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Characteristic Cycle and the Euler Number of a Constructible Sheaf on a Surface

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Dedicated to Professor Kunihiko Kodaira, on the occasion of 100th anniversary

Abstract. We define the characteristic cycle of a constructible sheaf on a smooth surface in the cotangent bundle. We prove that the intersection number with the 0-section equals the Euler number and that the total dimension of vanishing cycles at an isolated characteristic point is also computed as an intersection number.

For a constructible sheaf on a smooth algebraic variety in positive characteristic, an analogy between the wild ramification of an étale sheaf and the irregularity of a $\mathcal{D}$-module in characteristic 0 suggests that the characteristic cycle is defined as a cycle of the cotangent bundle. Its intersection product with the 0-section is expected to give the characteristic class [4] and the Euler number for a proper variety consequently. At an isolated characteristic point (see the last paragraph of Section 1 for the definition) of a fibration to a curve, the intersection number with the section defined by a non-vanishing differential form of the curve is expected to be equal to the total dimension of nearby cycles.

In a tamely ramified case, the characteristic cycle has an elementary definition in arbitrary dimension (1.4). For a sheaf on a curve, the characteristic cycle is determined by the Swan conductor at the boundary (1.6). For a sheaf on a surface, Deligne and Laumon define the characteristic cycle implicitly in [20] (see also [16, Letter 3 (b)]) using the total dimension of the nearby cycles and compute the Euler number, under the "non-feroce" assumption.

To remove the assumption, Deligne further sketched a global method, extending that in [6], in a letter [8] and in unpublished notes [9] with more

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detail. The method fits in an approach of Beilinson using the Radon transform [5].

In this article, we define the characteristic cycle of a sheaf on a surface in general in Definition 3.8, by combining the approach using the Radon transform and the non-logarithmic version [25] of ramification theory developed in a joint work with Abbes, following the ideas in [8] and [9]. We prove that the intersection number with the 0-section equals the Euler number in general in Theorem 3.19 and that that with the section defined by a fibration to a curve computes the total number of nearby cycles at an isolated characteristic point in Theorem 3.17, as suggested in [9]. We also show in Proposition 3.20 that it is the same as that defined in [25, Definition 3.5] as long as the latter is defined. The relation with the characteristic class defined in [4] is still to be clarified.

The definition goes as follows. First, by studying the ramification of the Radon transform using the ramification theory developed in [25], we define the characteristic cycle that a priori may depend on the choice of a projective embedding. Using a deformation argument [9] and the dimension formula for the nearby cycles by Deligne and Laumon [19], we show that the characteristic cycle thus defined computes the total dimension of nearby cycles at an isolated characteristic point. We deduce from this that the characteristic cycle is in fact independent of the choice of a projective embedding.

The deformation argument relies on the stability of nearby cycles under small deformation of fibrations. This in turn follows from a generalization of Hensel's lemma due to Elkik [10] together with the vanishing of the limit of nearby cycles for a certain sequence of blow-up and the stability of the dimension of nearby cycles. The last fact is based on a generalization by Kato [17], [14] of the formula [19] used above and the stability of the ramification of restrictions to curves.

We prove that the Euler number equals the intersection number of the characteristic cycle with the 0-section, applying the Grothendieck-Ogg-Shafarevich formula computing the Euler number of a sheaf on a curve [13] two times. The equality implies that the difference with the characteristic cycle defined in [25] is controlled by a divisor numerically equivalent to zero. By using a finite covering trivializing the ramification except at one irreducible component of the ramification divisor, we conclude that this divisor is in fact zero and derive the coincidence of the two definitions.

We briefly describe the content of each section. After briefly recalling the ramification theory developed in [25] in Section 1, we prove in Section 2.1 the stability of the ramification of the restrictions to curves in Propositions 2.1 and 2.4. We also show a continuity of the total dimension of nearby cycles in Proposition 2.6. Using a generalization of Hensel's lemma due to Elkik [10] recalled in Section 2.2, we prove the stability Theorem 2.14 of nearby cycles under small deformation of fibrations in Section 2.3.

