let ∇^b be a Bott connection on $Q(\mathcal{F})$, ∇^s the flat connection with respect to s, θ the trivialization of $Q^*(\mathcal{F})$ dual to s, and an infinitesimal deformation $\{\sigma\}$ of θ which represents an infinitesimal deformation of $\mathcal{F}$. Let ω^b and ω^s be connection forms of ∇^b and ∇^s with respect to s, and $\dot{\omega}$ an infinitesimal deformation of ω^b with respect to σ. We set $\omega^{(2)} = \begin{pmatrix} \omega^b & O \\ \dot{\omega} & \omega^b \end{pmatrix}$, $\omega^{(2)}_0 = \begin{pmatrix} \omega^b & O \\ O & \omega^b \end{pmatrix}$, $R = d\omega^b + \omega^b \wedge \omega^b$ and $R^{(2)} = d\omega^{(2)} + \omega^{(2)} \wedge \omega^{(2)}$. Let $\widetilde{D}\chi^F$ be the algebra homomorphism from $(E(\mathcal{F})^* \otimes Q(\mathcal{F})) \times (\mathrm{DW}_q \otimes \mathbb{C})$ to $\Omega^*(M)$ determined by the conditions that

$$\begin{aligned}
\widetilde{D}\chi^F(c_i) &= c_i(R),\\
\widetilde{D}\chi^F(u_i) &= \Delta_{c_i}(\omega^b, \omega^s),\\
\widetilde{D}\chi^F(\dot{c}_i) &= b_i(R^{(2)}),\\
\widetilde{D}\chi^F(\dot{u}_i) &= \Delta_{b_i}(\omega^{(2)}, \omega^{(2)}_0).
\end{aligned}$$

In the real case, we define $\widetilde{D}\chi^F_{\mathbb{R}}$ by the conditions that

$$\begin{aligned}
\widetilde{D}\chi^F_{\mathbb{R}}(c_i) &= c_i(R),\\
\widetilde{D}\chi^F_{\mathbb{R}}(h_i) &= \Delta_{c_i}(\omega^b, \omega^s),\\
\widetilde{D}\chi^F_{\mathbb{R}}(\dot{c}_i) &= b_i(R^{(2)}),\\
\widetilde{D}\chi^F_{\mathbb{R}}(\dot{h}_i) &= \Delta_{b_i}(\omega^{(2)}, \omega^{(2)}_0).
\end{aligned}$$

Then, by repeating arguments of the same kind of as in the proof of Theorem 5.19, we can show that the mappings induced on the cohomology are independent of choices. □

Note that a homomorphism from $H^*(\mathrm{DW}_q^{\mathbb{C}})$ to $H^*(M;\mathbb{C})$ is induced by $D\chi^F$.

Remark 5.34. Example 5.11 of [4] shows that $D\chi^F$ indeed depends on the homotopy type of s. It is shown by examining $\dot{u}_1 u_1 c_1^q$ of a transversely holomorphic foliation of which the complex normal bundle is trivial. It is also shown that the effect of the change of trivialization is indeed valued in $\mathbb{C}$, which suggests that there are no direct analogue of Fuks-Lodder-Kotschick class in the category of transversely holomorphic foliations unless the triviality of normal bundles is assumed, because this fact implies that the imaginary part of the Fuks-Lodder-Kotschick class hardly makes sense. There is indeed no direct analogue in $H^*(\mathrm{DWU}_1)$ calculated in Example 5.17. On the other hand, it seems unknown in the real case if there is a family or infinitesimal deformation of a foliation of which $\dot{h}_1 h_1 c_1^q$ is non-trivial even if $q = 1$. Similarly, it is unknown if the classes $\dot{u}_1 \dot{\bar{u}}_1 c_1 \bar{c}_1$, $\widetilde{u}_1 \dot{u}_1 c_1 \bar{c}_1$, $\widetilde{u}_1 \dot{\bar{u}}_1 c_1 \bar{c}_1$ and $\widetilde{u}_1 \dot{u}_1 \dot{\bar{u}}_1 c_1 \bar{c}_1 \in H^*(\mathrm{DWU}_1)$ in Example 5.17 can be non-trivial for some infinitesimal deformation or not (the above-mentioned example in [4] does not work).

