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On Moderate Degenerations of Polarized Ricci-Flat Kähler Manifolds

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On the occasion of the centennial celebration of Professor Kunihiko Kodaira

Abstract. We study degenerations of smooth projective varieties with trivial canonical bundle, and discuss equivalences that the finiteness of the Weil-Petersson distance, the uniform boundedness of diameters with respect to Kähler-Einstein metrics, and that the limit variety has canonical singularities at worst.

1. Introduction

We discuss relations among various geometric properties along degenerations of smooth projective varieties with trivial canonical bundle, such as the finiteness of the Weil-Petersson distance, the uniform boundedness of diameters with respect to Ricci-flat Kähler metrics, the volume non-collapsing property, and that the limit variety has canonical singularities at worst. This picture was initiated by Wang [W1], [W2] and continued by Tosatti [To] in connection with recent developments of Gromov-Hausdorff convergence theory in Kähler geometry by Tian, Donaldson-Sun [DS], and of the minimal model theory by Birkar-Cascini-Hacon-M$^{\mathrm{C}}$Kernan [BCHM] and others, especially Lai [L] ([Fu] in more explicit terms). We complete the picture in this paper. To state our results, we explain our situation and recall basic definitions.

Set up 1.1. (1) Let X be a reduced and irreducible complex space admitting a projective surjective holomorphic map $f : X \to C$ with connected fibers, to a Riemann surface C with a special point $0 \in C$. Suppose that $X^o := X \setminus f^{-1}(0)$ is smooth, f is smooth on X^o, and that $X_t := f^{-1}(t)$ is

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n-dimensional and has a trivial canonical bundle, i.e. $K_{X_t} = \mathcal{O}_{X_t}$ for any $t \in C^o := C \setminus \{0\}$. Let $X_0 := f^*(0)$ be the special/central fiber, which may be non-reduced. The symbol t will also stand for a local coordinate of C centered at 0.

(2) Let L be a holomorphic line bundle on X^o which is f-ample over C^o, and denote by $L_t = L|_{X_t}$ for $t \in C^o$. According to Yau, there exists a unique *Ricci-flat Kähler form* ω_t on X_t in the cohomology class $c_1(L_t)$ for $t \in C^o$. (If one prefers, one can start with a line bundle L on X which is f-ample over C, and with normal X.)

1.2. There are fundamental geometric intrinsic objects/properties attached to a family of varieties in 1.1. For our purpose here, we may suppose that C is a disk in $\mathbb{C}$ or an open Riemann surface.

(1) Let $X' \to X$ be the normalization, and let $f' : X' \to C$ be the induced morphism. We denote by $K_{X'}$ the canonical sheaf, which is reflexive of rank 1, defined by $j_* K_{X'_{\mathrm{reg}}}$ with $j : X'_{\mathrm{reg}} \to X'$ is the open immersion of the regular part, and by $K_{X'/C} := K_{X'} \otimes f'^* K_C^{-1}$. Then $f'_* K_{X'/C}$ becomes a line bundle and hence trivial on C. We take a frame $\eta \in H^0(C, f'_* K_{X'/C})$ so that $\eta \mathcal{O}_C \cong f'_* K_{X'/C}$. There are naturally induced homomorphism $f'^* f'_* K_{X'/C} \to K_{X'/C}$ and isomorphism $H^0(C, f'_* K_{X'/C}) \cong H^0(X', K_{X'/C})$. Thus there exists

$$\Omega \in H^0(X', K_{X'/C})$$

corresponding to η such that $H^0(X', K_{X'/C}) = \Omega f'^* H^0(C, \mathcal{O}_C)$. We denote the restriction by $\Omega_t = \Omega|_{X_t} \in H^0(X_t, K_{X_t})$ for any $t \in C^o$, which we regard as a nowhere vanishing holomorphic n-form, and $(-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t$ as a non-degenerate volume form on X_t. We can ask whether or not the volume $\int_{X_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t$ is uniformly bounded in $t \in C^o$. This property does not depend on the generator Ω (i.e. η).

(2) We recall a classical definition, [W1, 0.7] for example. We consider a smooth $(1,1)$-form

$$\omega_{\mathrm{WP}} := \frac{\sqrt{-1}}{2\pi} \overline{\partial} \partial \log \left(\int_{X_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t \right)$$

on C^o, where Ω_t is as above. By Griffiths' computation on the curvature of the Hodge (line) bundle $f_* K_{X^o/C^o}$, ω_{WP} is a semi-positive $(1,1)$-form on C^o. This ω_{WP}, or the corresponding (pseudo-)metric tensor, is called the

Weil-Petersson (pseudo-)metric on C^o. Thus we can discuss whether or not 0 is at finite distance from C^o with respect to ω_{WP} (from any reference point $q \in C^o$). We will refer as $d_{\mathrm{WP}}(C^o, 0) < \infty$ or $d_{\mathrm{WP}}(C^o, 0) = \infty$. By [W1, 1.2], $d_{\mathrm{WP}}(C^o, 0) = \infty$ is equivalent to that ω_{WP} is quasi-isometric to the Poincaré metric at the boundary 0.

(3) Let L and ω_t be as in 1.1(2). Let $B_{\omega_t}(x, r)$ be the geodesic ball of radius r centered at $x \in X_t$ and let $\mathrm{Vol}_{\omega_t} B_{\omega_t}(x, r)$ be the volume with everything respect to ω_t. Let also $\mathrm{diam}\,(X_t, \omega_t)$ be the diameter. We consider the following *volume non-collapsing property* (with respect to L or ω_t) [DS, (1.2)]: *There exists a constant* $\alpha > 0$ *such that, for any* $t \neq 0$*, any* $x \in X_t$*, and any* $0 < r \le \mathrm{diam}\,(X_t, \omega_t)$*, a uniform estimate*

$$\mathrm{Vol}_{\omega_t} B_{\omega_t}(x, r) \geq \alpha r^{2n}$$

holds. This property is in fact equivalent to the following *uniform diameter bound* ([To, 1.1(e) $\Leftrightarrow$ (f)]): *There exists a constant* $\alpha > 0$ *such that*

$$\mathrm{diam}\,(X_t, \omega_t) \leq \alpha$$

holds for all $t \neq 0$*.*

(4) We finally recall a relation. The Ricci-flat Kähler form ω_t satisfies a Monge-Ampère equation

$$\omega_t^n = e^{c_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t$$

for a normalizing constant $c_t \in \mathbb{R}$ satisfying $c_1(L_t)^n = e^{c_t} \int_{X_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t$, where $c_1(L_t)^n$ is independent of $t \neq 0$.

Wang [W1, 2.3] proved that, if X_0 (is normal and) has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$, then 0 is at finite Weil-Petersson distance from C^o; $d_{\mathrm{WP}}(C^o, 0) < \infty$. He conjectured some sort of converse [W1, 2.4]: the finiteness $d_{\mathrm{WP}}(C^o, 0) < \infty$ implies X_0 has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$, possibly after a finite base change and a birational modification, and he proved it under some sort of relative minimal model conjecture holds [W2, 1.2]. Recently [To, 1.2] proves such a kind of converse using the semi-stable minimal model theory from [Fu]. Our first result is to prove a more precise version of the converse without using the semi-stable minimal model theory.

