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The Generalized Hodge and Bloch Conjectures are Equivalent for General Complete Intersections, II

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This paper is dedicated to the memory of Kunihiko Kodaira, on the occasion of his centenary

Abstract. We prove an unconditional (but slightly weakened) version of the main result of [13], which was, starting from dimension 4, conditional to the Lefschetz standard conjecture. Let X be a variety with trivial Chow groups, (i.e. the cycle class map to cohomology is injective on $CH(X)_{\mathbb{Q}}$). We prove that if the cohomology of a general hypersurface Y in X is "parameterized by cycles of dimension c", then the Chow groups $CH_i(Y)_{\mathbb{Q}}$ are trivial for $i \leq c-1$.

Let X be a smooth complex projective variety. We will say that X has geometric coniveau $\geq c$ if the transcendental cohomology of X, that is, the orthogonal complement with respect to Poincaré duality of the "algebraic cohomology" of X generated by classes of algebraic cycles,

$$H^*(X,\mathbb{Q})_{tr} := H^*(X,\mathbb{Q})^{\perp}_{alg},$$

is supported on a closed algebraic subset $W \subset X$, with $\operatorname{codim} W \geq c$.

According to the generalized Hodge conjecture [7], X has geometric coniveau $\geq c$ if and only if X has Hodge coniveau $\geq c$, where we define the Hodge coniveau of X as the minimum over k of the Hodge coniveaux of the Hodge structures $H^k(X,\mathbb{Q})_{tr}$. Here we recall that the Hodge coniveau of a weight k Hodge structure $(L, L^{p,q})$ is the integer $c \leq k/2$ such that

$$L_{\mathbb{C}} = L^{k-c,c} \oplus L^{k-c-1,c+1} \oplus \ldots \oplus L^{c,k-c}$$

with $L^{k-c,c} \neq 0$. As the Hodge coniveau is computable by looking at the Hodge numbers, we know conjecturally how to compute the geometric coniveau.

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A fundamental conjecture on algebraic cycles is the generalized Bloch conjecture (see [17, Conjecture 1.10]), which was formulated by Bloch [1] in the case of surfaces, and can be stated as follows:

CONJECTURE 0.1. *Assume X has geometric coniveau $\geq c$. Then the cycle class map*

$$CH_i(X)_{\mathbb{Q}} \to H^{2n-2i}(X, \mathbb{Q}),\ n = \dim X,$$

is injective for any $i \leq c-1$.

Concrete examples are given by hypersurfaces in projective space, or more generally complete intersections. For a smooth complete intersection Y of r hypersurfaces in $\mathbb{P}^n$, the Hodge coniveau of Y is equal to the Hodge coniveau of $H^{n-r}(Y, \mathbb{Q})_{tr}$, the last space being for the very general member Y, except in a small number of cases, equal to the Hodge coniveau of $H^{n-r}(Y, \mathbb{Q})_{prim}$. The latter is computed by Griffiths:

THEOREM 0.2. *If $Y \subset \mathbb{P}^n$ is a complete intersection of hypersurfaces of degrees $d_1 \leq \ldots \leq d_r$, the Hodge coniveau of $H^{n-r}(Y, \mathbb{Q})_{prim}$ is $\geq c$ if and only if $n \geq \sum_i d_i + (c-1)d_r$.*

Conjecture 0.1, combined with the Grothendieck-Hodge conjecture, thus predicts that for such a Y, the Chow groups $CH_i(Y)_{\mathbb{Q}}$ are equal to $\mathbb{Q}$ for $i \leq c-1$, a result which is essentially known only for coniveau 1 (then Y is a Fano variety, so $CH_0(Y) = \mathbb{Z}$) and a small number of particular cases for coniveau ≥ 2, e.g. cubic hypersurfaces of dimension ≤ 8 or complete intersections of quadrics [9]. Note that weaker statements are known, thanks to the work of Paranjape [10] or Esnault-Levine-Viehweg [5], saying that for fixed multidegree $d_1 \leq \ldots \leq d_r$ and for fixed i, the Chow groups $CH_i(Y)_{\mathbb{Q}}$ are equal to $\mathbb{Q}$ for Y as above and very large n.

We will say that a smooth projective variety X has trivial Chow groups if for any i, the cycle class map $CH_i(X)_{\mathbb{Q}} \to H^{2n-2i}(X, \mathbb{Q})$, $n = \dim X$, is injective. By [8], this implies that the whole rational cohomology of X is algebraic, that is, consists of cycle classes. The class of such varieties includes projective spaces and more generally toric varieties, Grassmannians, projective bundles over a variety with trivial Chow groups, see [13] for further discussion of this notion.

In [13], we proved Conjecture 0.1 for very general complete intersections of very ample hypersurfaces in a smooth projective variety X with trivial Chow groups, *assuming the Lefschetz standard conjecture*. More precisely, the results proved in loc. cit. are unconditional in the case of complete intersections surfaces and threefolds, for which the Lefschetz standard conjecture is not needed. They have been improved later on for families of surfaces in [14], where the geometric setting is much more general: instead of the universal family of complete intersection surfaces in some X, we consider any family of smooth projective surfaces $\mathcal{S} \to B$ satisfying the condition that $\mathcal{S} \times_B \mathcal{S} \to B$ has a smooth projective completion which is rationally connected or more generally has trivial CH_0 group.

The purpose of this paper is to prove unconditionally, in the geometric setting of general complete intersections Y in a variety X with trivial Chow groups, a slightly weaker form of Conjecture 0.1, which is equivalent to it in dimension 2, 3, or assuming the Lefschetz standard conjecture.

Assume Y has dimension m and geometric coniveau c. Then there exist a smooth projective variety W with $\dim W = m - c$, and a morphism $j: W \to Y$ such that $j_* : H^{m-2c}(W, \mathbb{Q}) \to H^m(Y, \mathbb{Q})_{tr}$ is surjective. This follows from the definition of the geometric coniveau and from Deligne's results on mixed Hodge structures [4] (see [17, proof of Theorem 2.39]). Let us now introduce a stronger notion, which we will reformulate later on in a more geometric form (see Lemma 1.1).

DEFINITION 0.3. Let Y be smooth projective of dimension m. We will say that the degree m cohomology of Y (or its primitive part with respect to a polarization) is parameterized by algebraic cycles of dimension c if

a) There exist a smooth projective variety T of dimension $m - 2c$ and a correspondence $P \in CH^{m-c}(T \times Y)_{\mathbb{Q}}$, such that

$$P^* : H^m(Y, \mathbb{Q}) \to H^{m-2c}(T, \mathbb{Q})$$

is injective (or equivalently: $P_* : H^{m-2c}(T, \mathbb{Q}) \to H^m(Y, \mathbb{Q})$ is surjective), resp.

$$P^* : H^m(Y, \mathbb{Q})_{prim} \to H^{m-2c}(T, \mathbb{Q})$$

is injective.

b) Furthermore P^* is compatible up to a coefficient with the intersection forms: for some rational number $N \neq 0$, $< P^*\alpha, P^*\beta >_T = N < \alpha, \beta >_Y$ for any α, $\beta \in H^m(Y, \mathbb{Q})$, (resp. for any α, $\beta \in H^m(Y, \mathbb{Q})_{prim}$).

Remark 0.4. The condition a) in Definition 0.3 obviously implies that $H^m(Y,\mathbb{Q})$ has geometric coniveau $\geq c$, since it vanishes away from the image in Y of the support of P, which is of dimension $\leq m-c$. The more precise condition that $H^m(Y,\mathbb{Q})$ comes from the cohomology of a variety T of dimension $\leq m-2c$ is formulated explicitly in [12], where it is shown that the two conditions are equivalent assuming the Lefschetz standard conjecture. Our definition is still stronger since we also impose the condition b) concerning the comparison of the intersection forms.

Remark 0.5. Assume

(i) The Hodge structure on $H^m(Y,\mathbb{Q})_{prim}$ is simple and does not admit other polarizations than the multiples of the one given by $<,>_Y$.

(ii) The Hodge structure on $H^m(Y,\mathbb{Q})_{prim}$ is exactly of Hodge coniveau c.

Then the nontriviality of $P^*: H^m(Y,\mathbb{Q})_{prim}\to H^{m-2c}(T,\mathbb{Q})$ implies its injectivity by the simplicity of the Hodge structure and also the condition b) of compatibility with the cup-product. Indeed, by assumption, $H^{m-c,c}(Y)_{prim}\neq 0$ hence by injectivity of P^*, we get nonzero classes $P^*\alpha\in H^{m-2c,0}(T)$. By the second Hodge-Riemann bilinear relations [17, 2.2.1], we then have $<P^*\alpha,P^*\overline{\alpha}>_T\neq 0$. Thus the pairing $<P^*\alpha,P^*\beta>_T$ on the Hodge structure $H^m(Y,\mathbb{Q})_{prim}$ is nondegenerate and polarizes this Hodge structure. Hence it must be by uniqueness a nonzero rational multiple of the pairing $<,>_Y$ and thus, condition b) is automatically satisfied in this case.

