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The Fatou Property of Block Spaces

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Abstract. Around thirty years ago, block spaces, which are the predual of Morrey spaces, had been considered. However, it seems that there is no proof that block spaces satisfy the Fatou property. In this paper the Fatou property for block spaces is verified and the predual of block spaces is characterized.

1. Introduction

The purpose of this paper is to verify the Fatou property for block spaces, which in turn yields a characterization for the predual of block spaces. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is a properly wider space than the Lebesgue space $L^p(\mathbb{R}^n)$ when $0 < q < p < \infty$ and this space works well with the fractional integral operators (cf. [8, 9, 10, 11]). We first recall the definition of Morrey spaces and consider block spaces which are Morrey spaces if we pass to the predual.

1.1. Morrey spaces

Let $0 < q \le p < \infty$ be two real parameters. For $f \in L^q_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} := \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}-\frac{1}{q}} \left(\int_Q |f(x)|^q \, dx\right)^{\frac{1}{q}}$$

$$= \sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q |f(x)|^q \, dx\right)^{\frac{1}{q}},$$

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where we have used the notation $\mathcal{Q}$ to denote the family of all cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes and $|Q|$ to denote the volume of Q. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is defined to be the subset of all L^q locally integrable functions f on $\mathbb{R}^n$ for which $\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$ is finite. It is easy see that $\|\cdot\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$ becomes the norm if $q \geq 1$ and that $\|\cdot\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$ becomes the quasi norm otherwise. Letting $0 < r < q \leq p < \infty$ and using Hölder's inequality, we have

$$|Q|^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q |f(x)|^r \, dx \right)^{\frac{1}{r}} \leq |Q|^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q |f(x)|^q \, dx \right)^{\frac{1}{q}} \tag{1.1}$$

and, hence,

$$\|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \geq \|f\|_{\mathcal{M}^p_r(\mathbb{R}^n)}.$$

This tells us that

$$L^p(\mathbb{R}^n) = \mathcal{M}^p_p(\mathbb{R}^n) \subset \mathcal{M}^p_q(\mathbb{R}^n) \subset \mathcal{M}^p_r(\mathbb{R}^n) \text{ when } p \geq q > r > 0. \tag{1.2}$$

If we let $f(x) = |x|^{-n/p}$, then the cube $R = (-t/2, t/2)^n$, $t > 0$, attains its Morrey-norm. In fact, if $0 < q < p < \infty$, then

$$\begin{aligned} \sup_{Q \in \mathcal{Q}} |Q|^{\frac{1}{p} - \frac{1}{q}} \left(\int_Q \frac{1}{|x|^{nq/p}} \, dx \right)^{\frac{1}{q}} &\leq c|R|^{\frac{1}{p} - \frac{1}{q}} \left(\int_R \frac{1}{|x|^{nq/p}} \, dx \right)^{\frac{1}{q}} \\ &= O\left((n(1 - q/p))^{-1/q} \right) \end{aligned}$$

and f belongs to $\mathcal{M}^p_q(\mathbb{R}^n)$. Because then f does not belong to $L^p(\mathbb{R}^n)$, we see that the Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ is properly wider than the Lebesgue space $L^p(\mathbb{R}^n)$. The completeness of Morrey spaces follows easily by that of Lebesgue spaces.

If the sequence of nonnegative functions $\{f_k\}_{k=1}^{\infty} \subset \mathcal{M}^p_q(\mathbb{R}^n)$ satisfies $f_k(x) \uparrow f(x)$ (a.e. $x \in \mathbb{R}^n$), then we have

$$\|f_k\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \uparrow \|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \tag{1.3}$$

from the definition of the Morrey norm $\|\cdot\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$. However, the following property is different from that of Lebesgue spaces.

For any measurable set $E \subset \mathbb{R}^n$ such that $|E| < \infty$ and any $f \in L^p(\mathbb{R}^n)$, we have by Hölder's inequality

$$\left| \int_E f(x) \, dx \right| \leq \int_E |f(x)| \, dx = \int_{\mathbb{R}^n} \chi_E(x) |f(x)| \, dx \leq |E|^{\frac{1}{p'}} \|f\|_{L^p(\mathbb{R}^n)} < \infty,$$

where p' is the conjugate number defined by $1/p + 1/p' = 1$ and χ_E stands for the characteristic function of E. While, if $f \in \mathcal{M}^p_q(\mathbb{R}^n)$ then it follows from the definition of Morrey-norm that

$$\begin{aligned}\int_Q |f(x)|\,dx &= |Q| \left(\frac{1}{|Q|} \int_Q |f(x)|\,dx \right) \\ &\le |Q| \left(\frac{1}{|Q|} \int_Q |f(x)|^q\,dx \right)^{\frac{1}{q}} \\ &= |Q|^{1-\frac{1}{p}} \cdot |Q|^{\frac{1}{p}} \left(\frac{1}{|Q|} \int_Q |f(x)|^q\,dx \right)^{\frac{1}{q}} \\ &\le \|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} |Q|^{\frac{1}{p'}}.\end{aligned}$$

This implies that for any family of counterable open cubes $\{Q_j\}_{j=1}^{\infty}$ such that $E \subset \bigcup_j Q_j$, we have

$$\begin{aligned}\left| \int_E f(x)\,dx \right| &\le \int_E |f(x)|\,dx \\ &\le \sum_j \int_{Q_j} |f(x)|\,dx \le \|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \sum_j |Q_j|^{\frac{1}{p'}}.\end{aligned} \tag{1.4}$$

In general, for two real parameters $0 < r \le \infty$ and $0 < d \le 1$, the Hausdorff capacity or the Hausdorff content of the set E is defined by

$$H^d_r(E) := \inf \sum_j |Q_j|^d,$$

where the infimum is taken over all counterable cubes $\{Q_j\}_{j=1}^{\infty}$ which cover E with the side-length less than r. Using this definition, we have by (1.4)

$$\left| \int_E f(x)\,dx \right| \le H^{1/p'}_{\infty}(E) \|f\|_{\mathcal{M}^p_q(\mathbb{R}^n)}. \tag{1.5}$$

Of course, $|E| < \infty$ does not always imply $H^{1/p'}_{\infty}(E) < \infty$. Thus, we cannot conclude from the fact that $|E| < \infty$ that the left-hand side of this inequality is finite.