6. Determination of $H^*(\mathrm{DWO}_2)$ and Comparison with $H^*(\mathrm{DWU}_1)$

We will first compute $H^*(\mathrm{DWO}_2)$. This is again done by means of spectral sequences. We have

$$\begin{aligned}
I_2 &= I(c_1^3, c_1c_2, c_2^2),\\
I_{2,2} &= I(c_1^3, c_1c_2, c_2^2, c_1^2\dot{c}_2, c_2\dot{c}_2, c_1\dot{c}_2^2, c_2\dot{c}_2^2, \dot{c}_2^3),\\
DI_q &= I(c_1^3, c_1c_2, c_2^2, c_1^2\dot{c}_2, c_2\dot{c}_2, c_1\dot{c}_2^2, c_2\dot{c}_2^2, \dot{c}_2^3, c_1^2\dot{c}_1, c_1\dot{c}_2 + c_2\dot{c}_1, c_2\dot{c}_2)\\
&= I(c_1^3, c_1c_2, c_2^2, c_1^2\dot{c}_2, c_2\dot{c}_2, c_1\dot{c}_2^2, c_2\dot{c}_2^2, \dot{c}_2^3, c_1^2\dot{c}_1, c_1\dot{c}_2 + c_2\dot{c}_1),
\end{aligned}$$

where $I(f_1, \dots, f_r)$ denotes the ideal generated by $f_1, \dots, f_r$.

Let

$$\begin{aligned}
A &= \{f \in \mathrm{DWO}_2 \mid f \text{ does not involve } h_1\},\\
B_0 &= \{f \in A \mid f \text{ does not involve } \dot{h}_1 \text{ or } \dot{h}_2\},\\
B_1 &= \{f \in A \mid f \text{ does not involve } \dot{h}_2\}.
\end{aligned}$$

Then, A, B_0 and B_1 are closed under d. As vector spaces, we have

$$\begin{aligned} B_0 &= \langle 1, c_1, c_1^2, c_2, \dot{c}_1^k, \dot{c}_2, \dot{c}_2^2, \dot{c}_1^k \dot{c}_2, \dot{c}_1^k \dot{c}_2^2, c_1 \dot{c}_1^k, c_2 \dot{c}_1^k (= -c_1 \dot{c}_1^k \dot{c}_2) \mid k > 0 \rangle, \\ B_1 &= B_0 \oplus \dot{h}_1 B_0. \end{aligned}$$

By examining the long exact sequence associated with $0 \to B_0 \to B_1 \to B_1/B_0 \to 0$, we see that

$$H^r(B_1) = \langle 1, c_1, c_1^2, c_2, \dot{c}_2, \dot{c}_2^2 \rangle \oplus \langle \dot{h}_1 c_1^2 \rangle.$$

Next, we examine the long exact sequence associated with $0 \to B_1 \to A \to A/B_1 \to 0$. Note that $H^*(A/B_1) \cong \dot{h}_2 H^*(B_1)$. The result is

$$H^*(A) = \langle 1, c_1, c_1^2, c_2, \dot{h}_1 c_1^2 \rangle \oplus \langle \dot{h}_2 c_1 + \dot{h}_1 c_2, \dot{h}_2 c_1^2, \dot{h}_2 c_2, \dot{h}_2 \dot{c}_2^2, \dot{h}_1 \dot{h}_2 c_1^2 \rangle.$$