After some preliminaries on the universal family of hyperplane sections in Section 3.1, we study the ramification of the Radon transform and define the characteristic cycle in Definition 3.8, depending on projective embedding in Section 3.2. We prove in Proposition 3.13 and Theorem 3.17 a formula computing the total dimension of nearby cycles as an intersection number with the characteristic cycle and deduce that it is in fact independent of a projective embedding. Finally in Section 3.3, we prove the equality for the Euler number in Theorem 3.19 and the equality of the characteristic cycle defined using the Radon transform with that defined in [25] in Proposition 3.20.

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Contents

1. Brief Summary of Ramification Theory

We briefly recall the ramification theory from [25]. Let K be a complete discrete valuation field with not necessarily perfect residue field F of characteristic $p > 0$. The filtration $(G^r_K)_r$ by (non-logarithmic) ramification groups of the absolute Galois group $G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$ is defined as a decreasing filtration by closed normal subgroups indexed by rational numbers $r \geqq 1$ [2], [3]. For a rational number $r \geqq 1$, define $G^{r+}_K \subset G^r_K$ to be the closure $\overline{\bigcup_{s>r} G^s_K}$. The subgroup $I_K = G^1_K \subset G_K$ is the inertia subgroup and $P_K = G^{1+}_K \subset G^1_K$ is its p-Sylow subgroup called the wild inertia subgroup. Assume that K is of characteristic p. Then, the graded piece $\mathrm{Gr}^r G_K = G^r_K/G^{r+}_K$ is known to be an abelian group annihilated by p [25, Corollary 2.28.1] for $r > 1$.

Let Λ be a finite field of characteristic $\ell \neq p$ and M be a finite Λ-module with continuous G_K-action. Then, there exists a unique decomposition $M = \bigoplus_{r \geqq 1} M^{(r)}$ called the slope decomposition characterized by the condition that the G^{r+}_K-fixed part $M^{G^{r+}_K}$ equals $\bigoplus_{s \leqq r} M^{(s)}$ for $r \geqq 1$. We define the total dimension of M by

$$\dim \mathrm{tot}_K M = \sum_{r \geqq 1} r \cdot \dim_\Lambda M^{(r)}. \tag{1.1}$$

In the classical case where the residue field F is perfect, it is equal to the sum $\dim M + \mathrm{Sw}_K M$ of the dimension and the Swan conductor [13, Section 4]. Further, if K is of characteristic p and if Λ contains a primitive p-th root of unity, the r-th piece $M^{(r)}$ for $r > 1$ is decomposed as $\bigoplus_{\chi\colon \mathrm{Gr}^r G_K \to \Lambda^\times} \chi^{\oplus n(\chi)}$ by characters of the abelian group $\mathrm{Gr}^r G_K$ annihilated by p, since the group algebra $\Lambda[G]$ is decomposed as $\prod_{\chi\colon G \to \Lambda^\times} \Lambda$ for a finite abelian group G annihilated by p.

We consider the case where X is a smooth scheme of dimension d over a perfect field k of characteristic $p > 0$ and $K = \mathrm{Frac}(\hat{\mathcal{O}}_{X,\xi})$ is the local field at the generic point ξ of a smooth irreducible divisor D. The residue field F is the function field $\kappa(\xi)$ of the divisor D and the residue field $\bar{F}$

of $K^{\rm sep}$ is an algebraic closure of F. For a rational number r, let $I(r)$ denote the fractional ideal $\{a \in K^{\rm sep} \mid {\rm ord}_K a \geqq -r\}$ and define an $\bar F$-vector space $L(r) = I(r) \otimes_{\mathcal{O}_{K^{\rm sep}}} \bar F$ of dimension 1. Then, the dual $({\rm Gr}^r G_K)^\vee = Hom_{\mathbf{F}_p}({\rm Gr}^r G_K, \mathbf{F}_p)$ of the $\mathbf{F}_p$-vector space ${\rm Gr}^r G_K$ is canonically identified as a subgroup of the $\bar F$-vector space $\Omega^1_{X/k,\xi} \otimes_{\mathcal{O}_{X,\xi}} L(r)$ by the canonical injection

