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Matrix Analogues of Some Properties for Bessel Matrix Functions

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Abstract. In this study, we obtain some recurrence relations for Bessel matrix functions of the first kind. Then we derive a new integral representation of these matrix functions. We lastly give new relations between Bessel matrix functions of the first kind and Laguerre matrix polynomials.

1. Introduction

In recent years, the theory of special matrix functions has become an active research area in applied mathematics. In this theory, Gamma and Beta matrix functions [10], Hermite matrix polynomials [2], Laguerre matrix polynomials [9], Jacobi matrix polynomials [4], Chebyshev matrix polynomials [3] and Bessel matrix polynomials [12] were studied. Furthermore, Bessel matrix functions were introduced by Jódar et al. in [6, 8, 7, 14]. It was considered that these function satisfy Bessel type differential equation

$$t^2 X''(t) + tX'(t) + \left(t^2 I - A^2\right) X(t) = 0,\ 0 < t < \infty$$

where A is a matrix in $\mathbb{C}^{r\times r}$ and unknown $X(t)$ is a $\mathbb{C}^{r\times 1}$-valued function. Jódar et al. gave different solutions of Bessel type differential equation according to a matrix A. After that study, Jódar et al. defined Bessel matrix functions of first kind and second kind. By using the kinds of Bessel matrix functions, the general solutions of this equation were obtained. Then, Çekim and Erkuş-Duman derived some integral representations of Bessel matrix functions in [1]. In current study, after having obtained the various properties by using these functions, we give relations between Bessel matrix functions of the first kind and Laguerre matrix polynomials.

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2. Preliminaries and Some Facts

For the sake of clarity, in the presentation of the next results, we recall some concepts and properties of the matrix functional calculus. Throughout this paper, we denote the set of all eigenvalues for a matrix $A \in \mathbb{C}^{r\times r}$ by $\sigma(A)$.

Definition 1. A matrix $A \in \mathbb{C}^{r\times r}$ is called a positive stable matrix if $Re(\lambda) > 0$ for all $\lambda \in \sigma(A)$.

Theorem 1 ([5]). *If $f(z)$ and $g(z)$ are holomorphic functions in an open set Ω of the complex plane, and if A is a matrix in $\mathbb{C}^{r\times r}$ for which $\sigma(A) \subset \Omega$, then*

$$f(A)g(A) = g(A)f(A).$$

Definition 2 ([9]). For a positive stable matrix P in $\mathbb{C}^{r\times r}$, the Gamma matrix function $\Gamma(P)$ is defined by

$$\Gamma(P) = \int_0^\infty e^{-t}\, t^{P-I}\, dt, \tag{2.1}$$

where $t^{P-I} = \exp\left[(P-I)\ln t\right]$.

Furthermore, the reciprocal scalar Gamma function, $\Gamma^{-1}(z) = 1/\Gamma(z)$, is an entire function of the complex variable z. Thus, for any $A \in \mathbb{C}^{r\times r}$, the Riesz-Dunford functional calculus [5] shows that $\Gamma^{-1}(A)$ is well defined and, indeed, it is the inverse of $\Gamma(A)$. Hence, if $\sigma(A)$ $(A \in \mathbb{C}^{r\times r})$ does not contain 0 or a negative integer,

$$\Gamma^{-1}(A) = A(A+I)(A+2I)...(A+kI)\Gamma^{-1}(A+(k+1)I). \tag{2.2}$$

Definition 3 ([9]). For positive stable matrices P and Q in $\mathbb{C}^{r\times r}$, the Beta matrix function $\mathcal{B}(P,Q)$ is defined by

$$\mathcal{B}(P,Q) = \int_0^1 t^{P-I}\ (1-t)^{Q-I}\ dt. \tag{2.3}$$

THEOREM 2 ([9]). *Let P,Q, $P+Q$ be positive stable matrices in $\mathbb{C}^{r\times r}$ and $PQ = QP$. Then*

$$\mathcal{B}\left(P,Q\right) = \Gamma\left(P\right)\Gamma\left(Q\right)\Gamma^{-1}\left(P+Q\right). \tag{2.4}$$

THEOREM 3 ([11]). *Let P and Q be matrices in $\mathbb{C}^{r\times r}$ where*

$PQ = QP$ *and*
$P+nI, Q+nI,\ P+Q+nI$ *are invertible for all $n \in \mathbb{N}$.*