Finally, we consider the long exact sequence associated with $0 \to A \to \mathrm{DWO}_2 \to \mathrm{DWO}_2/A \to 0$, where $H^*(\mathrm{DWO}_2/A) \cong h_1 H^*(A)$. We obtain

$$H^r(\mathrm{DWO}_2) = \begin{cases} \langle 1 \rangle, & r = 0, \\ \langle c_2 \rangle, & r = 4, \\ \langle h_1 c_1^2, h_1 c_2, \dot{h}_1 c_1^2, \dot{h}_1 c_2 + \dot{h}_2 c_1 \rangle, & r = 5, \\ \langle \dot{h}_1 h_1 c_1^2 \rangle, & r = 6, \\ \langle \dot{h}_2 c_2 \rangle, & r = 7, \\ \langle \dot{h}_2 h_1 c_1^2, \dot{h}_2 h_1 c_2, \dot{h}_1 \dot{h}_2 c_1^2 \rangle, & r = 8, \\ \langle \dot{h}_1 \dot{h}_2 h_1 c_1^2 \rangle, & r = 9, \\ \langle \dot{h}_2 \dot{c}_2^2 \rangle, & r = 11, \\ \langle \dot{h}_2 h_1 \dot{c}_2^2 \rangle, & r = 12, \\ 0, & \text{otherwise.} \end{cases}$$

Up to multiplications of constants, the class c_2 is the first Pontrjagin class, $h_1 c_1^2$ and $\dot{h}_1 c_1^2$ are the Godbillon-Vey class and its infinitesimal derivative, $h_1 c_2$ is one of the 'classical' secondary classes in $H^5(\mathrm{WO}_2)$, and $\dot{h}_1 h_1 c^2$ is the Fuks-Lodder-Kotschick class. In general, we can compute $H^*(\mathrm{DWO}_q)$ etc., by means of spectral sequences as above. It seems however difficult to obtain a set of basis as a vector space such as the Vey basis for $H^*(\mathrm{WO}_q)$ or $H^*(\mathrm{W}_q)$.

In what follows, we denote $c_i, \bar{c}_i \in \mathrm{WU}_q \subset \mathrm{W}_q^{\mathbb{C}}$ by $v_i, \bar{v}_i$ in order to avoid confusions. Given a transversely holomorphic foliation, we can forget the transverse holomorphic structure [1], [2]. This corresponds to the natural maps $B\Gamma_q^{\mathbb{C}} \to B\Gamma_{2q}$ and $B\overline{\Gamma}_q^{\mathbb{C}} \to B\overline{\Gamma}_{2q}$. Accordingly, we have homomorphisms

$$\begin{aligned} \lambda\colon H^*(\mathrm{WO}_{2q}) &\to H^*(\mathrm{WU}_q), \\ \widehat{\lambda}\colon H^*(\mathrm{W}_{2q}) &\to H^*(\mathrm{W}_q \otimes \mathbb{C}). \end{aligned}$$

The same can be done for $H^*(\mathrm{DWU}_q)$ and $H^*(\mathrm{DWO}_{2q})$. The relevant maps are

$$\begin{aligned} D\lambda\colon H^*(\mathrm{DWO}_{2q}) &\to H^*(\mathrm{DWU}_q), \\ D\widehat{\lambda}\colon H^*(\mathrm{DW}_{2q}) &\to H^*(\mathrm{DW}_q^{\mathbb{C}}). \end{aligned}$$

They are defined by DGA-homomorphisms $\widetilde{D}\lambda\colon \mathrm{DWO}_{2q} \to \mathrm{DWU}_q$ and $\widetilde{D}\widehat{\lambda}\colon \mathrm{DW}_{2q} \to \mathrm{DW}_q^{\mathbb{C}}$ such that

$$\begin{aligned} \widetilde{D}\lambda(c_i) &= (\sqrt{-1})^i \sum_{k=0}^{i} (-1)^k v_{i-k}\bar{v}_k, \\ \widetilde{D}\lambda(h_{2i+1}) &= \frac{(-1)^i}{2}\sqrt{-1} \sum_{k=0}^{2i+1} (-1)^k \widetilde{u}_{2i-k+1}(v_k + \bar{v}_k), \\ \widetilde{D}\lambda(\dot{c}_i) &= (\sqrt{-1})^i \sum_{k=0}^{i} (-1)^k (\dot{v}_{i-k}\bar{v}_k + v_{i-k}\dot{\bar{v}}_k), \\ \widetilde{D}\lambda(\dot{h}_{2i+1}) &= \frac{(-1)^i}{2}\sqrt{-1} \sum_{k=0}^{2i} (-1)^k (\dot{\widetilde{u}}_{2i-k+1}(v_k + \bar{v}_k) + \widetilde{u}_{2i-k+1}(\dot{v}_k + \dot{\bar{v}}_k)), \end{aligned}$$