THEOREM 1.3. *Suppose in 1.1(1) that the morphism $f : X \to C$ is log-canonical, and that $K_{X/C} \sim_{\mathbb{Q}} 0$. Suppose further that* 0 *is at finite Weil-Petersson distance from C^o, i.e. $d_{\mathrm{WP}}(C^o, 0) < \infty$. Then X_0 has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$.*

In our setting, $f : X \to C$ is log-canonical, if X is normal and the pair (X, X_0) has log-canonical singularities ([KM, 7.1] or 2.1 here), for example if X is smooth and X_0 is reduced and (not necessarily simple) normal crossing. Here $K_{X/C} \sim_{\mathbb{Q}} 0$ means that the corresponding Weil divisor $K_{X/C}$ satisfies $mK_{X/C} \sim 0$ for some integer $m > 0$. It will turn out that $K_{X/C} \sim_{\mathbb{Q}} 0$ is equivalent to $K_{X/C} \sim 0$ (see 2.4). If X is smooth with $K_{X/C} = \mathcal{O}_X$ and X_0 is simple normal crossing, 1.3 is a direct consequence of [W1, 2.1] together with the adjunction formula. To obtain 1.3, we use, other than [W1, 2.1], (non-)uniruledness criteria of varieties such as [HM] and [Ta], which largely rely on an extension technique of pluricanonical forms.

Tosatti [To] introduces a new perspective in the study of Wang, related to the work of Donaldson-Sun [DS] on the stability problem in Kähler geometry (Donaldson-Tian-Yau conjecture). He shows that if X_0 has canonical singularities, then the volume non-collapsing property holds, and he asks about the converse. We answer his question by proving 1.3 and the following

THEOREM 1.4. *Suppose in 1.1 that the volume non-collapsing property with respect to L holds. Then* 0 *is at finite Weil-Petersson distance from C^o, i.e. $d_{\mathrm{WP}}(C^o, 0) < \infty$.*

This together with 1.3 gives an algebro-geometric characterization of volume non-collapsing property for families of Calabi-Yau type manifolds, which was mentioned by Donaldson-Sun [DS, p. 65]. A theorem [DS, 1.2] says, at least in our setting 1.1, that after embedding these X_t ($t \neq 0$) into $\mathbb{P}^N$ and taking a projective transform of X_t, there exists a limit X_∞ as $t \to 0$ in a Hilbert scheme of varieties in $\mathbb{P}^N$, moreover X_∞ turns out to be a normal projective variety with log-terminal singularities at worst ([DS, 4.15], which actually proves X_∞ has canonical singularities at worst and $K_{X_\infty} = \mathcal{O}_{X_\infty}$). The limit variety X_∞ and our X_0 may be different, however we can compare the period maps of manifolds converging to these two limits (one is our $f : X^o \to C^o$). For a family converging to X_∞, we can apply

[W1, 2.3] and obtain the finiteness of the Weil-Petersson distance. We then deduce the finiteness $d_{\mathrm{WP}}(C^o, 0) < \infty$ for $f : X \to C$.

In the rest of this introduction, we list a number of corollaries corresponding to various situations of interests, for the reader's convenience (cf. [To, 1.1, 1.2]). Our contribution is limited to 1.3 and 1.4. Other statements are mainly due to [W1], [To], [RZ] (see §4). The question of Tosatti [To] is an implication (f) $\Rightarrow$ (a) in 1.5. It seems that it is difficult to prove it directly. See also [To, §1] for a nice overview of consequences when X_0 has canonical singularities at worst.

Corollary 1.5. *Suppose in 1.1 (and 1.2) that the morphism $f : X \to C$ is log-canonical, and that $K_{X/C} \sim_{\mathbb{Q}} 0$. Then the following properties are equivalent:*

(a) X_0 has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$.

(b) 0 is at finite Weil-Petersson distance from C^o, i.e. $d_{\mathrm{WP}}(C^o, 0) < \infty$.

(c) There is a constant $\alpha > 0$ such that $\int_{X_t} (-1)^{n^2/2}\Omega_t \wedge \overline{\Omega}_t \le \alpha$ for all $t \ne 0$.

(d) There is a constant $\alpha > 0$ such that $\omega_t^n \ge \alpha^{-1}(-1)^{n^2/2}\Omega_t \wedge \overline{\Omega}_t$ on X_t for all $t \ne 0$.

(e) There is a constant $\alpha > 0$ such that $\operatorname{diam}(X_t, \omega_t) \le \alpha$ *for all $t \ne 0$.*

(f) The volume non-collapsing property with respect to L holds.

Corollary 1.6. *Suppose in 1.1 that the morphism $f : X \to C$ is log-canonical. Then the following property (a') and the properties (b), (c), (d), (e), (f) in 1.5 are equivalent:*

(a') There exists an (in fact unique) irreducible component F of X_0 such that $H^0(\widetilde{F}, K_{\widetilde{F}}) \ne 0$ for a smooth projective variety $\widetilde{F}$ birational to F.
Suppose further that X and C are quasi-projective, then the above equivalent properties are equivalent to the following:

(a) X_0 has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$, after possibly a birational modification along the central fiber.

The quasi-projectivity assumption is required when we apply a relative minimal model program in [L] (and in [Fu]). The meaning of (a) in 1.6 is that, there exists another $f' : X' \to C$ satisfying 1.1(1) which is birational to $f : X \to C$ and isomorphic over C^o, and whose central fiber $X'_0 = f'^*(0)$ has canonical singularities at worst and $K_{X'_0} = \mathcal{O}_{X'_0}$.

Corollary 1.7. *In 1.1, the properties in (b), (e), (f) in 1.5 are equivalent.*
Suppose further that X and C are quasi-projective, then the above equivalent properties are equivalent to the following:

(a) X_0 has canonical singularities at worst and $K_{X_0} = \mathcal{O}_{X_0}$, after possibly a finite base change and a birational modification along the central fiber.

In particular (e) and (f) do not depend on the polarization L, as (b) does not. When X has singularities worse than canonical, $\Omega|_{X^o} \in H^0(X^o, K_{X^o/C^o})$ does not behave well by birational transforms for example, and this is the reason why the conditions (c) and (d) are not mentioned in 1.7. The meaning of (a) in 1.7 is that, there exist a finite morphism $\tau : C' \to C$ and a morphism $f'' : X'' \to C'$ satisfying 1.1(1) over C', which is a birational modification of $f' : X' = X \times_C C' \to C'$ along the central fiber, and whose central fiber $X_0'' := f''^*(\tau^{-1}(0))$ has canonical singularities at worst and $K_{X_0''} = \mathcal{O}_{X_0''}$. Note that the properties 1.2 (2) and (3) are not affected by taking finite base changes $C' \to C$ (and consider a family $X \times_C C' \to C'$), by taking birational modifications $X' \dashrightarrow X$ along the central fiber (and consider a family $X' \to C$), and combinations of these.

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2. Use of a Frame Work of Minimal Model Theory

We shall prove 1.3 in this section. We recall some basics in the classification theory of algebraic varieties, at least in our setting.