Then, we have

$$\mathcal{B}\left(P,Q\right) = \Gamma\left(P\right)\Gamma\left(Q\right)\Gamma^{-1}\left(P+Q\right). \tag{2.5}$$

Moreover, let us consider the Bessel functions of the first kind of order υ, defined by

$$J_\upsilon(t) = \left(\frac{t}{2}\right)^\upsilon \sum_{m\geq 0} \frac{(-1)^m}{m!}\Gamma^{-1}(\upsilon+m+1)\left(\frac{t}{2}\right)^{2m},\ 0<t<\infty$$

which is an entire function of the parameter υ [13]. Thus, if H is a Jordan block of the following form

$$H = \begin{bmatrix} \upsilon & 1 & 0 & \dots & 0 \\ 0 & \upsilon & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \upsilon \end{bmatrix} \in \mathbb{C}^{r\times r}, \tag{2.6}$$

we can write the image by means of the matrix functional calculus acting on the matrix H and the function of υ, $J_\upsilon(t)$, the Bessel matrix functions of the first kind of order H is defined as follows:

$$J_H(t) = \left(\frac{t}{2}\right)^H \sum_{m\geq 0} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+1)I)\left(\frac{t}{2}\right)^{2m},\ 0<t<\infty \tag{2.7}$$

where $\upsilon \notin \mathbb{Z}^-$ [8]. Let us take $A \in \mathbb{C}^{r\times r}$ satisfying

$$\sigma(A) \cap \mathbb{Z}^- = \phi. \tag{2.8}$$

In [8], Jódar et al. defined Bessel matrix functions of the first kind of order A as the following:

$$J_A(t) = \left(\frac{t}{2}\right)^A \sum_{m\geq 0} \frac{(-1)^m}{m!}\Gamma^{-1}(A+(m+1)I)\left(\frac{t}{2}\right)^{2m}, \; 0 < t < \infty. \tag{2.9}$$

Now, let us consider the general case. Let A be a matrix satisfying the condition (2.8) and $H = \operatorname{diag}(H_1, ..., H_k)$ be the Jordan canonical form of A, where H_i is a Jordan block such that

$$H_i = \begin{bmatrix} \upsilon_i & 1 & 0 & \dots & 0 \\ 0 & \upsilon_i & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & \upsilon_i \end{bmatrix} \in \mathbb{C}^{p_i \times p_i}, \tag{2.10}$$

$$p_i \geq 1, \; p_1 + p_2 + ... + p_k = r.$$

Here if $p_i = 1$, we can write $H_i = (\upsilon_i)$ where $\upsilon_i \notin \mathbb{Z}^-$ for $1 \leq i \leq k$. Thus, the Bessel matrix functions of the first kind can be written in the form

$$J_H(t) = \operatorname{diag}_{1\leq i\leq k}(J_{H_i}(t)).$$

Furthermore, if P is an invertible matrix in $\mathbb{C}^{r\times r}$ and $H = \operatorname{diag}(H_1, ..., H_k)$ is the Jordan canonical form of A such that

$$H = \operatorname{diag}(H_1, ..., H_k) = PAP^{-1},$$

we have the Bessel matrix functions of the first kind of order A in (2.9) (see [8]).

3. Some Properties for Bessel Matrix Functions of the First Kind

In this section, we derive some recurrence relations for Bessel matrix functions of the first kind. Let H be a Jordan block defined as in (2.6).

Thus, if H satisfies the condition (2.8), and $0 < z < \infty$, we have the following properties. Using (2.7) and differentiating with respect to z, we have

$$\begin{aligned}
&\frac{d}{dz}[z^H J_H(z)] \\
&\quad = \frac{d}{dz}\left(z^H \left(\frac{z}{2}\right)^H \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+1)I)\left(\frac{z}{2}\right)^{2m}\right) \\
&\quad = 2^{H-I}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+1)I)(2mI+2H)\left(\frac{z}{2}\right)^{(2m-1)I+2H} \\
&\quad = z^H\left(\left(\frac{z}{2}\right)^{H-I} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+mI)\left(\frac{z}{2}\right)^{2m}\right) \\
&\quad = z^H J_{H-I}(z).
\end{aligned}$$