where $v_0 = \bar{v}_0 = 1$, $\dot{v}_0 = \dot{\bar{v}} = 0$, and

$$\begin{aligned} \widetilde{D}\widehat{\lambda}(c_i) &= (\sqrt{-1})^i \sum_{k=0}^{i} (-1)^k v_{i-k}\bar{v}_k, \\ \widetilde{D}\widehat{\lambda}(h_{2i+1}) &= \frac{(-1)^i}{2}\sqrt{-1} \sum_{k=0}^{2i+1} (-1)^k \widetilde{u}_{2i-k+1}(v_k + \bar{v}_k), \end{aligned}$$

$$\widetilde{D}\widehat{\lambda}(h_{2i}) = \frac{(-1)^i}{2}\sum_{k=0}^{2i}(-1)^k(u_{2i-k}\bar{v}_k + \bar{u}_k v_{2i-k}),$$

$$\widetilde{D}\widehat{\lambda}(\dot{c}_i) = (\sqrt{-1})^i\sum_{k=0}^{i}(-1)^k(\dot{v}_{i-k}\bar{v}_k + v_{i-k}\dot{\bar{v}}_k),$$

$$\widetilde{D}\widehat{\lambda}(\dot{h}_{2i+1}) = \frac{(-1)^i}{2}\sqrt{-1}\sum_{k=0}^{2i+1}(-1)^k(\dot{\widetilde{u}}_{2i-k+1}(v_k + \bar{v}_k) + \widetilde{u}_{2i-k+1}(\dot{v}_k + \dot{\bar{v}}_k)),$$

$$\widetilde{D}\widehat{\lambda}(\dot{h}_{2i}) = \frac{(-1)^i}{2}\sum_{k=0}^{2i}(-1)^k(\dot{u}_{2i-k}\bar{v}_k + \dot{\bar{u}}_k v_{2i-k} + u_{2i-k}\dot{\bar{v}}_k + \bar{u}_k\dot{v}_{2i-k}).$$

We have the following version of Lemma 3.1 of [1].

Lemma 6.1.

1) *If $\mathcal{F}$ is a transversely holomorphic foliation, then there is a natural homomorphism $\lambda_\Theta\colon H^*(M;\Theta_{\mathcal{F}}) \to H^*(M;\Theta_{\mathcal{F}_\mathbb{R}})$, where $\mathcal{F}_\mathbb{R}$ is the foliation $\mathcal{F}$ but the transverse holomorphic structure forgotten.*

2) *The homomorphisms $\widetilde{D}\lambda$ and $\widetilde{D}\widehat{\lambda}$ induce on the cohomology the homomorphisms $D\lambda$ and $D\widehat{\lambda}$ such that $D\chi_\mathbb{R}(\lambda_\Theta(\sigma),\alpha) = D\chi(\sigma, D\lambda(\alpha))$ and $D\chi_\mathbb{R}^F(\lambda_\Theta(\sigma),\alpha) = D\chi(\sigma, D\widehat{\lambda}(\alpha))$.*