Assumption (i) above, which is a Noether-Lefschetz type statement, is easy to prove in practice for the very general member of the family of hypersurfaces or complete intersections, using a Mumford-Tate groups argument. Thus the remark above says that in practice and for the very general member of a family, the hard point to check in Definition 0.3 is a), as b) follows.

Actually, we will use in the paper a reformulation (which is in fact a weakening) of Definition 0.3 (see Lemma 1.1). Namely, assuming that the cohomology of Y splits as the orthogonal direct sum

$$H^*(Y,\mathbb{Q})=K\bigoplus_{\perp}H^m(Y,\mathbb{Q})_{prim},\ K\subset H^*(Y,\mathbb{Q})_{alg},$$

our set of conditions a) and b) for primitive cohomology is equivalent to the fact that there is a cohomological decomposition of the diagonal

$$[\Delta_Y] = [Z] + \sum_i \alpha_i [Z_i \times Z_i'] \text{ in } H^{2m}(Y \times Y, \mathbb{Q}),$$

where Z_i, Z_i' are algebraic subvarieties of Y, $\dim Z_i + \dim Z_i' = m$, and Z is an m-cycle of $Y \times Y$ which is supported on $W \times W$, where $W \subset Y$ is a closed algebraic subset with $\dim W \leq m - c$.

The main result we prove in this paper is:

THEOREM 0.6. *Let X be a smooth projective n-fold with trivial Chow groups and let L be a very ample line bundle on X. Assume that for the general hypersurface $Y \in |L|$, the cohomology group $H^{n-1}(Y, \mathbb{Q})_{prim}$ is nonzero and parameterized by algebraic cycles of dimension c in the sense of Definition 0.3.*

Then for any smooth member Y of $|L|$, the cycle class map

$$CH_i(Y)_{\mathbb{Q}} \to H^{2n-2-2i}(Y, \mathbb{Q}),\ n = \dim X,$$

is injective for any $i \leq c - 1$.

REMARK 0.7. One can more generally consider a very ample vector bundle E on X and the smooth varieties $Y \subset X$ of codimension $r = \operatorname{rank} E$ obtained as zero loci of sections of E. This however immediately reduces to the hypersurface case by replacing X with $\mathbb{P}(E^*)$, (see [17, 4.1.2] for details).

REMARK 0.8. The condition that $H^{n-1}(Y, \mathbb{Q})_{prim}$ is nonzero is not very restrictive: very ample hypersurfaces with no nonzero primitive cohomology are rather rare (even if they exist, for example odd dimensional quadrics in projective space). Typically, if X is defective, that is, its projective dual is not a hypersurface, its hyperplane sections have no nonzero primitive cohomology. We refer to [18], [19] for the study of this phenomenon.

We will give in section 2 one concrete application of Theorem 0.6. It concerns hypersurfaces obtained as hyperplane sections of the Grassmannian $G(3, 10)$ which were studied in [3].

We will finally conclude the paper explaining how to modify the assumptions of Theorem 0.6 in order to cover cases where the line bundle L is not very ample (see Proposition 3.1, Theorem 3.3). This is necessary if we want to apply these methods to submotives of G-invariant hypersurfaces cut-out by a projector of G, where G is a finite group acting on X.

Let us say a word on the strategy of the proof. First of all, our assumption can be reformulated by saying that an adequate correction $\Delta_{Y,prim}$ of the diagonal Δ_Y of Y by a cycle restricted from $X \times X$ is cohomologous to a cycle Z supported on $W \times W$, where $W \subset Y$ is a closed algebraic subset of codimension $\geq c$.

We then deduce from the fact that this last property is satisfied by a general $Y \in |L|$ that an adequate correction $\Delta_{Y,prim}$ of the diagonal Δ_Y of Y by a cycle restricted from $X \times X$ is *rationally equivalent* to a cycle Z supported on $W \times W$, where $W \subset Y$ is a closed algebraic subset of codimension $\geq c$. We finally use the following lemma (see [13]):

Lemma 0.9. *Assume X has trivial Chow groups and that we have a decomposition*

$$\Delta_Y = Z_1 + Z_2 \text{ in } CH^{n-1}(Y \times Y)_{\mathbb{Q}}, \tag{1}$$

where Z_1 is the restriction of a cycle on $X \times X$ and Z_2 is supported on $W \times W$, with $\operatorname{codim} W \geq c$, *then $CH_i(Y)_{\mathbb{Q},hom} = 0$ for $i \leq c-1$.*

Proof. For any $z \in CH_i(Y)_{\mathbb{Q},hom}$, let both sides of (1) act on z. We then get

$$z = Z_{1*}z + Z_{2*}z \text{ in } CH_i(Y)_{\mathbb{Q}}.$$

As Z_1 is the restriction of a cycle on $X \times X$, the map Z_{1*} on $CH_i(Y)_{\mathbb{Q},hom}$ factors through $j_* : CH_i(Y)_{\mathbb{Q},hom} \to CH_i(X)_{\mathbb{Q},hom}$ and $CH_i(X)_{\mathbb{Q},hom}$ is 0 by assumption. On the other hand, if $i \leq c-1$, $Z_{2*}z = 0$ because the projection of the support of Z_2 to Y is of codimension $\geq c$ so does not meet a general representative of z. $\square$

1. Proof of Theorem 0.6

We establish a few preparatory lemmas before giving the proof of the main theorem. Let X be a smooth projective variety of dimension n with trivial Chow groups, and L be a very ample line bundle on X. Let $Y \subset X$ be a smooth member of $|L|$.

We start with the following lemma:

Lemma 1.1. *Let T be a smooth projective variety of dimension $n-1-2c$ and $P \in CH^{n-1-c}(T \times Y)_{\mathbb{Q}}$ such that*

$$P^* : H^{n-1}(Y, \mathbb{Q})_{prim} \to H^{n-2c-1}(T, \mathbb{Q})$$

is compatible with cup-product up to a coefficient, that is

$$(2) \qquad (P^*\alpha, P^*\beta)_T = N(\alpha, \beta)_Y, \ \forall \alpha, \ \beta \in H^{n-1}(Y, \mathbb{Q})_{prim},$$

for some $N \neq 0$. Then

$$(3) \qquad (P, P)_*([\Delta_T]) = N[\Delta_Y] + [\Gamma] + [\Gamma_1] \text{ in } H^{2n-2}(Y \times Y, \mathbb{Q}),$$

where the cycle Γ is the restriction to $Y \times Y$ of a cycle with $\mathbb{Q}$-coefficients on $X \times X$, and the cycle Γ_1 is 0 if $n-1$ is odd, and of the form $\sum_i \alpha_i Z_i \times Z'_i$, $\dim Z_i = \dim Z'_i = \frac{n-1}{2}$ *if $n-1$ is even.*

Here $(P, P) \in CH^{2n-2}(T \times T \times Y \times Y)_{\mathbb{Q}}$ is just the product $P \times P \subset T \times Y \times T \times Y \cong T \times T \times Y \times Y$ if P is the class of a subvariety, and is defined as $pr_{13}^* P \cdot pr_{24}^* P$ in general.

Proof. Indeed, let $\Gamma' := (P, P)_*(\Delta_T) \in CH_{n-1}(Y \times Y)_{\mathbb{Q}}$. Observe that $\Gamma' = {}^tP \circ P$ in $CH_{n-1}(Y \times Y)_{\mathbb{Q}}$. As $P^* : H^{n-1}(Y, \mathbb{Q})_{prim} \to H^{n-2c-1}(Z, \mathbb{Q})$ satisfies (2), we find that the cycle class $[\Gamma'] \in H^{2n-2}(Y \times Y, \mathbb{Q})$ satisfies the property that

$$[\Gamma']_* = P_* \circ P^* : H^*(Y, \mathbb{Q}) \to H^*(Y, \mathbb{Q})$$

induces

$$NI_d : H^{n-1}(Y, \mathbb{Q})_{prim} \to H^{n-1}(Y, \mathbb{Q})_{prim}$$

via the composite map

$$End\,(H^{n-1}(Y, \mathbb{Q})) \stackrel{rest}{\to} Hom\,(H^{n-1}(Y, \mathbb{Q})_{prim}, H^{n-1}(Y, \mathbb{Q}))$$

$$\stackrel{proj}{\to} Hom\,(H^{n-1}(Y, \mathbb{Q})_{prim}, H^{n-1}(Y, \mathbb{Q})_{prim}),$$

where the projection $H^{n-1}(Y, \mathbb{Q}) \to H^{n-1}(Y, \mathbb{Q})_{prim}$ is the transpose with respect to the intersection pairing of the inclusion $H^{n-1}(Y, \mathbb{Q})_{prim} \to$

$H^{n-1}(Y,\mathbb{Q})$. As $H^{n-1}(Y,\mathbb{Q}) = H^{n-1}(Y,\mathbb{Q})_{prim} \oplus_\perp H^{n-1}(X,\mathbb{Q})_{|Y}$, it follows that

$$[\Gamma']_{*|H^{n-1}(Y,\mathbb{Q})_{prim}} = NId + \eta : H^{n-1}(Y,\mathbb{Q})_{prim} \to H^{n-1}(Y,\mathbb{Q}),$$

where η takes value in $H^{n-1}(X,\mathbb{Q})_{|Y}$.