$$\text{(1.2)} \qquad {\rm char}\colon ({\rm Gr}^r G_K)^\vee \to \Omega^1_{X/k,\xi} \otimes_{\mathcal{O}_{X,\xi}} L(r)$$

defined in [25, Corollary 2.28.2]. For a non-trivial character $\chi \in ({\rm Gr}^r G_K)^\vee$, let $F(\chi)$ denote a finite extension of F where ${\rm char}(\chi)$ regarded as an $\bar F$-linear mapping $L(-r) \to \Omega^1_{X/k,\xi} \otimes_{\mathcal{O}_{X,\xi}} \bar F$ descends to an $F(\chi)$-linear mapping. Then, it defines a line $L(\chi)$ in the fiber $T^*X \times_X {\rm Spec}\, F(\chi)$ of the cotangent bundle $T^*X = \mathbf{V}(\Omega^1_{X/k})$ at ${\rm Spec}\, F(\chi) \to \xi \in X$.

Let $U = X - D$ denote the complement and $j\colon U \to X$ be the open immersion. Let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on U. We assume that Λ contains a primitive p-th root of unity, fix an isomorphism $\mathbf{F}_p \to \mu_p(\Lambda)$ and identify $({\rm Gr}^r G_K)^\vee = Hom({\rm Gr}^r G_K, \Lambda^\times)$. Then, the characteristic cycle ${\rm Char}\, j_!\mathcal{F}$ is defined on a neighborhood of the generic point ξ of D as follows. Let $\bar\eta$ be the geometric point of U defined by the separable closure $K^{\rm sep}$ and let $M = \mathcal{F}_{\bar\eta}$ be the continuous representation of G_K defined by $\mathcal{F}$. Then the slope decomposition $M = \bigoplus_{r \geqq 1} M^{(r)}$ and the decomposition by characters $M^{(r)} = \bigoplus_{\chi \in ({\rm Gr}^r G_K)^\vee} \chi^{\oplus n(\chi)}$ for $r > 1$ are defined. Let $T^*_X X \subset T^*X$ denote the 0-section and $T^*_D X \subset T^*X$ the conormal bundle. We define the germ of the characteristic cycle ${\rm Char}\, j_!\mathcal{F}$ at ξ to be

$$\text{(1.3)} \qquad (-1)^d \Biggl({\rm rank}\, \mathcal{F} \cdot [T^*_X X] + \dim M^{(1)} \cdot [T^*_D X] + \sum_{r>1} r \cdot \sum_{\chi \in ({\rm Gr}^r G_K)^\vee} \frac{n(\chi)}{[F(\chi)\colon F]} [L(\chi)] \Biggr)$$

where $d = \dim X$. Let $({\rm Char}\, j_!\mathcal{F})^{\rm wild}_D$ denote the sum of the last term in the parantheses.

More generally, we consider the case where D is not necessarily an irreducible and smooth divisor. After removing closed subset of codimension

$\geqq 2$ if necessary, we assume that D is a divisor with simple normal crossings. Let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on $U = X - D$. Then, further after removing closed subset of codimension $\geqq 2$ if necessary, we may assume that the ramification of $\mathcal{F}$ along D is non-degenerate in the sense of [25, Definition 3.1] which we will not recall here.

Assuming that the ramification of $\mathcal{F}$ along D is non-degenerate, the characteristic cycle Char $j_!\mathcal{F}$ is defined as follows. Let $D_1, \ldots, D_m$ be the irreducible components of D and for a subset $I \subset \{1, \ldots, m\}$, let D_I denote the intersection $\bigcap_{i\in I} D_i$ and $T^*_{D_I}X \subset T^*X$ be the conormal bundle. If $\mathcal{F}$ is tamely ramified along D, the characteristic cycle Char $j_!\mathcal{F}$ is defined by

$$\text{Char } j_!\mathcal{F} = (-1)^d \left(\sum_{I\subset\{1,\ldots,m\}} \text{rank } \mathcal{F} \cdot [T^*_{D_I}X] \right). \tag{1.4}$$

