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Mordell-Weil Lattice of Higher Genus Fibration on a Fermat Surface

By Tetsuji Shioda

In memory of Professor Kunihiko Kodaira

Abstract. The Mordell-Weil lattice of higher genus fibration is studied for the axial fibration on a Fermat surface. The basic theorems (the rank, the height formula, etc) are obtained, and examples and various generalization will be discussed.

1. Introduction

For a positive integer m greater than 3, let X_m denote the Fermat surface of degree m in $\mathbf{P}^3$:

$$X_m : x_0^m + x_1^m + x_2^m + x_3^m = 0. \tag{1}$$

We work first in characteristic 0 (until we state otherwise) and let k be an algebraically closed field (e.g. $k = \mathbb{C}$). Then there are $3m^2$ lines (i.e. one-dimensional projective subspaces of $\mathbf{P}^3$) contained in X_m.

Fixing any line l_0 on $X = X_m$, one can define a higher genus fibration

$$f : X_m \to \mathbf{P}^1 \tag{2}$$

as follows. We call it the *axial fibration* on the Fermat surface with chosen axis l_0, following Masuda-Matsumoto [7].

First take any point $x \in X - l_0$. Then l_0 and $\{x\}$ span a plane in $\mathbf{P}^3$, say π_x. On the other hand, note that the set of planes in $\mathbf{P}^3$ which contain a given line l_0

$$B := \{H \mid l_0 \subset H \subset \mathbf{P}^3\} \simeq \mathbf{P}^1 \tag{3}$$

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has naturally a structure of $\mathbf{P}^1$. Thus the map $x \mapsto \pi_x$ defines a morphism

$$f' : X - l_0 \to \mathbf{P}^1 \tag{4}$$

which will be extended to the morphism f. For a moment, suppose X is defined by the equation:

$$X : x^m - y^m = z^m - w^m \tag{5}$$

and the line l_0 is given by

$$l_0 : x - y = 0, z - w = 0. \tag{6}$$

By considering the linear pencil of planes:

$$H_t : t(x-y) + (z-w) = 0 (t \in \mathbf{P}^1) \tag{7}$$

the set B can be identified with $\mathbf{P}^1$ by the correspondence $H_t \leftrightarrow t$. In terms of coordinates, we have for $\xi = (x : y : z : w) \in X$

$$f(\xi) = t = -\frac{z-w}{x-y} = -\frac{x^{m-1} + x^{m-2}y + \cdots + y^{m-1}}{z^{m-1} + z^{m-2}w + \cdots + w^{m-1}}. \tag{8}$$

This shows, in particular, that the morphism f restricts on the line l_0 to the map: for $\xi_0 = (x : x : z : z) \in l_0$, let $u = x/z$, then we have

$$f(\xi_0) = -(\frac{x}{z})^{m-1}, \quad t = -u^{m-1} \tag{9}$$

which is a ramified cover $\mathbf{P}^1 \to \mathbf{P}^1$ of degree $m-1$, totally ramified at the two points 0 and ∞. Note that the structure of the fibration is independent of the choice of l_0, since any two lines on X are transformed to each other by an automorphism of X.

The generic fibre Γ_t is a smooth plane curve of degree $m-1$ defined over the rational function field $k(t)$, and its genus g is given by

$$g = \frac{1}{2}(m-2)(m-3). \tag{10}$$

Note that g is greater than 1 for $m \geq 5$, while $g = 1$ for $m = 4$.

The aim of this paper is to study the above situation from the viewpoint of the Mordell-Weil lattices for higher genus fibration.

We have developped a theory of the Mordell-Weil lattices for higher genus fibration in [13, 15]. The basic idea is the same as the case of elliptic fibration (see [12]).

Let X be a smooth projective surface and consider a fibration on X

$$f : X \to \mathbf{P}^1 \tag{11}$$

such that the generic fibre Γ_t is a smooth curve of positive genus g. Let $J = J_t$ denote the Jacobian variety of the generic fibre over $k(t)$. Assume that X is a regular surface (e.g. any smooth surface in $\mathbf{P}^3$). Then, by [15, Theorem 3], we have

$$J(k(t)) \simeq NS(X)/T, \tag{12}$$

i.e. the Mordell-Weil group $M = J(k(t))$ is isomorphic to the quotient group of the Néron-Severi group $NS(X)$ by the trivial lattice T. In particular, it implies that the Mordell-Weil rank r is given by

$$r = \rho(X) - \mathrm{rk}T, \quad \mathrm{rk}T = 2 + \sum_v (m_v - 1) \tag{13}$$

where m_v denotes the number of irreducible components of a reducible fibre $f^{-1}(v)$ and the summation runs over the places v with such reducible fibres.

Moreover, by using the intersection theory on the surface X, we can define the height pairing on M, which puts a lattice structure on M modulo torsion. We call the resulting structure on M the *Mordell-Weil lattice* of (X, f) ([15, Theorems 7, 8]).

2. Main Results

Applying the above idea to the axial fibration (2) on the Fermat surface $X_m (m \geq 4)$, we obtain the main results of this paper:

THEOREM 1. *Let $J = J_t$ denote the Jacobian variety of Γ_t over $k(t)$, the generic fibre of the axial fibration on the Fermat surface X_m. Then the rank of the Mordell-Weil group $M := J(k(t))$ is given by the formula:*

$$r = \rho(X_m) - (4m - 2) \tag{14}$$

where $\rho(X_m)$ is the Picard number of X_m.

For the explicit formula of the Picard number $\rho(X_m)$, see Aoki [1] and Shioda [10]. For the convenience of the reader, we recall from ([10]) that, if $\rho_1(X_m)$ denotes the rank of the subgroup of $NS(X_m)$ spanned by the classes of lines, then we have

$$\rho_1(X_m) = 3(m-1)(m-2) + \begin{cases} 1 & (m : \text{odd}) \\ 2 & (m : \text{even}) \end{cases} \tag{15}$$

and the Picard number of X_m is of the form

$$\rho(X_m) = \rho_1(X_m) + \beta(m) \tag{16}$$

where $\beta(m)$ is a non-negative term which is at most linear in m. It is known that $\beta(m) = 0$ if and only if $GCD(m, 6) = 1$ or $m \leq 4$.

Now we take a second line l_1 which is disjoint from the given axis l_0, and choose it as the zero section O of $f : X \to \mathbf{P}^1$. It defines the origin of the group law of J. Let

$$\mathbb{L} = \{l \mid l \cap l_0 = \emptyset\} \tag{17}$$

denote the set of lines l which are disjoint from l_0. For each $l \in \mathbb{L}$, we have a section $P_l \in \Gamma_t(k(t)) \subset M$ (cf. Lemma 8).

THEOREM 2. *Assume that m is relatively prime to 6 or $m = 4$. The Mordell-Weil group $J(k(t))$ is a finitely generated abelian group of rank*

$$r = \rho_1(X_m) - (4m-2). \tag{18}$$

The Mordell-Weil group is generated by the sections $\{P_l\}$ associated with lines $l \in \mathbb{L}$.

The last statement is based on the recent results by Schütt-Shioda-van Luijk [9] for $m < 100$ and by Degtyarev [3] for any m under the condition that $(m, 6) = 1$.