To conclude, we use the orthogonal decomposition

$$H^*(Y,\mathbb{Q}) \cong H^*(X,\mathbb{Q})_{|Y} \bigoplus_{\perp} H^{n-1}(Y,\mathbb{Q})_{prim}$$

given by the Lefschetz theorem on hyperplane sections and the hard Lefschetz theorem on Y. The class of the symmetric cycle

$$\Gamma'' := \Gamma' - N\Delta_Y = (P,P)_*(\Delta_T) - N\Delta_Y \tag{4}$$

acts as 0 on $H^{n-1}(Y,\mathbb{Q})_{prim}$, hence by the orthogonal decomposition above, it lies in

$$H^*(X,\mathbb{Q})_{|Y} \otimes H^*(X,\mathbb{Q})_{|Y} \bigoplus H^{n-1}(Y,\mathbb{Q})_{prim} \otimes H^{n-1}(X,\mathbb{Q})_{|Y}$$

$$\bigoplus H^{n-1}(X,\mathbb{Q})_{|Y} \otimes H^{n-1}(Y,\mathbb{Q})_{prim}.$$

Finally we use the fact that X has trivial Chow groups, so that its cohomology is algebraic by [8]; hence $H^*(X,\mathbb{Q})_{|Y} \otimes H^*(X,\mathbb{Q})_{|Y}$ consists of classes of cycles on $Y \times Y$ restricted from $X \times X$. In the decomposition above, we thus find that

$$[\Gamma''] = [\Gamma] + \eta + \eta', \tag{5}$$

for some classes $\eta \in H^{n-1}(Y,\mathbb{Q})_{prim} \otimes H^{n-1}(X,\mathbb{Q})_{|Y}$, $\eta' \in H^{n-1}(X,\mathbb{Q})_{|Y} \otimes H^{n-1}(Y,\mathbb{Q})_{prim}$, and $[\Gamma] \in H^*(X,\mathbb{Q})_{|Y} \otimes H^*(X,\mathbb{Q})_{|Y}$ for some algebraic cycle Γ on $Y \times Y$ restricted from $X \times X$. Note that if $n-1$ is odd, then $H^{n-1}(X,\mathbb{Q})_{|Y} = 0$, so $\eta = \eta' = 0$ and we get

$$[\Gamma''] = [(P,P)_*(\Delta_T)] - N[\Delta_Y] = [\Gamma]$$

so the lemma is proved in this case.

When $n-1$ is even, for $\gamma \in H^{n-1}(X,\mathbb{Q})_{|Y}$, we have

$$(\eta+\eta')_*(\gamma) = \eta'_*(\gamma),\ (\eta+\eta')^*(\gamma) = \eta^*(\gamma).$$

As $\eta+\eta'$ is an algebraic class on $Y\times Y$ and γ is also algebraic, we conclude that $\eta^*(\gamma)$ is algebraic on Y for any $\gamma\in H^{n-1}(X,\mathbb{Q})_{|Y}$ and similarly for ${\eta'}_*(\gamma)$. It follows then from the fact that the intersection pairing of Y restricted to $H^{n-1}(X,\mathbb{Q})_{|Y}$ is nondegenerate that both classes η and η' can be written as $\sum_i \alpha_i Z_i\times Z'_i$, $\dim Z_i=\dim Z'_i=\frac{n-1}{2}$, which provides by (4) and (5) the desired cycle Γ_1 with class $\eta+\eta'$, satisfying (3). □

Corollary 1.2. *Under the same assumptions, there exist a closed algebraic subset $W\subset Y$ of codimension $\geq c$, an $n-1$-cycle $Z\subset W\times W$, with $\mathbb{Q}$-coefficients and an $n-1$-cycle Γ in $Y\times Y$ which is the restriction of an $n+1$-cycle in $X\times X$ such that*

$$[\Delta_Y]=[Z]+[\Gamma] \text{ in } H^{2n-2}(Y\times Y,\mathbb{Q}). \tag{6}$$

Proof. Indeed, if $n-1$ is odd, we have the equality of cohomology classes

$$(P,P)_*([\Delta_T])=N[\Delta_Y]+[\Gamma]$$

and $(P,P)_*([\Delta_T])$ is supported on $W\times W$, where W is the image of the support of P, hence has dimension $\leq n-1-c$.

When $n-1$ is even, we write as in (3)

$$(P,P)_*([\Delta_T])=N[\Delta_Y]+[\Gamma]+[\Gamma_1],$$

where $\Gamma_1=\sum_i\alpha_i Z_i\times Z'_i$, with $\dim Z_i=\frac{n-1}{2}$, and we take for W the union of the image of the support of P and of the Z_i and Z'_i. (This works because $c\leq\frac{n-1}{2}$.) □

Let now $B\subset|L|$ be the Zariski open set parameterizing smooth hypersurfaces Y_b in X with equation $\sigma_b\in\mathbb{P}(H^0(X,L))$ and let $\pi:\mathcal{Y}\to B$ be the universal family,

$$\mathcal{Y}=\{(t,x)\in B\times X,\ x\in\mathcal{Y}_t\},\ \pi=pr_1.$$

We will be mainly interested in the fibered self-product $\mathcal{Y}\times_B\mathcal{Y}$ where the relative diagonal $\Delta_{\mathcal{Y}}$ lies, but it is more convenient to blow it up in $\mathcal{Y}\times_B\mathcal{Y}$. The resulting variety $\widetilde{\mathcal{Y}\times_B\mathcal{Y}}$ was also considered in [13] and the following lemma was proved (we include the proof for completeness):

LEMMA 1.3. *The quasi-projective variety $\widetilde{\mathcal{Y}\times_B\mathcal{Y}}$ is a Zariski open set in a projective bundle M over the blow-up $\widetilde{X\times X}$ of $X\times X$ along its diagonal.*

PROOF. Indeed, a point in $\widetilde{\mathcal{Y}\times_B\mathcal{Y}}$ is a 4-uple (b,x_1,x_2,z) consisting of a point of B, two points x_1, x_2 in $\mathcal{Y}_b$, and a length 2 subscheme $z\subset\mathcal{Y}_b$ whose associated cycle is x_1+x_2. There is thus a morphism p from $\widetilde{\mathcal{Y}\times_B\mathcal{Y}}$ to $\widetilde{X\times X}$ which parameterizes triples (x_1,x_2,z) where x_1, x_2 are two points of X, and $z\subset X$ is a subscheme of length 2 whose associated cycle is x_1+x_2. The fiber of p over (x_1,x_2,z) is clearly the set of $b\in B$ such that $\sigma_{b|z}=0$. Thus $\widetilde{\mathcal{Y}\times_B\mathcal{Y}}$ is Zariski open in the variety

$$M:=\{(\sigma,(x_1,x_2,z)),\ \sigma_{|z}=0\}\subset\mathbb{P}(H^0(X,L))\times\widetilde{X\times X}.$$

The very ampleness of L guarantees that M is a projective bundle over $\widetilde{X\times X}$. $\square$

We now assume that the main assumption of Theorem 0.6 holds, namely that there exist for general $b\in B$ a smooth projective variety T_b of dimension $n-1-2c$ and a correspondence with $\mathbb{Q}$-coefficients $P_b\in CH^{n-1-c}(T_b\times\mathcal{Y}_b)_{\mathbb{Q}}$ of codimension $n-1-c$ (a family of c-cycles in Y_b parameterized by T_b) such that

$$P_b^*:H^{n-1}(Y_b,\mathbb{Q})_{prim}\to H^{n-2c-1}(T_b,\mathbb{Q})$$

is compatible with cup-product up to a coefficient $N\neq 0$. We then have the following result in the same spirit as Proposition 2.7 in [13], which is very simple but nevertheless a key point in the whole argument.

LEMM 1.4. *Under the same assumptions, there exist a smooth quasi-projective variety $\mathcal{T}\to B$ projective over B, of relative dimension $n-1-2c$, and a codimension $n-1-c$ cycle $\mathcal{P}\in CH^{n-1-c}(\mathcal{T}\times_B\mathcal{Y})_{\mathbb{Q}}$ such that the map $\mathcal{P}_b^*:H^{n-1}(\mathcal{Y}_b,\mathbb{Q})_{prim}\to H^{n-2c-1}(\mathcal{T}_b,\mathbb{Q})$ is compatible with cup-product up to a coefficient $N'\neq 0$, for any $b\in B$ such that the fiber $\mathcal{T}_b$ is smooth of dimension $n-1-2c$.*

Here we denote by $\mathcal{P}_b\in CH^{n-1-c}(\mathcal{T}_b\times\mathcal{Y}_b)_{\mathbb{Q}}$ the restriction to $\mathcal{T}_b\times\mathcal{Y}_b$ of the cycle $\mathcal{P}$.

