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The Magnus Representation and Homology Cobordism Groups of Homology Cylinders

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Abstract. A *homology cylinder* over a compact manifold is a homology cobordism between two copies of the manifold together with a boundary parametrization. We study abelian quotients of the homology cobordism group of homology cylinders. For homology cylinders over general surfaces, it was shown by Cha, Friedl and Kim that their homology cobordism groups have infinitely generated abelian quotient groups by using Reidemeister torsion invariants. In this paper, we first investigate their abelian quotients again by using another invariant called the Magnus representation. After that, we apply the machinery obtained from the Magnus representation to higher dimensional cases and show that the homology cobordism groups of homology cylinders over a certain series of manifolds regarded as a generalization of surfaces have big abelian quotients. In the proof, a homological localization, called the *acyclic closure*, of a free group and its automorphism group play important roles and our result also provides some information on these groups from a group-theoretical point of view.

1. Introduction

Let $\Sigma_{g,1}$ be a compact connected oriented surface of genus g with one boundary component. A *homology cylinder* over $\Sigma_{g,1}$ is a homology cobordism between two copies of $\Sigma_{g,1}$ together with a boundary parametrization. The set of all isomorphism classes of homology cylinders has a natural product operation given by stacking, so that it forms a monoid denoted by $\mathcal{C}_{g,1}$. The study of the monoid $\mathcal{C}_{g,1}$ was initiated by Goussarov [7] and Habiro [9] in their theory of clasper surgery and finite type invariants of 3-dimensional manifolds. In their study, a quotient group of the monoid by clasper surgery equivalence was introduced and its structure was intensively clarified in Massuyeau-Meilhan [19, 20] and Habiro-Massuyeau [10].

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On the other hand, Garoufalidis and Levine [4] introduced another equivalence relation by considering homology cobordisms of homology cylinders and they defined the quotient group $\mathcal{H}_{g,1}$ called the *homology cobordism group* of homology cylinders. Investigating the structure of the group $\mathcal{H}_{g,1}$ is considered to be important because it provides an efficient way of understanding the set of homology cobordism classes of 3-dimensional manifolds as the braid group contributes to knot theory. In this paper, we focus on this group $\mathcal{H}_{g,1}$ as well as its generalization to higher dimensional cases.

Another motivation for studying the monoid and groups of homology cylinders comes from the fact that they include the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_{g,1}$. They share many properties. For example, Garoufalidis-Levine [4], Levine [18] and Habegger [8] gave a deep relationship to the theory of Johnson homomorphisms used originally in the study of $\mathcal{M}_{g,1}$ and its subgroups. In pursuing more relationships between $\mathcal{M}_{g,1}$ and $\mathcal{H}_{g,1}$, it should be an important step to determine and compare their abelianizations. As for $\mathcal{M}_{g,1}$, it was first shown by Harer [11] that the abelianization is trivial (namely, the group $\mathcal{M}_{g,1}$ is perfect) except a few low genus cases. On the other hand, the abelianization of $\mathcal{H}_{g,1}$ has not yet been determined (see Section 5 for details).

In our previous paper [26], we introduced two kinds of invariants of homology cylinders, the Magnus representation and the Reidemeister torsion. Both invariants are crossed homomorphisms from the monoid $\mathcal{C}_{g,1}$ to some groups of matrices. It was observed that the Magnus representation factors through $\mathcal{H}_{g,1}$, while the Reidemeister torsion does not so. In the same paper, we found many abelian quotients of *sub*monoids of $\mathcal{C}_{g,1}$ and *sub*groups of $\mathcal{H}_{g,1}$. However no information was extracted on abelian quotients of the whole monoid and group. It had been conjectured that $\mathcal{H}_{g,1}$ was perfect for general g as in the case of $\mathcal{M}_{g,1}$.

After that, however, Cha, Friedl and Kim [2] succeeded in showing that the abelianization of $\mathcal{H}_{g,1}$ is infinitely generated by using a version of the Reidemeister torsion which we will call the *H-torsion* in this paper. In fact, they took an appropriate reduction of the torsion invariant so that the resulting map becomes a homology cobordism invariant homomorphism.

The purpose of the first half of this paper is to use (the determinant of) the Magnus representation together with Cha-Friedl-Kim's reduction technique to investigate again abelian quotients of $\mathcal{H}_{g,1}$. In Section 5, we will show that there exists a relationship between the invariant obtained from the Magnus representation and Cha-Friedl-Kim's torsion invariant.

Our invariant using the Magnus representation can be easily applied to homology cylinders over higher dimensional manifolds, which seem to have their own interest. Investigating this invariant is the purpose of the second half. The main theorem (Theorem 6.1) is that the homology cobordism groups of homology cylinders over a certain series of manifolds regarded as a higher dimensional generalization of surfaces have abelian quotients isomorphic to the free abelian group of infinite rank. For the proof, we use a purely group-theoretical description of the Magnus representation as the representation of the automorphism group of the *acyclic closure* of a free group. This description was first given by Le Dimet [15] in the context of string links, and then we clarified its relationship to homology cylinders in our previous paper [24]. The definition of the acyclic closure, which is originally due to Levine [16, 17], and its fundamental properties are reviewed in Section 7. We prove the theorem by constructing first an epimorphism from the homology cobordism group onto the automorphism group of the acyclic closure of a free group (Theorem 7.7) and then showing that this automorphism group has an abelian quotient isomorphic to the free abelian group of infinite rank (Theorem 8.5).

All manifolds are assumed to be smooth throughout this paper, while similar statements hold for other categories. We use the same notation to write a continuous map and the induced homomorphisms on fundamental groups and homology groups. Also, all homology groups are with $\mathbb{Z}$-coefficients.

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2. Homology Cylinders over a Manifold

We begin by giving the definition of homology cylinders over a manifold. Originally, homology cylinders are introduced and studied for surfaces by Goussarov [7], Habiro [9], Garoufalidis and Levine [4, 18]. The definition below is a natural generalization to it.

Let X be a compact oriented connected k-dimensional manifold. We assume for simplicity that the boundary ∂X of X is connected or empty.

DEFINITION 2.1. A *homology cylinder over* X consists of a compact oriented $(k+1)$-dimensional manifold M with two embeddings $i_+, i_- : X \hookrightarrow \partial M$ such that:

(i) i_+ is orientation-preserving and i_- is orientation-reversing,

(ii) $\partial M = i_+(X) \cup i_-(X)$ and $i_+(X) \cap i_-(X) = i_+(\partial X) = i_-(\partial X)$,

(iii) $i_+|_{\partial X} = i_-|_{\partial X}$,

(iv) $i_+, i_- : H_*(X) \to H_*(M)$ are isomorphisms.

We denote a homology cylinder by (M, i_+, i_-) or simply M. The boundary ∂M of M is the double of X if $\partial X \neq \emptyset$. Otherwise it is the disjoint union of two copies of X.

Two homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over X are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism $f : M \xrightarrow{\cong} N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$. We denote by $\mathcal{C}(X)$ the set of all isomorphism classes of homology cylinders over X. We define a product operation on X by

$$(M, i_+, i_-) \cdot (N, j_+, j_-) := (M \cup_{i_- \circ (j_+)^{-1}} N, i_+, j_-)$$

for (M, i_+, i_-), $(N, j_+, j_-) \in \mathcal{C}(X)$, which endows $\mathcal{C}(X)$ with a monoid structure. The unit is the trivial homology cylinder $(X \times [0,1], \mathrm{id} \times 1, \mathrm{id} \times 0)$, where collars of $i_+(X) = (\mathrm{id} \times 1)(X)$ and $i_-(X) = (\mathrm{id} \times 0)(X)$ are stretched half-way along $(\partial X) \times [0,1]$ so that $i_+(\partial X) = i_-(\partial X)$.

Example 2.2. For a self-diffeomorphism φ of X which restricts to the identity map on a neighborhood of ∂X, we can construct a homology cylinder by setting

$$(X \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$$

with the same treatment of the boundary as above. It is easily checked that the isomorphism class of $(X \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$ depends only on the isotopy (fixing a neighborhood of ∂X pointwise) class of φ and that this construction gives a monoid homomorphism from the diffeotopy group $\mathcal{M}(X)$ to $\mathcal{C}(X)$.

Remark 2.3. The homomorphism $\mathcal{M}(X) \to \mathcal{C}(X)$ is *not* necessarily injective. In fact, if $[\varphi] \in \operatorname{Ker}(\mathcal{M}(X) \to \mathcal{C}(X))$, the definition of the homomorphism only says that φ is a *pseudo isotopy* over X.

We also introduce *homology cobordisms* of homology cylinders, which define an equivalence relation among homology cylinders.

Definition 2.4. Two homology cylinders (M, i_+, i_-) and (N, i_+, i_-) over X are said to be *homology cobordant* if there exists a compact oriented $(k+2)$-dimensional manifold W such that:

(1) $\partial W = M \cup (-N)/(i_+(x) = j_+(x),\, i_-(x) = j_-(x)) \quad x \in X$,

(2) the inclusions $M \hookrightarrow W$, $N \hookrightarrow W$ induce isomorphisms on the homology group,

where $-N$ denotes the manifold N with the opposite orientation.

We denote by $\mathcal{H}(X)$ the quotient set of $\mathcal{C}(X)$ with respect to the equivalence relation of homology cobordism. The monoid structure of $\mathcal{C}(X)$ induces a group structure of $\mathcal{H}(X)$. We call $\mathcal{H}(X)$ the *homology cobordism group* of homology cylinders over X.

3. Homology Cylinders over a Surface

Let $\Sigma_{g,1}$ be a compact oriented surface of genus g with one boundary component. We take a base point p of $\Sigma_{g,1}$ on the boundary $\partial\Sigma_{g,1}$ and $2g$ oriented loops $\gamma_1, \gamma_2, \dots, \gamma_{2g}$ as in Figure 1. These loops form a spine R_{2g} of $\Sigma_{g,1}$ and they give a basis of $\pi_1(\Sigma_{g,1})$, a free group of rank $2g$. The boundary loop ζ is given by $\zeta = [\gamma_1, \gamma_{g+1}][\gamma_2, \gamma_{g+2}] \cdots [\gamma_g, \gamma_{2g}]$. We denote the first homology group $H_1(\Sigma_{g,1})$ by H for simplicity. The group H can be identified with $\mathbb{Z}^{2g}$ by choosing $\{\gamma_1, \gamma_2, \dots, \gamma_{2g}\}$ as a basis of H, where we write γ_j again for γ_j as an element of H. This basis is a symplectic basis with respect to the intersection pairing on H.