Next, we consider the case where the ramification of $\mathcal{F}$ along D is non-degenerate and totally wild; for every irreducible component D_i of D, the tame part $\mathcal{F}^{(1)}_{\bar{\eta}_i}$ is 0. Then the germ of cycle $(\text{Char } j_!\mathcal{F})^{\text{wild}}_{D_i}$ in (1.3) for each irreducible component D_i of D is defined as a cycle of T^*X and the characteristic cycle is defined by the equality

$$\text{Char } j_!\mathcal{F} = (-1)^d \left(\text{rank } \mathcal{F} \cdot [T^*_X X] + \sum_i (\text{Char } j_!\mathcal{F})^{\text{wild}}_{D_i} \right). \tag{1.5}$$

In general, the characteristic cycle Char $j_!\mathcal{F}$ is characterized by (1.4) and (1.5) together with the additivity for exact sequence on $\mathcal{F}$ and the compatibility by étale pull-back. Define the singular support $SS(j_!\mathcal{F}) \subset T^*X$ to be the union of the underlying set of the components of the characteristic cycle Char $j_!\mathcal{F}$. If $\dim X = 1$, we have

$$\text{Char } j_!\mathcal{F} = -\Big(\text{rank } \mathcal{F} \cdot [T^*_X X] + \sum_{x\in D} (\text{rank } \mathcal{F} + \text{Sw}_x \mathcal{F}) \cdot [T^*_x X] \Big). \tag{1.6}$$

Going back to the general dimension, the total dimension divisor is defined by

$$DT j_!\mathcal{F} = \sum_i \dim \text{tot}_{K_i} \mathcal{F}_{\bar{\eta}_i} \cdot D_i \tag{1.7}$$

where the geometric point $\bar{\eta}_i$ is defined by a separable closure of the local field K_i at the generic point of an irreducible component D_i of D. Note that in the definition of the total dimension divisor we do not need to assume that ramification is non-degenerate.

More generally, for a constructible complex $\mathcal{K}$ of Λ-module on X such that the restriction of the cohomology sheaf $\mathcal{H}^q\mathcal{K}$ is locally constant on U for every integer q, the Artin divisor $a(\mathcal{K})$ is defined by

$$a(\mathcal{K}) = \sum_q (-1)^q \left(DT j_! j^* \mathcal{H}^q(\mathcal{K}) - \sum_i \dim \mathcal{H}^q(\mathcal{K})_{\bar{\xi}_i} \cdot D_i \right) \tag{1.8}$$

where $\bar{\xi}_i$ is a geometric point dominating the generic point of an irreducible component D_i of D.

Let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on $U = X - D$ with non-degenerate ramification along a divisor D with simple normal crossings. Let C be a smooth curve over k and $C \to X$ be an immersion over k such that the intersection $C \cap D$ is finite. We say that the immersion $C \to X$ is non-characteristic at a closed point $x \in C \cap D$ with respect to $j_!\mathcal{F}$ if the tangent vector of C at x is not annihilated by any nonzero differential form in the fiber of $SS(j_!\mathcal{F})$ at x. If $C \to X$ is non-characteristic at x, the total dimension divisor is compatible with the pull-back [25, Proposition 3.8]:

$$(DT j_!\mathcal{F}, C)_x = \dim \mathrm{tot}_x \mathcal{F}|_C. \tag{1.9}$$

Let C be a smooth curve over k and $f\colon X \to C$ be a smooth morphism over k. We say that $f\colon X \to C$ is *non-characteristic* with respect to $j_!\mathcal{F}$ if the section of T^*X defined by the pull-back by f of a non-vanishing differential form on C does not intersect with the singular support $SS(j_!\mathcal{F})$. If $\dim X = 2$, it is equivalent to that for every closed point c of C, the immersion $X \times_C c \to X$ is non-characteristic. We say that $x \in X$ is a characteristic point of $f\colon X \to C$ with respect to $j_!\mathcal{F}$ if $f\colon X \to C$ is not non-characteristic on a neighborhood of x. A morphism $f\colon X \to C$ non-characteristic with respect to $j_!\mathcal{F}$ is universally locally acyclic relatively to $j_!\mathcal{F}$, if either $\mathcal{F}$ is tamely ramified along D or $\mathcal{F}$ is totally wildly ramified along D and $f|_D\colon D \to C$ is flat by [25, Proposition 3.15]. In particular, the complex of vanishing cycles $\phi(j_!\mathcal{F}, f)$ on the geometric fiber $X_{\bar{c}}$ is 0 for every geometric closed point $\bar{c}$ of C.