PROOF. The reason is very simple: Using our assumption and a Hilbert schemes or Chow varieties argument, we can certainly construct data $\mathcal{T}'$, $\mathcal{P}'$ as above over a finite cover U', say of degree N_0, of a Zariski open set U of B. We then consider $\mathcal{T}'$ as a family over U which we denote by $\mathcal{T}_U$, and $\mathcal{P}'$ as a relative correspondence over U between $\mathcal{T}_U$ and $\mathcal{Y}_U$ which we denote by $\mathcal{P}_U \in CH^{n-1-c}(\mathcal{T}_U \times_U \mathcal{Y}_U)_{\mathbb{Q}}$. For a general point $u \in U$, the fiber of $\mathcal{T}_U$ over u is the disjoint union of the fibers $\mathcal{T}'_{u'}$, where $u' \in U'$ maps to u, and the correspondence $\mathcal{P}_u$ is the disjoint union of the correspondences $\mathcal{P}'_{u'} \in CH^{n-1-c}(\mathcal{T}'_{u'} \times \mathcal{Y}_u)$, where $u' \in U'$ maps to u. Hence $\mathcal{P}_u^* : H^{n-1}(\mathcal{Y}_u, \mathbb{Q})_{prim} \to H^{n-2c-1}(\mathcal{T}_u, \mathbb{Q})$ multiplies the intersection form by NN_0, which proves the lemma with $N' = NN_0$ and B replaced by U. If we want to have $\mathcal{T}$ and $\mathcal{P}$ defined over the whole of B, we simply take a relative projective completion of $\mathcal{T}_U$, which we can assume to be smooth by desingularization, and we extend $\mathcal{P}_U$ by taking Zariski closures. □

COROLLARY 1.5. *Under the same assumptions, there exist a closed algebraic subset $\mathcal{W} \subset \mathcal{Y}$ of codimension $\geq c$ and a cycle $\mathcal{Z} \in CH^{n-1}(\mathcal{Y} \times_B \mathcal{Y})_{\mathbb{Q}}$ which is supported on $\mathcal{W} \times_B \mathcal{W}$, such that for any $b \in B$, the restricted cycle*

$$\mathcal{Z}_b - \Delta_{\mathcal{Y}_b}$$

is cohomologous in $\mathcal{Y}_b \times \mathcal{Y}_b$ to a cycle Γ_b coming from $X \times X$.

PROOF. With notation as in Lemma 1.4, we first define $\mathcal{W}_0 \subset \mathcal{Y}$ as the image of the support of $\mathcal{P}$ under the second projection. Then we define $\mathcal{Z}_0$ as $\frac{1}{N'}(\mathcal{P}, \mathcal{P})_*(\Delta_{\mathcal{T}/B})$, where $(\mathcal{P}, \mathcal{P}) \in CH^{2n-2}(\mathcal{T} \times_B \mathcal{T} \times_B \mathcal{Y} \times_B \mathcal{Y})_{\mathbb{Q}}$ denotes the relative correspondence $pr_{13}^*\mathcal{P} \cdot pr_{24}^*\mathcal{P}$ between $\mathcal{T} \times_B \mathcal{T}$ and $\mathcal{Y} \times_B \mathcal{Y}$, with

$$pr_{13}, pr_{24} : \mathcal{T} \times_B \mathcal{T} \times_B \mathcal{Y} \times_B \mathcal{Y} \to \mathcal{T} \times_B \mathcal{Y}$$

the two natural projections. If $n-1$ is odd, the conclusion then follows directly from Lemma 1.1, with $\mathcal{Z} = \mathcal{Z}_0$, $\mathcal{W} = \mathcal{W}_0$.

When $n-1$ is even, we argue as in the proof of Corollary 1.2, which says that for any $b \in B$, there exist cycles $Z_{i,b}$, $Z'_{i,b}$, $i \geq 1$, of dimension $\frac{n-1}{2}$ in $\mathcal{Y}_b$, a cycle Γ_b in $\mathcal{Y}_b \times \mathcal{Y}_b$ which is the restriction of a cycle in $X \times X$, and rational numbers α_i such that $\mathcal{Z}_{0,b} - \Delta_{\mathcal{Y}_b} - \Gamma_b$ is cohomologous in $\mathcal{Y}_b \times \mathcal{Y}_b$ to $\sum_i \alpha_i Z_{i,b} \times Z'_{i,b}$. The cycles $Z_{i,b}$, $Z'_{i,b}$, $i \geq 1$, can be defined over a generically finite cover $B' \to B$, giving families

$$\mathcal{Z}_i \subset \mathcal{Y}', \ \mathcal{Z}'_i \subset \mathcal{Y}'$$

with $\mathcal{Y}' := \mathcal{Y} \times_B B'$. Then, over B', we have the cycle $\mathcal{Z}'_0 \in CH^{n-1}(\mathcal{Y}' \times_{B'} \mathcal{Y}')_{\mathbb{Q}}$ defined as the pull-back of $\mathcal{Z}_0$, such that for any $b \in B'$,

$$[\mathcal{Z}'_{0,b}] - [\Delta_{\mathcal{Y}'_b} - \Gamma_b] = \sum_{i \geq 1} \alpha_i [\mathcal{Z}_{i,b} \times_{B'} \mathcal{Z}'_{i,b}].$$

Denote $\phi : \mathcal{Y}' \to \mathcal{Y}$, $(\phi, \phi) : \mathcal{Y}' \times_{B'} \mathcal{Y}' \to \mathcal{Y} \times_B \mathcal{Y}$ the natural morphisms,

$$\mathcal{W} := \mathcal{W}_0 \cup \phi(\cup_i \mathrm{Supp}\, \mathcal{Z}_i) \cup \phi(\cup_i \mathrm{Supp}\, \mathcal{Z}'_i),$$

and

$$\mathcal{Z} = \mathcal{Z}_0 - \frac{1}{\deg \phi}\, (\phi, \phi)_* (\sum_i \alpha_i \mathcal{Z}_i \times_{B'} \mathcal{Z}'_i).$$

Then $\mathcal{W}$ and $\mathcal{Z}$ satisfy the desired conclusion. □

Proof of Theorem 0.6. Recall the Zariski open inclusion

$$\widetilde{\mathcal{Y} \times_B \mathcal{Y}} \subset M$$

of Lemma 1.3, where $p : M \to \widetilde{X \times X}$ is a projective bundle over $\widetilde{X \times X}$. In both cases, the "$\sim$" means that we blow-up along the diagonal.

By Corollary 1.5, our assumptions give a subvariety $\mathcal{W} \subset \mathcal{Y}$ of codimension $\geq c$ and a cycle $\mathcal{Z} \in CH^{n-1}(\mathcal{Y} \times_B \mathcal{Y})_{\mathbb{Q}}$ which is supported on $\mathcal{W} \times_B \mathcal{W}$, such that for any $b \in B$, the cycle

$$\mathcal{Z}_b - \Delta_{\mathcal{Y}_b}$$

is cohomologous in $\mathcal{Y}_b \times \mathcal{Y}_b$ to a cycle Γ_b coming from $X \times X$. Note that we can clearly assume that Γ_b is the restriction to $\mathcal{Y}_b \times \mathcal{Y}_b$ of a cycle Γ' of $X \times X$, which is independent of b, since we are interested only in its cohomology class:

$$[\Gamma_b] = [\Gamma'_{|\mathcal{Y}_b \times \mathcal{Y}_b}] \text{ in } H^{2n-2}(\mathcal{Y}_b \times \mathcal{Y}_b, \mathbb{Q}).$$

In other words, the cycle

$$\mathcal{Z} - \Delta_{\mathcal{Y}/B} - p_0^*(\Gamma') \in CH^{n-1}(\mathcal{Y} \times_B \mathcal{Y})_{\mathbb{Q}},$$

where $p_0 : \mathcal{Y} \times_B \mathcal{Y} \to X \times X$ is the natural map, is cohomologous to 0 along the fibers of $\mathcal{Y} \times_B \mathcal{Y} \to B$.

We now blow-up the relative diagonal, pull-back these cycles to $\widetilde{\mathcal{Y} \times_B \mathcal{Y}}$ and extend them to M (see Lemma 1.3). This provides us with a cycle

$$R := \overline{\widetilde{\mathcal{Z}}} - \overline{\widetilde{\Delta_{\mathcal{Y}/B}}} - p^*(\Gamma') \in CH^{n-1}(M)_{\mathbb{Q}}, \tag{7}$$

which has the property that its restriction to $\widetilde{\mathcal{Y}_b \times \mathcal{Y}_b} \subset M$ is cohomologous to 0, for any $b \in B$. We prove now:

Proposition 1.6. *There exists a cycle $\gamma \in CH^{n-1}(X \times X)_{\mathbb{Q}}$ such that for any $b \in B$, $R - p^*\gamma$ maps to 0 in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ via the map $\tau_{b*} \circ i_b^*$, where $\tau_b : \widetilde{\mathcal{Y}_b \times \mathcal{Y}_b} \to \mathcal{Y}_b \times \mathcal{Y}_b$ is the blow-up of the diagonal and $i_b : \widetilde{\mathcal{Y}_b \times \mathcal{Y}_b} \to M$ is the inclusion map.*

Admitting the proposition temporarily, the proof of Theorem 0.6 concludes as follows: For any $b \in B$, the image $\tau_{b*} \circ i_b^*(R - p^*\gamma) \in CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ is by construction the cycle

$$\mathcal{Z}_b - \Delta_{\mathcal{Y}_b} - \Gamma'_{|\mathcal{Y}_b \times \mathcal{Y}_b} - \gamma_{|\mathcal{Y}_b \times \mathcal{Y}_b} \in CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}. \tag{8}$$

Proposition 1.6 says that the cycle (8) vanishes in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$, which can be rewritten as:

$$\Delta_{\mathcal{Y}_b} = \mathcal{Z}_b + \gamma'_{|\mathcal{Y}_b \times \mathcal{Y}_b} \text{ in } CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}, \tag{9}$$

for a cycle $\gamma' \in CH^{n-1}(X \times X)_{\mathbb{Q}}$. Recalling that for general $b \in B$, $\mathcal{Z}_b$ is supported on $\mathcal{W}_b \times \mathcal{W}_b$ with $\mathcal{W}_b \subset \mathcal{Y}_b$ closed algebraic of codimension $\geq c$, this implies by Lemma 0.9 that the cycle class map is injective on $CH_i(\mathcal{Y}_b)_{\mathbb{Q}}$ for general $b \in B$ and $i \leq c-1$.