Proof. Let ∇^b be a Bott connection on $Q(\mathcal{F})$. Let θ be a local trivialization of $Q^*(\mathcal{F})$ and ω be the connection form of ∇^b with respect to the dual of θ. A section $\sigma = (\sigma^j)$ of $\bigwedge^i E(\mathcal{F})^* \otimes Q(\mathcal{F})$ is a representative of a class in $H^i(M;\Theta_{\mathcal{F}})$ if and only if there is a $\mathfrak{gl}_q(\mathbb{C})$-valued 1-form μ such that $d\sigma + \omega\wedge\sigma + \mu\wedge\theta = 0$. If we choose $\theta_\mathbb{R} = \theta \oplus \bar{\theta}$ as a trivialization of $Q(\mathcal{F}_\mathbb{R})\otimes\mathbb{C} \cong Q(\mathcal{F}) \oplus \overline{Q(\mathcal{F})}$, then $\nabla^b_\mathbb{R} = \nabla^b \oplus \overline{\nabla^b}$ is a Bott connection on $Q(\mathcal{F}_\mathbb{R})\otimes\mathbb{C}$ and $\omega_\mathbb{R} = \omega\oplus\bar{\omega}$ is its connection form. Therefore, $\sigma_\mathbb{R} = \sigma\oplus\bar{\sigma}$ gives a $d_{\nabla^b_\mathbb{R}}$-closed form in $\bigwedge^i(E(\mathcal{F}_\mathbb{R})^*\otimes\mathbb{C})\otimes(Q(\mathcal{F})\otimes\mathbb{C})$. Note that $E(\mathcal{F}_\mathbb{R})\otimes\mathbb{C} = E(\mathcal{F})\cap\overline{E(\mathcal{F})}$. If we set $\mu_\mathbb{R} = \mu\oplus\bar{\mu}$, then $d\sigma_\mathbb{R} + \omega_\mathbb{R}\wedge\sigma_\mathbb{R} + \mu_\mathbb{R}\wedge\theta_\mathbb{R} = 0$. Similarly we can show that if σ is d_{∇^b}-exact, then $\sigma\oplus\bar{\sigma}$ is $d_{\nabla^b_\mathbb{R}}$-exact. Therefore, if we set $\lambda_\Theta(\sigma) = \sigma_\mathbb{R}$, then λ_Θ induces a homomorphism on the cohomology. Thus the part 1) is shown. The proof of the part 2) is essentially parallel to that of [1, Lemma 3.1] so that we give only the sketch. First we note that c_i and $\dot{c}_i$ calculated by using $\nabla^b_{\mathbb{R}}$ are equal to the right hand sides of defining relation of $\widetilde{D}\lambda(c_i)$ and $\widetilde{D}\lambda(\dot{c}_i)$ calculated by using $\nabla^b \oplus \overline{\nabla^b}$. Then, by integrating the relation, we see that $\widetilde{D}\lambda$ gives a desired homomorphism. The proof for $\widetilde{D}\widehat{\lambda}$ can be done in a parallel way. □

The following is a corollary to Proposition 5.15.

Lemma 6.2. *We have* $\operatorname{Ker}\lambda = \operatorname{Ker} D\widehat{\lambda} \cap H^*(\mathrm{WO}_{2q})$ *and* $\operatorname{Im}\lambda = \operatorname{Im} D\lambda \cap H^*(\mathrm{WU}_q)$.

If $q = 1$, then the mapping $D\lambda$ is given by the conditions that

$$\begin{aligned}
h_1 &\mapsto \sqrt{-1}\widetilde{u}_1,\\
c_1 &\mapsto \sqrt{-1}(v_1 - \bar{v}_1),\\
c_2 &\mapsto v_1\bar{v}_1,\\
\dot{h}_1 &\mapsto \sqrt{-1}(\dot{u}_1 - \dot{\bar{u}}_1),\\
\dot{h}_2 &\mapsto \dot{u}_1\bar{v}_1 + \dot{\bar{u}}_1 v_1,\\
\dot{c}_1 &\mapsto \sqrt{-1}(\dot{v}_1 - \dot{\bar{v}}_1),\\
\dot{c}_2 &\mapsto \dot{v}_1\bar{v}_1 + \dot{\bar{v}}_1 v_1.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\operatorname{Ker} D\lambda = \langle & c_2, h_1(c_1^2 - 2c_2), \dot{h}_1 c_1^2, \dot{h}_1 c_2 + \dot{h}_2 c_1, \dot{h}_2 c_2, \dot{h}_2 h_1 c_1^2, \dot{h}_2 h_1 c_2,\\
& \dot{h}_1\dot{h}_2 h_1 c_1^2, \dot{h}_2 \dot{c}_2^2, \dot{h}_1\dot{h}_2 c_1^2, \dot{h}_2 h_1 \dot{c}_2^2\rangle,\\
\operatorname{Im} D\lambda = \langle & 1, \widetilde{u}_1 v_1\bar{v}_1, (\dot{u}_1 - \dot{\bar{u}}_1)\widetilde{u}_1 v_1 \bar{v}_1\rangle.
\end{aligned}$$