We use the notation $\mathcal{M}_{g,1} := \mathcal{M}(\Sigma_{g,1})$, $\mathcal{C}_{g,1} := \mathcal{C}(\Sigma_{g,1})$ and $\mathcal{H}_{g,1} := \mathcal{H}(\Sigma_{g,1})$ following our previous papers. The diffeotopy group $\mathcal{M}_{g,1}$ is also

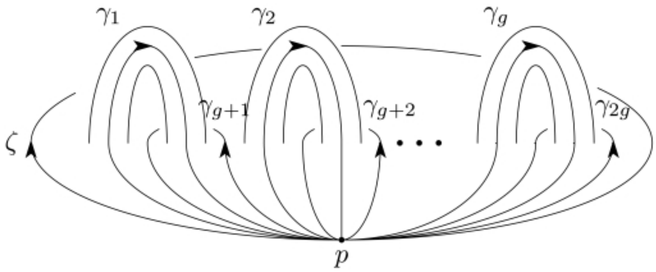

Fig. 1. Our basis of $\pi_1(\Sigma_{g,1})$.

called the *mapping class group* of $\Sigma_{g,1}$. It was shown by Garoufalidis-Levine [4, Section 2.4] that the homomorphism $\mathcal{M}_{g,1} \to \mathcal{C}_{g,1}$ and the composition $\mathcal{M}_{g,1} \to \mathcal{C}_{g,1} \to \mathcal{H}_{g,1}$ are injective.

Example 3.1 (Levine [18]). Let L be a string link of g strings, which is a generalization of a pure braid. We embed a g-holed disk D_g^2 into $\Sigma_{g,1}$ as a closed regular neighborhood of the union of the loops $\gamma_{g+1}, \gamma_{g+2}, \dots, \gamma_{2g}$ in Figure 1. Let C be the complement of an open tubular neighborhood of L in $D^2 \times [0,1]$. By choosing a framing of L, we can fix a diffeomorphism $h : \partial C \xrightarrow{\cong} \partial(D_g^2 \times [0,1])$. Then the manifold M_L obtained from $\Sigma_{g,1} \times [0,1]$ by removing $D_g^2 \times [0,1]$ and regluing C by h becomes a homology cylinder with the same boundary parametrizations i_+, i_- as the trivial homology cylinder.

The monoid $\mathcal{C}_{g,1}$ and the group $\mathcal{H}_{g,1}$ share many properties with the group $\mathcal{M}_{g,1}$. The most fundamental one is given by their action on H. Define a map

$$\sigma : \mathcal{C}_{g,1} \longrightarrow \mathrm{Aut}\,(H)$$

by assigning to $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ the automorphism $i_+^{-1} \circ i_-$ of H. This map extends the natural action of $\mathcal{M}_{g,1}$ on H and it is a monoid homomorphism. The image of φ consists of the automorphisms of H preserving the intersection pairing. Therefore, under the identification $H \cong \mathbb{Z}^{2g}$ mentioned above, we have an epimorphism

$$\sigma : \mathcal{C}_{g,1} \longrightarrow \mathrm{Sp}(2g, \mathbb{Z}).$$

We put $\mathcal{IC}_{g,1} := \operatorname{Ker}\sigma$, which is an analogue of the Torelli group $\mathcal{I}_{g,1} = \operatorname{Ker}(\sigma : \mathcal{M}_{g,1} \to \operatorname{Sp}(2g,\mathbb{Z}))$. We can see that σ induces a group homomorphism $\sigma : \mathcal{H}_{g,1} \to \operatorname{Sp}(2g,\mathbb{Z})$ and we denote its kernel by $\mathcal{IH}_{g,1}$.

4. Magnus Representation and H-torsion for Homology Cylinders

Here, we recall two kinds of invariants for homology cylinders from [5, 26], which are analogous to invariants for string links defined by Le Dimet [15] and Kirk-Livingston-Wang [13].

Since the group $H = H_1(\Sigma_{g,1})$ is free abelian, the group ring $\mathbb{Z}[H]$ is isomorphic to the Laurent polynomial ring of variables $\gamma_1, \gamma_2, \ldots, \gamma_{2g}$. We can embed $\mathbb{Z}[H]$ into the fractional field $\mathcal{K}_H := \mathbb{Z}[H](\mathbb{Z}[H] - \{0\})^{-1}$.

Let $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ be a homology cylinder. Since $H_1(M) \cong H_1(\Sigma_{g,1})$, the field $\mathcal{K}_{H_1(M)} := \mathbb{Z}[H_1(M)](\mathbb{Z}[H_1(M)] - \{0\})^{-1}$ is defined. We regard $\mathcal{K}_H$ and $\mathcal{K}_{H_1(M)}$ as local coefficient systems on $\Sigma_{g,1}$ and M respectively. By an argument using covering spaces, we have the following. We refer to [3, Proposition 2.10] and [13, Proposition 2.1] for the proof.

LEMMA 4.1. *$i_\pm : H_*(\Sigma_{g,1}, p; i_\pm^* \mathcal{K}_{H_1(M)}) \to H_*(M, p; \mathcal{K}_{H_1(M)})$ are isomorphisms of right $\mathcal{K}_{H_1(M)}$-vector spaces.*

This lemma plays an important role in defining our invariants below.

(I) Magnus representation

By using the spine R_{2g} taken in the previous section, we identify $\pi_1(\Sigma_{g,1}) = \langle \gamma_1, \ldots, \gamma_{2g} \rangle$ with a free group F_{2g} of rank $2g$. Since $R_{2g} \subset \Sigma_{g,1}$ is a deformation retract, we have

$$\begin{aligned} H_1(\Sigma_{g,1}, p\,; i_\pm^* \mathcal{K}_{H_1(M)}) &\cong H_1(R_{2g}, p\,; i_\pm^* \mathcal{K}_{H_1(M)}) \\ &= C_1(\widetilde{R_{2g}}) \otimes_{F_{2g}} i_\pm^* \mathcal{K}_{H_1(M)} \cong \mathcal{K}_{H_1(M)}^{2g} \end{aligned}$$

with a basis $\{\widetilde{\gamma_1} \otimes 1, \ldots, \widetilde{\gamma_{2g}} \otimes 1\} \subset C_1(\widetilde{R_{2g}}) \otimes_{F_{2g}} i_\pm^* \mathcal{K}_{H_1(M)}$ as a right free $\mathcal{K}_{H_1(M)}$-module, where $\widetilde{\gamma_i}$ is a lift of γ_i on the universal covering $\widetilde{R_{2g}}$. We denote by $\mathcal{K}_{H_1(M)}^{2g}$ the space of column vectors with $2g$ entries in $\mathcal{K}_{H_1(M)}$.

DEFINITION 4.2. (1) For $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$, we denote by $r'(M) \in \mathrm{GL}(2g, \mathcal{K}_{H_1(M)})$ the representation matrix of the right $\mathcal{K}_{H_1(M)}$-isomorphism

$$\mathcal{K}_{H_1(M)}^{2g} \cong H_1(\Sigma_{g,1}, p\,; i_-^* \mathcal{K}_{H_1(M)}) \xrightarrow[i_+^{-1} \circ i_-]{\cong} H_1(\Sigma_{g,1}, p\,; i_+^* \mathcal{K}_{H_1(M)}) \cong \mathcal{K}_{H_1(M)}^{2g}$$

(2) The *Magnus representation* for $\mathcal{C}_{g,1}$ is the map $r : \mathcal{C}_{g,1} \to \mathrm{GL}(2g, \mathcal{K}_H)$ which assigns to $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ the matrix $r(M) := {}^{i_+^{-1}} r'(M)$ obtained from $r'(M)$ by applying i_+^{-1} to each entry.

We call $r(M)$ the *Magnus matrix* for M. The map r has the following properties:

THEOREM 4.3 ([26, 25]). (1) (*Crossed homomorphism*) *For* $M_1, M_2 \in \mathcal{C}_{g,1}$, *we have*

$$r(M_1 \cdot M_2) = r(M_1) \cdot {}^{\sigma(M_1)} r(M_2).$$

In particular, the restriction of r *to* $\mathcal{IC}_{g,1}$ *is a homomorphism.*
(2) (*Symplecticity*) *For any* $M \in \mathcal{C}_{g,1}$, *we have the equality*

$$\overline{r(M)^T}\ \widetilde{J}\ r(M) = {}^{\sigma(M)} \widetilde{J},$$

where $\overline{r(M)^T}$ *is obtained from* $r(M)$ *by taking the transpose and applying the involution induced from the map* $(H \ni x \mapsto x^{-1} \in H)$ *to each entry, and* $\widetilde{J} \in \mathrm{GL}(2g, \mathbb{Z}[H])$ *is the matrix which appeared in Papakyriakopoulos' paper* [23]. *(The matrix* J *is mapped to the usual symplectic matrix by applying the trivializer* $\mathbb{Z}[H] \to \mathbb{Z}$ *to each entry.)*
(3) (*Homology cobordism invariance*) *The map* $r : \mathcal{C}_{g,1} \to \mathrm{GL}(2g, \mathcal{K}_H)$ *induces a crossed homomorphism* $r : \mathcal{H}_{g,1} \to \mathrm{GL}(2g, \mathcal{K}_H)$ *and its restriction to* $\mathcal{IH}_{g,1}$ *is a homomorphism.*

(II) H-torsion

Since the relative complex $C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})$ obtained from any smooth triangulation of $(M, i_+(\Sigma_{g,1}))$ is acyclic by Lemma 4.1, we can define its Reidemeister torsion

$$\tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})) \in \mathcal{K}_{H_1(M)}^{\times} / (\pm H_1(M)),$$

where $\mathcal{K}_{H_1(M)}^{\times} := \mathcal{K}_{H_1(M)} - \{0\}$ is the unit group of $\mathcal{K}_{H_1(M)}$. We refer to Milnor [21] and Turaev [29] for generalities of Reidemeister torsions.

DEFINITION 4.4. The *H-torsion* $\tau(H)$ of a homology cylinder $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$ is defined by

$$\tau(M) := {}^{i_+^{-1}}\tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})) \in \mathcal{K}_H^{\times}/(\pm H),$$

where $\mathcal{K}_H^{\times} = \mathcal{K}_H - \{0\}$ is the unit group of $\mathcal{K}_H$.