To conclude that this holds also for any $b \in B$, we can observe that (9) holds for any $b \in B$ and it is still true for any $b \in B$ that $\mathcal{Z}_b$ is rationally equivalent to a cycle supported on $\mathcal{W}'_b \times \mathcal{W}'_b$ with $\mathcal{W}'_b \subset \mathcal{Y}_b$ closed algebraic of codimension $\geq c$, even if $\mathcal{W}_b$ itself is not of codimension $\geq c$. □

Proof of Proposition 1.6. Let $\delta \in CH^1(M)$ be the class of the pull-back to M of the exceptional divisor of $\widetilde{X \times X}$ and let $h = c_1(\mathcal{O}_M(1)) \in CH^1(M)$, where $\mathcal{O}_M(1)$ refers to the projective bundle structure of M over $\widetilde{X \times X}$. Note that $M \subset |L| \times \widetilde{X \times X}$, where the first projection restricts on $\mathcal{Y} \times_B \mathcal{Y}$ to the natural map to B. Thus h is the inverse image of a line bundle on $|L|$ by the first projection $M \to |L|$ and it restricts to 0 in $CH^1(\widetilde{\mathcal{Y}_b \times \mathcal{Y}_b})$. The class δ restricts to the class δ_b of the exceptional divisor of $\widetilde{\mathcal{Y}_b \times \mathcal{Y}_b}$. Finally, note that

$$\tau_{b*}(\delta_b^k) = 0 \text{ in } CH(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}} \text{ for } 0 < k < n-1,$$

$$\tau_{b*}(\delta_b^{n-1}) = (-1)^{n-2}\Delta_{\mathcal{Y}_b} \text{ in } CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}.$$

The projective bundle formula tells us that $CH(M)$ is generated by the powers of h as a module over the ring $CH(\widetilde{X \times X})$. Next, as the diagonal restriction map $CH(X \times X) \to CH(X)$ is surjective, the blow-up formula tells us that $CH(\widetilde{X \times X})$ is generated over the ring $CH(X \times X)$ by the powers of δ.

It follows that codimension $n-1$ cycles on M can be written in the form

$$z = \sum_{r,s} h^r \delta^s p^*(\gamma_{r,s}),$$

where $r+s \leq n-1$ and $\gamma_{r,s} \in CH^{n-1-r-s}(X \times X)$. By the above arguments, we get

$$\tau_{b*} \circ i_b^*(z) = \gamma_{0,n-1}(-1)^{n-2}\Delta_{\mathcal{Y}_b} + \gamma_{0,0|\mathcal{Y}_b \times \mathcal{Y}_b} \text{ in } CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}},$$

where $\gamma_{0,n-1} \in CH^0(X \times X) = \mathbb{Z}$ is just a number.

We apply this analysis to the cycle R of (7), whose image in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ is by construction cohomologous to 0. Writing as above $R = \sum_{r,s} h^r \delta^s p^*(\gamma_{r,s})$, this gives us an equality

$$(10) \quad \tau_{b*} \circ i_b^*(R) = \gamma_{0,n-1}(-1)^{n-2}\Delta_{\mathcal{Y}_b} + \gamma_{0,0|\mathcal{Y}_b \times \mathcal{Y}_b} \text{ in } CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$$

and in particular an equality of cycle classes:

$$(11) \quad \gamma_{0,n-1}(-1)^{n-2}[\Delta_{\mathcal{Y}_b}] + [\gamma_{0,0|\mathcal{Y}_b \times \mathcal{Y}_b}] = 0 \text{ in } H^{2n-2}(\mathcal{Y}_b \times \mathcal{Y}_b, \mathbb{Q}).$$

Using our hypothesis that the primitive cohomology of $\mathcal{Y}_b$ is nonzero, (11) implies that $\gamma_{0,n-1} = 0$. Thus (10) gives us that the image of R in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ is equal to $\gamma_{0,0|\mathcal{Y}_b \times \mathcal{Y}_b}$. This proves the proposition, with $\gamma = \gamma_{0,0}$. □

2. An Application

Let us give one new application: In [3], Debarre and the author studied smooth members Y of $|\mathcal{L}|$, where $\mathcal{L}$ is the Plücker polarization on the Grassmannian $G(3,10)$. More precisely, let V_{10} be a 10-dimensional complex vector space. To a smooth hypersurface $Y \subset G(3,V_{10})$ defined by an element σ of $\bigwedge^3 V_{10}^* = H^0(G(3,V_{10}),\mathcal{L})$, we associated the subvariety $F(Y)$ of the Grassmannian $G(6,V_{10})$ of 6-dimensional vector subspaces of V_{10}, defined by

$$F(Y) := \{[W] \in G(6,V_{10}),\, W \subset V_{10},\, \sigma_{|W} = 0\}.$$

We proved in [3] that for general σ, $F(Y)$ is a smooth hyper-Kähler 4-fold. There is a natural correspondence $P \subset F(Y) \times Y$ defined by

$$P = \{([W],[W']) \in F(Y) \times Y,\, W' \subset W\}.$$

By the first projection $P \to F(Y)$, P is a bundle over $F(Y)$ into Grassmannians $G(3,6)$.

The following result is proved in [3]:

THEOREM 2.1. *The map $P^* : H^{20}(Y,\mathbb{Q})_{prim} \to H^2(F(Y),\mathbb{Q})$ is injective with image equal to $H^2(F(Y),\mathbb{Q})_{prim}$, (where "prim" refers now to the Plücker polarization). Furthermore, $h^{11,9}(Y) \neq 0$, the number of moduli of $F(Y)$ is 20, and this is equal to $h^{1,1}(F(Y)) - 1$.*

The variety Y itself has primitive Hodge numbers $h^{p,q}_{prim} = 0$ for $p > 11$ or $q > 11$, and

$$h^{11,9}(Y)_{prim} = h^{9,11}(Y)_{prim} = 1,\ h^{10,10}(Y)_{prim} = 20.$$

We have the following consequence (this illustrates Remark 0.5):

LEMMA 2.2. *There exists a number $\mu \neq 0$ such that for the general hypersurface Y as above (so that $F(Y)$ is smooth of dimension 4), we have*

$$(12)\qquad <\alpha,\beta>_Y = \mu < P^*\alpha, l^2 P^*\beta >_{F(Y)},\ \text{for}\ \alpha,\,\beta \in H^{20}(Y,\mathbb{Q})_{prim}$$

where $l = c_1(\mathcal{L}_{|F(Y)}) \in H^2(F(Y),\mathbb{Q})$.

Proof. Since the morphism $P^* : H^{20}(Y,\mathbb{Q})_{prim} \to H^2(F(Y),\mathbb{Q})$ is locally constant when Y deforms in the family, it suffices to prove the statement for a single very general Y. Since $F(Y)$ is a projective hyper-Kähler fourfold with 20 moduli and $h^{1,1}(F(Y)) = 21$, for very general Y, the Hodge structure on $H^2(F(Y),\mathbb{Q})_{prim}$ is simple, and admits a unique polarization up to a coefficient. Hence the same is true for the Hodge structure on $H^{20}(Y,\mathbb{Q})_{prim}$. Thus the polarizations on both sides of (12) must coincide via P^* up to a nonzero coefficient. □

Corollary 2.3. *The varieties Y as above have their cohomology parameterized by cycles of dimension* 9*.*

Proof. Indeed, let $T \subset F(Y)$ be the intersection of two general members of $\mathcal{L}_{|F(Y)}$. Then (12) says that the restricted correspondence $P_T := P_{|T\times Y}$ satisfies

$$< \alpha, \beta >_Y = \mu < P_T^*\alpha, P_T^*\beta >_T . \ \square$$

We now get the following conclusion:

Theorem 2.4. *The smooth hyperplane sections Y of $G(3,10)$ satisfy $CH_i(Y)_{\mathbb{Q},hom} = 0$ for $i < 9$.*

Proof. This follows indeed from Corollary 2.3 and Theorem 0.6, since we know that $H^{20}(Y,\mathbb{Q})_{prim}$ is nonzero by the condition $h^{11,9}(Y) \neq 0$. □

3. Comments on the "Very Ampleness" Assumption

The very ampleness assumption made previously is too restrictive since there are many more applications obtained by considering varieties X with the action of a finite group G preserving the line bundle L, and by studying G-invariant hypersurfaces $Y \in |L|$, and more precisely the submotive of Y determined by a projector $\pi \in \mathbb{Q}[G]$. It often happens that the coniveau of such a submotive is greater than the coniveau of the whole cohomology of Y.