We say that a closed point x is an isolated characteristic point of $f\colon X \to C$ with respect to $j_!\mathcal{F}$ if the restriction of f to a neighborhood of x is non-characteristic with respect to $j_!\mathcal{F}$ except possibly at x. This definition makes sense also in the case where there exists an neighborhood V of x such that $V - \{x\}$ is smooth over k and that $(D \cap V) - \{x\}$ is a divisor of $V - \{x\}$ with simple normal crossings. The condition that x is an isolated characteristic point of $f\colon X \to C$ with respect to $j_!\mathcal{F}$ implies that there exists an neighborhood V of x such that $V - \{x\}$ is smooth over C.

2. Stability of Nearby Cycles

2.1. Stability of ramification of the restrictions to curves

For morphisms $f\colon X \to S$ and $T \to S$ of schemes, let $(X, f) \times_S T$ denote the fibered product to indicate the morphism, if necessary. For morphisms $f\colon X \to S$ and $g\colon X \to S$ of schemes and a closed subscheme Z of X defined by an ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, we say that f and g are congruent to each other modulo $\mathcal{I}_Z$ and write $f \equiv g \bmod \mathcal{I}_Z$ if the restrictions $f|_Z\colon Z \to S$ and $g|_Z\colon Z \to S$ are the same.

PROPOSITION 2.1. *Let X be a normal surface over a perfect field k of characteristic $p > 0$ and let $f\colon X \to C$ be a flat morphism over k to a smooth curve C over k. Let $D \subset X$ be a closed subscheme quasi-finite over C, let u be a closed point of D and let $v = f(u) \in C$.*

1. *There exists an integer $N \geqq 1$ such that if a morphism $g\colon X \to C$ over k satisfies $g \equiv f \bmod \mathfrak{m}_u^N$, then $g\colon X \to C$ is flat at u, its restriction $g|_D\colon D \to C$ is quasi-finite at u. Further, if $f|_D\colon D \to C$ is flat at u (resp. and if $f|_{D - \{u\}}\colon D - \{u\} \to C$ is étale), then $g|_D\colon D \to C$ is flat at u (resp. and $g|_{D - \{u\}}\colon D - \{u\} \to C$ is étale on a neighborhood of u except at u).*

2. *Let Λ be a finite field of characteristic $\ell \neq p$ and $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on the complement $U = X - D$. Assume that u is* an isolated characteristic point *of $f\colon X \to C$ with respect to $j_!\mathcal{F}$. Then, there exists an integer $N \geqq 1$ such that if a morphism $g\colon X \to C$ over k satisfies $g \equiv f \bmod \mathfrak{m}_u^N$, then u is an isolated characteristic point of $g\colon X \to C$ with respect to $j_!\mathcal{F}$.*

3. *Further, there exists an integer $N \geqq 1$ such that if a morphism $g\colon X \to C$ over k satisfies $g \equiv f \bmod \mathfrak{m}_u^N$, there exist an étale neighborhood*

$V \to C$ of $v = f(u)$ such that the connected components $(D, f|_D)^0_V$ and $(D, g|_D)^0_V$ of $(D, f|_D) \times_C V$ and $(D, g|_D) \times_C V$ containing u are finite over V and that for every closed point $y \in V - \{v\}$, we have

$$\sum_{x \in (D,f|_D)^0_V, f(x)=y} \dim \mathrm{tot}_x(\mathcal{F}|_{f^{-1}(y)}) = \sum_{x \in (D,g|_D)^0_V, g(x)=y} \dim \mathrm{tot}_x(\mathcal{F}|_{g^{-1}(y)}). \tag{2.1}$$

In 3, an étale neighborhood $V \to C$ of v means an étale morphism equipped with $v \to V$ lifting $v \to C$.