Thus, for the Bessel matrix functions of the first kind, we have the first property

$$\frac{d}{dz}[z^H J_H(z)] = z^H J_{H-I}(z) \tag{3.1}$$

where $0 \notin \sigma(H)$. Similarly, we obtain

$$\begin{aligned}
\frac{d}{dz}[z^{-H} J_H(z)] &= \frac{d}{dz}\left(z^{-H} \left(\frac{z}{2}\right)^H \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+1)I)\left(\frac{z}{2}\right)^{2m}\right) \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m}{m!}2m\Gamma^{-1}(H+(m+1)I)2^{-2mI-H}z^{2m-1} \\
&= -\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+2)I)2^{-(2m+1)I-H}z^{2m+1} \\
&= -z^{-H}\left(\sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+2)I)\left(\frac{z}{2}\right)^{(2m+1)I+H}\right) \\
&= -z^{-H} J_{H+I}(z).
\end{aligned}$$

That is, we have the second property

$$\frac{d}{dz}[z^{-H} J_H(z)] = -z^{-H} J_{H+I}(z). \tag{3.2}$$

Then from (3.1), we obtain

$$zJ_H'(z) = zJ_{H-I}(z) - HJ_H(z) \tag{3.3}$$

where $0 \notin \sigma(H)$. From (3.2), we derive

$$zJ_H'(z) = -zJ_{H+I}(z) + HJ_H(z) \tag{3.4}$$

where $0 \notin \sigma(H)$. Also, by (3.3) and (3.4), we get

$$2J_H'(z) = J_{H-I}(z) - J_{H+I}(z) \tag{3.5}$$

and

$$2HJ_H(z) = z[J_{H-I}(z) + J_{H+I}(z)] \tag{3.6}$$

where $0 \notin \sigma(H)$. Furthermore, using (2.7), we find

$$J_H(z) = \frac{2}{z}(H-I)J_{H-I}(z) - J_{H-2I}(z) \tag{3.7}$$

where $0, 1 \notin \sigma(H)$.

Now, let us give a generalization for these properties. Let H_i be a matrix defined by (2.10) and $H = \mathrm{diag}(H_1, ..., H_k)$ be a matrix in $\mathbb{C}^{r\times r}$. Here v_i is not a negative integer for $1 \le i \le k$. For the matrix $H = \mathrm{diag}(H_1, ..., H_k)$, the properties (3.1)∼(3.7) can easily be obtained.

Also, if P is an invertible matrix in $\mathbb{C}^{r\times r}$ and $H = \mathrm{diag}(H_1, ..., H_k)$ is the Jordan canonical form of A such that

$$H = \mathrm{diag}(H_1, ..., H_k) = PAP^{-1},$$

we can prove the following theorem for the matrix A.

Theorem 4. *If A is a matrix in $\mathbb{C}^{r\times r}$ which satisfies the condition (2.8), and $0 < z < \infty$, we obtain*

$$\begin{aligned}
&\frac{d}{dz}[z^A J_A(z)] = z^A J_{A-I}(z) \textit{ where } 0 \notin \sigma(A),\\
&\frac{d}{dz}[z^{-A} J_A(z)] = -z^{-A} J_{A+I}(z),\\
&zJ_A'(z) = zJ_{A-I}(z) - AJ_A(z) \textit{ where } 0 \notin \sigma(A),
\end{aligned}$$

$$zJ_A^{'}(z) = -zJ_{A+I}(z) + AJ_A(z),$$
$$2J_A^{'}(z) = J_{A-I}(z) - J_{A+I}(z) \text{ where } 0 \notin \sigma(A),$$
$$2AJ_A(z) = z[J_{A-I}(z) + J_{A+I}(z)] \text{ where } 0 \notin \sigma(A),$$
$$J_A(z) = \frac{2}{z}(A-I)J_{A-I}(z) - J_{A-2I}(z) \text{ where } 0, 1 \notin \sigma(A).$$

4. An Integral Representation for Bessel Matrix Functions of the First Kind

In this section, we derive a new integral representation for Bessel matrix functions of the first kind. The integral is given as

$$\int_0^t (\sqrt{x(t-x)})^H J_H\left(\sqrt{x(t-x)}\right) dx \tag{4.1}$$

where H is a Jordan block matrix satisfying the condition (2.8), $0 < t < \infty$. Substituting (2.7) in (4.1) and choosing $x = ty$, it follows that

$$\begin{aligned}
&\int_0^t (\sqrt{x(t-x)})^H J_H\left(\sqrt{x(t-x)}\right) dx \\
&= t^{H+I} \int_0^1 (\sqrt{y(1-y)})^H \left(\frac{t\sqrt{y(1-y)}}{2}\right)^H \\
&\times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+1)I) \left(\frac{t\sqrt{y(1-y)}}{2}\right)^{2m} dy \\
&= t^{H+I}\left(\frac{t}{2}\right)^H \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}\Gamma^{-1}(H+(m+1)I)\left(\frac{t}{2}\right)^{2m} \\
&\times \int_0^1 y^{H+mI}(1-y)^{H+mI} dy.
\end{aligned}$$