Typically, the quintic Godeaux surfaces S studied in [15] are smooth quintic surfaces, so they have $h^{2,0}(S) \neq 0$. However they are invariant under the Godeaux action of $G = \mathbb{Z}/5\mathbb{Z}$ and the G-invariant part of $H^{2,0}$ is 0. If $\pi = \frac{1}{5}\sum_{g\in G} g$ is the projector onto the G-invariant part, we thus have $H^{2,0}(S)^{\pi} = 0$ so the Hodge coniveau of $H^2(S,\mathbb{Q})^{\pi}$ is 1. The Lefschetz theorem on $(1,1)$-classes then says that the cohomology $H^2(S,\mathbb{Q})^{\pi}$ consists of classes of 1-cycles and it easily implies that it is parameterized by 1-cycles in the sense of Definition 0.3. Similarly, the case of cubic fourfolds invariant under a finite group acting trivially on $H^{3,1}(X)$ is studied in [6]. In this case, the projector to be considered is $1-\pi_G$, where π_G is again the projector onto the G-invariant part. As $1-\pi_G$ acts as 0 on $H^{3,1}(X)$, the Hodge structure on $H^4(X,\mathbb{Q})^{1-\pi_G}_{prim}$ is trivial of type $(2,2)$. As the Hodge conjecture is satisfied by cubic fourfolds (see [2], [20], or [17] for the integral coefficients version), one gets that the cohomology $H^4(X,\mathbb{Q})^{1-\pi_G}_{prim}$ consists of classes of 2-cycles, and it implies as above that it is parameterized by 2-cycles in the sense of Definition 0.3, while for the whole cohomology $H^4(X,\mathbb{Q})$, it is only parameterized by 1-cycles.

On the other hand, the linear system of G-invariant hypersurfaces is clearly not very ample, so Theorem 0.6 a priori does not apply. Let us explain the variants of Theorem 0.6 which will apply to the situations above. First of all we have the following:

PROPOSITION 3.1. *Let X be smooth projective of dimension n with trivial Chow groups, and L be an ample line bundle on X. Assume that*

i) The cohomology $H^{n-1}(Y_t,\mathbb{Q})_{prim}$, $t\in B$, is nonzero and is parameterized by algebraic cycles of dimension c.

Then the conclusion of Theorem 0.6 still holds, namely $CH_i(Y_t)_{hom,\mathbb{Q}} = 0$ for $i \leq c-1$ if instead of assuming L very ample, we only assume

ii) The line bundle L is generated by global sections and the locus of points $(x,y)\in X\times X$ such that there exists $(x,y,z)\in \widetilde{X\times X}_{\Delta}$, where z is a length 2 subscheme of X with associated cycle $x+y$ imposing only one condition to $H^0(X,L)$, has codimension $> n$ in $X\times X$.

REMARK 3.2. Note that L being ample, the morphism $\phi_L : X \to \mathbb{P}^N$ given by sections of L is finite, so a priori the locus appearing in ii) has codimension $\geq n$ in $X\times X$. We want that, away from the diagonal, this locus has codimension $> n$, which is equivalent to saying that ϕ_L is generically 1-to-1 on its image. Our condition along the diagonal is automatic since it says that ϕ_L is generically an immersion.

Proof of Proposition 3.1. Indeed, going through the proof of Theorem 0.6, we see that we used the condition that L is very ample to say that $\mathcal{Y}\times_B\mathcal{Y}$ has a smooth projective completion

$$M=\{((x,y,z),f),\,(x,y,z)\in\widetilde{X\times X}_\Delta,\,f\in|L|,\,f_{|z}=0\}\tag{13}$$

which is a projective bundle over $\widetilde{X\times X}_\Delta$. If L is not very ample, then M defined in (13) is not anymore a projective bundle over $\widetilde{X\times X}_\Delta$ via the first projection but we can as in [6] overcome this problem by simply blow-up $\widetilde{X\times X}_\Delta$ along the sublocus where the length 2 subscheme z of X does not impose independent conditions to $|L|$, until we get a smooth projective variety $X'\to\widetilde{X\times X}_\Delta$ together with a projective bundle $M'\to X'$, where M' maps birationally to M and a Zariski open set M'_0 of M' admits a dominating proper map

$$\phi_0:M'_0\to\mathcal{Y}\times_B\mathcal{Y}.$$

Namely letting $\pi:M'\to|L|$ be the composition of the map $\tau':M'\to M$ and the second projection $M\to|L|$, we can define M'_0 as $\pi^{-1}(B)$ and ϕ_0 is simply the restriction to M'_0 of the composition ϕ of $\tau':M'\to M$ and of the natural map $((x,y,z),f)\mapsto((x,y),f)$ from M to $\overline{\mathcal{Y}}\times_{|L|}\overline{\mathcal{Y}}$, where $\overline{\mathcal{Y}}$ is the universal hypersurface over $|L|$.

Under assumption i), we conclude as in the proof of Theorem 0.6 that there is a cycle

$$R:=\overline{\widetilde{\mathcal{Z}}}-\overline{\widetilde{\Delta_{\mathcal{Y}/B}}}-p^*(\Gamma)\in CH^{n-1}(M')_{\mathbb{Q}},\tag{14}$$

where $\Gamma\in CH^{n-1}(X\times X)_{\mathbb{Q}}$, and $\mathcal{Z}$ is a codimension $n-1$ cycle in $\mathcal{Y}\times_B\mathcal{Y}$ which is supported on $\mathcal{W}\times_B\mathcal{W}$, $\operatorname{codim}\mathcal{W}\geq c$, such that the image $\tau''_{b*}\circ i_b^*(R)\in CH^{n-1}(\mathcal{Y}_b\times\mathcal{Y}_b)_{\mathbb{Q}}$ is cohomologous to 0, for any $b\in B$. Here the $\widetilde{\cdot}$ means that we take the pull-back of the considered cycles via ϕ_0^* and the $\overline{\cdot}$ means that we extend the cycles from M'_0 to M'. The map i_b is the inclusion of the fiber $\widetilde{\mathcal{Y}_b\times\mathcal{Y}_b}$ of π in M' and the map $\tau''_b:\widetilde{\mathcal{Y}_b\times\mathcal{Y}_b}\to\mathcal{Y}_b\times\mathcal{Y}_b$ is the restriction of ϕ_0 to $\widetilde{\mathcal{Y}_b\times\mathcal{Y}_b}$.

Recall now that M' is a projective bundle over X' which itself is obtained by blowing up $\widetilde{X\times X}_\Delta$ along subloci whose images in $X\times X$ are of codimension $>n$, hence of dimension $<n$ and thus intersect the general

$\mathcal{Y}_b \times \mathcal{Y}_b$ along a closed algebraic subset of dimension $< n-1$, since $|L|$ is base-point free. In particular, we have a morphism $p' : M' \to X \times X$, giving an inclusion $CH(X \times X)_{\mathbb{Q}} \to CH(M')_{\mathbb{Q}}$. It is immediate that the morphism $\tau''_{b*} \circ i_b^*$ is a morphism of $CH(X \times X)_{\mathbb{Q}}$-modules. By the general facts concerning the Chow groups of a projective bundle and a blow-up, we can write any element of $CH^{n-1}(M')_{\mathbb{Q}}$ as a polynomial with coefficients in the ring $CH(X \times X)_{\mathbb{Q}}$ in the following generators:

1. the class $h = c_1(\mathcal{O}_{M'}(1))$, where the line bundle $\mathcal{O}_{M'}(1)$ is the pull-back of $\mathcal{O}_{|L|}(1)$ to M' so that $h_{|\widetilde{\mathcal{Y}_b \times \mathcal{Y}_b}}$ is 0 and thus $i_b^*(h^k) = 0$ for all $k > 0$;

2. the class δ, which is the bull-back to M' of the exceptional divisor of $\widetilde{X \times X}_{\Delta}$ over the diagonal. The divisor δ restricts to the exceptional divisor of $\widetilde{\mathcal{Y}_b \times \mathcal{Y}_b}$ and the only power δ^k, $0 < k \leq n-1$ mapping to a nonzero element of $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ via $\tau''_{b*} \circ i_b^*$ is δ^{n-1}, since the other terms δ^k, with $k < n-1$ will be contracted to the diagonal of $\mathcal{Y}_b$ via the blow-down map $\widetilde{\mathcal{Y}_b \times \mathcal{Y}_b} \to \mathcal{Y}_b \times \mathcal{Y}_b$.

3. Cycles of codimension $\leq n-1$ supported on the other exceptional divisors of the blow-up map $X' \to \widetilde{X \times X}_{\Delta}$. Any such cycle will be sent to 0 in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ by the map $\tau''_{b*} \circ i_b^*$ since its intersection with $\widetilde{\mathcal{Y}_b \times \mathcal{Y}_b}$ is supported over a sublocus of $\mathcal{Y}_b \times \mathcal{Y}_b$ of dimension $< n-1$.