Definition 2.1 ([KM, 2.34, 7.1]). Let X be a normal variety and Δ be an effective $\mathbb{Q}$-Weil divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $\mu : X' \to X$ be a log-resolution of singularities of the pair (X, Δ), and write as $K_{X'} + \Delta' \sim_{\mathbb{Q}} \mu^*(K_X + \Delta) + \sum e_i E_i$, where Δ' is the strict transform of Δ, E_i are μ-exceptional prime divisors, $e_i \in \mathbb{Q}$, and where $\sim_{\mathbb{Q}}$ stands for the $\mathbb{Q}$-linear equivalence.

(1) Then X (with $\Delta = 0$) has *canonical singularities*, if (K_X is $\mathbb{Q}$-Cartier and) $e_i \geq 0$ holds for any i for any log-resolution of singularities $\mu : X' \to X$. Canonical singularities are rational singularities and satisfy $\mu_* K_{X'} = K_X$ for any resolution of singularities $\mu : X' \to X$ ([KM, 5.12]).

(2) The pair (X, Δ) is *log-canonical* (*lc* for short), if $e_i \geq -1$ holds for any i for some any log-resolution of singularities $\mu : X' \to X$.

(3) A morphism $f : X \to C$ (as in 1.1) is *log-canonical* (*lc* for short), if X is normal and the pair $(X, X_t = f^*(t))$ is log-canonical for any $t \in C$. Then, any fiber of f is reduced ([KM, 7.2(1)]), and X has canonical singularities at worst ([KM, 7.2(5)] with $B = 0$).

Following a classical convention, for a proper variety Y, we set $p_g(Y) = \dim H^0(Y', K_{Y'})$ for any smooth model Y' of Y. If Y has canonical singularities, then $p_g(Y) = \dim H^0(Y, K_Y)$.

We next explain a nice property of lc morphisms, which we will use several time.

Remark 2.2. Let $f : X \to C$ be as in 1.1 and suppose f is lc.

(1) We let in general $a : C' \to C$ be a finite morphism from a Riemann surface C', and let $b : X' := X \times_C C' \to X$ be the induced morphism. Then by [KM, 7.6], X' is normal and the morphism $f' : X' \to C'$ is lc, and also $K_{X'/C'} = b^* K_{X/C}$ (see the proof of [KM, 7.6]). In particular, any fiber of f' is reduced, and X' has canonical singularities. As f is smooth over $C \setminus \{0\}$, $X' \setminus (a \circ f')^{-1}(0)$ is smooth over $C' \setminus a^{-1}(0)$. By definition of the fiber product $X \times_C C'$, $f'^{-1}(0')$ and $f^{-1}(0)$ are the same for every $0' \in a^{-1}(0)$. Thus if we want to say something on X_0, we are free to take a finite base change.

(2) By the semi-stable reduction theorem ([KM, 7.17]), possibly after replacing C by an open subset containing 0, we can find a finite morphism $a : C' \to C$ from a Riemann surface C' such that $a^{-1}(0)$ consists of a point $0' \in C'$ and a resolution of singularities $\mu : X'' \to X' = X \times_C C'$ which is

isomorphic over $X' \setminus f'^{-1}(0')$, such that every fiber of the induced morphism $f'' : X'' \to C'$ is reduced and simple normal crossing, and such that f'' is smooth over $C' \setminus \{0'\}$.

$$\begin{array}{ccccc} X'' & \xrightarrow{\mu} & X' = X \times_C C' & \xrightarrow{b} & X \\ {\scriptstyle f''}\Big\downarrow & & {\scriptstyle f'}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ C' & = & C' & \xrightarrow[a]{} & C \end{array}$$

We let $X'_0 := f'^*(0')$ which is identified with X_0 via b. Locally around $0' \in C'$, the map $a : C' \to C$ is given by $t' \mapsto t = (t')^m$ with an integer $m > 0$. The condition that $a^{-1}(0)$ consists of a point, is not so important (just for a simplification of presentations). If C is a disk, it is clear. If C is quasi-projective, we can obtain this by a "covering lemma" ([KM, 2.67]).

We further note the following basic properties of Ω in 1.2.

REMARK 2.3. Suppose in 1.1 that $f : X \to C$ is lc, and C is non-compact. We take an $\Omega \in H^0(X, K_{X/C})$ (up to a choice of a frame) in 1.2.

(1) Let $X_0 = \sum_{i \in I} F_i$ be the irreducible decomposition as a Weil divisor. We consider $\Omega|_{X_{\mathrm{reg}}} \in H^0(X_{\mathrm{reg}}, K_{X/C})$. Then the zero divisor of $\Omega|_{X_{\mathrm{reg}}}$ is $\sum_{j \in J} r_j (F_j \cap X_{\mathrm{reg}})$ for some integers $r_j > 0$ for a subset of indexes $J \subset I$. As $H^0(X, K_{X/C}) = \Omega f^* H^0(C, \mathcal{O}_C)$ and X_0 is reduced, it has to be $J \neq I$. Thus $K_{X/C} \sim \sum_{j \in J} r_j F_j$ as a Weil divisor.

(2) As X has canonical singularities, we have $\mu_* K_{X'} = K_X$ for any resolution of singularities $\mu : X' \to X$, and hence $H^0(X', K_{X'/C}) \cong H^0(X, K_{X/C})$. Thus $\Omega \in H^0(X, K_{X/C})$ is canonically attached to $f : X \to C$, namely we can take $\Omega' \in H^0(X', K_{X'/C})$ such that $H^0(X', K_{X'/C}) = \Omega' f'^* H^0(C, \mathcal{O}_C)$, and $\mu_* \Omega' = \Omega$, and in particular $\Omega'|_{X'^o} = \Omega|_{X^o}$ via the isomorphism $\mu : X'^o \to X^o$, where $X'^o = X' \setminus f'^{-1}(0)$.

(3) The lc morphism $f : X \to C$ is nicely behaved under finite base changes $C' \to C$ as we saw in 2.2. By a flat base change $a : C' \to C$, we have a natural isomorphism $a^* f_* K_{X/C} \xrightarrow{\sim} f'_* b^* K_{X/C}$. The induced $f' : X' \to C'$ is lc, and $K_{X'/C'} = b^* K_{X/C}$, and hence $a^* f_* K_{X/C} \cong f'_* K_{X'/C'}$. A frame of $f_* K_{X/C}$ induces that of $a^* f_* K_{X/C}$ and hence of $f'_* K_{X'/C'}$. Thus $\Omega' := b^* \Omega \in H^0(X', K_{X'/C'})$ satisfies $H^0(X', K_{X'/C'}) = \Omega' f'^* H^0(C', \mathcal{O}_{C'})$.

Lemma 2.4. *(1) Suppose further in 1.1 that $f : X \to C$ is lc, C is non-compact, $K_{X/C} \sim_{\mathbb{Q}} 0$, i.e., $mK_{X/C} = \mathcal{O}_X$ for an integer $m > 0$. Then $K_{X/C} \sim 0$.*

(2) Let Y be a normal projective variety with canonical singularities such that $K_Y \sim_{\mathbb{Q}} 0$ and $H^0(Y, K_Y) \neq 0$ (i.e., $p_g(Y) \neq 0$). Then $K_Y = \mathcal{O}_Y$.