From (2.3) and (2.5), we have

$$\begin{aligned}
&\int_0^t (\sqrt{x(t-x)})^H J_H\left(\sqrt{x(t-x)}\right) dx \\
&= t^{H+I} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma^{-1}(H+(m+1)I) \left(\frac{t}{2}\right)^{2mI+H} \\
&\times B(H+(m+1)I, H+(m+1)I) \\
&= t^{H+I} \sum_{m=0}^{\infty} \left\{ \frac{(-1)^m}{m!} \Gamma^{-1}(H+(m+1)I) \left(\frac{t}{2}\right)^{2mI+H} \right. \\
&\left. \times\, \Gamma^2(H+(m+1)I)\Gamma^{-1}(2H+(2m+2)I) \right\}
\end{aligned}$$

where $2H+nI$ is invertible for all $n \in \mathbb{N}$. Also, by (2.2) and (2.7), we obtain

$$\begin{aligned}
&\int_0^t (\sqrt{x(t-x)})^H J_H\left(\sqrt{x(t-x)}\right) dx \\
&= \sqrt{\pi} t^{H+I} \sum_{m=0}^{\infty} \left\{ \frac{(-1)^m}{m!} \Gamma^{-1}(H+(m+1)I) \left(\frac{t}{2}\right)^{2mI+H} \Gamma^2(H+(m+1)I) \right. \\
&\left. \times\, 2^{-2H-(2m+1)I} \Gamma^{-1}(H+(m+1)I) \Gamma^{-1}\left(H+\tfrac{1}{2}I+(m+1)I\right) \right\} \\
&= \sqrt{\pi} t^{H+\frac{1}{2}I} 2^{-H} \left(\frac{t}{4}\right)^{H+\frac{1}{2}I} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \Gamma^{-1}(H+\frac{1}{2}I+(m+1)I) \left(\frac{t}{4}\right)^{2m} \\
&= \sqrt{\pi} t^{H+\frac{1}{2}I} 2^{-H} J_{H+\frac{1}{2}I}\left(\frac{t}{2}\right).
\end{aligned}$$

This result is summarized below:

THEOREM 5. *If H is a Jordan block matrix in $\mathbb{C}^{r\times r}$ which satisfies condition (2.8) and $2H+nI$ is invertible for all $n \in \mathbb{N}$, $0 < t < \infty$, we obtain the following integral representation:*

$$\int_0^t (\sqrt{x(t-x)})^H J_H\left(\sqrt{x(t-x)}\right) dx = \sqrt{\pi} t^{H+\frac{1}{2}I} 2^{-H} J_{H+\frac{1}{2}I}\left(\frac{t}{2}\right).$$

Now, we can give the generalization of Theorem 5.

COROLLARY 1. *If A is a matrix in $\mathbb{C}^{r\times r}$ which satisfies condition (2.8) and $2A+nI$ is invertible for all $n \in \mathbb{N}$, $0 < t < \infty$, we obtain the following integral representation:*

$$\int_0^t (\sqrt{x(t-x)})^A J_A\left(\sqrt{x(t-x)}\right) dx = \sqrt{\pi} t^{A+\frac{1}{2}I} 2^{-A} J_{A+\frac{1}{2}I}\left(\frac{t}{2}\right).$$

5. New Relations of Bessel Matrix Functions of the First Kind and Laguerre Matrix Polynomials

In this section, we obtain the new relations between Bessel matrix functions of the first kind and Laguerre matrix polynomials.