1.2. Block spaces

We shall define block spaces following [4]. Let $1 < q \le p < \infty$. We say that a function b on $\mathbb{R}^n$ is a (p', q')-block provided that b is supported on a cube $Q \in \mathcal{Q}$ and satisfies

$$\left(\int_Q |b(x)|^{q'} \, dx \right)^{\frac{1}{q'}} \le |Q|^{\frac{1}{p} - \frac{1}{q}}. \tag{1.6}$$

(The cube Q will be called the support cube of b.) The space $\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ is defined by the set of all functions f locally in $L^{q'}(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} := \inf \left\{ \|\{\lambda_k\}\|_{l^1} \, : \, f = \sum_k \lambda_k b_k \right\} < \infty,$$

where $\|\{\lambda_k\}_{k=1}^\infty\|_{l^1} = \sum_k |\lambda_k| < \infty$ and b_k is a (p', q')-block, and the infimum is taken over all possible decompositions of f. By the definition of the norm we see that the inclusion

$$L^{p'}(\mathbb{R}^n) = \mathcal{B}^{p'}_{p'}(\mathbb{R}^n) \supset \mathcal{B}^{p'}_{q'}(\mathbb{R}^n) \supset \mathcal{B}^{p'}_{r'}(\mathbb{R}^n) \text{ when } p \ge q > r > 1. \tag{1.7}$$

In [4, Theorem 1] and [16, Proposition 5] the following was proved.

PROPOSITION 1.1. *Let $1 < q \le p < \infty$. Then the predual space of $\mathcal{M}^p_q(\mathbb{R}^n)$ is $\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ in the following sense:*

If $g \in \mathcal{M}^p_q(\mathbb{R}^n)$, then $f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x) g(x) \, dx$ is an element of $\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)^*$ and

$$\left| \int_{\mathbb{R}^n} f(x) g(x) \, dx \in \mathbb{R} \right| \le \|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} \|g\|_{\mathcal{M}^p_q(\mathbb{R}^n)}, \quad (f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)).$$

Moreover, for any $L \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)^*$, there exists $g \in \mathcal{M}^p_q(\mathbb{R}^n)$ such that

$$L(f) = \int_{\mathbb{R}^n} f(x) g(x) \, dx, \quad (f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)),$$

and

$$\|L\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)^*} = \|g\|_{\mathcal{M}^p_q(\mathbb{R}^n)}.$$

See [7] for more details about the predual spaces. We refer [1, 2, 5] for recent development of the theory of the predual spaces.

In this paper we shall prove the following theorem which assert the Fatou property of block spaces.

THEOREM 1.2. *Let $1 < q \le p < \infty$. Suppose that f and f_k, $(k = 1, 2, \dots)$, are nonnegative, $\|f_k\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} \le 1$ and $f_k \uparrow f$ a.e. Then $f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} \le 1$.*

This quite simple fact can not be found in any literature as far as we know. Since the case when $p = q$ is clear from the Fatou lemma, Theorem 1.2 is significant only when $q < p$. Seemingly, it is clear that we have $f \in L^{p'}(\mathbb{R}^n)$. However, it is difficult to find an expression of f.

The letter C will be used for constants that may change from one occurrence to another. Constants with subscripts, such as C_1, C_2, do not change in different occurrences.

2. Proof of Theorem 1.2

In what follows we shall prove Theorem 1.2. We need the following lemmas.

LEMMA 2.1. *Let $1 < q \le p < \infty$. Then, a function f belongs to $\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ if and only if there exists $g \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ such that $|f(x)| \le g(x)$ (a.e. $x \in \mathbb{R}^n$).*

PROOF. Suppose that $f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$. Then there exist a sequence $\{\lambda_k\}_{k=1}^\infty \in l^1$ and a (p', q')-block b_k such that $f = \sum_k \lambda_k b_k$. Letting $g = \sum_k |\lambda_k||b_k|$, we have $g \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ and $|f| \le g$. Conversely, suppose that there exists $g \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ that satisfies $|f(x)| \le g(x)$. Decompose g as $g = \sum_k \lambda'_k b'_k$ where $\{\lambda'_k\}_{k=1}^\infty \in l^1$ and b'_k is a (p', q')-block. Then we see that

$$\chi_{\{y:\, g(y) \ne 0\}}(x) = \sum_k \lambda'_k \frac{1}{g(x)} b'_k(x) \chi_{\{y:\, g(y) \ne 0\}}(x)$$

and, hence,

$$f(x) = \sum_k \lambda'_k \frac{f(x)}{g(x)} b'_k(x) \chi_{\{y:\, g(y) \ne 0\}}(x).$$

Since $|f(x)|/g(x) \le 1$ as long as $g(x) \ne 0$, the function $(f/g)b'_k \chi_{\{y:\, g(y)\ne 0\}}$ becomes a (p',q')-block. This proves the lemma. $\square$

We denote by $\mathcal{D}$ the family of all dyadic cubes of the form $Q = 2^{-k}(i + [0,1)^n)$, $k \in \mathbb{Z}$, $i \in \mathbb{Z}^n$.