The map $\tau : \mathcal{C}_{g,1} \to \mathcal{K}_H^{\times}/(\pm H)$ has the following properties:

THEOREM 4.5. (1) (*Crossed homomorphism* [26]) *For $M_1, M_2 \in \mathcal{C}_{g,1}$, we have*

$$\tau(M_1 \cdot M_2) = \tau(M_1) \cdot {}^{\sigma(M_1)}\tau(M_2).$$

In particular, the restriction of τ to $\mathcal{IC}_{g,1}$ is a homomorphism.
(2) (Cha-Friedl-Kim [2, Theorem 3.10], Turaev [28, Theorem 1.11.2]) *If $M, N \in \mathcal{C}_{g,1}$ are homology cobordant, then there exists $q \in \mathcal{K}_H^{\times}$ such that*

$$\tau(M) = \tau(N) \cdot q \cdot \overline{q} \in \mathcal{K}_H^{\times}/(\pm H).$$

Note that the restriction of τ to $\mathcal{M}_{g,1}$ is trivial since $\Sigma_{g,1} \times [0,1]$ is simple homotopy equivalent to $\Sigma_{g,1} \times \{1\}$.

Explicit formulas for $r(M)$ and $\tau(M)$ are given in [5, Section 4], which are based on the formulas for the corresponding invariants of string links by Kirk-Livingston-Wang [13]. An *admissible presentation* of $\pi_1(M)$ is defined to be a presentation of the form

$$\langle i_-(\gamma_1), \dots, i_-(\gamma_{2g}), z_1, \dots, z_l, i_+(\gamma_1), \dots, i_+(\gamma_{2g}) \mid r_1, \dots, r_{2g+l} \rangle$$

for some integer $l \geq 0$. That is, it is a finite presentation with deficiency $2g$ whose generating set includes $i_-(\gamma_1), \dots, i_-(\gamma_{2g}), i_+(\gamma_1), \dots, i_+(\gamma_{2g})$ and is ordered as above. Such a presentation always exists. For any admissible presentation, we define $2g \times (2g+l)$, $l \times (2g+l)$ and $2g \times (2g+l)$ matrices A, B, C by

$$A = {}^{i_+^{-1}}\left(\overline{\left(\frac{\partial r_j}{\partial i_-(\gamma_i)}\right)}\right)_{\substack{1 \leq i \leq 2g \\ 1 \leq j \leq 2g+l}}, \quad B = {}^{i_+^{-1}}\left(\overline{\left(\frac{\partial r_j}{\partial z_i}\right)}\right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq 2g+l}},$$

$$C = {}^{i_+^{-1}}\left(\overline{\left(\frac{\partial r_j}{\partial i_+(\gamma_i)}\right)}\right)_{\substack{1\le i\le 2g\\ 1\le j\le 2g+l}}$$

over $\mathbb{Z}[H] \subset \mathcal{K}_H$.

PROPOSITION 4.6 ([5, Propositions 4.5, 4.6]). *For any homology cylinder $M = (M, i_+, i_-) \in \mathcal{C}_{g,1}$, the square matrix $\begin{pmatrix} A \\ B \end{pmatrix}$ is invertible over $\mathcal{K}_H$ and we have*

$$r(M) = -C \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} I_{2g} \\ 0_{(l,2g)} \end{pmatrix} \in \mathrm{GL}(2g, \mathcal{K}_H),$$
$$\tau(M) = \det \begin{pmatrix} A \\ B \end{pmatrix} \in \mathcal{K}_H^{\times}/(\pm H).$$

Example 4.7 ([26, Examples 4.4, 6.2]). Let L be the string link of 2 strings depicted in Figure 2. We can construct a homology cylinder $(M_L, i_+, i_-) \in \mathcal{C}_{2,1}$ as mentioned in Example 3.1.

An admissible presentation of $\pi_1(M_L)$ is given by

$$\left\langle \begin{array}{c} i_-(\gamma_1), \ldots, i_-(\gamma_4) \\ z \\ i_+(\gamma_1), \ldots, i_+(\gamma_4) \end{array} \middle| \begin{array}{l} i_+(\gamma_1)i_-(\gamma_3)^{-1}i_+(\gamma_4)i_-(\gamma_1)^{-1}, \\ [i_+(\gamma_1), i_+(\gamma_3)]i_+(\gamma_2)zi_-(\gamma_2)^{-1}[i_-(\gamma_3), i_-(\gamma_1)], \\ i_+(\gamma_4)i_-(\gamma_3)i_+(\gamma_4)^{-1}z^{-1}, \\ i_-(\gamma_3)i_+(\gamma_3)^{-1}i_-(\gamma_3)^{-1}z,\ i_-(\gamma_4)z^{-1}i_+(\gamma_4)^{-1}z \end{array} \right\rangle.$$

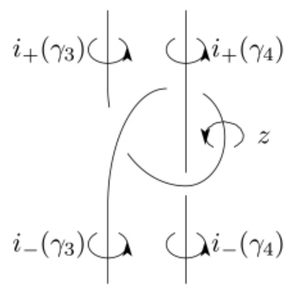

Fig. 2. The string link L.

By using Proposition 4.6, we have

$$r(M_L) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-\gamma_1^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_2^{-1}\gamma_3^{-1}\gamma_4^{-1}-\gamma_4^{-1}+1}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_3^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_4^{-1}(\gamma_4^{-1}-1)}{\gamma_3^{-1}+\gamma_4^{-1}-1} \\ \frac{\gamma_1^{-1}\gamma_3\gamma_4^{-1}}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{(1-\gamma_3^{-1})(\gamma_2^{-1}\gamma_3^{-1}-\gamma_2^{-1}-1)}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{\gamma_3^{-1}-1}{\gamma_3^{-1}+\gamma_4^{-1}-1} & \frac{-\gamma_3^{-1}\gamma_4^{-1}+\gamma_3^{-1}+2\gamma_4^{-1}-1}{\gamma_3^{-1}+\gamma_4^{-1}-1} \end{pmatrix},$$

$$\tau(M_L) = -1 + \gamma_3 - \gamma_3\gamma_4^{-1} = -\gamma_3(\gamma_3^{-1} + \gamma_4^{-1} - 1).$$

Note that

$$\det(r(M_L)) = \gamma_3^{-1}\gamma_4^{-1}\frac{\gamma_3 + \gamma_4 - 1}{\gamma_3^{-1} + \gamma_4^{-1} - 1}.$$

5. Abelian Quotients

In this section, we discuss abelian quotients of $\mathcal{C}_{g,1}$ and $\mathcal{H}_{g,1}$ by comparing them to the corresponding result for $\mathcal{M}_{g,1}$. First, as commented in [6], we point out that $\mathcal{C}_{g,1}$ has the monoid $\theta^3_{\mathbb{Z}}$ of homology 3-spheres as a big abelian quotient. In fact, we have a *forgetful* homomorphism $F : \mathcal{C}_{g,1} \to \theta^3_{\mathbb{Z}}$ defined by $F(M, i_+, i_-) = S^3 \sharp X_1 \sharp X_2 \sharp \cdots \sharp X_n$ for the prime decomposition $M = M_0 \sharp X_1 \sharp X_2 \sharp \cdots \sharp X_n$ of M where M_0 is the unique factor having non-empty boundary and $X_i \in \theta^3_{\mathbb{Z}}$ $(1 \leq i \leq n)$. The map F owes its well-definedness to the uniqueness of the prime decomposition of 3-manifolds and it is a monoid epimorphism.

The underlying 3-manifolds of homology cylinders obtained from $\mathcal{M}_{g,1}$ are all $\Sigma_{g,1} \times [0, 1]$ and, in particular, irreducible. Therefore it seems more reasonable to compare $\mathcal{M}_{g,1}$ with the submonoid $\mathcal{C}^{\mathrm{irr}}_{g,1}$ of $\mathcal{C}_{g,1}$ consisting of all (M, i_+, i_-) with M irreducible.

In contrast with the fact that $\mathcal{M}_{g,1}$ is a perfect group for $g \geq 3$ (see Harer [11]), many infinitely generated abelian quotients for monoids and homology cobordism groups of irreducible homology cylinders have been found until now. For example, we have the following results:

- In [26, Corollary 6.16], we showed that the submonoids $\mathcal{C}^{\mathrm{irr}}_{g,1} \cap \mathcal{IC}_{g,1}$ and $\mathrm{Ker}\,(\mathcal{C}^{\mathrm{irr}}_{g,1} \to \mathcal{H}_{g,1})$ have abelian quotients isomorphic to $(\mathbb{Z}_{\geq 0})^\infty$. The proof uses the H-torsion τ and its non-commutative generalization.

- Morita [22, Corollary 5.2] used what is called the trace maps to show that the group $\mathcal{IH}_{g,1}$, which coincides with the quotient of $\mathcal{C}_{g,1}^{\mathrm{irr}} \cap \mathcal{IC}_{g,1}$ by homology cobordisms, has an abelian quotient isomorphic to $\mathbb{Z}^\infty$.
- In a joint work with Goda in [6, Theorem 2.6], we showed that $\mathcal{C}_{g,1}^{\mathrm{irr}}$ has an abelian quotient isomorphic to $(\mathbb{Z}_{\geq 0})^\infty$ by using sutured Floer homology (a variant of Heegaard Floer homology). However, the projection map to this abelian quotient does not factor through $\mathcal{H}_{g,1}$.

By taking into account the similarity between the two groups $\mathcal{M}_{g,1}$ and $\mathcal{H}_{g,1}$, it had been conjectured that $\mathcal{H}_{g,1}$ was perfect. However, Cha-Friedl-Kim [2] found a method for extracting homology cobordism invariants of homology cylinders from the H-torsion $\tau : \mathcal{C}_{g,1} \to \mathcal{K}_H^\times/(\pm H)$, which is a crossed homomorphism, as follows.

First they consider the subgroup $A \subset \mathcal{K}_H^\times$ defined by

$$A := \{f^{-1} \cdot \varphi(f) \mid f \in \mathcal{K}_H^\times,\ \varphi \in \mathrm{Sp}(2g,\mathbb{Z})\},$$

by which we can obtain a *homomorphism*

$$\tau : \mathcal{C}_{g,1} \longrightarrow \mathcal{K}_H^\times/(\pm H \cdot A).$$

Note that $f = \overline{f}$ holds in $\mathcal{K}_H^\times/(\pm H \cdot A)$ since $-I_{2g} \in \mathrm{Sp}(2g,\mathbb{Z})$. Second, they use the equality mentioned in Theorem 4.5 (2). Namely, if we put

$$N := \{f \cdot \overline{f} \mid f \in \mathcal{K}_H^\times\},$$

then we obtain a homomorphism

$$\widetilde{\tau} : \mathcal{H}_{g,1} \longrightarrow \mathcal{K}_H^\times/(\pm H \cdot A \cdot N).$$

Note that $f^2 = f\overline{f} = 1$ holds for any $f \in \mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$.