We have $c_2, h_1(c_1^2 - 2c_2) \in \operatorname{Ker}\lambda$ and $1, \widetilde{u}_1 v_1\bar{v}_1 \in \operatorname{Im}\lambda$. Note that $2\sqrt{-1}\widetilde{u}_1 v_1 \bar{v}_1$ is the Godbillon-Vey class and that

$$\begin{aligned}
&(\dot{u}_1 - \dot{\bar{u}}_1)v_1\bar{v}_1 + \widetilde{u}_1\dot{v}_1\bar{v}_1 + \widetilde{u}_1 v_1 \dot{\bar{v}}_1\\
&= -d(\widetilde{u}_1(\dot{u}_1(2v_1 + \bar{v}_1) + \dot{\bar{u}}_1(2\bar{v}_1 + v_1))).
\end{aligned}$$

In general, the Godbillon-Vey class is equal to $\frac{(2q)!}{q!q!}\sqrt{-1}\widetilde{u}_1 v_1^q \bar{v}_1^q$, and

$$\begin{aligned}
&(\dot{u}_1 - \dot{\bar{u}}_1)v_1^q\bar{v}_1^q + q\widetilde{u}_1 v_1^{q-1}\dot{v}_1\bar{v}_1^q + q\widetilde{u}_1 v_1^q \bar{v}_1^{q-1}\dot{\bar{v}}_1\\
&= -d(\widetilde{u}_1 v_1^{q-1}\bar{v}_1^{q-1}(\dot{u}_1((q+1)v_1 + q\bar{v}_1) + \dot{\bar{u}}_1((q+1)\bar{v}_1 + qv_1))).
\end{aligned}$$

This gives another proof of the rigidity of the Godbillon-Vey class in the category of transversely holomorphic foliations.

We next study the mapping $D\widehat{\lambda}\colon H^*(\mathrm{DW}_2) \to H^*(\mathrm{DW}_1 \otimes \mathbb{C})$. We can show by similar arguments as above that

$$H^r(\mathrm{DW}_2) = \begin{cases} \langle 1 \rangle, & r = 0, \\ \langle h_1c_1^2, h_1c_2, \dot{h}_1c_1^2, \dot{h}_1c_2 + \dot{h}_2c_1 \rangle, & r = 5, \\ \langle \dot{h}_1h_1c_1^2 \rangle, & r = 6, \\ \langle h_2c_2, \dot{h}_2c_2 \rangle, & r = 7, \\ \langle \dot{h}_2h_1c_1^2, \dot{h}_2h_1c_2, \dot{h}_1\dot{h}_2c_1^2, & \\ \quad h_1h_2c_1^2, h_1h_2c_2, \dot{h}_1h_2c_1^2, h_2(\dot{h}_1c_2 + \dot{h}_2c_1) \rangle, & r = 8, \\ \langle \dot{h}_1\dot{h}_2h_1c_1^2, \dot{h}_1h_1h_2c_1^2 \rangle, & r = 9, \\ \langle \dot{h}_2h_2c_2 \rangle, & r = 10, \\ \langle \dot{h}_2\dot{c}_2^2, \dot{h}_2h_1h_2c_1^2, \dot{h}_2h_1h_2c_2, \dot{h}_1\dot{h}_2h_2c_1^2 \rangle, & r = 11, \\ \langle \dot{h}_2h_1\dot{c}_2^2, \dot{h}_1\dot{h}_2h_1h_2c_1^2 \rangle, & r = 12, \\ \langle \dot{h}_2h_2\dot{c}_2^2 \rangle, & r = 14, \\ \langle \dot{h}_2h_1h_2\dot{c}_2^2 \rangle, & r = 15, \\ 0, & \text{otherwise.} \end{cases}$$