LEMMA 2.2. *Let $1 < q \le p < \infty$ and $f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ with $\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} \le 1$. Then f can be decomposed as*

$$f = \sum_{Q\in\mathcal{D}} \lambda(Q) b(Q),$$

where $\lambda(Q)$ is a positive number with

$$\sum_{Q\in\mathcal{D}} \lambda(Q) \le 2\cdot 3^n$$

and $b(Q)$ is a (p',q')-block with $\operatorname{supp} b(Q) \subset 3Q$.

PROOF. First, decompose f as

$$f = \sum_{k\in K} \lambda_k b_k$$

where $K \subset \mathbb{N}$ is an index set, $\sum_{k\in K} |\lambda_k| \le 2$ and b_k is a (p',q')-block with the support cube Q_k. We divide K into the disjoint sets $K(Q) \subset \mathbb{N}$, $Q \in \mathcal{D}$, as

$$K = \bigcup_{Q\in\mathcal{D}} K(Q)$$

and $K(Q)$ fulfills, when $k \in K(Q)$,

$$Q_k \subset 3Q \text{ and } |Q_k| \ge |Q|.$$

We now rewrite f as

$$\begin{aligned} f &= \sum_{k\in K} \lambda_k b_k = \sum_{Q\in\mathcal{D}} \left(\sum_{k\in K(Q)} \lambda_k b_k \right) \\ &= \sum_{Q\in\mathcal{D}} \left\{ 3^n \sum_{k\in K(Q)} |\lambda_k| \right\} \cdot \left\{ \left(3^n \sum_{k\in K(Q)} |\lambda_k| \right)^{-1} \sum_{k\in K(Q)} \lambda_k b_k \right\} \\ &=: \sum_{Q\in\mathcal{D}} \lambda(Q) b(Q). \end{aligned}$$

It follows that

$$\sum_{Q\in\mathcal{D}} \lambda(Q) = 3^n \sum_{Q\in\mathcal{D}} \left(\sum_{k\in K(Q)} |\lambda_k| \right) = 3^n \sum_{k\in K} |\lambda_k| \le 2\cdot 3^n$$

and that

$$\begin{aligned}
&\left(3^n \sum_{k\in K(Q)} |\lambda_k|\right)^{-1} \left\| \sum_{k\in K(Q)} \lambda_k b_k \right\|_{L^{q'}(\mathbb{R}^n)} \\
&\le \left(3^n \sum_{k\in K(Q)} |\lambda_k|\right)^{-1} \sum_{k\in K(Q)} |\lambda_k| \|b_k\|_{L^{q'}(\mathbb{R}^n)} \\
&\le \left(3^n \sum_{k\in K(Q)} |\lambda_k|\right)^{-1} \sum_{k\in K(Q)} |\lambda_k| |Q_k|^{\frac{1}{p}-\frac{1}{q}} \\
&\le \left(3^n \sum_{k\in K(Q)} |\lambda_k|\right)^{-1} |Q|^{\frac{1}{p}-\frac{1}{q}} \sum_{k\in K(Q)} |\lambda_k| \\
&\le |3Q|^{\frac{1}{p}-\frac{1}{q}},
\end{aligned}$$

which imply that $b(Q)$ is a (p',q')-block with $\operatorname{supp} b(Q) \subset 3Q$. These complete the proof. □

Proof of Theorem 1.2. We may assume that $1 < q < p < \infty$ as we remarked just below the statement of the theorem. By Lemma 2.2 f_k can be decomposed as

$$f_k = \sum_{Q\in\mathcal{D}} \lambda_k(Q) b_k(Q),$$

where $\lambda_k(Q)$ is a positive number with

$$\sum_{Q\in\mathcal{D}} \lambda_k(Q) \le 2\cdot 3^n \tag{2.1}$$

and $b_k(Q)$ is a (p',q')-block with $\operatorname{supp} b_k(Q) \subset 3Q$ and

$$\|b_k(Q)\|_{L^{q'}(\mathbb{R}^n)} \le |3Q|^{\frac{1}{p}-\frac{1}{q}}. \tag{2.2}$$

Noticing (2.1), (2.2) and using the weak$*$-compactness of the Lebesgue space $L^{q'}(3Q)$, we now apply a diagonalization argument and, hence, we can select an infinite subsequence $\{f_{k_j}\}_{j=1}^{\infty} \subset \{f_k\}_{k=1}^{\infty}$ that satisfies the following:

$$f_{k_j} = \sum_{Q \in \mathcal{D}} \lambda_{k_j}(Q) b_{k_j}(Q),$$

$$\lim_{j \to \infty} \lambda_{k_j}(Q) = \lambda(Q), \tag{2.3}$$

$$\lim_{j \to \infty} b_{k_j}(Q) = b(Q) \text{ in the weak$*$-topology of } L^{q'}(3Q), \tag{2.4}$$

where $b(Q)$ is a (p', q')-block with $\operatorname{supp} b(Q) \subset 3Q$. We set

$$f_0 := \sum_{Q \in \mathcal{D}} \lambda(Q) b(Q).$$

Then, by the Fatou theorem and (2.1),

$$\sum_{Q \in \mathcal{D}} \lambda(Q) \le \liminf_{j \to \infty} \sum_{Q \in \mathcal{D}} \lambda_{k_j}(Q) \le 2 \cdot 3^n, \tag{2.5}$$

which implies $f_0 \in \mathcal{B}_{q'}^{p'}(\mathbb{R}^n)$.

We shall verify that

$$\lim_{j \to \infty} \int_{Q_0} f_{k_j}(x)\, dx = \int_{Q_0} f_0(x)\, dx \tag{2.6}$$

for all $Q_0 \in \mathcal{D}$. Once (2.6) is established, we will see that $f = f_0$ and $f \in \mathcal{B}_{q'}^{p'}(\mathbb{R}^n)$ by virtue of the Lebesgue differentiation theorem because at least we know that f_0 locally in $L^{q'}(\mathbb{R}^n)$.