Writing the cycle R in (14) using these generators, it follows from this enumeration that the analogue of Proposition 1.6 still holds in our situation, since the extra cycles in $CH^{n-1}(M')$ appearing in 3 above vanish in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$, so that we can simply, by modifying R if necessary, assume they do not appear. The classes of the form $\delta^k p^* Z$, for $0 < k < n-1$, can be ignored for the same reason and we conclude that

$$\tau''_{b*} \circ i_b^*(R) = \mu \Delta_{\mathcal{Y}_b} + \Gamma_{|\mathcal{Y}_b \times \mathcal{Y}_b} \text{ in } CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$$

for some cycle $\Gamma \in CH^{n-1}(X \times X)_{\mathbb{Q}}$. On the other hand, our assumption is that $\tau''_{b*} \circ i_b^*(R)$ is cohomologous to 0. The assumption made in i) that the cohomology $H^{n-1}(\mathcal{Y}_t, \mathbb{Q})_{prim}$, $t \in B$, is nonzero shows that the diagonal of $\mathcal{Y}_b$ is not cohomologous to the restriction of a cycle in $X \times X$, and it follows that $\mu = 0$.

As $\tau''_{b*} \circ i_b^*(R) = \mathcal{Z}_b - \Delta_{\mathcal{Y}_b}$ modulo a cycle restricted from $X \times X$, we thus conclude as in the proof of Theorem 0.6 that there is a codimension $n-1$ cycle γ in $X \times X$ such that $\gamma_{|\mathcal{Y}_b \times \mathcal{Y}_b} = \Delta_{\mathcal{Y}_b} - \mathcal{Z}_b$ in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ and the end of the proof of Proposition 3.1 then works exactly as in the proof of Theorem 0.6. □

Proposition 3.1 does not apply to the above mentioned situation where we replace $|L|$ by some G-invariant linear subsystem $|L|^G$ (or (χ, G)-invariant for some character χ), where G is a finite group acting on X, since then the (proper transforms of the) graphs of elements of $g \in G$ in $\widetilde{X \times X}_\Delta$ provide codimension n subvarieties of $\widetilde{X \times X}_\Delta$ along which the subscheme z imposes at most one condition to $|L|^G$. The best we can assume in this situation is the following:

(*) The linear system $|L|^G := \mathbb{P}(H^0(X, L)^G)$ has no base-points and the codimension $\leq n$ components of the locus of points in $\widetilde{X \times X}_\Delta$ parameterizing triples (x, y, z) such that the length 2 subscheme z with support $x + y$ imposes only one condition to $H^0(X, L)^G$ is the union of the (proper transforms of the) graphs of elements of $e \neq g \in G$ (and this equality is a scheme theoretic equality generically along each of these graphs).

Then we have the following variant of Theorem 0.6. Let X be smooth projective with trivial Chow groups, endowed with an ample line bundle L and an action of the finite group G such that L is G-linearized and satisfies (*). Let $\pi = \sum_g a_g g \in \mathbb{Q}[G]$ be a projector of G. For a general hypersurface $Y \in |L|^G$, Y is smooth and we assume that π^* acts on $H^{n-1}(Y, \mathbb{Q})_{prim}$ as the orthogonal projector $H^{n-1}(Y, \mathbb{Q}) \to H^{n-1}(Y, \mathbb{Q})^\pi$.

THEOREM 3.3. *Assume the following:*

(i) For the general hypersurface $Y \in |L|^G$ *the cohomology* $H^{n-1}(Y, \mathbb{Q})^\pi_{prim}$ *is parameterized by cycles of dimension* c.

(ii) The primitive components $g^* \in End_{\mathbb{Q}}(H^{n-1}(Y, \mathbb{Q})_{prim})$ *of the cohomology classes of the graphs of elements of* g *are linearly independent over* $\mathbb{Q}$.

Then the groups $CH_i(Y)^\pi_{\mathbb{Q},hom}$ *are trivial for* $i \leq c - 1$.

PROOF. The proof is a generalization of the proofs of Theorem 0.6 and Proposition 3.1. Let $B \subset |L|^G$ be the open set parameterizing smooth invariant hypersurfaces and let $\Delta_{\pi,b} = \sum_g a_g\Gamma_g \subset Y_b \times Y_b$, where Γ_g is the graph of g acting on Y_b; let $\Delta_{\pi,b,prim}$ be the primitive part of $\Delta_{\pi,b}$, obtained by correcting $\Delta_{\pi,b}$ by the restriction to $Y_b \times Y_b$ of a $\mathbb{Q}$-cycle of $X \times X$, in such a way that $[\Delta_{\pi,b,prim}]^*$ acts as the orthogonal projector onto $H^{n-1}(Y_b,\mathbb{Q})^{\pi}_{prim}$.

Our assumption that $H^{n-1}(Y_b,\mathbb{Q})^{\pi}_{prim}$ is parameterized by algebraic cycles of codimension c implies that there exist a codimension c closed algebraic subset $W_b \subset Y_b$ and a $n-1$-cycle $Z_b \subset W_b \times W_b$ such that Z_b is cohomologous to $\Delta_{\pi,b,prim}$ in $Y_b \times Y_b$.

We then spread these data over B and get a codimension c subvariety $\mathcal{W} \subset \mathcal{Y}$, where $f : \mathcal{Y} \to B$ is the universal family, and a cycle $\mathcal{Z}$ supported on $\mathcal{W} \times_B \mathcal{W}$ such that

$$\mathcal{Z} - \Delta_{\pi,\mathcal{Y}/B,prim}$$

has its restriction cohomologous to 0 on the fibers $\mathcal{Y}_b \times \mathcal{Y}_b$ of the map $(f,f) : \mathcal{Y} \times_B \mathcal{Y} \to B$.

We now have to prove the analogue of Proposition 1.6. As in the proof of Proposition 3.1, the difficulty comes from the fact that the variety

$$M := \{((x,y,z),\sigma) \in \widetilde{X \times X}_{\Delta} \times |L|^G,\ \sigma_{|z} = 0\}$$

is no longer a projective bundle over $\widetilde{X \times X}_{\Delta}$ due to the lack of very ampleness of the G-invariant linear system $|L|^G$. In the case of Proposition 3.1, we had a smooth projective model X' of $\widetilde{X \times X}_{\Delta}$ obtained by blowing-up $\widetilde{X \times X}_{\Delta}$ along subloci of codimension $> n$, on which we analyzed the conveniently defined extension R of the cycle $\mathcal{Z} - \Delta_{\pi,\mathcal{Y}/B,prim}$ (first by pull-back under blow-up to $\widetilde{\mathcal{Y} \times_B \mathcal{Y}}_{\Delta}$, and then by extension to the projective completion M'). In our new situation, the only new feature lies in the fact that in order to get the projective bundle $M' \to X'$, we have to blow-up in $\widetilde{X \times X}_{\Delta}$ the graphs of $g \in G$ which are of codimension n and intersect $\mathcal{Y}_b \times \mathcal{Y}_b$ along a codimension $n-1$ locus, namely the graph Γ_g of g acting on $\mathcal{Y}_b$. As in the proof of Proposition 3.1, further blow-ups may be needed in order to construct the model T', but they are over closed algebraic subsets of $X \times X$ of codimension $> n$.

For any codimension $n-1$ cycle of M' supported in an exceptional divisor of the map $X' \to X \times X$ over graph (g), its image in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ is a multiple of Γ_g.

With the same notations as in the proof of Proposition 3.1, we write our cycle $T \in CH^{n-1}(M')_{\mathbb{Q}}$ as a sum

$$T = P(h, \delta_g) + A,$$

where P is a polynomial in the variables h, δ_g, $g \in G$, whose coefficients are pull-backs of cycles on $X \times X$, and A is a cycle supported on an exceptional divisor of $X' \to X$ over a closed algebraic subset of $X \times X$ of codimension $> n$. Here $h = c_1(\mathcal{O}_{M'}(1))$, where the line bundle $\mathcal{O}_{M'}(1)$ comes from $\mathcal{O}_{|L|^G}(1)$ and thus restricts to 0 on the fibers of $M' \to |L|^G$, which are birationally equivalent to $\mathcal{Y}_b \times \mathcal{Y}_b$. The divisors δ_b are the exceptional divisors over the generic points of the graphs graph g.

We now recall that the cycle R maps, via the natural correspondence $\tau_{b*} \circ i_b^*$ between M' and $\mathcal{Y}_b \times \mathcal{Y}_b$, to $\mathcal{Z}_b - \Delta_{\pi,b,prim} \in CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$, where $\mathcal{Z}_b$ is supported on $\mathcal{W}_b \times \mathcal{W}_b$, with $\operatorname{codim} \mathcal{W}_b \subset \mathcal{Y}_b \geq c$.