Proof. (1) We take a frame $\Omega \in H^0(X, K_{X/C})$ as in 1.2. We have $(\Omega|_{X_{\mathrm{reg}}})^{\otimes m} \in H^0(X_{\mathrm{reg}}, mK_{X/C}) = H^0(X_{\mathrm{reg}}, \mathcal{O}_X) = H^0(X, \mathcal{O}_X) = f^*H^0(C, \mathcal{O}_C)$. As we saw in 2.3(1) that $\Omega|_{X_{\mathrm{reg}}}$ does not identically vanish along X_0, $\Omega|_{X_{\mathrm{reg}}}$ is nowhere zero on X_{reg}. In particular $\Omega|_{X_{\mathrm{reg}}}$ gives a trivialization $K_{X_{\mathrm{reg}}} = \mathcal{O}_{X_{\mathrm{reg}}}$. Hence $K_X = j_*K_{X_{\mathrm{reg}}} = \mathcal{O}_X$, where $j : X_{\mathrm{reg}} \to X$ is the open immersion.

(2) We take a non-zero $\Omega \in H^0(Y, K_Y)$. Suppose $mK_Y = \mathcal{O}_Y$ for an integer $m > 0$. Then $(\Omega|_{Y_{\mathrm{reg}}})^{\otimes m} \in H^0(Y_{\mathrm{reg}}, mK_Y) = H^0(Y_{\mathrm{reg}}, \mathcal{O}_Y) = H^0(Y, \mathcal{O}_Y) = \mathbb{C}$. Thus $\Omega|_{Y_{\mathrm{reg}}}$ is nowhere zero on Y_{reg}, and hence gives a trivialization $K_{Y_{\mathrm{reg}}} = \mathcal{O}_{Y_{\mathrm{reg}}}$. Then we have $K_Y = \mathcal{O}_Y$. $\square$

We now prove 1.3 and 1.6 (a') $\Rightarrow$ (a).

2.5 (Proof of 1.3). We use the semi-stable reduction process in 2.2 (1) and (2), and use the notations there.

$$\begin{array}{ccccc}
X'' & \xrightarrow{\mu} & X' = X \times_C C' & \xrightarrow{b} & X \\
{\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\
C' & = & C' & \xrightarrow[a]{} & C
\end{array}$$

As $X'_0 = f'^*(0') = X_0$, it is enough to show that X'_0 has canonical singularities and $K_{X'_0} = \mathcal{O}_{X'_0}$.

(3) Let $X''_0 := f''^*(0') = S_1 + S_2 + \ldots + S_k$ be the irreducible decomposition. By our assumption, $0'$ is at finite Weil-Petersson distance from $(C')^o$, i.e. $d_{\mathrm{WP}}((C')^o, 0) < \infty$. By Wang [W1, 2.1], $d_{\mathrm{WP}}((C')^o, 0) < \infty$ if and only if $p_g(S_1) = 1$ and $p_g(S_2) = \ldots = p_g(S_k) = 0$ (possibly after relabeling). In particular S_1 is not uniruled.

Since X' has canonical singularities, every μ-exceptional divisor in X'' is uniruled by [HM, 1.5]. Thus S_1 is not μ-exceptional, and then $\mu(S_1)$ is an irreducible component of X'_0, which is not uniruled. Noting $X'_0 = X_0$, there exists a unique irreducible component $F \subset X_0$ such that $p_g(F) \neq 0$. Note that we do not use $K_{X/C} = \mathcal{O}_X$ yet.

(4) We note that $K_{X'/C'} = b^*K_{X/C} = \mathcal{O}_{X'}$. As $K_{X'/C'} = \mathcal{O}_{X'}$, X'_0 has to be irreducible by [Ta, 1.1(2)] (otherwise all irreducible components of X'_0 are uniruled). Then by [Ta, 1.1(1.1)], X'_0 is normal (otherwise $X'_0 = X_0 = F$ is uniruled). We now note that X' has no codimension 2 singularities, since X' is smooth where the Cartier divisor X'_0 is smooth. Thus the assumptions in [Ta, 1.1(1.2)] are satisfied, and hence X'_0 has canonical singularities. By the adjunction formula, we have $K_{X'_0} = \mathcal{O}_{X'_0}$. $\square$

2.6 (Proof of 1.6 (a') $\Rightarrow$ (a)). We use the semi-stable reduction process in 2.2 (1) and (2), and use the notations there.

(3) By the condition (a') and [Ta, 1.1(0)], there exists a unique component $F \subset X_0$ with $p_g(F) \neq 0$. We take a log-resolution of singularities $\mu : Y \to X$ of the pair (X, X_0), which is isomorphic outside X_0 (and $\operatorname{Supp} Y_0$ is simple normal crossing, where $Y_0 = (f \circ \mu)^*(0)$). We note that the strict transform $\widetilde{F}$ of F in Y is not uniruled, and other irreducible components of Y_0 are uniruled by [Ta, 1.1(2)]. As f has reduced fibers, the multiplicity of Y_0 along $\widetilde{F}$ is one, too.

(4) We run a relative minimal model program [L, 4.4] for $g := f \circ \mu : Y \to C$ and obtain a relative minimal model $g' : Y' \to C$. Here Y' has ($\mathbb{Q}$-factorial) terminal singularities at worst, $K_{Y'} \sim_{\mathbb{Q}} 0$, and Y' is obtained from Y, say $\varphi : Y \dashrightarrow Y'$, by a finite composition of divisorial contractions and flips, and $\varphi : Y \dashrightarrow Y'$ is isomorphic over C^o. (This is not exactly the same as [L, 4.4]. However it is said that this is a common knowledge among experts. We refer [Fu, 1.5] which is written explicitly for the author's and the reader's conveniences.) Let $T = \operatorname{Supp} g'^*(0)$. Since $\widetilde{F}$ is not uniruled, $\widetilde{F}$ is not contracted by φ (a divisorial contraction contracts a uniruled divisor [KMM, 5-1-8]). In particular, T contains an irreducible component which is birational to $\widetilde{F}$. We then see that T is irreducible, because otherwise all irreducible components of T are uniruled ([Ta, 1.1(2)], [Fu, 1.5]). Thus T is birational to $\widetilde{F}$ and $T = g'^*(0)$ (with multiplicity one). In particular T is not uniruled and $p_g(T) \neq 0$. This can happen only when T is normal and has canonical singularities ([Ta, 1.1], [Fu, 1.5]). Then $K_T \sim_{\mathbb{Q}} \mathcal{O}_T$, and in fact $K_T = \mathcal{O}_T$ by virtue of 2.4. Notice that the quasi-projectivity of X and C is required when we apply [L] (and [Fu]). $\square$

REMARK 2.7. There may be other variants of 1.3 and 1.6 (a') $\Rightarrow$ (a), for example [To, 1.1 (b) $\Rightarrow$ (a), 1.2]. These can be obtained as combinations of a semi-stable reduction, a relative minimal model program, (non-) uniruledness criteria, adjunction formulas, ...

3. Use of Donaldson-Sun's Theory

3.1 (Proof of 1.4). We prove here 1.4, namely that the volume non-collapsing property implies the finiteness of the Weil-Petersson distance, i.e. $d_{\mathrm{WP}}(C^o, 0) < \infty$. Since our assertion is local around X_0, we may suppose that C is a unit disk in $\mathbb{C}$ with center $t = 0$, and that everything is defined on a slightly larger disk (or we may replace C be a smaller disk). We take a sequence of points $t_i \in C^o, i = 1, 2, \ldots,$ such that $t_i \to 0$ as $i \to \infty$. As we will see, the volume non-collapsing property along a sequence of points $t_i \in C^o$ with $\lim_{i\to\infty} t_i = 0$ is enough to conclude the finiteness of Weil-Petersson distance.