Let $\varepsilon > 0$ be given. We set

$$\begin{cases} \mathcal{D}_1(Q_0) := \{Q \in \mathcal{D} : \, Q \cap Q_0 \neq \emptyset, \, |3Q| \le c_1\}, \\ \mathcal{D}_2(Q_0) := \{Q \in \mathcal{D} : \, Q \cap Q_0 \neq \emptyset, \, |3Q| \in (c_1, c_2)\}, \\ \mathcal{D}_3(Q_0) := \{Q \in \mathcal{D} : \, Q \cap Q_0 \neq \emptyset, \, |3Q| \ge c_2\}, \end{cases}$$

where we have defined, keeping in mind that $1/p - 1/q < 0$,

$$\begin{cases} c_1^{\frac{1}{p}} = \dfrac{\varepsilon}{12 \cdot 3^n}, \\ c_2^{\frac{1}{p} - \frac{1}{q}} = \dfrac{\varepsilon}{12 \cdot 3^n |Q_0|^{1/q}}. \end{cases}$$

It follows that

$$\sum_{Q \in \mathcal{D}_1(Q_0)} \int_{Q_0} \left|\lambda_{k_j}(Q) b_{k_j}(Q)(x) - \lambda(Q) b(Q)(x)\right| \, dx$$
$$\leq \sum_{Q \in \mathcal{D}_1(Q_0)} (\lambda_{k_j}(Q) + \lambda(Q)) |3Q|^{\frac{1}{p}} \leq 4 \cdot 3^n c_1^{\frac{1}{p}} = \frac{\varepsilon}{3},$$

where we have used (2.1), (2.5) and

$$\int_{3Q} |b_{k_j}(Q)(x)| \, dx, \int_{3Q} |b(Q)(x)| \, dx \leq |3Q|^{\frac{1}{p}}$$

(see (1.6)). It follows from Hölder's inequality that

$$\sum_{Q \in \mathcal{D}_3(Q_0)} \int_{Q_0} \left|\lambda_{k_j}(Q) b_{k_j}(Q)(x) - \lambda(Q) b(Q)(x)\right| \, dx$$
$$\leq |Q_0|^{\frac{1}{q}} \sum_{Q \in \mathcal{D}_3(Q_0)} (\lambda_{k_j}(Q) + \lambda(Q)) |3Q|^{\frac{1}{p} - \frac{1}{q}} \leq 4 \cdot 3^n |Q_0|^{\frac{1}{q}} c_2^{\frac{1}{p} - \frac{1}{q}} = \frac{\varepsilon}{3},$$

where we have used the fact that $1/p - 1/q < 0$ and

$$\|b_{k_j}(Q)\|_{L^{q'}(3Q)}, \|b(Q)\|_{L^{q'}(3Q)} \leq |3Q|^{\frac{1}{p} - \frac{1}{q}}.$$

Finally,

$$\sum_{Q \in \mathcal{D}_2(Q_0)} \left| \int_{Q_0} \lambda_{k_j}(Q) b_{k_j}(Q)(x) \, dx - \int_{Q_0} \lambda(Q) b(Q)(x) \, dx \right|$$
$$\leq \sum_{Q \in \mathcal{D}_2(Q_0)} \left| \int_{Q_0} \lambda_{k_j}(Q) b_{k_j}(Q)(x) \, dx - \int_{Q_0} \lambda(Q) b_{k_j}(Q)(x) \, dx \right|$$
$$+ \sum_{Q \in \mathcal{D}_2(Q_0)} \left| \int_{Q_0} \lambda(Q) b_{k_j}(Q)(x) \, dx - \int_{Q_0} \lambda(Q) b(Q)(x) \, dx \right|.$$

From (2.3) and the fact that $\mathcal{D}_2(Q_0)$ contains the only finite number of dyadic cubes,

$$\sum_{Q \in \mathcal{D}_2(Q_0)} \left| \int_{Q_0} \lambda_{k_j}(Q) b_{k_j}(Q)(x) \, dx - \int_{Q_0} \lambda(Q) b_{k_j}(Q)(x) \, dx \right|$$
$$\leq c_2^{\frac{1}{p}} \sum_{Q \in \mathcal{D}_2(Q_0)} |\lambda_{k_j}(Q) - \lambda(Q)| \leq \frac{\varepsilon}{6}$$

for large j. From (2.4),

$$\sum_{Q \in \mathcal{D}_2(Q_0)} \left| \int_{Q_0} \lambda(Q) b_{k_j}(Q)(x)\,dx - \int_{Q_0} \lambda(Q) b(Q)(x)\,dx \right|$$
$$\le 2 \cdot 3^n \sup_{Q \in \mathcal{D}_2(Q_0)} \left| \int_{Q_0} b_{k_j}(Q)(x)\,dx - \int_{Q_0} b(Q)(x)\,dx \right| \le \frac{\varepsilon}{6}$$

for large j. These prove (2.6).

Since $f_k \uparrow f$ a.e., we must have by (2.6)

$$\int_{Q_0} f(x)\,dx = \int_{Q_0} f_0(x)\,dx$$

for all $Q_0 \in \mathcal{D}$. This yields $f = f_0$ a.e., by the Lebesgue differential theorem, and, hence, $f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$.