In our polynomial $P(h, \delta_g)$, only the terms of degree 0 in h can be mapped by $\tau_{b*} \circ i_b^*$ to a nonzero element in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)$ and concerning the powers of δ_g, only the terms of degree $n-1$ in δ_g can be mapped to a nonzero element in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$ (and they are then mapped to the class of Γ_g in $CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$). The monomials of degree $\leq n-1$ involving at least two of the δ_g will also be annihilated by $\tau_{b*} \circ i_b^*$ since their images will be supported on $\Gamma_g \cap \Gamma_{g'}$ which has dimension $< n-1$. Hence we can assume that

$$R = R_0 + \sum_g \lambda_g \delta_g^{n-1},$$

where R_0 is the pull-back to M' of a cycle on $X \times X$, without changing the image $\tau_{b*} \circ i_b^*(R) \in CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}}$. We thus have

$$\tau_{b*} \circ i_b^*(R) = R_{0|\mathcal{Y}_b \times \mathcal{Y}_b} + (-1)^n \sum_g \lambda_g \Gamma_g^{n-1} \text{ in } CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}} \tag{15}$$

We know that $\tau_{b*} \circ i_b^*(R)$ is cohomologous to 0 in $\mathcal{Y}_b \times \mathcal{Y}_b$. As we made the assumption that the endomorphisms $\Gamma_{g,b*} : H^{n-1}(\mathcal{Y}_b, \mathbb{Q})_{prim} \to H^{n-1}(\mathcal{Y}_b, \mathbb{Q})_{prim}$ are linearly independent, we conclude from (15) that all λ_g vanish, so that $R = R_0$. As we have

$$\tau_{b*} \circ i_b^*(R) = \mathcal{Z}_b - \Delta_{\pi,b,prim} \text{ in } CH^{n-1}(\mathcal{Y}_b \times Y_b)_{\mathbb{Q}},$$

we conclude that

$$\mathcal{Z}_b - \Delta_{\pi,b,prim} - R_{0|Y_b\times Y_b} = 0 \in CH^{n-1}(\mathcal{Y}_b \times \mathcal{Y}_b)_{\mathbb{Q}},$$

where we recall that $\mathcal{Z}_b$ is supported on $\mathcal{W}_b \times \mathcal{W}_b$ with $\operatorname{codim} \mathcal{W}_b \geq c$. The argument explained in the introduction then allows to conclude that $CH_i(\mathcal{Y}_b)^{\pi}_{hom,\mathbb{Q}} = 0$ for $i < c$. $\square$

We refer to [13] for further potential applications of the general strategy developed in Theorems 0.6, 3.3. Let us just mention one challenging example. In [15], the case of quintic hypersurfaces in $\mathbb{P}^4$ invariant under the involution acting by $(-1,-1,+1,+1,+1)$ on homogeneous coordinates is studied. The involution acts as the identity on $H^{3,0}(X)$ and it is proved that the antiinvariant part of $H^3(X,\mathbb{Q})$ is parameterized by 1-cycles. Theorem 3.3 above (applied to the blow-up of $\mathbb{P}^4$ along the line $\{X_2 = X_3 = X_4 = 0\}$ to avoid base-points) then implies that $CH_0(X)^-$ is equal to 0, a result which was already obtained in [15]. The next case to study would be that of a sextic hypersurface in $\mathbb{P}^5$ defined by an equation invariant under the involution i acting on homogeneous coordinates by

$$i^*(X_0,\dots,X_5) = (-X_0,-X_1,-X_2,X_3,X_4,X_5). \tag{16}$$

This involution acts by $-Id$ on $H^{4,0}(X)$ and thus the cohomology $H^4(X,\mathbb{Q})^+$ invariant under the involution has Hodge coniveau 1, so is expected to be parameterized by 1-cycles. Assuming this is true, then Theorem 3.3 would imply that the invariant part $CH_0(X)_0^+$ of the group of 0-cycles of degree 0 on X is 0. Indeed, Remark 0.5 applies to the very general invariant hypersurface in this case, by standard infinitesimal variations of Hodge structure arguments. This shows that if for the general invariant hypersurface X as above, $H^4(X,\mathbb{Q})^+$ is of geometric coniveau 1, then it is parameterized by 1-cycles in the sense of Definition 0.3.

This example is particularly interesting because it relates to the following question asked and studied in [16, Section 3]: For any variety Y, we have the map

$$\mu_Y : CH_0(Y)_{hom} \otimes CH_0(Y)_{hom} \to CH_0(Y\times Y)$$
$$z\otimes z' \mapsto p_1^*z\cdot p_2^*z',$$

and the map

$$\mu_Y^- : CH_0(Y)_{hom} \otimes CH_0(Y)_{hom} \to CH_0(Y\times Y),$$

$$z \otimes z' \mapsto p_1^* z \cdot p_2^* z' - p_1^* z' \cdot p_2^* z.$$

Let now S be a smooth projective $K3$ surface.

Question 3.4. Is it true that the map μ_S^- is 0?

This is implied by the generalized Bloch conjecture since the space $H^{4,0}(S \times S)^-$ of holomorphic 4-forms on $S \times S$ antiinvariant under the involution exchanging the factors is 0. (Note that there are nonzero antiinvariant holomorphic 2-forms on $S \times S$, but they are of the form $p_1^*\omega - p_2^*\omega$, where $\omega \in H^{2,0}(S)$, while the 0-cycles in the image of μ_S^- are annihilated by p_{1*} and p_{2*}.)

The precise relation between Question 3.4 and i-invariant CH_0 groups of sextic hypersurfaces invariant under an involution i of the type (16) is the following : the Shioda construction (see [11]) shows that if $C \subset \mathbb{P}^2$ is a plane curve of degree 6 defined by a polynomial equation $f(X_0, X_1, X_2)$, the sextic fourfold X defined by the equation $f(X_0, X_1, X_2) - f(Y_0, Y_1, Y_2)$ is rationally dominated by the product $\Sigma \times \Sigma$, where Σ is the sextic surface in $\mathbb{P}^3$ with equation $U^6 = f(X_0, X_1, X_2)$. The rational map $\Phi : \Sigma \times \Sigma \dashrightarrow X$ is explicitly given by

$$\Phi((x, u), (y, v)) = (vx, uy).$$

It makes X birationally equivalent to the quotient of $\Sigma \times \Sigma$ by $G = \mathbb{Z}/6\mathbb{Z}$, where we choose an isomorphism $g \mapsto \zeta$ between G and the group of 6th roots of unity and the actions of $g \in G$ on Σ and $\Sigma \times \Sigma$ are given by

$$g(x, u) = (x, \zeta u),$$

$$g((x, u), (y, v)) = (g(x, u), g(y, v)).$$

Note now that the $K3$ surface S, which is defined as the double cover of $\mathbb{P}^2$ ramified along C, is also the quotient of Σ by the action of $\mathbb{Z}/3\mathbb{Z} \subset \mathbb{Z}/6\mathbb{Z}$. Let $p : \Sigma \to S$ be the quotient map. We now have

Lemma 3.5. *Via the map $\Phi_* \circ (p, p)^*$, the group*

$$\mathrm{Im}\,(\mu_S : CH_0(S)_0 \otimes CH_0(S)_0 \to CH_0(S \times S))$$

embeds into $CH_0(X)_0$, and the image $\mathrm{Im}\,\mu_S^-$ embeds into the invariant part of $CH_0(X)_0$ under the involution i (which is of the type (16)) acting on coordinates by

$$i(X_0, X_1, X_2, Y_0, Y_1, Y_2) = \sqrt{-1}(Y_0, Y_1, Y_2, -X_0, -X_1, -X_2),$$

which leaves the equation of X invariant.

Proof. The Shioda rational map Φ is the quotient map by the group $G = \mathbb{Z}/6\mathbb{Z}$. So for a 0-cycle $z \in CH_0(\Sigma \times \Sigma)$, we have

$$\Phi^*(\Phi_*(z)) = \sum_{g \in G} g^* z.$$

Let now $z = \sum_i pr_1^* z_i \cdot pr_2^* z_i'$, $\deg z_i = \deg z_i' = 0$, be an element of $\mathrm{Im}\,\mu$. Then denoting by j the involution of S over $\mathbb{P}^2$, we have $j^* z_i = -z_i$, $j^* z_i' = -z_i'$, so that $(j, j)^*(z) = z$. It immediately follows that $(p, p)^* z$ is invariant under G, so that $\sum_{g \in G} g^*((p, p)^* z) = 6(p, p)^* z$, which proves the injectivity since $(p, p)^*$ is injective.

Let us now check that the cycles in $\Phi_*(\mathrm{Im}\,\mu^-)$ are invariant under i. Indeed, elements of $(p, p)^*(\mathrm{Im}\,\mu^-)$ are antiinvariant under the involution τ acting on $\Sigma \times \Sigma$ exchanging factors. On the other hand, elements of $\mathrm{Im}\,\mu^-$ are also antiinvariant under the involution (Id, j) acting on $S \times S$. It follows that for $z \in \mathrm{Im}\,\mu^-$, one has $\tau^*((p, p)^*((Id, j)^*(z))) = z$. Applying Φ_*, we get that $\Phi_*((p, p)^* z)$ is invariant under i. $\square$

In conclusion, if we were able to prove that for the sextic fourfolds X invariant under the involution i of the type (16), the i-invariant part of $H^4(X, \mathbb{Q})$ is parameterized by 1-cycles, then by Theorem 3.3, we would get that $CH_0(X)_0^+ = 0$ and by Lemma 3.5, we would conclude that the map μ_S^- is 0, thus solving Question 3.4 for $K3$ surfaces which are ramified double covers of $\mathbb{P}^2$.

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