Note, in particular, that the rank r above is greater than $6g$ if $m > 4$:

COROLLARY 3. *The ratio of the MW rank to genus is at least 6:*

$$\frac{r}{g} \geq 6 \quad \textit{for any} \quad m \geq 4 \tag{19}$$

with equality only if $m = 4$.

Going back to the general case, we now describe the height pairing on the Mordell-Weil lattice for the Fermat surface. To state it, we fix the following notation: for each reducible fibre $F_v := f^{-1}(v)$, suppose it has m_v irreducible components, say $\theta_{v,i} (0 \le i \le m_v - 1)$, where we denote by $\theta_{v,0}$ the identity component of F_v.

THEOREM 4. *Consider two sections $P = P_l$ and $Q = P_{l'}$ where l and l' are lines on the Fermat surface X_m which are disjoint from the axis l_0 (i.e. $l, l' \in \mathbb{L}$). Then the height pairing $\langle P, Q\rangle$ is given by the following formula:*

$$\langle P, Q\rangle = m - 2 + (l \cdot l_1) + (l' \cdot l_1) - (l \cdot l') - \sum_v contr_v(P, Q) \tag{20}$$

$$\langle P, P\rangle = 2(m - 2) + 2(l \cdot l_1) - \sum_v contr_v(P) \tag{21}$$

where $(l \cdot l_1)$, etc. denote the intersection number of two lines and the summation runs over v in

$$\Sigma_m = \{0, \infty\} \cup \begin{cases} \mu_{2m} & (m : even) \\ \mu_m & (m : odd) \end{cases} \tag{22}$$

which gives the position of reducible fibres of f (by [7], see Proposition 9 below).

The local contribution term $contr_v(P, Q)$ (and $contr_v(P) := contr_v(P, P)$) is defined as follows. Suppose l intersects $\theta_{v,i}$ and l' intersects $\theta_{v,j}$, Then

$$contr_v(P, Q) = \begin{cases} 0 & if\ i = 0\ or\ j = 0 \\ (-I_v)^{-1}_{i,j} & if\ i > 0\ and\ j > 0; \end{cases} \tag{23}$$

the latter means the (i, j)-entry of the matrix $(-I_v)^{-1}$, given by Lemma 18.

THEOREM 5. *Assume m is even and ≥ 4. Then we have*

$$\langle P_l, P_l\rangle = 2(m - 2) + 2(l \cdot l_1) - \frac{2}{m-1} n_0 - \frac{1}{m-2} n_1 \tag{24}$$

where n_0 (resp. n_1) is the number of $v \in \{0, \infty\}$ (resp. $v \in \mu_{2m}$) such that l hits a non-identity component at v.

THEOREM 6. *Assume m is odd and ≥ 5. Then we have*

$$\langle P_l, P_l\rangle = 2(m-2) + 2(l\cdot l_1) - \frac{2}{m-1}(n_0+n_3) - \frac{m-2}{(m-1)(m-3)}(n_2-n_3) \tag{25}$$

where n_0 (resp. n_2) is the number of $v \in \{0,\infty\}$ (resp. $v \in \mu_m$) such that l hits a non-identity component at v, and where n_3 is the number of $v \in \mu_m$ such that l hits a non-identity component at v which is a line and the identity component is also a line.

THEOREM 7. *Fix $m \geq 4$, and let $M = J(k(t))$ be, as before, the Mordell-Weil lattice of the axial fibration $f : X_m \to \mathbf{P}^1$ on the Fermat surface X_m. For any sections $P, Q \in M$, the height pairing $\langle P, Q\rangle$ have values in $\mathbb{Q}$ with a bounded denominator:*

$$\langle P, Q\rangle \in \frac{1}{D}\mathbb{Z}, \quad D = \begin{cases} (m-1)(m-2) & (m: \textit{even}) \\ (m-1)(m-3) & (m: \textit{odd}). \end{cases} \tag{26}$$

The proof will be given in §5, after we make some preliminary study on the axial fibration and the trivial lattice in the next two sections. In the later sections, we discuss some examples, and extension to positive characteristic case and to more general surfaces.

3. Preliminaries

Let us go back to the coordinate system $\{x_i\}$ in which the Fermat surface X_m is given by the equation (1). Let l_0 denote the line on X_m:

$$l_0 : x_0 + \epsilon\, x_1 = 0, \quad x_2 + \epsilon\, x_3 = 0 \tag{27}$$

where we set throughout this paper

$$\epsilon := 1 \text{ if } m \text{ is odd, and } \epsilon := e^{\pi i/m} \text{ if } m \text{ is even.} \tag{28}$$

We let $k = \mathbb{C}$ for simplicity. Letting $\zeta = e^{2\pi i/m}$ (for any fixed m), we have the factorization:

$$x^m + y^m = \Pi_{i=1}^m (x + \epsilon\, \zeta^{i-1}\, y). \tag{29}$$

Thus we find $3m^2$ lines on X_m, denoted by $L1[i,j]$ $(1 \le i, j \le m)$, etc., as follows:

$$\begin{aligned} L1[i,j] &: x_0 + \epsilon\,\zeta^{i-1} x_1 = 0, \quad x_2 + \epsilon\,\zeta^{j-1} x_3 = 0, \\ L2[i,j] &: x_0 + \epsilon\,\zeta^{i-1} x_2 = 0, \quad x_1 + \epsilon\,\zeta^{j-1} x_3 = 0, \\ L3[i,j] &: x_0 + \epsilon\,\zeta^{i-1} x_3 = 0, \quad x_1 + \epsilon\,\zeta^{j-1} x_2 = 0. \end{aligned} \tag{30}$$

The axial fibration on X_m with axis l_0

$$f : X_m \to \mathbf{P}^1 \tag{31}$$

is now defined by

$$f(x) = t = -\frac{x_2 + \epsilon\, x_3}{x_0 + \epsilon\, x_1} = -\frac{\Pi'(x_0 + \epsilon\,\zeta^{i-1}\, x_1)}{\Pi'(x_2 + \epsilon\,\zeta^{i-1}\, x_3)} \tag{32}$$

where Π' means a product over $i = 2$ to m.

Let H_t denote the hyperplane of $\mathbf{P}^3$ defined by

$$H_t : t(x_0 + \epsilon\, x_1) + (x_2 + \epsilon\, x_3) = 0 \quad (t \in \mathbf{P}^1) \tag{33}$$

which forms a linear pencil of planes containing the line l_0.

For each t, the intersection of H_t with $X = X_m$ is a reducible curve:

$$X \cap H_t := l_0 + F_t \tag{34}$$

and the residual part F_t is a plane curve in $H_t \simeq \mathbf{P}^2$ which can be identified with the fibre $f^{-1}(t)$. Thus, for any point $v \in \mathbf{P}^1$, the fibre $F_v = f^{-1}(v)$ over v is a possibly reducible plane curve of degree $m-1$ in $H_v \simeq \mathbf{P}^2$.