Since we have verified $f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$, by the Hahn-Banach theorem, we see that there exists an $L \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)^*$ such that

$$L(f) = \|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} \text{ and } \|L\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)^*} = 1,$$

which imply by Proposition 1.1

$$\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} = \sup\left\{ \left| \int_{\mathbb{R}^n} f(x) g(x)\,dx \right| : \|g\|_{\mathcal{M}^p_q(\mathbb{R}^n)} = 1 \right\}.$$

There holds also

$$\sup\left\{ \left| \int_{\mathbb{R}^n} f_k(x) g(x)\,dx \right| : \|g\|_{\mathcal{M}^p_q(\mathbb{R}^n)} = 1 \right\}$$
$$= \|f_k\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} \le 1, \quad (k = 1, 2, \dots).$$

These yield

$$\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} = \sup\left\{ \left| \int_{\mathbb{R}^n} f_k(x) g(x)\,dx \right| : k = 1, 2, \dots,\ \|g\|_{\mathcal{M}^p_q(\mathbb{R}^n)} = 1 \right\} \le 1.$$

This completes the proof of the theorem. $\square$

3. Banach Function Spaces

To state our next result, we need terminology from the theory of the Banach function spaces introduced in the book [3]. We place ourselves in the setting of a σ-finite measure space (R, μ). Let $\mathbb{M}^+$ be the cone of all μ-measurable functions on R assuming their values lie in $[0, \infty]$.

Definition 3.1. A mapping $\rho : \mathbb{M}^+ \to [0, \infty]$ is called a "Banach function norm" if, for all $f, g, f_n, (n = 1, 2, 3, \dots)$, in $\mathbb{M}^+$, for all constants $a \geq 0$ and for all μ-measurable subsets E of R, the following properties hold:

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ μ-a.e.; $\rho(af) = a\rho(f)$; $\rho(f + g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f$ μ-a.e. $\Rightarrow \rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \uparrow f$ μ-a.e. $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

(P4) $\mu(E) < \infty \Rightarrow \rho(\chi_E) < \infty$;

(P5) $\mu(E) < \infty \Rightarrow \int_E f \, d\mu \leq C_E \rho(f)$ for some constant C_E, $0 < C_E < \infty$, depending on E and ρ but independent of f.

Let $\mathbb{M}$ denote the collection of all extended scalar-valued (real or complex) μ-measurable functions on R. As usual, any two functions coinciding μ-a.e. shall be identified.

Definition 3.2. Let ρ be a Banach function norm. The collection $X = X(\rho)$ of all functions f in $\mathbb{M}$ for which $\rho(|f|) < \infty$ is called a "Banach function space". For each $f \in X$, define

$$\|f\|_X := \rho(|f|).$$

Let $1 < q < p < \infty$. The Morrey space $\mathcal{M}^p_q(\mathbb{R}^n)$ and the block space $\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ are not Banach function spaces, since the norm $\|\cdot\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$ fails property P5 and the norm $\|\cdot\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)}$ fails property P4.

Example 3.3. Here, we exhibit an example showning that $\|\cdot\|_{\mathcal{M}^p_q(\mathbb{R}^n)}$ fails property P5. From this example, we learn that the norm $\|\cdot\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)}$ fails property P4 when $1<q<p<\infty$. For simplicity, we let $n=1$ and $1<q<p=2$; other cases are dealt analogously.

Let us consider the sequence

$$(a_1, a_2, \dots) = \left(1, \frac{1}{4}, \frac{1}{4}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}, \dots\right),$$

that is, a_j is a decreasing sequence and 4^{-k} appears 2^k times for $k=0,2,\dots$. We may write $a_j = 4^{-[\log_2 j]}$, where $[a]$ stands the largest integer not greater than a.

Let $\alpha(2,q) \gg 1$. We define

$$E := \bigcup_{j=1}^{\infty} (\alpha(2,q)^j, \alpha(2,q)^j + a_j).$$

Then $|E| = 2$. Define

$$f(t) := \sum_{j=1}^{\infty} \frac{1}{(a_j)^{1/2}} \chi_{(\alpha(2,q)^j, \alpha(2,q)^j + a_j)}(t), \quad (t \in \mathbb{R}).$$

Then

$$\|f\|_{\mathcal{M}^2_q(\mathbb{R})} = \sup_j \frac{(a_j)^{1/2}}{(a_j)^{1/2}} = 1.$$

Meanwhile,

$$\int_E f(t)\,dt = \sum_{j=1}^{\infty} 4^{j/2-j} \cdot 2^j = \infty.$$

This example also reads $\chi_E \notin \mathcal{B}^2_{q'}(\mathbb{R})$ by a duality argument.

DEFINITION 3.4. If ρ is a Banach function norm, its "associate norm" ρ' is defined on $\mathbb{M}^+$ by

$$\rho'(g) := \sup\left\{\int_R fg\,d\mu : f \in \mathbb{M}^+, \rho(f) \le 1\right\}, \quad (g \in \mathbb{M}^+). \tag{3.1}$$

We have the following property:

THEOREM 3.5 ([3, Theorem 2.2]). *Let ρ be a Banach function norm. Then the associate norm ρ' is itself a Banach function norm.*

With Theorem 3.5 in mind, we define the associate space X' of X.

DEFINITION 3.6. Let ρ be a Banach function norm and let $X = X(\rho)$ be the Banach function space determined by ρ as in Definition 3.2. Let ρ' be the associate norm of ρ. The Banach function space $X(\rho')$ determined by ρ' is called the "associate space" of X and is denoted by X'.

THEOREM 3.7 ([3, Theorem 2.7]). *Every Banach function space X coincides with its second associate space X''. In other words, a function f belongs to X if and only if it belongs to X'', and in that case*

$$\|f\|_X = \|f\|_{X''}, \quad (f \in X = X'').$$

4. Application of Theorem 1.2

In what follows we shall apply Theorem 1.2 and characterize the predual of block spaces.