The structure of $\mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$ is given as follows. Recall that $\mathcal{K}_H = \mathbb{Z}[H](\mathbb{Z}[H] - \{0\})^{-1}$. The ring $\mathbb{Z}[H]$ is a Laurent polynomial ring of $2g$ variables and it is a unique factorization domain. Thus every Laurent polynomial f is factorized into irreducible polynomials uniquely up to multiplication by a unit in $\mathbb{Z}[H]$. In particular, for every irreducible polynomial $\lambda \in \mathbb{Z}[H]$, we can count the exponent of λ in the factorization of f. This counting naturally extends to that for elements in $\mathcal{K}_H^\times$ by using negative numbers for denominators. Under the identification by $\pm H \cdot A \cdot N$, an element in $\mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$ is determined by the exponents of all $\mathrm{Sp}(2g,\mathbb{Z})$-orbits of irreducible polynomials (up to multiplication by a unit in $\mathbb{Z}[H]$) modulo 2. Note that the action of $\mathrm{Sp}(2g,\mathbb{Z})$ keeps the irreducibility of a polynomial in $\mathbb{Z}[H]$ and the number of its monomials unchanged. Therefore $\mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\infty$. Finally by using infinitely many $(\mathbb{Z}/2\mathbb{Z})$-torsion elements of the knot concordance group, they show the following:

THEOREM 5.1 (Cha-Friedl-Kim [2]). *The image of the homomorphism*

$$\widetilde{\tau} : \mathcal{H}_{g,1} \longrightarrow \mathcal{K}_H^\times/(\pm H \cdot A \cdot N)$$

is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\infty$ *and it splits.*

REMARK 5.2. In [2], Cha-Friedl-Kim further applied the above method to homology cylinders over a compact oriented surface $\Sigma_{g,n}$ of genus g with $n \geq 2$ boundaries and showed that the abelianization of the homology cobordism group $\mathcal{H}(\Sigma_{g,n})$ has infinite rank.

Now we try to investigate abelian quotients of $\mathcal{H}_{g,1}$ by using the Magnus representation r. It looks easier to extract information of $\mathcal{H}_{g,1}$ from the representation r together with Cha-Friedl-Kim's idea, since r itself is an homology cobordism invariant as mentioned in Theorem 4.3 (3). Consider two maps

$$\widehat{r} : \mathcal{H}_{g,1} \xrightarrow{r} \mathrm{GL}(2g, \mathcal{K}_H) \xrightarrow{\det} \mathcal{K}_H^\times \longrightarrow \mathcal{K}_H^\times/(\pm H),$$
$$\widetilde{r} : \mathcal{H}_{g,1} \xrightarrow{\widehat{r}} \mathcal{K}_H^\times/(\pm H) \longrightarrow \mathcal{K}_H^\times/(\pm H \cdot A).$$

While $\widehat{r}$ is a crossed homomorphism, its restriction to $\mathcal{IH}_{g,1}$ and $\widetilde{r}$ are homomorphisms. Note that both $\mathcal{K}_H^\times/(\pm H)$ and $\mathcal{K}_H^\times/(\pm H \cdot A)$ are isomorphic to $\mathbb{Z}^\infty$.

THEOREM 5.3. (1) *For* $(M, i_+, i_-) \in \mathcal{C}_{g,1}$*, the equality*

$$\widehat{r}(M) = \overline{\tau(M)} \cdot (\tau(M))^{-1} \quad \in \mathcal{K}_H^\times/(\pm H)$$

holds.
(2) *For* $g \geq 1$*, the homomorphism* $\widetilde{r} : \mathcal{H}_{g,1} \to \mathcal{K}_H^\times/(\pm H \cdot A)$ *is trivial.*

(3) *For $g \geq 2$, the image of the homomorphism $\widehat{r}|_{\mathcal{IH}_{g,1}} : \mathcal{IH}_{g,1} \to \mathcal{K}_H^{\times}/(\pm H)$ is isomorphic to $\mathbb{Z}^{\infty}$.*

PROOF. (1) Let

$$\langle i_-(\gamma_1), \dots, i_-(\gamma_{2g}), z_1, \dots, z_l, i_+(\gamma_1), \dots, i_+(\gamma_{2g}) \mid r_1, \dots, r_{2g+l} \rangle$$

be an admissible presentation of $\pi_1(M)$. We calculate the matrices A, B, C as in the previous section. By the formula in Proposition 4.6, we have an equality

$$\begin{pmatrix} r(M) & Z \end{pmatrix} = -C \begin{pmatrix} A \\ B \end{pmatrix}^{-1}$$

for some $2g \times l$ matrix Z. It follows that $r(M)A = -ZB - C$. By taking the determinant of

$$\begin{pmatrix} r(M) & 0_{(2g,l)} \\ 0_{(l,2g)} & I_l \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} r(M)A \\ B \end{pmatrix} = \begin{pmatrix} -ZB - C \\ B \end{pmatrix},$$

we have the equality

$$\det(r(M)) \det \begin{pmatrix} A \\ B \end{pmatrix} = \det \begin{pmatrix} -ZB - C \\ B \end{pmatrix} = \det \begin{pmatrix} -C \\ B \end{pmatrix} = \det \begin{pmatrix} C \\ B \end{pmatrix}.$$

Again by Proposition 4.6, we have $\det \begin{pmatrix} A \\ B \end{pmatrix} = \tau(M)$ and it is easy to see that

$$\det \begin{pmatrix} C \\ B \end{pmatrix} = {}^{\sigma(M)}\tau(M^{-1}),$$

where $M^{-1} = (-M, i_-, i_+) \in \mathcal{C}_{g,1}$. Recall that $\tau(M)$ is the pullback of the torsion of the complex $C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})$ by i_+, and $\tau(M^{-1})$ is that of $C_*(M, i_-(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})$ by i_-. These complexes are related by the Poincaré duality. By the duality of Reidemeister torsions (see Milnor [21] or Turaev[29]), we have

$$\begin{aligned} {}^{i_-}\tau(M^{-1}) &= \tau(C_*(M, i_-(\Sigma_{g,1}); \mathcal{K}_{H_1(M)})) \\ &= \overline{\tau(C_*(M, i_+(\Sigma_{g,1}); \mathcal{K}_{H_1(M)}))} = \overline{{}^{i_+}\tau(M)} \in \mathcal{K}_{H_1(M)}/(\pm H_1(M)). \end{aligned}$$

Therefore we have ${}^{\sigma(M)}\tau(M^{-1}) = \overline{\tau(M)}$. Our claim follows from this.

(2) As mentioned above, the action of $\mathrm{Sp}(2g,\mathbb{Z})$ implies that $f = \overline{f}$ for any $f \in \mathcal{K}_H^\times/(\pm H \cdot A)$. Then the claim immediately follows from (1). (We may also use the symplecticity of the image of r mentioned in Theorem 4.3 (2).)

(3) We use the homology cylinder $M_L \in \mathcal{C}_{2,1}$ in Example 4.7. While $M_L \notin \mathcal{IC}_{2,1}$, we can adjust it by some $g_1 \in \mathcal{M}_{2,1}$ so that $M_L \cdot g_1 \in \mathcal{IC}_{2,1}$. Since $\widehat{r}$ is trivial on $\mathcal{M}_{2,1}$, we have

$$\widehat{r}(M_L \cdot g_1) = \widehat{r}(M_L) = \frac{\gamma_3 + \gamma_4 - 1}{\gamma_3^{-1} + \gamma_4^{-1} - 1} \in \mathcal{K}_H^\times/(\pm H).$$

Take $f \in \mathcal{M}_{2,1}$ such that $\sigma(f) \in \mathrm{Sp}(4,\mathbb{Z})$ maps

$$\gamma_1 \longmapsto \gamma_1 + \gamma_4, \quad \gamma_2 \longmapsto \gamma_2, \quad \gamma_3 \longmapsto \gamma_2 + \gamma_3, \quad \gamma_4 \longmapsto \gamma_4.$$

Consider $f^m \cdot M_L \in \mathcal{C}_{2,1}$ and adjust it by some $g_m \in \mathcal{M}_{2,1}$ so that $f^m \cdot M_L \cdot g_m \in \mathcal{IC}_{2,1}$. Then we have

$$\widehat{r}(f^m \cdot M_L \cdot g_m) = {}^{\sigma(f^m)}\widehat{r}(M_L) = \frac{\gamma_2^m\gamma_3 + \gamma_4 - 1}{\gamma_2^{-m}\gamma_3^{-1} + \gamma_4^{-1} - 1} \in \mathcal{K}_H^\times/(\pm H).$$

Since $\gamma_2^m\gamma_3 + \gamma_4 - 1$ is a degree 1 polynomial with respect to the variable γ_3 and the coefficient of γ_3 is a monomial, we see that it is irreducible. By applying the involution, the irreducibility of $\gamma_2^{-m}\gamma_3^{-1} + \gamma_4^{-1} - 1$ follows. It is easily checked that

$$\begin{aligned} &\gamma_2^m\gamma_3 + \gamma_4 - 1 \neq \gamma_2^{-m}\gamma_3^{-1} + \gamma_4^{-1} - 1 \\ &\gamma_2^m\gamma_3 + \gamma_4 - 1 \neq \gamma_2^k\gamma_3 + \gamma_4 - 1 \quad (m \neq k) \end{aligned}$$

as elements of $\mathcal{K}_H^\times/(\pm H)$ by considering the ratios among monomials, which are invariant under the multiplication of any element of $\pm H$. Therefore we conclude that the values

$$\left\{ \frac{\gamma_2^m\gamma_3 + \gamma_4 - 1}{\gamma_2^{-m}\gamma_3^{-1} + \gamma_4^{-1} - 1} \right\}_{m=0}^{\infty}$$

generate an infinitely generated subgroup of $\mathcal{K}_H^\times/(\pm H)$. This completes the proof when $g = 2$. We can use the above computation for $g \geq 3$ by a stabilization. □

From the above theorem, we observe that it seems not easy to find new abelian quotients of $\mathcal{H}_{g,1}$ by using the Magnus representation.