Proof. Replacing X by a neighborhood of u, we may assume that $X - \{u\}$ is smooth over k and that the reduced part of $D - \{u\}$ is a smooth divisor of $X - \{u\}$.

1. Let $t \in \mathcal{O}_{C,v}$ be a uniformizer and $N \geqq 1$ be an integer such that $f^*t \notin \mathfrak{m}_u^N$. Then if $g \equiv f \bmod \mathfrak{m}_u^N$, we have $g^*t \notin \mathfrak{m}_u^N$ and g is flat at u.

Let $N \geqq 2$ be an integer such that $\mathfrak{m}_u^{N-1}$ annihilates $(D, f|_D) \times_C v$ on a neighborhood of u. If $g \equiv f \bmod \mathfrak{m}_u^N$, the elementary lemma below implies then $(D, g|_D) \times_C v$ is also annihilated by $\mathfrak{m}_u^{N-1}$. Hence $g|_D \colon D \to C$ is quasi-finite at u. Assume $f|_D \colon D \to C$ is flat at u. Then, the pull-back by f of a uniformizer $t \in \mathcal{O}_{C,v}$ forms a regular sequence of the local Cohen-Macaulay ring $\mathcal{O}_{D,u}$ of dimension 1 and so is the pull-back by g. Hence $g|_D \colon D \to C$ is also flat at u. Further if $f|_D \colon D \to C$ is étale except at u, let $N \geqq 2$ be an integer such that $\mathfrak{m}_u^{N-1}$ annihilates $\Omega^1_{D/C,u}$ with respect to $f|_D$. If $g \equiv f \bmod \mathfrak{m}_u^N$, then $\mathfrak{m}_u^{N-1}$ also annihilates $\Omega^1_{D/C,u}$ with respect to $f|_D$ and hence $g|_D \colon D \to C$ is étale on a neighborhood of u except at u.

Lemma 2.2. *Let A be a noetherian ring, $I \subset A$ be an ideal, M be an A-module of finite type and $N \geqq 1$ be an integer. If the canonical surjection $M/I^N M \to M/I^{N-1}M$ is an isomorphism, then M is annihilated by I^{N-1} on a neighborhood of* Spec A/I.

Proof. By the assumption, the A-module $I^{N-1}M$ satisfies $I \cdot I^{N-1}M = I^{N-1}M$. Hence, it follows from Nakayama's lemma. □

2. Let $\pi\colon X' \to X$ be a resolution. Namely, X' is a smooth surface over k, π is proper and $X' - \pi^{-1}(u) \to X - \{u\}$ is an isomorphism. The singular support $SS(j_!\mathcal{F})$ is defined as a closed subset of $T^*(X - \{u\})$. Let $SS(j_!\mathcal{F})' \subset T^*X'$ denote the closure of $SS(j_!\mathcal{F})$ and regard it as a reduced closed subscheme. Let $E \subset X'$ denote the inverse image $\pi^{-1}(u) = X' \times_X u$.

Let $t \in \mathcal{O}_{C,v}$ be a uniformizer and let $df\colon X' \to T^*X'$ denote the section defined by f^*dt on a neighborhood of E. By the assumption that u is an isolated characteristic point of $f\colon X \to C$ with respect to $j_!\mathcal{F}$, the intersection $(X', df) \times_{T^*X'} SS(j_!\mathcal{F})'$ of the image of the section df and the singular support $SS(j_!\mathcal{F})'$ is a subset of the inverse image $T^*X' \times_{X'} E$. Let $N \geqq 2$ be an integer such that $(X', df) \times_{T^*X'} SS(j_!\mathcal{F})'$ is a closed subscheme annihilated by $\mathcal{I}_E^{N-2}$. Since $g \equiv f \bmod \mathfrak{m}_u^N$ implies $dg \equiv df \bmod \mathfrak{m}_u^{N-1}\Omega^1_{X/k}$, the intersection $(X', dg) \times_{T^*X'} SS(j_!\mathcal{F})'$ is also annihilated by $\mathcal{I}_E^{N-2}$ on a neighborhood of $T^*X' \times_{X'} E$ by Lemma 2.2 and u is an isolated characteristic point of $g\colon X \to C$ with respect to $j_!\mathcal{F}$.