THEOREM 4.1. *Let $1 < q \le p < \infty$. Then the associate space $\mathcal{M}^p_q(\mathbb{R}^n)'$ coincides with the block space $\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$.*

PROOF. We see that $\mathcal{B}^{p'}_{q'}(\mathbb{R}^n) \subset \mathcal{M}^p_q(\mathbb{R}^n)'$ by Proposition 1.1. So we shall verify the converse. Suppose that a measurable function f satisfies

$$\sup\left\{\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx\right| : \|g\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \le 1\right\} \le 1. \tag{4.1}$$

Then we first see that $|f(x)| < \infty$ (a.e. $x \in \mathbb{R}^n$). Splitting f into its real and imaginary parts and each of these into its positive and negative parts, we may assume without loss of generality that $f \ge 0$. For $k = 1, 2, \dots$, set $Q_k := (-k, k)^n$ and let

$$f_k(x) := \min(f(x), k)\chi_{Q_k}(x) \quad (x \in \mathbb{R}^n).$$

We notice that $f_k \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ and $\|f_k\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} \le 1$ by Lemma 2.1, Proposition 1.1 and (4.1). Since $f_k \uparrow f$ a.e., it follows from Theorem 1.2 that $f \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ and $\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} \le 1$. This proves the theorem. □

By Theorems 3.7 and 4.1 (or directly Proposition 1.1), one sees that

$$\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)' = \mathcal{M}^p_q(\mathbb{R}^n)'' = \mathcal{M}^p_q(\mathbb{R}^n).$$

Furthermore, from the fact that $\mathcal{M}^p_q(\mathbb{R}^n)'' = \mathcal{M}^p_q(\mathbb{R}^n)$, we are able to characterize the predual of block spaces following the argument in [3].

Definition 4.2. Let $1 < q \le p < \infty$. The closure in $\mathcal{M}^p_q(\mathbb{R}^n)$ of the set of all finite linear combination of the characteristic functions of sets of finite measure is denoted by $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Theorem 4.3. *Let $1 < q \le p < \infty$. Then the predual space of $\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ is $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$ in the following sense:*

If $g \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$, then $f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n) \mapsto \int_{\mathbb{R}^n} f(x)g(x)\,dx$ is an element of $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*$ and

$$\left|\int_{\mathbb{R}^n} f(x)g(x)\,dx \in \mathbb{R}\right| \le \|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)}\|g\|_{\mathcal{M}^p_q(\mathbb{R}^n)}, \quad (f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)).$$

Moreover, for any $L \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*$, there exists $g \in \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$ such that

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx, \quad (f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)),$$

and that

$$\|L\|_{\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*} = \|g\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)}.$$

Proof. The first assertion is clear. So we shall prove that $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^* \subset \mathcal{B}^{p'}_{q'}(\mathbb{R}^n)$. Thanks to Theorem 4.1, we need only show

$$\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^* \subset \mathcal{M}^p_q(\mathbb{R}^n)'.$$

Suppose that L belongs to $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*$. We shall exhibit a function g in $\mathcal{M}^p_q(\mathbb{R}^n)'$ such that we have

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx, \quad (f \in \widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)). \tag{4.2}$$

If we let $Q_i := i + [0,1)^n$, $i \in \mathbb{Z}^n$, then the sequence $\{Q_i\}_{i\in\mathbb{Z}^n}$ forms disjoint subsets of $\mathbb{R}^n$, each of which has measure one and whose union is all of $\mathbb{R}^n$. For each $i \in \mathbb{Z}^n$, let $\mathcal{A}_i$ denote the Lebesgue measurable subsets of Q_i and define a set-function λ_i on $\mathcal{A}_i$ by

$$\lambda_i(A) = L(\chi_A), \quad (A \in \mathcal{A}_i).$$

Notice that $\lambda_i(A)$ is well-defined for all $A \in \mathcal{A}_i$ because χ_A belongs to $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

We claim that λ_i is countably additive on $\mathcal{A}_i$. Indeed, let $(A_k)_{k=1}^{\infty}$ be a sequence of disjoint sets from $\mathcal{A}_i$ and let

$$B_l = \bigcup_{k=1}^{l} A_k, \quad (l = 1, 2, \dots), \qquad A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{l=1}^{\infty} B_l.$$

It follows from (1.2) and the Lebesgue dominated convergence theorem that

$$\|\chi_A - \chi_{B_l}\|_{\mathcal{M}^p_q(\mathbb{R}^n)} \le \|\chi_A - \chi_{B_l}\|_{L^p(Q_i)} \to 0 \text{ as } l \to \infty.$$

The continuity and linearity of L give

$$\lambda_i(A) = L(\chi_A) = \lim_{l\to\infty} L(\chi_{B_l}) = \lim_{l\to\infty} \sum_{k=1}^{l} L(\chi_{A_k}) = \sum_{k=1}^{\infty} \lambda_i(A_k),$$

which establishes the claim.

Since $|\lambda_i(A)| \le \|L\|_{\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)^*}$ for all $A \in \mathcal{A}_i$ and $\lambda_i(A) = 0$ for all $A \in \mathcal{A}_i$ such that $|A| = 0$, by the Radon-Nikodym theorem, there is a unique measurable function g_i on Q_i such that

$$L(\chi_A) = \lambda_i(A) = \int_{\mathbb{R}^n} \chi_A(x)g_i(x)\,dx, \quad (A \in \mathcal{A}_i).$$

Since the sets Q_i are disjoint we may define a function g on all of $\mathbb{R}^n$ by setting $g = g_i$ on each Q_i. Clearly,

$$L(\chi_E) = \int_{\mathbb{R}^n} \chi_E(x)g(x)\,dx \tag{4.3}$$

for all characteristic functions of sets of finite measure χ_E.