6. Generalization to Higher-Dimensional Cases

In the remaining sections, we apply the argument in the previous section to higher dimensional cases and see that the determinant of the Magnus representation works well for them.

For $k \geq 2$ and $n \geq 1$, we put

$$X_n^k := \mathop{\#}_{n}(S^1 \times S^{k-1}).$$

The manifold X_n^k may be regarded as a generalization of a closed surface since $X_n^2 = \Sigma_{n,0}$.

Suppose $k \geq 3$. Then we have $\pi_1(X_n^k) \cong \pi_1(X_n^k - \operatorname{Int} D^k) \cong F_n$, where $\operatorname{Int} D^k$ is an open k-ball. We choose a base point p of $X_n^k - \operatorname{Int} D^k$ from the boundary and take an ordered basis $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of F_n (and $H_1 := H_1(F_n) \cong \mathbb{Z}^n$).

Similarly to Lemma 4.1, we can check that

$$i_\pm : H_*(X_n^k - \operatorname{Int} D^k, p; i_\pm^* \mathcal{K}_{H_1(M)}) \to H_*(M, p; \mathcal{K}_{H_1(M)})$$

are isomorphisms of right $\mathcal{K}_{H_1(M)}$-vector spaces for any homology cylinder (M, i_+, i_-) over $X_n^k - \operatorname{Int} D^k$, where $\mathcal{K}_{H_1} = \mathbb{Z}[H_1](\mathbb{Z}[H_1] - \{0\})^{-1}$. Hence we can define the Magnus representation

$$r : \mathcal{C}(X_n^k - \operatorname{Int} D^k) \longrightarrow \mathrm{GL}(n, \mathcal{K}_{H_1})$$

by the same procedure as before. The map r is a crossed homomorphism and induces $r : \mathcal{H}(X_n^k - \operatorname{Int} D^k) \to \mathrm{GL}(n, \mathcal{K}_{H_1})$. Consider the composition

$$\widetilde{r} : \mathcal{H}(X_n^k - \operatorname{Int} D^k) \xrightarrow{r} \mathrm{GL}(n, \mathcal{K}_{H_1}) \xrightarrow{\det} \mathcal{K}_{H_1}^\times \longrightarrow \mathcal{K}_{H_1}^\times / (\pm H_1 \cdot A') \cong \mathbb{Z}^\infty,$$

where $A' := \{f^{-1} \cdot \varphi(f) \mid f \in \mathcal{K}_{H_1}^\times,\ \varphi \in \mathrm{Aut}(H_1)\}$, which gives a homomorphism.

Now we mention the main result in the remaining sections. Note that we have a surjective homomorphism $\mathcal{H}(X_n^k - \operatorname{Int} D^k) \twoheadrightarrow \mathcal{H}(X_n^k)$ by gluing a small trivial cylinder along the boundary, which corresponds to capping the boundary of $X_n^k - \operatorname{Int} D^k$ by a k-ball D^k.

THEOREM 6.1. *For any $k \geq 3$ and $n \geq 2$, we have:*

(1) *The image of the homomorphism* $\widetilde{r}$ *is an infinitely generated subgroup of* $\mathbb{Z}^\infty$. *In particular,* $H_1(\mathcal{H}(X_n^k - \operatorname{Int} D^k))$ *has infinite rank.*

(2) *The homomorphism* $\widetilde{r}$ *factors through* $\mathcal{H}(X_n^k)$. *Therefore* $H_1(\mathcal{H}(X_n^k))$ *has infinite rank.*

For the proof, which occupies Sections 7 and 8, we use the action of $\mathcal{H}(X_n^k - \operatorname{Int} D^k)$ on the group called the *acyclic closure* of F_n. This action may be regarded as a generalization of the action of the diffeotopy group $\mathcal{M}(X_n^k - \operatorname{Int} D^k)$ on $F_n = \pi_1(X_n^k - \operatorname{Int} D^k)$. Recall that the Magnus representation was originally defined for automorphisms of F_n by using the Fox derivatives. The Magnus representation for homology cylinders we have seen is an extension of this representation by using twisted homology. In the following sections, we describe an equivalent definition of our Magnus representation by using the extended Fox derivatives first given by Le Dimet [15].

7. The Acyclic Closure of a Group

The notion of the acyclic closure (or HE-closure in [17]) of a group was defined as a variation of the algebraic closure of a group by Levine [16, 17]. We summarize here the definition and fundamental properties. We also refer to Hillman's book [12] and Cha's paper [1].

Definition 7.1. Let G be a group, and let $F_m = \langle x_1, x_2, \dots, x_m \rangle$ be a free group of rank m.
(i) $w = w(x_1, x_2, \dots, x_m) \in G * F_m$, a word in $x_1, x_2, \dots, x_m$ and elements of G, is said to be *acyclic* if

$$w \in \operatorname{Ker}\left(G * F_m \xrightarrow{\text{proj}} F_m \longrightarrow H_1(F_m)\right).$$

(ii) Consider the following "equation" with variables $x_1, x_2, \dots, x_m$:

$$\begin{cases} x_1 &= w_1(x_1, x_2, \dots, x_m) \\ x_2 &= w_2(x_1, x_2, \dots, x_m) \\ &\vdots \\ x_m &= w_m(x_1, x_2, \dots, x_m) \end{cases}.$$

When all words $w_1, w_2, \ldots, w_m \in G * F_m$ are acyclic, we call such an equation an *acyclic system* over G.
(iii) A group G is said to be *acyclically closed* (AC, for short) if every acyclic system over G with m variables has a unique "solution" in G for any $m \geq 0$, where a "solution" means a homomorphism φ that makes the diagram

$$\begin{array}{ccc} G & & \\ \downarrow & \overset{\mathrm{id}}{\searrow} & \\ \dfrac{G * F_m}{\langle\langle x_1 w_1^{-1}, \ldots, x_m w_m^{-1} \rangle\rangle} & \xrightarrow[\varphi]{} & G \end{array}$$

commutative, where $\langle\langle x_1 w_1^{-1}, \ldots, x_m w_m^{-1} \rangle\rangle$ denotes the normal closure in $G * F_m$.

Example 7.2. Let G be an abelian group. For $g_1, g_2, g_3 \in G$, consider the equation

$$\begin{cases} x_1 = g_1 x_1 g_2 x_2 x_1^{-1} x_2^{-1} \\ x_2 = x_1 g_3 x_1^{-1} \end{cases},$$

which is an acyclic system. Then we have a unique solution $x_1 = g_1 g_2$, $x_2 = g_3$.

As we see from this example, all abelian groups are AC. Moreover, it is shown in [16, Proposition 1] that AC groups are closed under taking intersections, direct products, central extensions, direct limits and inverse limits. In particular, all nilpotent groups are AC.

Let us define the acyclic closure of a group.

PROPOSITION 7.3 ([16, Proposition 3]). *For any group G, there exists a pair of a group G^{acy} and a homomorphism $\iota_G : G \to G^{\mathrm{acy}}$ satisfying the following properties:*

(1) *G^{acy} is an AC-group.*

(2) *Let $f : G \to A$ be a homomorphism and suppose that A is an AC-group. Then there exists a unique homomorphism $f^{\mathrm{acy}} : G^{\mathrm{acy}} \to A$ which satisfies $f^{\mathrm{acy}} \circ \iota_G = f$.*

Moreover such a pair is unique up to isomorphism.

DEFINITION 7.4. We call ι_G (or G^{acy}) obtained above the *acyclic closure* of G.

Taking the acyclic closure of a group is functorial, namely, for each group homomorphism $f : G_1 \to G_2$, we have the induced homomorphism $f^{\mathrm{acy}} : G_1^{\mathrm{acy}} \to G_2^{\mathrm{acy}}$ by applying the universal property of G_1^{acy} to the homomorphism $\iota_{G_2} \circ f$, and the composition of homomorphisms induces that of the corresponding homomorphisms on acyclic closures.

The most important properties of the acyclic closure are the following, where a homomorphism is said to be 2-*connected* if it induces an isomorphism on the first (group) homology and an epimorphism on the second homology.

PROPOSITION 7.5 ([16, Proposition 4]). *For any group G, the acyclic closure $\iota_G : G \to G^{\mathrm{acy}}$ is 2-connected.*

PROPOSITION 7.6 ([16, Proposition 5]). *Let G_1 be a finitely generated group and G_2 be a finitely presentable group. For each 2-connected homomorphism $f : G_1 \to G_2$, the induced homomorphism $f^{\mathrm{acy}} : G_1^{\mathrm{acy}} \to G_2^{\mathrm{acy}}$ on acyclic closures is an isomorphism.*

From Proposition 7.5 and Stallings' theorem [27], the nilpotent quotients of a group and those of its acyclic closure are isomorphic. Note that the homomorphism ι_G is not necessarily injective: consider a perfect group G and the 2-connected homomorphism $G \to \{1\}$. As for a free group F_n, its residual nilpotency shows that $\iota_{F_n} : F_n \to F_n^{\mathrm{acy}}$ is injective. We write $\gamma_i \in F_n^{\mathrm{acy}}$ again for the image of $\gamma_i \in F_n$ by $\iota_{F_n} : F_n = \langle \gamma_1, \gamma_2, \ldots, \gamma_n \rangle \hookrightarrow F_n^{\mathrm{acy}}$.

Now we return to our discussion on homology cylinders. For each homology cylinder $(M, i_+, i_-) \in \mathcal{C}(X_n^k - \mathrm{Int}\, D^k)$, the homomorphisms $i_\pm : F_n = \pi_1(X_n^k - \mathrm{Int}\, D^k) \to \pi_1(M)$ are 2-connected. Hence we have a commutative diagram

$$\begin{array}{ccccc}
F_n & \xrightarrow{i_-} & \pi_1(M) & \xleftarrow{i_+} & F_n \\
{\scriptstyle \iota_{F_n}}\downarrow & & {\scriptstyle \iota_{\pi_1(M)}}\downarrow & & \downarrow{\scriptstyle \iota_{F_n}} \\
F_n^{\mathrm{acy}} & \underset{\cong}{\xrightarrow{i_-^{\mathrm{acy}}}} & \pi_1(M)^{\mathrm{acy}} & \underset{\cong}{\xleftarrow{i_+^{\mathrm{acy}}}} & F_n^{\mathrm{acy}}
\end{array}$$

by Proposition 7.6. From this, we obtain a monoid homomorphism

$$\mathrm{Acy} : \mathcal{C}(X_n^k - \mathrm{Int}\, D^k) \longrightarrow \mathrm{Aut}\,(F_n^{\mathrm{acy}})$$

defined by $\mathrm{Acy}(M, i_+, i_-) = (i_+^{\mathrm{acy}})^{-1} \circ i_-^{\mathrm{acy}}$ and we can check that it induces a group homomorphism

$$\mathrm{Acy} : \mathcal{H}(X_n^k - \mathrm{Int}\, D^k) \longrightarrow \mathrm{Aut}\,(F_n^{\mathrm{acy}}).$$

For homology cylinders over the closed manifold X_n^k, we have similar homomorphisms

$$\mathrm{Acy} : \mathcal{C}(X_n^k) \longrightarrow \mathrm{Out}\,(F_n^{\mathrm{acy}}),$$
$$\mathrm{Acy} : \mathcal{H}(X_n^k) \longrightarrow \mathrm{Out}\,(F_n^{\mathrm{acy}})$$

using the outer automorphism group $\mathrm{Out}\,(F_n^{\mathrm{acy}}) := \mathrm{Aut}\,(F_n^{\mathrm{acy}})/\mathrm{Inn}\,(F_n^{\mathrm{acy}})$ of F_n^{acy}.