(1) We first make a simplification. We reduce to the case that L is a line bundle on X (not only on X^o), L is f-very ample over C, and $f_*L \cong U \otimes \mathcal{O}_C$ for an $(N+1)$-dimensional $\mathbb{C}$-vector space U, where $U \cong H^0(X_t, L_t)$ for any $t \in C$.

It is clear that the volume non-collapsing property with respect to L holds if and only if it holds with respect to $L^k (= L^{\otimes k})$ for some (any) integer $k > 0$. In particular we are free to pass to a higher power of L. We may suppose that L is f-very ample over C^o, for example we can take L^k with $k = O(n^3)$ by the effective very ampleness result of [AS]. Note that in any event, f_*L is locally free on C^o, and any locally free sheaf on an open Riemann surface is trivial ([Fo, 30.4]). Then as in [Ht, III.9.8], the embedded $f : X^o \to C^o$ in $\mathbb{P}(f_*L) \cong \mathbb{P}^N \times C^o$ can be extended (by taking the closure in $\mathbb{P}^N \times C$) as a family $f' : X' \to C$ in $\mathbb{P}^N \times C$ satisfying 1.1 and L on X^o also extends as an f'-very ample line bundle L' on X' (that is a restriction of the relative $\mathcal{O}(1)$). Hence without loss of generality, by replacing $f : X \to C$ and L by $f' : X' \to C$ and L', we may suppose from the beginning that L is f-very ample on X, as the properties in questions in 1.4 depend only on $f : X^o \to C^o$ and on L on X^o. Then by taking a higher power of this new L if necessary, we may suppose there are no higher cohomology groups, i.e., $H^q(X_t, L_t) = 0$ for any $t \in C$ (including $t = 0$)

and $q > 0$, and that $f_*L \cong U \otimes \mathcal{O}_C$ for an $(N+1)$-dimensional $\mathbb{C}$-vector space U, where $U \cong H^0(X_t, L_t)$ for any $t \in C$. We obtain an embedding $\Psi : X \to (\mathbb{P}(f_*L) \cong)\mathbb{P}(U) \times C$ over C. Let $\Phi = \mathrm{pr}_1 \circ \Psi : X \to \mathbb{P}(U)$, where $\mathrm{pr}_1 : \mathbb{P}(U) \times C \to \mathbb{P}(U)$ is the first projection. We will also use the notation $\mathbb{P}^N$ instead of $\mathbb{P}(U)$ in what follows.

(2) Let $\mathrm{Hilb}_P(\mathbb{P}^N)$ be the Hilbert scheme of closed subschema in $\mathbb{P}^N$ with Hilbert polynomial P satisfying $P(m) = h^0(X_t, L_t^m)$ for all sufficiently large integer m. Here $P(x) = \frac{d}{n!}x^n +$ (lower order terms), with $d = c_1(L_t)^n$. Let $\mathrm{Hilb} \subset \mathrm{Hilb}_P(\mathbb{P}^N)$ be the reduced structure of $\mathrm{Hilb}_P(\mathbb{P}^N)$. For every $t \in C$, the image $\Phi(X_t) \subset \mathbb{P}^N$ defines a point $\Phi(X_t) \in \mathrm{Hilb}$ (by a slight abuse of notation). By the universal property of the Hilbert scheme, we have a morphism

$$h : C \to \mathrm{Hilb}, \text{ given by } t \mapsto \Phi(X_t),$$

such that $X \cong h^*(\mathrm{Univ})$ over C, where $\mathrm{Univ} \to \mathrm{Hilb}$ is the universal family (induced from the one over $\mathrm{Hilb}_P(\mathbb{P}^N)$) ([Kol, I.1.4]). Let $\mathrm{Hilb}^o \subset \mathrm{Hilb}$ be the Zariski open subset parameterizing smooth subvarieties in it (cf. [C, 1.24]). We have a morphism $h : C^o \to \mathrm{Hilb}^o$ by restriction.

(3) We take a Hermitian metric a_t on L_t for $t \in C^o$, whose curvature form satisfies $\omega_t = \frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial \log a_t$. By Donaldson and Sun [DS, 1.2], especially around [DS, Lemma 4.3] and the argument after [DS, 4.13] (using the volume non-collapsing condition), possibly after replacing L by a higher power and passing to a subsequence of t_is, we can find an orthonormal basis of $H^0(X_{t_i}, L_{t_i})$ with respect to ω_t and a_t for each i, such that a sequence $T_{t_i}(X_{t_i}) \in \mathrm{Hilb}$ convergent to a point $W \in \mathrm{Hilb}$, where $T_{t_i} : X_{t_i} \to \mathbb{P}^N$ is the embedding with respect to the orthonormal basis of $H^0(X_{t_i}, L_{t_i}) \cong U$. Furthermore, by taking a higher power of L and passing to a subsequence if necessary, Donaldson and Sun show that W is normal ([DS, 4.12] in general) and has "log-terminal" singularities in the Kähler-Einstein cases [DS, 4.15]. (The study of W is done by exploiting the Gromov-Hausdorff convergence $(X_{t_i}, \omega_{t_i}) \to (X_\infty, \omega_\infty)$.)

We show that W has canonical singularities at worst and $K_W = \mathcal{O}_W$ in our situation. Since $K_{X_t} = \mathcal{O}_{X_t}$ for any $t \in C^o$, the proof of [DS, 4.15] shows that there is a nowhere vanishing holomorphic n-form Θ on the smooth locus W_{reg} (and it is L^2 on W_{reg}). That gives a trivialization $K_{W_{\mathrm{reg}}} = \mathcal{O}_{W_{\mathrm{reg}}}$, and then $K_W = j_*(K_{W_{\mathrm{reg}}}) = j_*(\mathcal{O}_{W_{\mathrm{reg}}}) = \mathcal{O}_W$ as we already know W is normal, where $j : W_{\mathrm{reg}} \to W$ is the open immersion. Thus W has $K_W = \mathcal{O}_W$ and is "log-terminal" by [DS, 4.15] (it is not clear K_W is $\mathbb{Q}$-Cartier in [DS, 4.15]). This concludes that W has canonical singularities at worst.

(4) We let $G = SL(N+1, \mathbb{C}) = SL(U)$, and realize it as an affine subvariety of $\mathbb{C}^{(N+1)^2}$: namely the space of all $(N+1) \times (N+1)$ matrices by our choice of a basis of the vector space U to give $U \cong \mathbb{C}^{N+1}$. We take a compactification $\mathbb{P}^{(N+1)^2}$ of $\mathbb{C}^{(N+1)^2}$, and take the Zariski closure $\overline{G}$ of G in $\mathbb{P}^{(N+1)^2}$ for convenience. The group G acts on Hilb in a natural manner. We denote by $G_x = \{(g, gx); g \in G\} \subset G \times \mathrm{Hilb}$ (which is a graph $G_x \cong G$) and by O_x the G-orbit of every $x \in \mathrm{Hilb}$. Then the projection $\mathrm{pr}_H : G \times \mathrm{Hilb} \to \mathrm{Hilb}$ induces a surjective morphism $G_x \to O_x$ for $x \in \mathrm{Hilb}$. (Although we do not use it, the morphism $G_{h(t)} \to O_{h(t)}$ is in fact finite for $t \in C^o$. See 3.2.) For every $x \in \mathrm{Hilb}$, the Zariski closure $\overline{G_x}$ in $\overline{G} \times \mathrm{Hilb}$ is irreducible, and $\mathrm{pr}_H(\overline{G_x}) = \overline{\mathrm{pr}_H(G_x)}$ which is $\overline{O_x}$ (and it is independent of the choice of a compactification of G). We shall denote by "pr_{-}" the projection to the factor under consideration.