We first show that g belongs to $\mathcal{M}_q^p(\mathbb{R}^n)'$. Choose and fix f in $\mathcal{M}_q^p(\mathbb{R}^n)$. Let, for $l = 1, 2, \dots$,

$$f_l(x) := \sum_{k=1}^{4^l} \frac{k}{2^l} \chi_{F_{k,l}}(x),$$

where

$$F_{k,l} := \left\{ x \in \mathbb{R}^n : |x| < 2^l, \frac{k}{2^l} \le |f(x)| < \frac{k+1}{2^l} \right\}.$$

If we suppose for the moment that g is real-valued, then $f_l \cdot \mathrm{sgn}\,(g)$ becomes a finite linear combination of characteristic functions of sets of finite measure. Hence, we may apply (4.3) and use the linearity of L to obtain

$$\int_{\mathbb{R}^n} f_l(x)|g(x)|\,dx = L(f_l \cdot \mathrm{sgn}\,(g)) \le \|L\|_{\widetilde{\mathcal{M}}_q^p(\mathbb{R}^n)^*} \|f_l\|_{\mathcal{M}_q^p(\mathbb{R}^n)}.$$

Letting $l \to \infty$, we have

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \le \|L\|_{\widetilde{\mathcal{M}}_q^p(\mathbb{R}^n)^*} \|f\|_{\mathcal{M}_q^p(\mathbb{R}^n)}$$

from the monotone convergence theorem and the Fatou property of Morrey norm. This means that g belongs to $\mathcal{M}_q^p(\mathbb{R}^n)'$. If g is complex-valued, then the same argument applied separately to the real and imaginary parts of g shows that each of these is in $\mathcal{M}_q^p(\mathbb{R}^n)'$ and, hence, that g again belongs to $\mathcal{M}_q^p(\mathbb{R}^n)'$.

Write, for a function f which can be written as a finite linear combination of characteristic functions of sets of finite measure,

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx$$

and observe the continuity of both sides on $\mathcal{M}_q^p(\mathbb{R}^n)$. Then we conclude that (4.2) holds. This complete the proof of the theorem. $\square$

REMARK 4.4. Let $1 < q \le p < \infty$. Let $\mathcal{C}_0$ be the class of continuous functions with compact support in $\mathbb{R}^n$. The Zorko space $\mathcal{Z}_q^p(\mathbb{R}^n)$ is defined by the closure in $\mathcal{M}_q^p(\mathbb{R}^n)$ of $\mathcal{C}_0$. In [2], Adams and Xiao pointed out (without detailed proof) $\mathcal{Z}_q^p(\mathbb{R}^n)$ is the predual of $\mathcal{B}_{q'}^{p'}(\mathbb{R}^n)$. In [6], Izumi, Sato and

Yabuta gave a detailed proof of this fact on the unit circle. The idea used in the proof of Theorem 1.2 comes from their nice paper.

We shall use $\{E_k\}_{k=1}^{\infty}$ to denote an arbitrary sequence of measurable subsets of $\mathbb{R}^n$. We shall write $E_k \to \emptyset$ a.e., if the characteristic functions χ_{E_k} converge to 0 pointwise a.e. Notice that the sets E_k are not required to have finite measure.

DEFINITION 4.5. Let $1 < q \le p < \infty$. A function f in $\mathcal{M}_q^p(\mathbb{R}^n)$ is said to have "absolutely continuous norm" in $\mathcal{M}_q^p(\mathbb{R}^n)$ if $\|f\chi_{E_k}\|_{\mathcal{M}_q^p(\mathbb{R}^n)} \to 0$ for every sequence $\{E_k\}_{k=1}^{\infty}$ satisfying $E_k \to \emptyset$ a.e. The set of all functions in $\mathcal{M}_q^p(\mathbb{R}^n)$ of absolutely continuous norm is denoted by $\widehat{\mathcal{M}}_q^p(\mathbb{R}^n)$.

THEOREM 4.6. *Let $1 < q \le p < \infty$. Then*

$$\widetilde{\mathcal{M}}_q^p(\mathbb{R}^n) = \widehat{\mathcal{M}}_q^p(\mathbb{R}^n).$$

PROOF. By [3, Theorem 3.13], we need only verify that the characteristic function χ_E has absolutely continuous norm for every set E of finite measure. Let $\{F_k\}_{k=1}^{\infty}$ be an arbitrary sequence for which $F_k \to \emptyset$ a.e. Then it follows from (1.2) and the Lebesgue dominated convergence theorem that

$$\|\chi_E\chi_{F_k}\|_{\mathcal{M}_q^p(\mathbb{R}^n)} \le \|\chi_E\chi_{F_k}\|_{L^p(\mathbb{R}^n)} \to 0 \text{ as } k \to \infty,$$

which proves the theorem. $\square$

Recently there are many closed subspaces of Morrey spaces. We refer to [13] as well as [15, Definition 2.23]. Also, it might be interesting to compare Theorem 4.3 with [12, Theorem 1.3], where they considered function spaces called smoothness Morrey spaces. We refer to [14] for an exhaustive detail of smoothness Morrey spaces.

5. Miscellaneous

Example 5.1. Let $1 < q < p < \infty$. We show that $\widetilde{\mathcal{M}}_q^p(\mathbb{R}^n)$ and $\mathcal{M}_q^p(\mathbb{R}^n)$ are different spaces. In fact, the former is narrower than the latter. We prove this by giving an example when $n = 1$; other cases can be dealt similarly.