Theorem 7.7. *For any $k \geq 3$ and $n \geq 2$, the homomorphisms*

$$\mathrm{Acy} : \mathcal{H}(X_n^k - \mathrm{Int}\, D^k) \longrightarrow \mathrm{Aut}\,(F_n^{\mathrm{acy}}),$$
$$\mathrm{Acy} : \mathcal{H}(X_n^k) \longrightarrow \mathrm{Out}\,(F_n^{\mathrm{acy}})$$

are surjective.

Proof. It suffices to show the surjectivity of the upper one. Given an element $\varphi \in \mathrm{Aut}(F_n^{\mathrm{acy}})$, we produce a homology cylinder $M = (M, i_+, i_-) \in \mathcal{H}(X_n^k - \mathrm{Int}\, D^k)$ satisfying $\mathrm{Acy}(M) = \varphi$. The construction below is based on the argument in Garoufalidis-Levine [4, Theorem 3] and its generalization in [24, Theorem 6.1].

First we take two continuous maps $f_+, f_- : X_n^k - \mathrm{Int}\, D^k \to K(F_n^{\mathrm{acy}}, 1)$ corresponding to homomorphisms $\iota_{F_n}, \varphi \circ \iota_{F_n} : F_n \to F_n^{\mathrm{acy}}$, respectively. Since $\partial(X_n^k - \mathrm{Int}\, D^k) = S^{k-1}$ is simply connected, we can combine these maps and obtain a map $f := f_+ \cup f_- : X_{2n}^k = (X_n^k - \mathrm{Int}\, D^k) \cup (-(X_n^k - \mathrm{Int}\, D^k)) \to K(F_n^{\mathrm{acy}}, 1)$. Let $i_+, i_- : X_n^k - \mathrm{Int}\, D^k \to X_{2n}^k$ be the corresponding embeddings onto the domains of f_+ and f_-. The manifold X_{2n}^k is the boundary of $M_0 := \natural_{2n}\, S^1 \times D^k$, the boundary connected sum of $2n$ copies of $S^1 \times D^k$. Since $\pi_1(M_0) \cong \pi_1(X_{2n}^k)$, we can extend f to the continuous map $\Phi : M_0 \to K(F_n^{\mathrm{acy}}, 1)$. Note that $H_1(M_0) \cong \mathbb{Z}^{2n}$ and $H_i(M_0) = 0$ for all $i \geq 2$.

Since $f \circ i_+ = \iota_{F_n} : F_n \to F_n^{\mathrm{acy}}$ induces an isomorphism on the first homology, we have $H_1(M_0) \cong i_+(H_1(X_n^k - \mathrm{Int}\, D^k)) \oplus \mathrm{Ker}\, \Phi$. To obtain a homology cylinder satisfying $\mathrm{Acy}(M) = \varphi$, we perform surgery to M_0 to kill $\mathrm{Ker}\, \Phi \cong \mathbb{Z}^n$ with keeping Φ on $X_{2n}^k = \partial M_0$. Take an element $\alpha \in H_1(M_0)$ from a basis of $\mathrm{Ker}\, \Phi$.

(Case 1) Suppose there exists a representative $C \in \pi_1(M_0)$ of α by a simple closed curve with $\Phi(C) = 1 \in F_n^{\mathrm{acy}}$. Let W_1 be the $(k+2)$-manifold obtained from $M_0 \times [0,1]$ by attaching a 2-handle $S^1 \times D^{k+1}$ to $M_0 \times \{1\} \subset \partial(M_0 \times [0,1])$ with any framing. We have

$$\pi_1(W_1) = \pi_1(M_0)/\langle\langle C \rangle\rangle,$$

where $\langle\langle C \rangle\rangle$ denotes the normal closure of the subgroup generated by C. The relative chain complex $C_*(W_1, M_0 \times [0,1])$ associated with the handle decomposition has only one generator in degree 2 and its homology class hits $\alpha \in H_1(M_0 \times [0,1]) \cong H_1(M_0)$. Therefore $H_1(W_1) \cong H_1(M_0)/\langle \alpha \rangle \cong \mathbb{Z}^{2n-1}$ and $H_i(W_1) \cong H_i(M_0)$ if $i \neq 1$. Since $\Phi(C) = 1$, we can extend Φ to W_1. We write $\Phi : W_1 \to K(F_n^{\mathrm{acy}}, 1)$ again for the extension.

Consider W_1 to be a cobordism between $M_0 = M_0 \times \{0\}$ and a new manifold M_1. That is, $\partial W_1 = M_0 \cup (-M_1)$. By duality, the cobordism W_1 is obtained from $M_1 \times [0,1]$ by attaching a k-handle. Since $k \geq 3$, it follows that $H_1(M_1) \cong H_1(W_1)$ with $\mathrm{Ker}\, \Phi|_{M_1} \cong \mathbb{Z}^{n-1}$.

(Case 2) Suppose there does not exist a representative $C \in \pi_1(M_0)$ of α by a simple closed curve with $\Phi(C) = 1 \in F_n^{\mathrm{acy}}$. In this case, we replace (M_0, i_+, i_-) by another manifold $(M_{0.5}, i_+, i_-)$ which is homology bordant to M_0 over $K(F_n^{\mathrm{acy}}, 1)$ and for which we can take a simple closed curve representing $\alpha \in H_1(M_{0.5}) \cong H_1(M_0)$ and its image by Φ is trivial in F_n^{acy}. Then we can apply the same argument as Case 1 to $M_{0.5}$.

Such a manifold $M_{0.5}$ is given as follows. The homomorphism $i_+ : F_n \to \pi_1(M_0)$ induces a homomorphism $i_+^{\mathrm{acy}} : F_n^{\mathrm{acy}} \to \pi_1(M_0)^{\mathrm{acy}}$ satisfying $i_+^{\mathrm{acy}} \circ \iota_{F_n} = \iota_{\pi_1(M_0)} \circ i_+$. Similarly we have $\Phi^{\mathrm{acy}} : \pi_1(M_0)^{\mathrm{acy}} \to F_n^{\mathrm{acy}}$ satisfying $\Phi = \Phi^{\mathrm{acy}} \circ \iota_{\pi_1(M_0)}$. Then

$$\Phi^{\mathrm{acy}} \circ i_+^{\mathrm{acy}} \circ \iota_{F_n} = \Phi^{\mathrm{acy}} \circ \iota_{\pi_1(M_0)} \circ i_+ = \iota_{F_n}.$$

By the universality of the acyclic closure, we have $\Phi^{\mathrm{acy}} \circ i_+^{\mathrm{acy}} = \mathrm{id}_{F_n^{\mathrm{acy}}}$. In particular, Φ^{acy} is onto.

Take a simple closed curve C representing $\alpha \in \operatorname{Ker}\Phi$. Since $\Phi(\alpha) = 0 \in H_1(F_n^{\mathrm{acy}})$, we can write $\Phi(C) = \prod_{i=1}^{l}[h_{i1}, h_{i2}]$ with $h_{ij} \in F_n^{\mathrm{acy}}$. We take an acyclic system

$$S \,:\, x_i = w_i(x_1, x_2, \ldots, x_m) \quad (i = 1, 2, \ldots, m)$$

over $\pi_1(M_0)$ whose solution in $\pi_1(M_0)^{\mathrm{acy}}$ includes

$$\{i_+^{\mathrm{acy}}(h_{11}), i_+^{\mathrm{acy}}(h_{12}), \ldots, i_+^{\mathrm{acy}}(h_{l1}), i_+^{\mathrm{acy}}(h_{l2})\}.$$

We attach a 1-handle to $M_0 \times \{1\} \subset \partial(M_0 \times [0,1])$ for each variable x_i and write x_i again for the added generator on the fundamental group of the resulting cobordism. We also attach a 2-handle along the loop $x_i w_i^{-1}$ for each $i = 1, 2, \ldots, m$ with any framing. We denote the resulting cobordism by $W_{0.5}$. Then

$$\pi_1(W_{0.5}) = (\pi_1(M) * \langle x_1, x_2, \ldots, x_m\rangle)/\langle\langle x_1 w_1^{-1}, x_2 w_2^{-1}, \ldots, x_m w_m^{-1}\rangle\rangle.$$

We define a homomorphism $\Phi_S : \pi_1(W_{0.5}) \to \pi_1(M_0)^{\mathrm{acy}}$ which lifts $\iota_{\pi_1(M_0)}$ by sending x_i to the corresponding solution of S. The composite $\Phi^{\mathrm{acy}} \circ \Phi_S : \pi_1(W_{0.5}) \to F_n^{\mathrm{acy}}$ induces a continuous map $\Phi : W_{0.5} \to K(F_n^{\mathrm{acy}}, 1)$ which extends $\Phi : M_0 \to K(F_n^{\mathrm{acy}}, 1)$.

The relative chain complex $C_*(W_{0.5}, M_0 \times [0,1])$ given by the handle decomposition has its non-trivial part in degree 1 and 2 generated by the above newly added handles. The acyclicity of the system S says that the boundary of the 2-handle associated with the relation $x_i w_i^{-1}$ is of the form

$$[x_i] + (\text{1-handles in } M_0 \times [0,1]) \equiv [x_i] \in C_1(W_{0.5}, M_0 \times [0,1]).$$

Therefore $H_*(W_{0.5}, M_0 \times [0,1]) \cong H_*(W_{0.5}, M_0) = 0$ holds.