We consider the Zariski closure of all G-orbits $O_{h(t)}$ over C:

$$A = \overline{\bigcup_{t \in C} \{t\} \times O_{h(t)}} = \overline{\bigcup_{t \in C} \{t\} \times \overline{O_{h(t)}}} \subset C \times \mathrm{Hilb}.$$

This can also be described as follows. We first consider the graph $\Lambda \cong C \times G$ of a morphism $1 \times \widetilde{h} : C \times G \to C \times G \times \mathrm{Hilb}$, given by $(t, g) \mapsto (t, g, gh(t))$, and take the Zariski closure $\overline{\Lambda}$ in $C \times \overline{G} \times \mathrm{Hilb}$. Then A is obtained by the projection $A = \mathrm{pr}_{C \times H}(\overline{\Lambda})$ to the factor $C \times \mathrm{Hilb}$. In particular A is irreducible and projective over C under the projection $\alpha := \mathrm{pr}_C|_A : A \to C$.

$$\begin{array}{ccccc}
C \times \overline{G} \times \mathrm{Hilb} & \xrightarrow{\mathrm{pr}_{C \times H}} & C \times \mathrm{Hilb} & \supset & A \\
\Big\downarrow \mathrm{pr}_C & & \Big\downarrow \mathrm{pr}_C & & \Big\downarrow \alpha \\
C & =\!= & C & = & C
\end{array}$$

The graph $\Lambda = \cup_{t \in C}(\{t\} \times G_{h(t)})$ is an irreducible variety of $\dim \Lambda = \dim G + 1$, and it is Zariski open in $\overline{\Lambda}$. Let $\lambda : \overline{\Lambda} \to C$ be the projection. As $\overline{\Lambda}$ is irreducible, every general fiber of λ is a union of varieties of dimension $\dim G$. Note $\lambda^{-1}(t) = (\lambda^{-1}(t) \cap \Lambda) \amalg (\lambda^{-1}(t) \cap (\overline{\Lambda} \setminus \Lambda)) = (\{t\} \times G_{h(t)}) \amalg (\lambda^{-1}(t) \cap (\overline{\Lambda} \setminus \Lambda))$, and $\dim(\overline{\Lambda} \setminus \Lambda) \leq \dim \overline{\Lambda} - 1 = \dim G$. Thus we have

$\dim(\lambda^{-1}(t) \cap (\overline{\Lambda} \setminus \Lambda)) < \dim G$ for general $t \in C$. As $\{t\} \times \overline{G_{h(t)}} \subset \lambda^{-1}(t)$ and $\dim \overline{G_{h(t)}} = \dim G$, we can see $\lambda^{-1}(t) = \{t\} \times \overline{G_{h(t)}}$ for general $t \in C$ (so that $\dim(\lambda^{-1}(t) \cap (\overline{\Lambda} \setminus \Lambda)) < \dim G$). By shrinking C if necessary, we may suppose $\lambda^{-1}(t) = \{t\} \times \overline{G_{h(t)}}$ for any $t \in C^o$. Since the map $\overline{\Lambda} \to A$ induces a surjection $\lambda^{-1}(t) \to \alpha^{-1}(t)$ and since $\mathrm{pr}_H(\overline{G_{h(t)}}) = \overline{O_{h(t)}}$ for any $t \in C$, we have $\alpha^{-1}(t) = \{t\} \times \overline{O_{h(t)}}$ for any $t \in C^o$ as well. We have also observed that general fibers of λ and α are irreducible. As C is normal, Zariski's main theorem shows both λ and α have connected fibers. The last point will not be used. We finally set

$$A^o := A \cap (C^o \times \mathrm{Hilb}^o),$$

which is non-empty and Zariski open in A (recall $\mathrm{Hilb}^o \subset \mathrm{Hilb}$ parametrizes smooth subvarieties).

(5) We will denote by $0_C \in C$ the special point 0 to avoid any risk of confusion. By [DS] as we explained above, we can find a sequence $g_i \in G$ such that $g_i h(t_i) \in \mathrm{Hilb}$ converges to a point $W \in \mathrm{Hilb}$, moreover W is normal with canonical singularities at worst and with $K_W = \mathcal{O}_W$. As $(t_i, g_i h(t_i)) \in A$ and A is closed, we have the limit $(0_C, W) \in A$. We do not know if $W \in \overline{O_{h(0)}}$, though $(0_C, W) \in \alpha^{-1}(0_C)$.

We take a general irreducible curve

$$B' \subset A$$

passing through the point $(0_C, W) \in A$ and $B' \cap A^o \neq \emptyset$. We note that $\{b' \in B', b' \in A \setminus A^o\}$ consists of isolated points in B', namely $\{b' \in B', b' \in A^o\}$ is non-empty Zariski open in B'. As $\alpha (= \mathrm{pr}_C|_A) : A \to C$ is projective and surjective, we may suppose that $\alpha|_{B'} : B' \to C$ is finite (and surjective). We take the normalization $\nu : B \to B'$, and obtain a finite morphism $\tau = \alpha|_{B'} \circ \nu : B \to C$ as a composition. Then, after shrinking C to a smaller disk (and shrink everything accordingly) and taking an irreducible component of B' containing $(0_C, W)$, we may suppose that $\{b' \in B', b' \in A^o\} = B' \setminus \{(0_C, W)\}$, and that $\nu^{-1}((0_C, W)) \subset B$ consists of a point, say $0_B \in B$, and the morphism $\tau : (B, 0_B) \to (C, 0_C)$ is isomorphic to a (ramified) covering $(\Delta, 0) \to (\Delta, 0)$ given by $s \mapsto t = s^k$ for an integer $k > 0$. (A minor remark is that, after shrinking the original B' according to the shrinking of C, B' may be reducible.) Let

$$g : Y := (B \times_{\mathrm{Hilb}} \mathrm{Univ})_{\mathrm{red}} \to B$$

be the induced family from the universal family Univ $\to$ Hilb via the composition $B \to A \subset (C \times \mathrm{Hilb}) \to \mathrm{Hilb}$, where everything is the given one and the last one is the projection. Here $(-)_{\mathrm{red}}$ stands for the reduced structure. Noting that $\nu(s) \in B' \cap \mathrm{Hilb}^o$ for every $s \in B^o := B \setminus \{0_B\}$, we see that $g : Y \to B$ satisfies 1.1(1) with the central fiber W over 0_B (we borrow the argument in [C, 1.26, p. 572 bottom]). Every other fiber $Y_s, s \in B^o$, is isomorphic to some of $X_t, t \in C^o$.