Lemma 8. *For any line $l \neq l_0$ on X, the following alternative holds: (i) if l intersects l_0, then l is a component of a reducible singular fibre of f. (ii) If l is disjoint from l_0, then it defines a section of f; call it $P_l \in \Gamma_t(k(t)) \subset J(k(t))$.*

Proof. If l intersects l_0, l_0 and l span a plane, say H_v for some $v \in \mathbf{P}^1$. Then l is contained in the fibre F_v, i.e. l is a component of a reducible fibre F_v. This proves (i). Next, assume that l is disjoint from l_0. Take any point $x \in l$. and set $f(x) = v$. By definition, the plane H_v is spanned by $\{x\}$ and

l_0. Since x is the unique point of intersection of l and H_v, we see that the $v \mapsto x$ defines a section of the fibration f. This proves (ii). □

The singular fibres of the axial fibration have been studied in detail by [7]. The shapes and the positions of singular fibres are stated in [7, Theorem 2]. There are irreducible singular fibres in case $m > 4$, but we focus on the reducible singular fibres in this paper.

We keep the previous notation: for any point $v \in \mathbf{P}^1$, $F_v = f^{-1}(v)$ denotes the fibre over v and m_v denotes the number of irreducible components of F_v. Let

$$\Sigma_m = Red(f) := \{v \in \mathbf{P}^1 | m_v > 1\} \tag{35}$$

be the support of reducible singular fibres of $f : X_m \to \mathbf{P}^1$. The results on the reducible fibres of $f : X_m \to \mathbf{P}^1$ are summarized as follows:

PROPOSITION 9 ([7]). *Fix $m > 3$. Then we have*

$$\Sigma_m = \{0, \infty\} \cup \begin{cases} \mu_{2m} & (m : \textit{even}) \\ \mu_m & (m : \textit{odd}) \end{cases} \tag{36}$$

where μ_N denotes the set $\{\alpha \in k | \alpha^N = 1\}$ of N-th roots of unity. More precisely,
(i) if $v = 0$ or ∞, then F_v is a union of $m-1$ lines meeting at one point ($m_v = m-1$);
(ii) if m is even and $v \in \mu_{2m}$, then F_v is a union of a line and an irreducible plane curve of degree $m-2$ ($m_v = 2$) meeting at $m-2$ points, and (iii) if m is odd and $v \in \mu_m$, then F_v is a union of two lines and an irreducible curve of degree $m-3$ ($m_v = 3$).

Let us describe some typical reducible fibres F_v:

LEMMA 10. *For $v = 0$ or $v = \infty$, F_v is a union of $m-1$ lines meeting at one point:*

$$F_0 = \sum_{i=2}^{m} L1[i,1], \quad F_\infty = \sum_{j=2}^{m} L1[1,j]. \tag{37}$$

PROOF. This is clear from the expression (32). □

LEMMA 11. *(i) Assume that m is odd and $v = 1 \in \mu_m$. Then we have*

$$F_1 = L2[1,1] + L3[1,1] + C \tag{38}$$

where C is a smooth irreducible plane curve of degree $m-3$.
(ii) Assume that m is even. For $v = 1$ or $v = \epsilon \in \mu_{2m}$, we have respectively

$$F_1 = L3[1,m] + C', \quad F_\epsilon = L2[1,1] + C'', \tag{39}$$

where C' are C'' are smooth irreducible plane curves of degree $m-2$.

PROOF. Look at the plane H_v at $v = 1$, i.e. (34) with $t = 1$. (i) For m odd, H_1 is defined by the equation:

$$H_1 : x_0 + x_1 + x_2 + x_3 = 0. \tag{40}$$

It contains, besides $l_0 = L1[1,1]$, two more lines on X:

$$L2[1,1] = \{x_0 + x_2 = 0, x_1 + x_3 = 0\},$$
$$L3[1,1] = \{x_0 + x_3 = 0, x_1 + x_2 = 0\}.$$

Hence the above two lines are two irreducible components of F_1, showing (38). (ii) Next, for m even, H_1 is defined by the equation $x_0+\epsilon x_1+x_2+\epsilon x_3 = 0$. It contains, besides $l_0 = L1[1,1]$, another line:

$$\begin{aligned}\{x_0 + \epsilon\, x_3 = 0, \epsilon\, x_1 + x_2 = 0\} &= \{x_0 + \epsilon\, x_3 = 0, x_1 + \epsilon\zeta^{-1} x_2 = 0\} \\ &= L3[1,m],\end{aligned}$$

since $\epsilon^2 = e^{2\pi i/m} = \zeta$ for m even. Hence the first equation of (39) follows; the second one is similarly shown.

The curves C, C', C'' above are smooth so that necessarily irreducible as they are plane curves. □

The next two lemmas show that there exist two cyclic automorphism groups of the Fermat surface X_m preserving the axial fibration which act transitively on the set of reducible fibres over μ_m for m odd (or over μ_{2m} for m even). As a consequence, all the line components of reducible fibres

F_v ($v \in \mu_m$ or μ_{2m}) can be determined from Lemma 11 as in Proposition 14 below.

LEMMA 12. *For any m, the map*

$$g := g_\zeta : x = (x_0, x_1, x_2, x_3) \mapsto x' = (x_0, x_1, \zeta x_2, \zeta x_3) \tag{41}$$

defines an automorphism of X_m of order m which preserves the axis l_0. It induces an automorphism $\bar{g}$ of $\mathbf{P}^1$ such that

$$\bar{g} : t \mapsto t' = \zeta t. \tag{42}$$

PROOF. By definition, we have

$$t' = -\frac{\zeta x_2 + \epsilon \zeta x_3}{x_0 + \epsilon x_1} = \zeta\, t$$

as claimed. □

LEMMA 13. *Assume m is even. Then the map*

$$\gamma := \gamma_\epsilon : x = (x_0, x_1, x_2, x_3) \mapsto x'' = (x_0, x_1, \epsilon^2 x_3, x_2) \tag{43}$$

defines an automorphism of X_m of order $2m$ which preserves the axis l_0 and which satisfies $\gamma^2 = g$. It induces an automorphism $\bar{\gamma}$ of $\mathbf{P}^1$ such that

$$\bar{\gamma} : t \mapsto t'' = \epsilon t. \tag{44}$$

PROOF. We have

$$t'' = -\frac{(\epsilon^2 x_3) + \epsilon x_2}{x_0 + \epsilon x_1} = -\frac{\epsilon (x_2 + \epsilon x_3)}{x_0 + \epsilon x_1} = \epsilon\, t. \ \square$$

PROPOSITION 14. *(i) Assume that m is odd. For any $v \in \mu_m$, we have*

$$F_v = L2[i,i] + L3[i,i] + C_i \quad \text{for } v = \zeta^{i-1} \quad (1 \le i \le m) \tag{45}$$

with some irreducible curve C_i of degree $m-3$.
(ii) Assume that m is even. For any $v \in \mu_{2m}$, we have either

$$F_v \;=\; L3[i, i-1] + C'_i \quad \text{for } v = \zeta^{i-1}, \text{ or} \tag{46}$$

$$F_v \;=\; L2[i\,, i\,] + C''_i \quad \text{for } v = \epsilon\zeta^{i-1} (1 \le i \le m) \tag{47}$$

with some irreducible curves C'_i or C''_i of degree $m-2$.