Consider $W_{0.5}$ to be a cobordism between $M_0 = M_0 \times \{0\}$ and a new manifold $M_{0.5}$. The dual handle decomposition of $W_{0.5}$ is obtained from $M_{0.5} \times [0,1]$ by attaching k- and $(k+1)$-handles. This shows that the inclusion $M_{0.5} \hookrightarrow W_{0.5}$ induces an isomorphism $\pi_1(M_{0.5}) \cong \pi_1(W_{0.5})$. By the Poincaré-Lefschetz duality, $H_*(W_{0.5}, M_{0.5}) \cong H^{(k+2)-*}(W_{0.5}, M_0) = 0$. Therefore we see that M_0 and $M_{0.5}$ are homology bordant over $K(F_n^{\mathrm{acy}}, 1)$ by the bordism $W_{0.5}$ and Φ. Note that this bordism preserves the direct sum decomposition $H_1(M_0) \cong i_+(H_1(X_n^k - \operatorname{Int} D^k)) \oplus \operatorname{Ker}\Phi$, namely we also have $H_1(M_{0.5}) \cong i_+(H_1(X_n^k - \operatorname{Int} D^k)) \oplus \operatorname{Ker}\Phi$ and we can take $\overline{\alpha} \in \operatorname{Ker}\Phi \subset H_1(M_{0.5})$ which corresponds to α.

Recall the simple closed curve C taken at the beginning of this argument. Since $\pi_1(M_{0.5}) \to \pi_1(W_{0.5})$ is an isomorphism, there exists a simple closed curve $\overline{C} \subset M_{0.5}$ which attains C in $\pi_1(W_{0.5})$. Now $h_{ij} \in F_n^{\mathrm{acy}}$ are in the image of $\Phi_S : \pi_1(W_{0.5}) \to F_n^{\mathrm{acy}}$, so that we can take $\overline{h_{ij}} \in \pi_1(M_{0.5})$ attaining h_{ij}. Then the simple closed curve $\overline{C}\left(\prod_{i=1}^{l}[\overline{h_{i1}}, \overline{h_{i2}}]\right)^{-1}$ represents $\overline{\alpha}$ and is mapped by Φ to the trivial element of F_n^{acy}. The manifold $M_{0.5}$ and the map $\Phi : M_{0.5} \to K(F_n^{\mathrm{acy}}, 1)$ are what we are looking for in this case.

By iterating the above procedure, we succeed in killing $\operatorname{Ker}\Phi \subset H_1(M_0)$ with keeping $f = \Phi|_{X_{2n}^k} : X_{2n}^k = \partial M_0 \to K(F_n^{\mathrm{acy}}, 1)$ unchanged. That is, we get a manifold M_n which is bordant to M_0 over $K(F_n^{\mathrm{acy}}, 1)$ by a bordism W_n and a map $\Phi : W_n \to K(F_n^{\mathrm{acy}}, 1)$ such that the kernel of $\Phi|_{M_n} : H_1(M_n) \to H_1(K(F_n^{\mathrm{acy}}, 1))$ is trivial. Since $\Phi \circ i_+ = \iota_{F_n}$, the maps $i_+ : H_1(X_n^k - \operatorname{Int} D^k) \to H_1(M_n)$ and $\Phi : H_1(M_n) \to H_1(K(F_n^{\mathrm{acy}}, 1))$ are isomorphisms.

Let us show that (M_n, i_+, i_-) is a homology cylinder over $X_n^k - \operatorname{Int} D^k$. The bordism W_n is obtained from $M_0 \times [0,1]$ by attaching 1-and 2-handles with the number of 2-handles greater than that of 1-handles by n. The dual handle decomposition is obtained from $M_n \times [0,1]$ by attaching their dual k-and $(k+1)$-handles. Therefore we have

$$\chi(M_n) + (-1)^k n = \chi(W_n) = \chi(M_0) + n = 1 - n,$$

where $\chi(\cdot)$ denotes the Euler characteristic, and

$$H_i(M_n) \cong H_i(W_n) \cong H_i(M_0) = 0$$

if $2 \leq i \leq k-2$. Since M_n is a compact $(k+1)$-dimensional manifold with non-empty boundary, it is homotopy equivalent to a k-dimensional CW-complex. Hence $H_i(M_n) = 0$ for $i \geq k+1$ and $H_k(M_n)$ is free. The inclusion $i_+ : X_n^k - \operatorname{Int} D^k \hookrightarrow M_n$ is decomposed to $X_n^k - \operatorname{Int} D^k \hookrightarrow \partial M_n \hookrightarrow M_n$, which shows that $H_1(M_n, \partial M_n) = 0$. By the Poincaré-Lefschetz duality, we have $H^k(M_n) \cong H_1(M_n, \partial M_n) = 0$. It follows that $H_k(M_n) = 0$ and $H_{k-1}(M_n)$ is free. Comparing with $\chi(M_n) = 1 - n + (-1)^{k-1} n$, we have $H_{k-1}(M_n) \cong \mathbb{Z}^n$. Finally we check that $i_+ : H_{k-1}(X_n^k - \operatorname{Int} D^k) \to H_{k-1}(M_n)$ is an isomorphism. The source and target are both isomorphic to $\mathbb{Z}^n$. The homomorphism $\varphi \circ \iota_{F_n} : \pi_1(X_n^k - \operatorname{Int} D^k) \to F_n^{\mathrm{acy}}$ is 2-connected and factors through $\pi_1(X_n^k - \operatorname{Int} D^k) \xrightarrow{i_-} \pi_1(M_n) \xrightarrow{\Phi} F_n^{\mathrm{acy}}$. From this, we see that $H_1(M_n, i_-(X_n^k - \operatorname{Int} D^k)) = 0$. The Poincaré-Lefschetz duality shows that

$$\begin{aligned} H_k(M_n, i_+(X_n^k - \operatorname{Int} D^k)) &\cong H^1(M_n, i_-(X_n^k - \operatorname{Int} D^k)) \\ &\cong \operatorname{Hom}(H_1(M_n, i_-(X_n^k - \operatorname{Int} D^k)), \mathbb{Z}) = 0 \end{aligned}$$

and it follows that $H_{k-1}(M_n, i_+(X_n^k - \operatorname{Int} D^k))$ is a torsion module. In fact, it is trivial because the universal coefficient theorem and the Poincaré-Lefschetz duality say that

$$\begin{aligned} H_{k-1}(M_n, i_+(X_n^k - \operatorname{Int} D^k)) &\cong H^k(M_n, i_+(X_n^k - \operatorname{Int} D^k)) \\ &\cong H_1(M_n, i_-(X_n^k - \operatorname{Int} D^k)) = 0. \end{aligned}$$

Hence $M := (M_n, i_+, i_-)$ is a homology cylinder over $X_n^k - \operatorname{Int} D^k$. Now we have a commutative diagram:

$$\begin{array}{ccccc} X_n^k - \operatorname{Int} D^k & \xrightarrow{f_-} & K(F_n^{\mathrm{acy}}, 1) & \xleftarrow{f_+} & X_n^k - \operatorname{Int} D^k \\ & {\scriptstyle i_-}\searrow & \uparrow{\scriptstyle \Phi} & \swarrow{\scriptstyle i_+} & \\ & & M_n & & \end{array}$$

From this diagram, we see that

$$\operatorname{Acy}(M) = (i_+^{\mathrm{acy}})^{-1} \circ i_-^{\mathrm{acy}} = (f_+^{\mathrm{acy}})^{-1} \circ f_-^{\mathrm{acy}} = \varphi.$$

This completes the proof. □

REMARK 7.8. Theorem 7.7 is considered to be an analogue of a part of Laudenbach's theorem [14, Théorème 4.3] that the natural action of the diffeotopy group $\mathcal{M}(X_n^3 - \operatorname{Int} D^3)$ on $\pi_1(X_n^3 - \operatorname{Int} D^3) = F_n$ gives an epimorphism $\mathcal{M}(X_n^3 - \operatorname{Int} D^3) \twoheadrightarrow \operatorname{Aut}(F_n)$, where the same statement holds for $X_n^k - \operatorname{Int} D^k$ with $k \geq 4$.

REMARK 7.9. For homology cylinders over a surface $\Sigma_{g,1}$, we can also define a homomorphism $\operatorname{Acy} : \mathcal{H}_{g,1} \to \operatorname{Aut}(F_{2g}^{\mathrm{acy}})$, where $F_{2g} = \pi_1(\Sigma_{g,1})$. In this case, however, it was shown in [24, Theorem 6.1] that Acy is *not* surjective. In fact, the image is given by

$$\{\varphi \in \operatorname{Aut}(F_{2g}^{\mathrm{acy}}) \mid \varphi(\zeta) = \zeta \in F_{2g}^{\mathrm{acy}}\},$$

where $\zeta \in F_{2g} \subset F_{2g}^{\mathrm{acy}}$ is the word corresponding to the boundary loop of $\Sigma_{g,1}$. This may be regarded as an analogue of the Dehn-Nielsen theorem for the action of the mapping class group $\mathcal{M}_{g,1}$ on F_{2g}.

8. The Magnus Representation Revisited

Now we give an alternative description of the Magnus representation for homology cylinders and use it to finish the proof of Theorem 6.1. For that, we recall the extended free derivatives originally defined by Le Dimet [15]. Precisely speaking, the derivatives given below are a reduced version to commutative rings.

Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be a basis of a free group F_n. The definition of the extended free derivatives is derived from the following lemma. The proof is almost the same as that of [15, Proposition 1.1].

PROPOSITION 8.1. *The homomorphism*

$$\chi : \mathcal{K}_{H_1}^n \longrightarrow I(F_n^{\mathrm{acy}}) \otimes_{\mathbb{Z}[F_n^{\mathrm{acy}}]} \mathcal{K}_{H_1}$$

sending $(a_1, \ldots, a_n)^T \in \mathcal{K}_{H_1}^n$ *to* $\displaystyle\sum_{i=1}^n (\gamma_i^{-1} - 1) \otimes a_i$ *is a right* $\mathcal{K}_{H_1}$*-isomorphism, where* $I(F_n^{\mathrm{acy}})$ *is the kernel of the trivializer* $\mathbb{Z}[F_n^{\mathrm{acy}}] \to \mathbb{Z}$.