(6) We now look at the period maps (of arbitrary weight with polarizations given by $\mathcal{O}_{\mathbb{P}^N}(1)$). We follow [GS, §3(b)] for a general discussion. Our main interest is, needless to say, the Hodge sub-line-bundle $(f|_{X^o})_* K_{X^o/C^o} \subset R^n(f|_{X^o})_*\mathbb{C}$ of weight n part. For the smooth family $f : X^o \to C^o$ with polarizations given by $L = \Phi^*\mathcal{O}_{\mathbb{P}^N}(1)$, we have a period map $\phi_f : C^o \to \Gamma\backslash D$ ([GS, p. 57], [W1, 0.2]). Also on Hilb^o (which parametrizes smooth members), we have a period map $\phi_H : \mathrm{Hilb}^o \to \Gamma\backslash D$. We note that, on $\overline{O_{h(t)}} \cap \mathrm{Hilb}^o$ with $t \in C^o$ (not only on $O_{h(t)}$), the period map ϕ_H takes a constant value $\phi_f(t)$. Thus, on $A^o (\subset C^o \times \mathrm{Hilb}^o)$, the right hand side of the following diagram is commutative:

$$\begin{tikzcd} B^o \arrow[r, "\nu"] \arrow[d, "\tau"'] & A^o \arrow[r, "\mathrm{pr}_H"] \arrow[d, "\alpha"'] & \mathrm{Hilb}^o \arrow[d, "\phi_H"] \\ C^o \arrow[r, equal] & C^o \arrow[r, "\phi_f"'] & \Gamma\backslash D \end{tikzcd}$$

Here $\alpha = \mathrm{pr}_C : A^o \to C^o$, resp. $\mathrm{pr}_H : A^o \to \mathrm{Hilb}^o$, is (the restriction of) the projection. By construction, the left hand side of the diagram is commutative. On the other hand, by definition of the family $g : Y \to B$, the period map $\phi_g : B^o \to \Gamma\backslash D$ over B^o is given by the composition along the upper right: $B^o \to A^o \to \mathrm{Hilb}^o \to \Gamma\backslash D$.

By the commutativity of the diagram above, we have $\phi_g = \phi_f \circ \tau : B^o \to C^o \to \Gamma\backslash D$. (To be honest, the author does not know whether $\phi_H : \mathrm{Hilb}^o \to \Gamma\backslash D$ is holomorphic or not as Hilb^o may be singular. However we can define ϕ_H set theoretically and obtain $\phi_g = \phi_f \circ \tau$, which is what we need.) In particular the Weil-Petersson form on B^o (which is the curvature form of the Hodge line bundle with the canonical L^2-metric) is the pull-back of the one on C^o by the finite morphism $\tau : B \to C$. As the central fiber W of $g : Y \to B$ has canonical singularities with $K_W = \mathcal{O}_W$, we have the finiteness of the Weil-Petersson distance, i.e. $d_{\mathrm{WP}}(B^o, 0_B) < \infty$ by Wang [W1, 2.3]. This is equivalent to the finiteness $d_{\mathrm{WP}}(C^o, 0_C) < \infty$. This completes the proof of 1.4. □

REMARK 3.2 (Oguiso). In the proof above, we made a comment on the finiteness of the morphism $G_x \to O_x$ for every $t \in C^o$. This indeed follows from the following fact (we learned from Oguiso). *Let M be a smooth projective variety with $K_M = \mathcal{O}_M$, and let L be an ample line bundle on M. Then $I_L = \{g \in \mathrm{Aut}\,(M);\ g^*L = L\}$ is finite.*

This (must be well-known and) is proved for Abelian varieties in [GH, p. 326] and for simply connected smooth projective variety M with $K_M = \mathcal{O}_M$ in [O, 2.4]. For the general case, without conditions on $\pi_1(M)$ or on $H^1(M, \mathcal{O}_M)$, we pass to the so-called Bogomolov decomposition of such types of manifolds, and after possibly taking a finite étale Galois cover, we can then reduce our assertion to the above mentioned primitive cases.

4. Proof of Corollaries

We shall prove the corollaries in the introduction.

4.1. The structure of proofs of 1.5, 1.6, 1.7 under 1.1 is as follows.

(1) If $f : X \to C$ is lc (as long as Ω is nicely defined in a functorial way), (c) and (d) in 1.6 (and hence in 1.5) are equivalent, see [To, 1.1(c) $\Leftrightarrow$ (d)].
(e) and (f) in 1.7 (and hence in 1.5 and 1.6) are equivalent, see [To, 1.1(e) $\Leftrightarrow$ (f)].
(f) implies (b) in 1.7 (and hence in 1.5 and 1.6) by 1.4.

(2) On 1.5. Our contributions are (f) $\Rightarrow$ (b) by 1.4 and (b) $\Rightarrow$ (a) by 1.3. The rests have been done by others as noticed in [To, 1.1]:
(a) $\Rightarrow$ (b) by [W1, 2.3].
(b) $\Rightarrow$ (c) by Gross in [RZ, Appendix B.1(ii)] (see also 4.5)
(c) $\Rightarrow$ (e)=(f) by [RZ, 2.1] (see also 4.3).

(3) On 1.6. Our main contribution is (f) $\Rightarrow$ (b) by 1.4. The other properties have almost already been obtained verbatim in prior works. Some minor adjustments of the implications listed above for 1.5 are sufficient to conclude.

(a') $\Rightarrow$ (b) follows from [W1, 2.1] after a semi-stable reduction 2.2.
(b) $\Rightarrow$ (a') is already done in the proof of 1.3, Step (3).
(b) $\Rightarrow$ (c) by a version of Gross in [RZ, Appendix B.1(ii)], see 4.5.
(c) $\Rightarrow$ (e)=(f) by a version of [RZ, 2.1], see 4.3.
(a) $\Rightarrow$ (b) by [W1, 2.3].
(a') $\Rightarrow$ (a) See 2.6. A relative minimal model theory [L] ([Fu]) is used.

(4) On 1.7. As those properties are stable under finite base changes and birational modifications along the special fiber, by taking a semi-stable reduction, we may suppose that X is smooth and X_0 is reduced and simple normal crossing. Thus 1.7 is reduced to 1.6.

We shall prove some variants of results in [RZ], which complete the proof of corollaries. The first one is a variant of [RZ, 2.1].

PROPOSITION 4.2. *Suppose that $\pi : M \to \Delta$ is as in 1.1(1), $\Delta \subset \mathbb{C}$ is a disk, M is normal, and that there exists an irreducible component F of the central fiber $M_0 = \pi^*(0)$ with multiplicity 1 such that $\Omega \in H^0(M, K_{M/\Delta})$ (as in 1.2) does not vanish identically along F. Let L be a line bundle on $M \setminus M_0$ which is relatively ample over $\Delta \setminus \{0\}$. Let $\widetilde{g}_t$ (for $t \neq 0$) be the unique Ricci-flat Kähler metric on $M_t = \pi^{-1}(t)$ with Kähler form $\widetilde{\omega}_t \in c_1(L)|_{M_t} \in H^{1,1}(M_t, \mathbb{R})$. Then there is a constant $D > 0$ independent of $t \neq 0$ such that the diameter of $(M_t, \widetilde{g}_t)$ satisfies*

$$\operatorname{diam}_{\tilde{g}_t}(M_t) \leq 2 + D \int_{M_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t$$

for all $t \neq 0$.