4. Trivial Lattice T

In general, the trivial lattice T of a fibred surface $f : X \to B$ (B a base curve) is defined as the sublattice of the Néron-Severi lattice $NS(X)$ of X:

$$T = U \oplus \Sigma_{v \in Red(f)} T_v \tag{49}$$

where U is the unimodular lattice spanned by the fibre class F and the zero-section (O). For each reducible fibre F_v, T_v denotes the sublattice spanned by the non-identity components, say $\theta_{v,i} (1 \le i \le m_v - 1)$, omitting the identity component $\theta_{v,0}$, intersecting (O), from F_v. We denote by I_v the intersection matrix of T_v, and set

$$\det T_v = |\det(-I_v)|, \quad \det T = \prod_v \det T_v. \tag{50}$$

Going back to the case of Fermat surface $X = X_m (m > 3)$ with axial fibration f, we choose any line l_1 disjoint from the axis l_0 (e.g. $l_1 = L1[2, 2]$, (30)) as the the zero-section.

Proposition 15. *For any m, the trivial lattice T of (X_m, f) has the following rank:*

$$\text{rk } T = 4m - 2. \tag{51}$$

Proof. By Proposition 9, we have $m_v = m - 1$ for $v = 0, \infty$ for any m, and $m_v = 3$ for $v \in \mu_m$ in case m is odd. Hence, if m is odd, we have

$$\text{rk} T = 2 + 2(m - 1 - 1) + m(3 - 1) = 4m - 2.$$

On the other hand, if m is even, we have $m_v = 2$ for each $v \in \mu_{2m}$. Hence we have

$$\text{rk} T = 2 + 2(m - 1 - 1) + (2m)(2 - 1) = 4m - 2$$

again. Thus rk$T$ equals $4m - 2$ for any m. □

Next we compute the intersection matrix I_v for each v. First we note that the self-intersection number of a line l on X_m (or on any smooth surface of degree m in $\mathbf{P}^3$) is equal to

$$l^2 = 2 - m, \tag{52}$$

This is well-known and it follows from the adjunction formula. Now we claim:

PROPOSITION 16.

$$\det T_v = \begin{cases} (m-1)^{m-3} & \textit{(i) if } v = 0, \infty \\ (m-2) & \textit{(ii) if } m \textit{ is even and } v \in \mu_{2m} \\ (m-1)(m-3) & \textit{(iii) if } m \textit{ is odd and } v \in \mu_m. \end{cases} \tag{53}$$

PROOF. (i) If $v = 0$ or ∞, then F_v is a union of $m-1$ lines, so that I_v is a square matrix of size $m-2$ for which all the diagonal entries are equal to $2-m$ and all non-diagonal entriries are equal to 1. It is easy to compute $\det(-I_v)$ as above.
(ii) if m is even and $v \in \mu_{2m}$, then $F_v = l + C$ for some line and degree $(m-2)$ curve. The total intersection matrix $\tilde{I}_v$ is given by

$$\tilde{I}_v = \begin{pmatrix} l^2 & lC \\ lC & C^2 \end{pmatrix} = \begin{pmatrix} 2-m & m-2 \\ m-2 & 2-m \end{pmatrix}.$$

Hence T_v has rank 1 and $\det = m-2$.
(iii) If m is odd and $v \in \mu_m$, then $F_v = l + l' + C'$ for two lines l, l' and degree $(m-3)$ curve C'. The total intersection matrix $\tilde{I}_v$ is given by

$$\tilde{I}_v = \begin{pmatrix} l^2 & ll' & lC' \\ ll' & l'^2 & l'C' \\ lC' & l'C' & C'^2 \end{pmatrix} = \begin{pmatrix} 2-m & 1 & m-3 \\ 1 & 2-m & m-3 \\ m-3 & m-3 & C'^2 \end{pmatrix}. \tag{54}$$

Here we have $C'^2 = -2(m-3)$, since $C'F_v = 0$. It follows that the intersection matrix I_v is either one of the following two, depending on which of l (or l') or C' is the identity component of F_v:

$$I_v = \begin{pmatrix} 2-m & m-3 \\ m-3 & -2(m-3) \end{pmatrix} \text{ or } \begin{pmatrix} 2-m & 1 \\ 1 & 2-m \end{pmatrix}. \tag{55}$$

Whichever the identity compoent may be, we have $\det I_v = (m-1)(m-3)$ in case (iii), as asserted. □

PROPOSITION 17. *The trivial lattice T of the Fermat surface X_m has the following determinant:*

$$\det T = \begin{cases} (m-1)^{2(m-3)}(m-2)^{2m} & (m : \textit{even}) \\ (m-1)^{3(m-2)}(m-3)^m & (m : \textit{odd}). \end{cases} \tag{56}$$

The following information about the inverse matrix of $-I_v$ will be used for computation of height formula:

LEMMA 18. *(i) If $v = 0$ or ∞, the (i,j)-entry $c_{i,j}$ of the inverse matrix $(-I_v)^{-1}$ is*

$$c_{i,i} = \frac{2}{m-1}, \quad c_{i,j} = \frac{1}{m-1}(i \neq j). \tag{57}$$

(ii) If m is even and $v \in \mu_{2m}$, then

$$(-I_v)^{-1} = (\frac{1}{m-2}). \tag{58}$$

(iii) If m is odd and $v \in \mu_m$, then

$$(-I_v)^{-1} = \begin{pmatrix} \frac{2}{m-1} & \frac{1}{m-1} \\ \frac{1}{m-1} & \frac{m-2}{(m-1)(m-3)} \end{pmatrix} \text{ or } \begin{pmatrix} \frac{m-2}{(m-1)(m-3)} & \frac{1}{(m-1)(m-3)} \\ \frac{1}{(m-1)(m-3)} & \frac{m-2}{(m-1)(m-3)} \end{pmatrix}. \tag{59}$$

PROOF. This can be checked easily from the expression of I_v given in the proof of Proposition 16. □

5. Proof of the Main Results Stated in §2

First we prove the theorems on the rank of our MWL:

PROOF OF THEOREM 1. This is an immediate consequence of the the main theorem of higher genus fibration [15, Theorem 3] and Proposition 15. In other words, it follows from (13) and (51). □

PROOF OF THEOREM 2. The first assertion of Theorem is a special case of Theorem 1 such that the degree m is relatively prime to 6, which has been known to imply that $NS(X_m)$ is generated by the classes of lines up to finite index (see [10]). In the recent work [9] and [3], it is shown that this index is equal to 1. In other words, under the condition $(m, 6) = 1$ or

$m = 4$, $NS(X_m)$ is generated by the classes of lines. (The case $m = 4$ was proven in 1970's by Inose and Mizukami, as explained in [9, §6]).

As a quotient group of $NS(X_m)$, the MW group $M = NS(X_m)/T$ is generated by the sections P_l associated with lines l on X_m. Note that the line l_0 is linearly equivalent to a $\mathbb{Z}$-linear combination of other lines which are mapped to either a section or 0 in M. For example, the following divisors are equal in $NS(X_m)$ as they are hyperplane sections:

$$D_1 = \sum_{j=1}^{m} L1[1, j], \quad D_2 = \sum_{j=1}^{m} L1[2, j].$$

Hence

$$l_0 = L1[1, 1] \equiv D_2 - \sum_{j=2}^{m} L1[1, j].$$

This proves Theorem 2. $\square$

PROOF OF COROLLARY 3. With the notation used in the theorem, we have obviously $\rho(X_m) \geq \rho_1(X_m)$ for any $m > 4$, and hence

$$r - 6g = \rho(X_m) - \mathrm{rk}T - 6g \geq \rho_1(X_m) - (4m - 2) - 3(m - 2)(m - 3) = 2m - 9.$$

Thus it follows that $r - 6g > 0$ for any $m > 4$. For $m = 4$, we have $r = 6, g = 1$ as is shown in the next section. $\square$

Next we prove the theorems about the height pairing of our MWL:

PROOF OF THEOREMS 4, 5, 6. Following the general theory [15, Theorem 7] of MWL of higher genus fibration and applying it to the axial fibration on the Fermat surface $f : X_m \to \mathbf{P}^1$, we define the height pairing and obtain the explicit formula (20) for $\langle P, Q\rangle$, for $P = P_l, Q = P_{l'}$ in $\Gamma_t(k(t)) \subset M = J(k(t))$, in which the term $contr_v(P, Q)$ is defined as in (23).