Let A be a matrix in $\mathbb{C}^{r\times r}$ satisfying the condition (2.8) and λ be a complex number whose real part is positive. Then, the Laguerre matrix polynomials $L_n^{(A,\lambda)}(x)$ are defined by [9]:

$$L_n^{(A,\lambda)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!(n-k)!}(A+I)_n \left[(A+I)_k\right]^{-1} (\lambda x)^k \quad , \quad n \in \mathbb{N}. \tag{5.1}$$

These matrix polynomials have the following generating matrix function:

$$\sum_{n=0}^{\infty} L_n^{(A,\lambda)}(x) t^n = (1-t)^{-A-I} e^{-\frac{\lambda x t}{1-t}} \; ; \; x \in \mathbb{C} \; , t \in \mathbb{C}, \; |t| < 1. \tag{5.2}$$

Recall that, if $M \in \mathbb{C}^{r\times r}$ is a positive stable matrix and $n \in \mathbb{Z}^+$, from [10], we have

$$\Gamma(M) = \lim_{n\to\infty} (n-1)! \, (M)_n^{-1} \, n^M. \tag{5.3}$$

We are in a position to give the first relation of Bessel matrix functions of the first kind and Laguerre matrix polynomials.

THEOREM 6. *Bessel matrix functions of the first kind and Laguerre matrix polynomials satisfy the relation*

$$\lim_{n\to\infty} \left\{ n^{-H} L_n^{(H,\lambda)}\left(\frac{x}{n}\right)\right\} = (\lambda x)^{-\frac{1}{2}H} J_H(2\sqrt{\lambda x}) \tag{5.4}$$

where x and λ are positive real numbers, $n \in \mathbb{Z}^+$, and H is both Jordan block and positive stable matrix.

PROOF. Using (5.1),(5.3) and (2.7), respectively, we get

$$\begin{aligned}
&\lim_{n\to\infty} \left\{ n^{-H} L_n^{(H,\lambda)} \left(\frac{x}{n}\right) \right\} \\
&= \lim_{n\to\infty} \left\{ n^{-H} \frac{(H+I)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k \lambda^k}{k!} (H+I)_k^{-1} \left(\frac{x}{n}\right)^k \right\} \\
&= \Gamma^{-1}(H+I) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} (H+I)_k^{-1} (\lambda x)^k \\
&= (\lambda x)^{\frac{1}{2}H} (\lambda x)^{-\frac{1}{2}H} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \Gamma^{-1}(H+(k+1)I)(\lambda x)^k \\
&= (\lambda x)^{-\frac{1}{2}H} J_H(2\sqrt{\lambda x}).
\end{aligned}$$

Hence, the proof is completed. □

COROLLARY 2. *Bessel matrix functions of the first kind and Laguerre matrix polynomials satisfy the following relation*

$$\lim_{n\to\infty} \left\{ n^{-A} L_n^{(A,\lambda)} \left(\frac{x}{n}\right) \right\} = (\lambda x)^{-\frac{1}{2}A} J_A(2\sqrt{\lambda x})$$

where x and λ are positive real numbers, $n \in \mathbb{Z}^+$, and A is a positive stable matrix.

Lastly, we present the second relation of Bessel matrix functions of the first kind and Laguerre matrix polynomials.

THEOREM 7. *Bessel matrix functions of the first kind and Laguerre matrix polynomials satisfy the relation*

$$\sum_{n=0}^{\infty} \Gamma^{-1}(H+(n+1)I) L_n^{(H,\lambda)}(x) t^n = e^t (\lambda x t)^{-\frac{1}{2}H} J_H(2\sqrt{\lambda x t}) \tag{5.5}$$

where H is a Jordan block satisfying the condition (2.8), and x, λ and t are positive real numbers.

PROOF. Using (5.1) and (2.7), we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty} \Gamma^{-1}(H+(n+1)I) L_n^{(H,\lambda)}(x) t^n \\
&= \sum_{n=0}^{\infty} \frac{\Gamma^{-1}(H+I)}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{k!} (H+I)_k^{-1} (\lambda x)^k t^n \\
&= \sum_{n,k=0}^{\infty} \frac{(-1)^k}{n!\ k!} \Gamma^{-1}(H+(k+1)I)(\lambda x)^k t^{n+k} \\
&= e^t (\lambda x t)^{-\frac{1}{2}H} J_H(2\sqrt{\lambda x t}),
\end{aligned}$$

which completes the proof. □

COROLLARY 3. *Bessel matrix functions of the first kind and Laguerre matrix polynomials satisfy the relation*

$$\sum_{n=0}^{\infty} \Gamma^{-1}(A+(n+1)I) L_n^{(A,\lambda)}(x) t^n = e^t (\lambda x t)^{-\frac{1}{2}A} J_A(2\sqrt{\lambda x t})$$

where x, λ and t are positive real numbers, and A is a matrix satisfying the condition (2.8).

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