DEFINITION 8.2. For $1 \le i \le n$, the *extended free derivative*

$$\frac{\partial}{\partial \gamma_i} : F_n^{\mathrm{acy}} \longrightarrow \mathcal{K}_{H_1}$$

with respect to γ_i is the map assigning to $v \in F_n^{\mathrm{acy}}$ the i-th component of $\overline{\chi^{-1}((v^{-1} - 1) \otimes 1)} \in \mathcal{K}_{H_1}$.

In Le Dimet [15, Proposition 1.3], the formulas

$$\frac{\partial \gamma_j}{\partial \gamma_i} = \delta_{i,j}, \qquad \frac{\partial (gh)}{\partial \gamma_i} = \frac{\partial g}{\partial \gamma_i} + g \frac{\partial h}{\partial \gamma_i}, \qquad \frac{\partial g^{-1}}{\partial \gamma_i} = -g^{-1} \frac{\partial g}{\partial \gamma_i}$$

for $g, h \in F_n^{\mathrm{acy}}$ are given. By them, we see that the extended free derivatives coincide with the original ones if we restrict them to F_n.

DEFINITION 8.3. The *Magnus representation* for $\mathrm{Aut}(F_n^{\mathrm{acy}})$ is the map

$$r : \mathrm{Aut}(F_n^{\mathrm{acy}}) \to M(n, \mathcal{K}_{H_1})$$

assigning to $\varphi \in \mathrm{Aut}(F_n^{\mathrm{acy}})$ the matrix

$$r(\varphi) := \left(\overline{\left(\frac{\partial \varphi(\gamma_j)}{\partial \gamma_i} \right)} \right)_{i,j}.$$

It is not difficult to check that the Magnus representation r is a crossed homomorphism and hence the image of r is included in the set $\mathrm{GL}(n, \mathcal{K}_{H_1})$. When we compute the Magnus matrix $r(\mathrm{Acy}(M))$ for a homology cylinder $M \in \mathcal{H}(X_n^k - \mathrm{Int}\, D^k)$, we shall meet the same formula as Proposition 4.6. This shows that Definition 8.3 gives an alternative description of the Magnus representation.

Example 8.4. Let f be a 2-connected endomorphism of F_n. By Proposition 7.6, the endomorphism f is uniquely extended to an automorphism f^{acy} of F_n^{acy}. In this case, the Magnus matrix $r(f^{\mathrm{acy}})$ is obtained by applying the original free derivatives to f.

Consider the composition

$$\widetilde{r} : \mathrm{Aut}(F_n^{\mathrm{acy}}) \stackrel{r}{\longrightarrow} \mathrm{GL}(n, \mathcal{K}_{H_1}) \stackrel{\det}{\longrightarrow} \mathcal{K}_{H_1}^{\times} \longrightarrow \mathcal{K}_{H_1}^{\times}/(\pm H_1 \cdot A') \cong \mathbb{Z}^{\infty},$$

where $A' := \{f^{-1} \cdot \varphi(f) \mid f \in \mathcal{K}_{H_1}^{\times},\ \varphi \in \mathrm{Aut}(H_1)\}$ as before. The map $\widetilde{r}$ is a homomorphism.

THEOREM 8.5. *For any $n \geq 2$, we have:*

(1) *The image of the homomorphism $\widetilde{r}$ is an infinitely generated subgroup of $\mathbb{Z}^{\infty}$. In particular, $H_1(\mathrm{Aut}\,(F_n^{\mathrm{acy}}))$ has infinite rank.*

(2) *The homomorphism $\widetilde{r}$ factors through* $\mathrm{Out}\,(F_n^{\mathrm{acy}})$. *Therefore $H_1(\mathrm{Out}\,(F_n^{\mathrm{acy}}))$ has infinite rank.*

PROOF. (1) Consider a homomorphism $f_m : F_n \to F_n$ defined by

$$f_m(\gamma_1) = (\gamma_1 \gamma_2^{-1} \gamma_1^{-1} \gamma_2^{-1})^m \gamma_1 \gamma_2^{2m}, \qquad f_m(\gamma_i) = \gamma_i \ (2 \leq i \leq n)$$

for each $m \geq 1$. The homomorphism f_m is 2-connected and therefore it induces an automorphism f_m^{acy} of F_n^{acy}. By using the original free derivatives (see Example 8.4), we see that the Magnus matrix $r(f_m^{\mathrm{acy}})$ is given by the lower triangular matrix

$$\begin{pmatrix} 1-\gamma_2+\gamma_2^2-\gamma_2^3+\cdots+\gamma_2^{2m} & 0 & \multirow{2}{*}{$0_{(2,n-2)}$} \\ * & 1 & \\ \multicolumn{2}{c}{0_{(n-2,2)}} & I_{n-2} \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} \widetilde{r}(f_m^{\mathrm{acy}}) &= 1-\gamma_2+\gamma_2^2-\gamma_2^3+\cdots+\gamma_2^{2m} \\ &= 1+(-\gamma_2)+(-\gamma_2)^2+(-\gamma_2)^3+\cdots+(-\gamma_2)^{2m}. \end{aligned}$$

By a well known fact on the cyclotomic polynomials, we see that the polynomial $\widetilde{r}(f_m^{\mathrm{acy}})$ is irreducible if $2m+1$ is prime. Moreover, the polynomials $\{\widetilde{r}(f_m^{\mathrm{acy}}) \mid 2m+1 \text{ is prime}\}$ are independent in the module $\mathcal{K}_{H_1}^{\times}/(\pm H_1 \cdot A')$ because their degrees are distinct. Therefore the claim for $H_1(\mathrm{Aut}\,(F_n^{\mathrm{acy}}))$ follows.

(2) It suffices to show that the composition

$$\Psi : F_n^{\mathrm{acy}} \twoheadrightarrow \mathrm{Inn}\,(F_n^{\mathrm{acy}}) \hookrightarrow \mathrm{Aut}\,(F_n^{\mathrm{acy}}) \xrightarrow{\widetilde{r}} \mathcal{K}_{H_1}^{\times}/(\pm H_1 \cdot A')$$

is trivial. The restriction of the homomorphism Ψ to F_n is trivial because the determinant of the Magnus matrix of an automorphism of F_n is in H_1. Therefore Ψ is an extension of the trivial map from F_n to the abelian group $\mathcal{K}_{H_1}^{\times}/(\pm H_1 \cdot A')$ which is AC. Then by Proposition 7.3, we see that the map Ψ is also trivial. □

Proof of Theorem 6.1. The Magnus representation mentioned in Section 6 is the composition of the homomorphism Acy and the above $\widetilde{r}$. Therefore our claims immediately follow from Theorems 7.7 and 8.5. □

The isomorphisms f_m in the proof of Theorem 8.5 can also be used to show the following.

Theorem 8.6. *The acyclic closure F_n^{acy} is not finitely generated for any $n \geq 2$.*

Proof. Suppose F_n^{acy} had a finite generating set $\{g_1, g_2, \ldots, g_l\}$. Then the above formulas for the extended free derivatives imply that the image of the derivative $\frac{\partial}{\partial \gamma_i}$ for each i is in the subring R of $\mathcal{K}_{H_1}$ obtained from $\mathbb{Z}[H_1]$ by adding $\left\{\frac{\partial g_1}{\partial \gamma_i}, \frac{\partial g_2}{\partial \gamma_i}, \dots, \frac{\partial g_l}{\partial \gamma_i}\right\}$. In particular, there are only finitely many irreducible polynomials which appear as factors of the denominators of reduced expressions for elements in R. However, for the automorphism f_m^{acy} with $2m+1$ prime constructed in the proof of Theorem 8.5, the $(1,1)$-entry of $\widetilde{r}((f_m^{\mathrm{acy}})^{-1}) = (\widetilde{r}(f_m^{\mathrm{acy}}))^{-1}$ is

$$\overline{\left(\frac{\partial (f_m^{\mathrm{acy}})^{-1}(\gamma_1)}{\partial \gamma_1}\right)} = \frac{1}{1-\gamma_2+\gamma_2^2-\gamma_2^3+\cdots+\gamma_2^{2m}}.$$

This contradicts to the property of R just mentioned. □

REMARK 8.7. It is easy to see that the derivative $\dfrac{\partial}{\partial \gamma_i} : F_n^{\mathrm{acy}} \to \mathcal{K}_{H_1}$ factors through the metabelian quotient

$$F_n^{\mathrm{acy}}/[[F_n^{\mathrm{acy}}, F_n^{\mathrm{acy}}], [F_n^{\mathrm{acy}}, F_n^{\mathrm{acy}}]]$$

of F_n^{acy}. Our proof of Theorem 8.6 shows that this metabelian quotient is also infinitely generated for any $n \geq 2$, and therefore it is not isomorphic to that of F_n. This fact contrasts with the nilpotent quotients of F_n and F_n^{acy} which are isomorphic by Stallings' theorem.

As mentioned in Section 7, the acyclic closure of a group is a variation of the algebraic closure of a group. The argument in this section can be applied to the algebraic closure F_n^{alg} of a free group F_n. In fact, as shown in [16, Proposition 5], an automorphism of F_n^{alg} is induced from a *normally surjective* 2-connected endomorphism of F_n. The 2-connected endomorphisms f_m of F_n constructed in the proof of Theorem 8.5 are normally surjective. Indeed, $\gamma_1 = (\gamma_1\gamma_2^{-1}\gamma_1^{-1}\gamma_2^{-1})^{-m} f_m(\gamma_1)\gamma_2^{-2m}$ is in the normal closure of the image of f_m because $\gamma_1\gamma_2^{-1}\gamma_1^{-1}\gamma_2^{-1} = (\gamma_1\gamma_2^{-1}\gamma_1^{-1})\gamma_2^{-1}$ is in it. Thus f_m induces an automorphism f_m^{alg} of F_n^{alg}. By Le Dimet's original construction, we can define the Magnus representation for $\mathrm{Aut}\,(F_n^{\mathrm{alg}})$. The remaining argument goes parallel to the case of F_n^{acy}. Consequently, we have the following.

THEOREM 8.8. *For any $n \geq 2$, we have:*
(1) $H_1(\mathrm{Aut}\,(F_n^{\mathrm{alg}}))$ *and* $H_1(\mathrm{Out}\,(F_n^{\mathrm{alg}}))$ *have infinite rank.*

(2) *The algebraic closure* F_n^{alg} *and its metabelian quotient* $F_n^{\mathrm{alg}}/[[F_n^{\mathrm{alg}}, F_n^{\mathrm{alg}}], [F_n^{\mathrm{alg}}, F_n^{\mathrm{alg}}]]$ *are not finitely generated.*

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