In the course of proof of Proposition 16, we have computed the intersection matrix I_v for each sublattice T_v. Thus, if a reducible fibre $F_v = \theta_{v,0} + \cdots + \theta_{v,m_v-1}$ is given with m_v irreducible components, then I_v is the intersection matrix of $m_v - 1$ non-identity components $(\theta_{v,i} \cdot \theta_{v,j})_{1 \leq i,j \leq m_v-1}$, omitting the identity component $\theta_{v,0}$.

We remark that T_v and I_v depend on the choice of the 0-section of f. For example, in the case (iii) where m is odd and $v \in \mu_m$, we have $F_v = l + l' + C'$ with two lines l, l' and degree $(m-3)$ curve C'. Hence, if (iii-1) the identity component $\theta_{v,0}$ is one of two lines, then we have

$$(60) \quad I_v = \begin{pmatrix} 2-m & m-3 \\ m-3 & -2(m-3) \end{pmatrix}, \ (-I_v)^{-1} = \begin{pmatrix} \frac{2}{m-1} & \frac{1}{m-1} \\ \frac{1}{m-1} & \frac{m-2}{(m-1)(m-3)} \end{pmatrix}.$$

But, if (iii-2) the identity component $\theta_{v,0}$ is the curve C', then we have

$$(61) \quad \begin{aligned} I_v &= \begin{pmatrix} 2-m & 1 \\ 1 & 2-m \end{pmatrix}, \\ (-I_v)^{-1} &= \begin{pmatrix} \frac{m-2}{(m-1)(m-3)} & \frac{1}{(m-1)(m-3)} \\ \frac{1}{(m-1)(m-3)} & \frac{m-2}{(m-1)(m-3)} \end{pmatrix}. \end{aligned}$$

In this way, we have shown Theorem 4.

Furthermore, by writing down the explicit values of $(-I_v)^{-1}$ for each v and collecting the terms (with respect to v) with the same value $contr_v(P)$, we complete the proof of Theorem 5 for m even, and Theorem 6 for m odd. □

Proof Theorem 7. We claim that each value of the height pairing $\langle P, Q \rangle$ is a rational number with bounded denominator $D = (m-1)(m-2)$ for m even or $D = (m-1)(m-3)$ for m odd. This is clear from the height formulas in the above theorems for $P = P_l, Q = P_{l'}$, i.e. for the sections associated with lines l, l' on X_m.

As $D\langle P, Q \rangle$ are integers for generators P, Q as proved, and as the pairing is bilinear, $D\langle x, y \rangle$ is also integer for all $x, y \in M$. □

6. Examples

Now that we have proven all the theorems announced in §2, let us apply them to have more definite results for the Mordell-Weil lattice M constructed from the axial fibration on the Fermat surface X_m with low degree

m. With the notation of lines (30), we fix the axis l_0 (27) and the zero-section l_1 as follows:

$$l_0 = L1[1,1], \quad l_1 = L1[2,2]. \tag{62}$$

Most of the computation below is carried out by the use of a computer with *Mathematica*. Note that, by definition, we set

$$\det M := \det(M/M_{tor}). \tag{63}$$

Namely $\det M$ denotes the determinant of the lattice $\bar{M} = M/M_{tor}$ which is a lattice in the true sense of the word. The Mordell-Weil group M will probably be torsion-free, but it is an open question[1] in case $m > 4$; it is equivalent to the claim that the trivial lattice T is a primitive sublattice in the Néron-Severi lattice NS.

6.1. The case $m = 4$

The Fermat quartic surface $X = X_4$ is a K3 surface with Picard number 20, and the axial fibration gives an elliptic fibration on it with a section. In the standard notation due to Kodaira [5], the two reducible fibres F_0 and F_∞ (see Lemma 10) are of type IV and the eight reducible fibres F_v at $v^8 = 1$ (cf. Proposition 14 (ii)) are of type I_2. Thus the trivial lattice is a direct sum of the hyperbolic lattice U and two copies of A_2 plus eight copies of A_1:

$$T = U \oplus A_2^{\oplus 2} \oplus A_1^{\oplus 8}, \quad \mathrm{rk} T = 14, \quad \det T = 2^8 3^2.$$

Hence, by Theorem 1, the Mordell-Weil rank is equal to $r = 20 - 14 = 6$.

Next, by Theorem 2, the MWL M is generated by the sections P_l associated to lines $l \in \mathbb{L}$. We can find a set of 6 lines which generate the MWL, by computing the height determinants of 6 lines. For example, take the following 6 lines:

$$\{L3[1,1], L2[3,2], L2[3,1], L3[3,3], L3[3,4], L1[2,3]\}.$$

[1]Solved. See "Added in Proof" at the end of the paper.

The height matrix of the corresponding sections is

$$\begin{vmatrix} 2/3 & 1/3 & 1/3 & -1/3 & 5/6 & 2/3 \\ 1/3 & 4/3 & 1/6 & 1/3 & 2/3 & 5/3 \\ 1/3 & 1/6 & 5/3 & 1/3 & 2/3 & 4/3 \\ -1/3 & 1/3 & 1/3 & 2/3 & -1/6 & 2/3 \\ 5/6 & 2/3 & 2/3 & -1/6 & 5/3 & 4/3 \\ 2/3 & 5/3 & 4/3 & 2/3 & 4/3 & 10/3 \end{vmatrix}$$

and its determinant is equal to $1/36$. Further any other section P_l is checked to be a $\mathbb{Z}$-linear combination of these by computing the height pairing.

Thus we have

PROPOSITION 19. *In case $m = 4$, we have*

$$\mathrm{rk} M = 6, \quad \det M = \frac{1}{36}.$$

The generic fibre Γ_t of f is a plane cubic over $k(t)$. In transforming its defining equation into the Weierstrass equation, we find:

PROPOSITION 20. *The Fermat quartic surface with axial fibration is isomorphic to the elliptic K3 surface:*

$$E = F_{1,0}^{(4)} : y^2 = x^3 - 3x + t^4 + \frac{1}{t^4}. \tag{64}$$

Note that this is the base change via $t \to t^4$ of the Inose's fibration

$$F_{1,0}^{(1)} : y^2 = x^3 - 3x + t + \frac{1}{t}. \tag{65}$$

on the singular K3 surface S with discriminant 4. One can directly compute the MW group $E(k(t))$ and reprove the above Proposition 19 in the framework of MWL of elliptic surfaces (cf. [12], [16], [17]). Moreover, we know that $M \simeq F_{1,0}^{(4)}(k(t))$ is torsion-free in this case (see [16, Lemma 6.2]).