PROOF. To infer this variant from [RZ, 2.1], it is enough to consider the following two points in the proof. There need to exist a point $p \in M_0$ such that, around the point $p \in M_0$, both the morphism $\pi : M \to \Delta$ is smooth (in the paragraph just before [RZ, Lemma 2.2]), and the fiberwise volume form $(-1)^{n^2/2}\Omega_t \wedge \overline{\Omega}_t$ is uniformly bounded from below (the bottom line in [RZ, p. 242]). If we take a general point $p \in F \subset M_0$ in 4.2, the above mentioned technical points are satisfied and the proof of [RZ, 2.1] goes through.

One may wonder about the role of π-ample line bundle $\mathcal{L}$ on M in [RZ, 2.1]. Let us take an auxiliary π-ample line bundle $\mathcal{L}$ on M, which gives an embedding $M \to \mathbb{P}^N \times \Delta$ over Δ such that $\mathcal{L}^m = (\mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^N}(1))|_M$ for some $m \geq 1$, where $\mathrm{pr}_1 : \mathbb{P}^N \times \Delta \to \mathbb{P}^N$ is the projection. Let $\omega_t = \frac{1}{m}\omega_{FS}|_{M_t}$ be the pull-back of the Fubini-Study metric via the induced embedding $M_t \to \mathbb{P}^N$ (see [RZ, p. 241 top]). In [RZ, 2.1], L and $\mathcal{L}$ are the same. However in fact, we can separate their roles: one (L) is giving polarizations for the Ricci-flat Kähler form $\widetilde{\omega}_t$, another ($\mathcal{L}$) is as an auxiliary π-ample line bundle and giving a reference metric ω_t. We note for example that [RZ, p. 242, line 7]: $\int_{M_t} \widetilde{\omega}_t \wedge \omega_t^{n-1} = (L|_{M_t}) \cdot (\mathcal{L}|_{M_t})^{n-1} =: \overline{C}$ is independent of $t \neq 0$. The roles of $\mathcal{L}$ and ω_t are not crucial. $\square$

In our terms, 4.2 is

COROLLARY 4.3. *Suppose in 1.1 that X is normal and X_0 is reduced. Let $\Omega \in H^0(X, K_{X/C})$ be as in 1.2. Then there is a constant $D > 0$ independent of $t \in C^o$ such that*

$$\mathrm{diam}\,(X_t, \omega_t) \leq 2 + D \int_{X_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t$$

holds for all $t \in C^o$.

PROOF. We know Ω does not vanish identically along X_0 by 2.3, and hence can apply 4.2. $\square$

We next bound $\int_{X_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t$.

PROPOSITION 4.4 ([RZ, Appendix B.1(ii)]). *Suppose in 1.1 that X is smooth and X_0 is reduced and has simple normal crossing (then it is well-known the monodromy acting on $R^n(f|_{X^o})_* \mathbb{C}$ is unipotent), and let $\Omega \in H^0(X, K_{X/C})$ be as in 1.2. Suppose that there exists an irreducible component F of X_0 such that Ω does not vanish identically on F and $H^0(F, K_F) \neq 0$. Then there is a constant $\alpha > 0$ such that $\int_{X_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t \leq \alpha$ for all $t \neq 0$.*

PROOF. The outline of the proof is as follows. We can start with the argument from the latter half of [RZ, p. 264], recalling some standard material concerning the limiting mixed Hodge structure and the nilpotent orbit theorem. In [RZ], the initial data (M, M_0, Ω) is resolved by a semi-stable reduction, and reduced to $(\widetilde{M}, \widetilde{M}_0, \eta^*\Omega)$, which corresponds to our (X, X_0, Ω) here. The $X_0 \subset \widetilde{M}_0$ (the strict transform of M_0) in [RZ] corresponds to our $F \subset X_0$. The special future that Ω does not vanish identically on F and $H^0(F, K_F) \neq 0$, only enters in the final five lines in the proof. Then the argument there is essentially due to Wang [W1, 1.1, 2.1]. □

Using the above version 4.4 of the [RZ] statement, we get

PROPOSITION 4.5. *Suppose in 1.1 that $f : X \to C$ is lc. If $d_{\mathrm{WP}}(C^o, 0) < \infty$, then there is a constant $\alpha > 0$ such that $\int_{X_t} (-1)^{n^2/2} \Omega_t \wedge \overline{\Omega}_t \leq \alpha$ for all $t \neq 0$.*

PROOF. There are some overlaps with 2.5 and 2.6. Let $X_0 = \sum_{i \in I} F_i$ be the irreducible decomposition. As in 2.3, we can write as $K_{X/C} \sim \sum_{j \in J} r_j F_j$ as a Weil divisor, where $J \subset I$ is a set of indexes with $J \neq I$, and r_j are positive integers. A basic fact is that Ω does not vanish identically along F_i for any $i \in I \setminus J$. By [Ta, 1.1(2)], all F_j with $j \in J$ are uniruled.

We use the semi-stable reduction as in 2.2 (1) and (2). We consider $\Omega' := b^*\Omega \in H^0(X', K_{X'/C'})$ and $\Omega'' \in H^0(X'', K_{X''/C'})$ such that $\mu_*\Omega'' = \Omega'$.

$$\begin{array}{ccccc} X'' & \xrightarrow{\mu} & X' = X \times_C C' & \xrightarrow{b} & X \\ {\scriptstyle f''}\big\downarrow & & {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ C' & = & C' & \xrightarrow[a]{} & C \end{array}$$

(3) Let $X_0'' := f''^*(0') = S_1 + S_2 + \ldots + S_k$ be the irreducible decomposition. By Wang [W1, 2.1], $0'$ is at finite Weil-Petersson distance from $(C')^o$ if and only if $p_g(S_1) = 1$ and $p_g(S_2) = \ldots = p_g(S_k) = 0$ (possibly after relabeling). In particular S_1 is not uniruled. Since X' has canonical singularities, every μ-exceptional divisor in X'' is uniruled by [HM, 1.5].

(4) Noting that $X_0' = f'^*(0')$ and $X_0 = f^*(0)$ are identified via the map b, we can see that $\Omega'' \in H^0(X'', K_{X''/C'})$ does not vanish identically along every irreducible component of X_0'' which correspond to F_i with $i \in I \setminus J$. (Recall $K_{X'/C'} = b^*K_{X/C}$. The ramification of $b : X' \to X$ affects on $K_{X'}$, but not on $K_{X'/C'}$.) Every irreducible component of X_0'' which correspond

to F_j with $j \in J$ is uniruled. Thus there exists a (unique) irreducible component $F''(= S_1)$ of X''_0 which corresponds to some of F_i with $i \in I \setminus J$ and $H^0(F'', K_{F''}) \neq 0$. Then we can apply 4.4 for $f'' : X'' \to C'$ with Ω'' and $F'' \subset X''_0$, and obtain a uniform bound for $\Omega''_{t'}$, $t' \in C'^o$. If $a(t') = t$ for $t' \in C'^o$, we have $\Omega''_{t'} = \Omega_t$ in $H^0(X_t, K_{X_t})$ say. We then have the same uniform bound for Ω_t, $t \in C^o$. □

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