Set

$$E := \bigcup_{k=1}^{\infty} (k - 1 + k^{\frac{p}{p-q}},\, k + k^{\frac{p}{p-q}}).$$

Then we see that χ_E belongs to $\mathcal{M}^p_q(\mathbb{R}^n)$ but does not belong to $\widetilde{\mathcal{M}}^p_q(\mathbb{R}^n)$.

Example 5.2. Let $1 < q < p < \infty$ and $L : \mathcal{M}^p_q(\mathbb{R}^n) \to \mathbb{R}$ be a bounded linear functional. Then in view of the embedding $L^p(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_q(\mathbb{R}^n)$, one has a function $g \in L^{p'}(\mathbb{R}^n)$ such that

$$L(f) = \int_{\mathbb{R}^n} f(x) g(x)\, dx, \quad (f \in L^p(\mathbb{R}^n)).$$

However, it can happen that L is not zero even when $g \equiv 0$; One can show this by an example.

Recall the set E defined in Example 5.1. Set, for $k = 1, 2, \dots,$

$$I_k := (k - 1 + k^{\frac{p}{p-q}},\, k + k^{\frac{p}{p-q}}).$$

Then,

$$\lim_{k \to \infty} \int_{I_k} \chi_E(x)\, dx = 1.$$

With this in mind, let us define a closed subspace H by

$$H := \left\{ f \in \mathcal{M}^p_q(\mathbb{R}) : \lim_{k \to \infty} \int_{I_k} f(x)\, dx \text{ exists} \right\}.$$

Then, from the definition of the norm, we have

$$\lim_{k \to \infty} \left| \int_{I_k} f(x)\, dx \right| \le \|f\|_{\mathcal{M}^p_q(\mathbb{R})}.$$

Consequently, it follows from the Hahn-Banach theorem that the mapping

$$f \in H \mapsto \lim_{k \to \infty} \int_{I_k} f(x)\, dx \in \mathbb{R}$$

extends to a continuous linear functional L. Observe that $L(\chi_E) = 1$ and, hence, $L \neq 0$. Meanwhile, L annihilates any compactly supported function in $\mathcal{M}^p_q(\mathbb{R})$ because such a function belongs to H. Therefore, if one considers a function g satisfying

$$L(f) = \int_{\mathbb{R}^n} f(x)g(x)\,dx$$

for all $f \in L^p(\mathbb{R}^n)$, then one obtains $g \equiv 0$ by virtue of the Lebesgue dominated convergence theorem.

We end this paper with the following proposition.

Proposition 5.3. *Let $1 < q \le p < \infty$. Suppose that $f \in L^{q'}(\mathbb{R}^n)$ has compact support. Then there exists a finite sequence $\{\lambda_j\}_{j=1}^N$ of nonnegative numbers and $\{b_j\}_{j=1}^N$ of (p', q')-blocks such that*

$$f = \sum_{j=1}^N \lambda_j b_j \text{ and } \sum_{j=1}^N \lambda_j \le 8\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)}.$$

Proof. The proof will be complete once we show that there exists a finite sequence $\{\lambda_j\}_{j=1}^N$ of nonnegative numbers and $\{b_j\}_{j=1}^N$ of (p', q')-blocks such that

$$f = \sum_{j=1}^N \lambda_j b_j \text{ and } \sum_{j=1}^N \lambda_j \le 2\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)}$$

when f is positive.

We know that, as is illustrated by the proof of Lemma 2.1, there exist an infinite sequence $\{\Lambda_j\}_{j=1}^\infty$ of nonnegative numbers and an infinite sequence $\{B_j\}_{j=1}^\infty$ of nonnegative (p', q')-blocks such that

$$f = \sum_{j=1}^\infty \Lambda_j B_j \text{ and } \sum_{j=1}^\infty \Lambda_j \le \frac{3}{2}\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)}.$$

Suppose that the support of f is engulfed by a large cube Q_0. By using the characteristic functions, we may as well assume that B_j is supported on Q_0. Then we have

$$f = \sum_{j=1}^{N-1} \Lambda_j B_j + \sum_{j=N}^\infty \Lambda_j B_j$$

and

$$\sum_{j=1}^{N-1} \Lambda_j + |Q_0|^{\frac{1}{q}-\frac{1}{p}} \left\| \sum_{j=N}^{\infty} \Lambda_j B_j \right\|_{L^{q'}(\mathbb{R}^n)}$$
$$\leq \frac{3}{2}\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)} + |Q_0|^{\frac{1}{q}-\frac{1}{p}} \left\| f - \sum_{j=1}^{N-1} \Lambda_j B_j \right\|_{L^{q'}(\mathbb{R}^n)}.$$

By the monotone convergence theorem, we see that

$$\sum_{j=1}^{N-1} \Lambda_j + |Q_0|^{\frac{1}{q}-\frac{1}{p}} \left\| \sum_{j=N}^{\infty} \Lambda_j B_j \right\|_{L^{q'}(\mathbb{R}^n)} \leq 2\|f\|_{\mathcal{B}^{p'}_{q'}(\mathbb{R}^n)}$$

as long as N is sufficient large. Thus, if we define $\lambda_1, \lambda_2, \dots, \lambda_N$ and $b_1, b_2, \dots, b_N$ as

$$\lambda_j = \Lambda_j,\ b_j = B_j \text{ when } j = 1, 2, \dots, N-1,$$

and

$$\Lambda_N = |Q_0|^{\frac{1}{q}-\frac{1}{p}} \left\| \sum_{j=N}^{\infty} \Lambda_j B_j \right\|_{L^{q'}(\mathbb{R}^n)}, \quad b_N = \begin{cases} \dfrac{1}{\lambda_N} \sum_{j=N}^{\infty} \Lambda_j B_j & \lambda_N \neq 0 \\ 0 & \lambda_N = 0, \end{cases}$$

then we obtain the desired decomposition. □

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