REMARK. Kuwata has kindly pointed out that his thesis [6] treated the Fermat quartic surface as the above elliptic fibration.

6.2. The case $m = 5$

The Fermat quintic surface $X = X_5$ is a smooth surface of general type in $\mathbf{P}^3$, containing $3m^2 = 75$ lines, and the axial fibration defines a genus 3 fibration $f : X \to \mathbf{P}^1$ with a section. We consider its MWL M.

PROPOSITION 21. *In case $m = 5$, we have*

$$r = \mathrm{rk}M = 19, \quad \det M = \frac{5^{12}}{2^{21}}.$$

PROOF. We outline below how to prove this, by applying our Theorems 1, 2, 4 and 6. First, since the Picard number is $\rho(X_5) = 37$ and $\mathrm{rk}T = 4 \cdot 5 - 2 = 18$, the Mordell-Weil rank is equal to $r = 37 - 18 = 19$ by Theorem 1.

In order to compute the height pairing, recall that the reducible fibres F_v are given as follows by Lemma 10 and Proposition 14:

$$\begin{aligned} &F_0 = \textstyle\sum_{i=2}^m L1[i,1], && \theta_{0,0} = L1[2,1] \\ (66)\quad &F_\infty = \textstyle\sum_{j=2}^m L1[1,j], && \theta_{\infty,0} = L1[1,2] \\ &F_v = L2[i,i] + L3[i,i] + C_i, && \theta_{v,0} = L2[i,i] \ (v = \zeta^{i-1} \in \mu_5). \end{aligned}$$

All the irreducible components are lines, except for the case $v \in \mu_5$ where C_i's are some irreducible conics. Note that the identity component $\theta_{v,0}$ of each reducible fibre F_v is a line given as above. This can be easily seen by checking that the intersection number of such a line with the line $l_1 = L1[2,2]$ (chosen for the zero-section) is equal to 1.

Now we are ready to compute the height pairing $\langle P_l, P_{l'} \rangle$ for any two lines l, l' disjoint from the axis, by Theorem 4 or by Theorem 6 if $l = l'$.

By Theorem 2, there should be a set of $r = 19$ lines, say Ω, such that the corresponding sections $P_i (1 \le i \le r)$ are linearly independent so that the height matrix has non-zero determinant:

$$\det H(\Omega) = \det(\langle P_i, P_j \rangle) \neq 0.$$

As a candidate of such, we try the following set of 19 lines:

$$\begin{aligned} \Omega = \{&L1[2,3], L1[2,4], L1[2,5], L1[3,2], L1[3,3], L1[3,4], L1[4,2], \\ &L2[1,2], L2[1,3], L2[1,4], L2[2,1], L2[2,3], L2[2,4], \\ &L3[1,2], L3[1,3], L3[1,4], L3[5,1], L3[5,2], L3[5,3]\}. \end{aligned} \tag{67}$$

Then a direct computation shows that the height matrix $H(\Omega)$ for this set Ω is given by the matrix below and that its determinant is equal to

$$\det H(\Omega) = \frac{5^{12}}{2^{21}}.$$

In particular, this implies that $P_i (1 \le i \le r) \in M$ are linearly independent. Thus, given any other line $l \in L$, the section $P = P_l$ should have some multiple νP which is an integral linear combination of P_i's. A direct height computation shows that we have $\nu = 1$ for any l. Thus we conclude that $P_i (1 \le i \le r)$ generate M (modulo torsion) and so we have