The mapping $D\widehat{\lambda}$ is given by the conditions that

$$\begin{aligned} h_1 &\mapsto \sqrt{-1}(u_1 - \bar{u}_1), \\ h_2 &\mapsto \frac{1}{2}(u_1\bar{v}_1 + \bar{u}_1v_1), \\ c_1 &\mapsto \sqrt{-1}(v_1 - \bar{v}_1), \\ c_2 &\mapsto v_1\bar{v}_1, \\ \dot{h}_1 &\mapsto \sqrt{-1}(\dot{u}_1 - \dot{\bar{u}}_1), \\ \dot{h}_2 &\mapsto \dot{u}_1\bar{v}_1 + \dot{\bar{u}}_1v_1, \\ \dot{c}_1 &\mapsto \sqrt{-1}(\dot{v}_1 - \dot{\bar{v}}_1), \\ \dot{c}_2 &\mapsto \dot{v}_1\bar{v}_1 + \dot{\bar{v}}_1v_1. \end{aligned}$$

Therefore,

$$\begin{aligned}
&\operatorname{Ker} D\widehat{\lambda} \\
&= \langle h_1(c_1^2 - 2c_2), \dot{h}_1 c_1^2, \dot{h}_1 c_2 + \dot{h}_2 c_1, h_2 c_2, \dot{h}_2 c_2, \dot{h}_2 h_1 c_1^2, \dot{h}_2 h_1 c_2, \dot{h}_1 \dot{h}_2 c_1^2, \\
&\quad h_1 h_2 c_1^2, h_1 h_2 c_2, \dot{h}_1 h_2 c_1^2, h_2(\dot{h}_1 c_2 + \dot{h}_2 c_1), \dot{h}_1 \dot{h}_2 h_1 c_1^2, \dot{h}_1 h_1 h_2 c_1^2, \\
&\quad \dot{h}_2 h_2 c_2, \dot{h}_2 \dot{c}_2^2, \dot{h}_2 h_1 h_2 c_1^2, \dot{h}_2 h_1 h_2 c_2, \dot{h}_1 \dot{h}_2 h_2 c_1^2, \\
&\quad \dot{h}_2 h_1 \dot{c}_2^2, \dot{h}_1 \dot{h}_2 h_1 h_2 c_1^2, \dot{h}_2 h_2 \dot{c}_2^2, \dot{h}_2 h_1 h_2 \dot{c}_2^2 \rangle, \\
&\operatorname{Im} D\widehat{\lambda} \\
&= \langle 1, (u_1 - \bar{u}_1) v_1 \bar{v}_1, (\dot{u}_1 - \dot{\bar{u}}_1)(u_1 - \bar{u}_1) v_1 \bar{v}_1 \rangle.
\end{aligned}$$

As we mentioned in Remark 5.34, we know that the class $\dot{u}_1 u_1 v_1$ can be non-trivial. The class does not belong to $\operatorname{Im} D\widehat{\lambda}$, which implies that the non-triviality is not derived from deformations of real foliations. On the other hand, the image of the Fuks-Lodder-Kotschick class $3\dot{h}_1 h_1 c_1^2$ is equal to $-6(\dot{u}_1 - \dot{\bar{u}}_1)(u_1 - \bar{u}_1)v_1\bar{v}_1$ which is non-trivial in $H^*(\mathrm{DW}_1 \otimes \mathbb{C})$ and $H^*(\mathrm{DWU}_1)$. However, we do not know any example of which $(\dot{u}_1 - \dot{\bar{u}}_1)(u_1 - \bar{u}_1)v_1\bar{v}_1$ is non-trivial.

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