$$\det M = \frac{5^{12}}{2^{21}}. \square$$

$$\begin{vmatrix}
\frac{45}{8} & \frac{15}{8} & \frac{5}{2} & \frac{25}{8} & \frac{5}{2} & \frac{5}{2} & \frac{25}{8} & 2 & \frac{5}{2} & \frac{11}{4} & \frac{11}{4} & 2 & \frac{5}{2} & \frac{5}{2} & \frac{3}{2} & \frac{13}{4} & \frac{5}{2} & \frac{3}{2} & \frac{13}{4} \\
\frac{15}{8} & \frac{45}{8} & \frac{5}{2} & \frac{25}{8} & \frac{15}{4} & \frac{5}{4} & \frac{25}{8} & 3 & \frac{7}{4} & \frac{5}{2} & \frac{11}{4} & 3 & \frac{7}{4} & \frac{3}{2} & \frac{9}{4} & \frac{7}{2} & \frac{3}{2} & \frac{9}{4} & \frac{7}{2} \\
\frac{5}{2} & \frac{5}{2} & 5 & \frac{15}{4} & \frac{15}{4} & \frac{5}{4} & \frac{15}{4} & 3 & \frac{11}{4} & \frac{7}{4} & \frac{5}{2} & 3 & \frac{11}{4} & 2 & \frac{9}{4} & \frac{13}{4} & 2 & \frac{9}{4} & \frac{13}{4} \\
\frac{25}{8} & \frac{25}{8} & \frac{15}{4} & \frac{45}{8} & \frac{5}{2} & \frac{5}{4} & \frac{15}{8} & \frac{11}{4} & \frac{11}{4} & \frac{5}{2} & 2 & \frac{11}{4} & \frac{11}{4} & \frac{5}{2} & \frac{3}{2} & \frac{13}{4} & \frac{5}{2} & \frac{3}{2} & \frac{13}{4} \\
\frac{5}{2} & \frac{15}{4} & \frac{15}{4} & \frac{5}{2} & 5 & \frac{5}{4} & \frac{15}{4} & \frac{11}{4} & \frac{9}{4} & \frac{9}{4} & \frac{11}{4} & \frac{11}{4} & \frac{9}{4} & \frac{3}{2} & \frac{5}{2} & 3 & \frac{3}{2} & \frac{5}{2} & 3 \\
\frac{5}{2} & \frac{5}{4} & \frac{5}{4} & \frac{5}{4} & \frac{5}{4} & \frac{5}{2} & \frac{5}{2} & \frac{3}{4} & \frac{3}{2} & 1 & \frac{7}{4} & \frac{3}{4} & \frac{3}{2} & 1 & \frac{3}{4} & \frac{7}{4} & 1 & \frac{3}{4} & \frac{7}{4} \\
\frac{25}{8} & \frac{25}{8} & \frac{15}{4} & \frac{15}{8} & \frac{15}{4} & \frac{5}{2} & \frac{45}{8} & \frac{11}{4} & \frac{5}{2} & \frac{7}{4} & 3 & \frac{11}{4} & \frac{5}{2} & \frac{3}{2} & \frac{9}{4} & \frac{7}{2} & \frac{3}{2} & \frac{9}{4} & \frac{7}{2} \\
2 & 3 & 3 & \frac{11}{4} & \frac{11}{4} & \frac{3}{4} & \frac{11}{4} & \frac{17}{4} & \frac{9}{8} & \frac{9}{8} & \frac{7}{4} & 2 & \frac{17}{8} & \frac{3}{4} & \frac{15}{8} & \frac{23}{8} & 2 & \frac{15}{8} & \frac{9}{4} \\
\frac{5}{2} & \frac{7}{4} & \frac{11}{4} & \frac{11}{4} & \frac{9}{4} & \frac{3}{2} & \frac{5}{2} & \frac{9}{8} & \frac{15}{4} & 1 & \frac{17}{8} & \frac{9}{8} & \frac{13}{8} & \frac{13}{8} & \frac{1}{2} & \frac{11}{4} & \frac{13}{8} & \frac{9}{8} & \frac{11}{4} \\
\frac{11}{4} & \frac{5}{2} & \frac{7}{4} & \frac{5}{2} & \frac{9}{4} & 1 & \frac{7}{4} & \frac{9}{8} & 1 & \frac{15}{4} & \frac{17}{8} & \frac{9}{4} & 1 & \frac{13}{8} & \frac{7}{4} & \frac{3}{2} & \frac{13}{8} & \frac{9}{8} & \frac{17}{8} \\
\frac{11}{4} & \frac{11}{4} & \frac{5}{2} & 2 & \frac{11}{4} & \frac{7}{4} & 3 & \frac{7}{4} & \frac{17}{8} & \frac{17}{8} & \frac{17}{4} & \frac{3}{2} & \frac{9}{8} & \frac{3}{4} & \frac{15}{8} & \frac{23}{8} & 2 & \frac{15}{8} & \frac{9}{4} \\
2 & 3 & 3 & \frac{11}{4} & \frac{11}{4} & \frac{3}{4} & \frac{11}{4} & 2 & \frac{9}{8} & \frac{9}{4} & \frac{3}{2} & \frac{17}{4} & \frac{9}{8} & 2 & \frac{15}{8} & \frac{9}{4} & \frac{11}{8} & \frac{15}{8} & \frac{23}{8} \\
\frac{5}{2} & \frac{7}{4} & \frac{11}{4} & \frac{11}{4} & \frac{9}{4} & \frac{3}{2} & \frac{5}{2} & \frac{17}{8} & \frac{13}{8} & 1 & \frac{9}{8} & \frac{9}{8} & \frac{15}{4} & \frac{13}{8} & \frac{9}{8} & \frac{11}{4} & 1 & \frac{7}{4} & \frac{17}{8} \\
\frac{5}{2} & \frac{3}{2} & 2 & \frac{5}{2} & \frac{3}{2} & 1 & \frac{3}{2} & \frac{3}{4} & \frac{13}{8} & \frac{13}{8} & \frac{3}{4} & 2 & \frac{13}{8} & \frac{13}{4} & \frac{3}{8} & \frac{11}{8} & 1 & \frac{3}{8} & \frac{5}{2} \\
\frac{3}{2} & \frac{9}{4} & \frac{9}{4} & \frac{3}{2} & \frac{5}{2} & \frac{3}{4} & \frac{9}{4} & \frac{15}{8} & \frac{1}{2} & \frac{7}{4} & \frac{15}{8} & \frac{15}{8} & \frac{9}{8} & \frac{3}{8} & \frac{13}{4} & \frac{3}{2} & \frac{11}{8} & \frac{9}{8} & \frac{3}{2} \\
\frac{13}{4} & \frac{7}{2} & \frac{13}{4} & \frac{13}{4} & 3 & \frac{7}{4} & \frac{7}{2} & \frac{23}{8} & \frac{11}{4} & \frac{3}{2} & \frac{23}{8} & \frac{9}{4} & \frac{11}{4} & \frac{11}{8} & \frac{3}{2} & \frac{21}{4} & \frac{19}{8} & \frac{21}{8} & \frac{25}{8} \\
\frac{5}{2} & \frac{3}{2} & 2 & \frac{5}{2} & \frac{3}{2} & 1 & \frac{3}{2} & 2 & \frac{13}{8} & \frac{13}{8} & 2 & \frac{11}{8} & 1 & 1 & \frac{11}{8} & \frac{19}{8} & \frac{13}{4} & \frac{3}{8} & \frac{11}{8} \\
\frac{3}{2} & \frac{9}{4} & \frac{9}{4} & \frac{3}{2} & \frac{5}{2} & \frac{3}{4} & \frac{9}{4} & \frac{15}{8} & \frac{9}{8} & \frac{9}{8} & \frac{15}{8} & \frac{15}{8} & \frac{7}{4} & \frac{3}{8} & \frac{9}{8} & \frac{21}{8} & \frac{3}{8} & \frac{13}{4} & \frac{3}{2} \\
\frac{13}{4} & \frac{7}{2} & \frac{13}{4} & \frac{13}{4} & 3 & \frac{7}{4} & \frac{7}{2} & \frac{9}{4} & \frac{11}{4} & \frac{17}{8} & \frac{9}{4} & \frac{23}{8} & \frac{17}{8} & \frac{5}{2} & \frac{3}{2} & \frac{25}{8} & \frac{11}{8} & \frac{3}{2} & \frac{21}{4}
\end{vmatrix}.$$

Observe that in this case $\det M$ is not equal to $\det NS/\det T$, as

$$\det NS = 5^{12}, \quad \det T = 2^{23}.$$

cf. [9, p.1950] for the former.

6.3. The case $m = 6$

For $m = 6$, we have $g = 6$ fibration on X_6 whose Picard number is $\rho(X_6) = 86$, which happens to be maximal in char 0, equal to the Hodge number $h^{1,1}$. The Mordell-Weil rank is equal to $r = 86 - (4 \cdot 6 - 2) = 64$ by Theorem 1.

In this case, however, Theorem 2 does not hold, since $\rho_1(X_6) = 62 < \rho(X_6)$ and the lines do not generate the Néron-Severi group of X_6. Some new idea will be needed to treat the MWL for $m = 6$.

6.4. The case $m = 7$

PROPOSITION 22. *In case $m = 7$, we have*

$$\mathrm{rk} M = 65, \quad \det M = \frac{7^{48}}{2^{29} 3^{15}}.$$

This can be verified in the same way as in the case $m = 5$. Note that here we have $\det M = \det NS / \det T$, because

$$\det NS = 7^{48}, \quad \det T = 2^{29} 3^{15}.$$

6.5. A conjecture

CONJECTURE 23. *Assume that m is prime to 6. Then we conjecture that*

$$\det M = \nu^2 \ \frac{\det NS(X_m)}{\det T} = \nu^2 \frac{m^{3(m-3)^2}}{(m-1)^{3(m-2)}(m-3)^m} \tag{68}$$

for some integer ν.

7. Positive Characteristic

Now we turn our attention to positive characteristic p.

We remark that we can obtain similar results for the axial fibration of the Fermat surface $X_m(p)$ of degree m in positive characteristic p too, *provided* we assume that (i) $X_m(p)$ is smooth, (ii) the generic fibre Γ_t is a smooth plane curve of genus $g = (m-2)(m-3)/2$, and (iii) the trivial lattice T

is the same as in the case of char 0. In particular, we state the following result in the supersingular case:

THEOREM 24. *Assume that* $p^\nu \equiv -1 \mod m$ *for some integer* ν *and that* Γ_t *is smooth. Then the Mordell-Weil group* $J(k(t))$ *is a finitely generated abelian group of rank*

$$r = b_2(m) - (4m - 2) = m(m^2 - 4m + 2) \ (m \geq 4) \tag{69}$$

where $b_2(m)$ *denotes the second Betti number of a smooth surface of degree* m*:*

$$b_2(m) = (m - 1)(m^2 - 3m + 3) + 1. \tag{70}$$

For example, for m=4 and $p \equiv -1 \mod 4$, $p > 3$, the Mordell-Weil rank of the elliptic fibration $f : X_4(p) \to \mathbf{P}^1$ is equal to $r = 8$. We have the same Weierstrass equation as before (65) for any $p > 3$.

REMARK. Recently Katsura [4] has examined the axial fibration on $X_m(p)$ in case $m = q + 1$ for q a power of p, which he calls a Lefschetz fibre space. He derives that the Mordell-Weil group is a finite p-group in that case. Note that it is not a contradiction to the above theorem of ours, because the generic fibre is not smooth there. Also Rams-Schütt [8] have studied the axial fibrations on quartic surfaces, and in particular, they discuss some genus 1 fibrations which is a quasi-elliptic fibration in characteristics $p = 3$.

8. Variants and Generalization

The method of axial fibration works in much more generality. In fact, we have:

THEOREM 25. *Let* X *be an arbitrary smooth surface of degree* m *in* $\mathbf{P}^3$ *which contains a line, say* l_0. *Then the axial fibration* $f : X \to \mathbf{P}^1$ *is defined, whose fibres are plane curves of degree* $m - 1$. *We assume that the generic fibre* Γ_t *is a smooth curve over* $k(t)$ *of genus* $g = (m - 2)(m - 3)/2$.

Then the Jacobian variety J *of* Γ_t *has the Mordell-Weil group* $M = J(k(t))$ *which is finitely generated and of rank* r *equal to*

$$r = \rho(X) - \mathrm{rk}\, T$$

where $\rho(X)$ is the Picard number of X and T denotes the trivial lattice of the axial fibration f. One can make M into a lattice, MWL, in the same way as in the case of Fermat surfaces discussed in the above.

In the above statement, the characteristic of the base field k is arbitrary, as far as the smoothness assumption of X and of Γ_t is satisfied.

Thus we are led to a new construction method of many MWL of higher genus fibrations in case $m = \deg(X)$ is greater than 4. On the other hand, this approach is equally meaningful in the case $m = 4$, which provides K3 surfaces with elliptic (or quasi-elliptic) fibrations, of geometric origin.

Let us illustrate this with an example in some detail.

Example. Fix $m \geq 4$ and consider the Klein surface of degree m in $\mathbf{P}^3$:

(71) $$Z_m : x_0 x_1^{m-1} + x_1 x_2^{m-1} + x_2 x_3^{m-1} + x_3 x_0^{m-1} = 0.$$

We have a pair of disjoint lines

$$l_0 : x_0 = 0, x_2 = 0 \quad \text{and} \quad l_1 : x_1 = 0, x_3 = 0.$$

Taking the line l_0 as the axis and the line l_1 as the zero-section, we have the axial fibration on the Klein surface:

$$f : Z_m \to \mathbf{P}^1,$$

defined by

$$f(\xi) = t = \frac{x_2}{x_0} = -\frac{x_1^{m-1} + x_0^{m-2} x_3}{x_3^{m-1} + x_2^{m-2} x_1}.$$

The Klein surface $Z_m = Z_m(p)$ is smooth if and only if

$$p \not| \, d := (m-1)^4 - 1 = m(m-2)d_0, \quad d_0 = 1 + (m-1)^2.$$

For any m and p, the generic fibre Γ_t of f is smooth, and the singular fibres at $t = 0$ and ∞ are irreducible. Further the other singular fibres are at the following d_0 places:

$$t^{d_0} = (-1)^m, \text{ i.e. } \quad t^{1+(m-1)^2} = (-1)^m.$$

For the Picard number formula for $\rho(Z_m)$ in char. 0, see [11, p.424]. In particular, if m is odd, it says that

(72) $$\rho(Z_m) = m^2 - m + 1.$$

Now we discuss a few cases of small degrees.

m $= 4$. We consider the Klein quartic $X = Z_4$. It is a smooth K3 surface in every characteristic different from $p = 2$ and $p = 5$. Assume $p \neq 2, 5$. Then $f : Z_4 \to \mathbf{P}^1$ is an elliptic fibration such that the ten singular fibre at $t^{10} = 1$ are all reducible, of Kodaira type I_2, each of which is a union of a line and a conic. Thus the trivial lattice has rank 12. On the other hand, we have $\rho(Z_4) = 20$ (in char 0). Hence we find that the MWL has rank

$$r = 20 - 12 = 8.$$

Now the generic fibre (a plane cubic) of (Z_4, f)

$$\Gamma_t : x_1^3 + x_3 x_0^2 + t(x_3^3 + t^2 x_1 x_0^2) = 0 \tag{73}$$

can be transformed to the Weierstrass form:

$$E : y^2 = x^3 - 3x - (t^5 + \frac{1}{t^5}). \tag{74}$$

Note that this is again a base change via $\tau = t^5$ of the Inose's fibration $S = F^{(1)}_{1,0}$. According to Artin-Rodriguez-Villegas-Tate [2], the Weierstrass transformation above works over the base scheme $\mathbf{P}^1_{\mathbb{Z}}$.

The structure of the Mordell-Weil lattice $M = E(k(t))$ can be described as follows: (i) in char 0 and in the ordinary case $p \equiv 1 \mod 4$ (but $p > 5$) , we have

$$r = 8, \quad \det = \frac{5^2}{2^8},$$

(ii) in the supersingular case $p \equiv -1 (\mod 4)$, $p > 3$, we have

$$r = 10, \quad \det = \frac{p^2}{2^{10}}.$$

m $= 5$. Finally we briefly mention the case $m = 5$; the Klein quintic surface Z_5 with genus $g = 3$ fibration. To our surprise, we find that there are seventeen (!) singular fibres at $t^{17} = -1$, which are all reducible and a union of a line and a plane cubic curve. Thus the trivial lattice has rank 19. On the other hand, we have $\rho(Z_5) = 21$ (in char 0), hence the MWL has rank

$$r = 21 - 19 = 2.$$

More generally, a similar argument works to prove:

PROPOSITION 26. *For the Klein surface of any odd degree $m > 3$, the MW rank is equal to*

$$r(Z_m) = m - 3$$

in char 0.

By (72), the Picard number formula for the Klein surface of odd degree m is equal to $\rho(Z_m) = m^2 - m + 1$. On the other hand, there are reducible singular fibres at the $d_0 = (m-1)^2 + 1$ places v defined by $t^{d_0} = -1$. Each such fibre ontains a line and has $m_v = 2$ irreducible components. This implies that $\mathrm{rk}T = 2 + d_0$, and hence the assertion by Theorem 10.

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Added in Proof. With the same notation as in §6, we have

$$\det \bar{M} = \frac{\det NS}{\det T'} = \frac{\det NS}{\det T}|M_{tor}|^2,$$

where T' denotes the primitive closure of T in NS. The proof will be given elsewhere.

Therefore the results stated in §6 imply that the Mordell-Weil group M is torsion-free in case $m = 4$ and $m = 7$, but, in case $m = 5$, M has a *non-trivial torsion*: $|M_{tor}| = 2$. In the latter case, J is the Jacobian of a plane quartic, and we can verify the existence of a non-trivial 2-torsion by using the bitangents of the plane quartic.

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Department of Mathematics
Rikkyo University
Tokyo 171-8501, Japan
E-mail: shioda@rikkyo.ac.jp