3. Shrinking X if necessary, we may assume that u is the unique point in the fiber of $f|_D\colon D \to C$. Since a quasi-finite scheme over a henselian discrete valuation ring is the disjoint union of a finite scheme and a flat scheme, there exist an étale neighborhood of $V \to C$ of $v = f(u)$ such that the connected components $(D, f|_D)^0_V$ and $(D, g|_D)^0_V$ are finite. In the rest of proof, we assume that they are finite and flat.

Let $DT(j_!\mathcal{F}, f)^0_V$ denote the part of the pull-back of $DT(j_!\mathcal{F})$ to $(X, f)_V = (X, f) \times_C V$ supported on $(D, f|_D)^0_V$ and similarly for $DT(j_!\mathcal{F}, g)^0_V$. Since u is an isolated characteristic point of $f\colon X \to C$ with respect to $j_!\mathcal{F}$, the left hand side of (2.1) is equal to the degree of $DT(j_!\mathcal{F}, f)^0_V$ over V by (1.9). We will take an integer $N \geqq 1$ satisfying the conditions in 1. Then, for a morphism $g\colon X \to C$ over k satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, the point u is an isolated characteristic point of $g\colon X \to C$ with respect to $j_!\mathcal{F}$ and the right hand side is also equal to the degree of $DT(j_!\mathcal{F}, g)^0_V$ over V.

Let $N \geqq 1$ be an integer such that $DT(j_!\mathcal{F}, f)^0_V \times_V v$ is annihilated by $\mathfrak{m}_u^{N-1}$. If $g \equiv f \bmod \mathfrak{m}_u^N$, then $DT(j_!\mathcal{F}, g)^0_V \times_V v$ is equal to $DT(j_!\mathcal{F}, f)^0_V \times_V v$ and is also annihilated by $\mathfrak{m}_u^{N-1}$ by Lemma 2.2. Since the degree of $DT(j_!\mathcal{F}, f)^0_V$ over V is equal to the length of the scheme $DT(j_!\mathcal{F}, f)^0_V \times_V v$, it is equal to the degree of $DT(j_!\mathcal{F}, g)^0_V$ and the assertion follows. □

The following example shows that in Proposition 2.1 and Theorem 2.14, one cannot drop the assumption of non-charactericity.

Example 2.3. Let $X = \mathbf{A}^2 = \mathrm{Spec}\, k[x, y]$ be the affine plane over an algebraically closed field k of characteristic $p > 2$ and $U = X - D$ be the complement of the y-axis D. Let $u = (0, 0)$ denote the origin of $X = \mathbf{A}^2$. Assume that Λ contains a primitive p-th root of unity and let $\mathcal{F}$ be the locally constant constructible sheaf of Λ-modules of rank 1 on U defined by the Artin-Schreier equation $z^p - z = \dfrac{y}{x^p}$. Then, the singular support $SS(j_!\mathcal{F})$ is the union of the zero-section $T^*_X X$ and the sub line bundle over D spanned by the section dy.

Let $f\colon X \to C = \mathbf{A}^1 = \mathrm{Spec}\, k[t]$ be the smooth morphism defined by $t \mapsto y$. It is *characteristic* with respect to $j_!\mathcal{F}$ at every point of D. The restriction $f|_D\colon D \to C$ is an isomorphism. For $c \in \mathbf{A}^1(k)$, the Swan conductor $\mathrm{Sw}_{(0,c)}(j_!\mathcal{F}|_{f^{-1}(c)})$ of the restriction to the fiber is 0 for $c = 0$ and 1 for $c \neq 0$. Hence by [19], we have $\dim \phi^1_u(j_!\mathcal{F}, f) = 1$.

Let $n \geqq 2$ be an integer and $g\colon X \to C = \mathbf{A}^1 = \mathrm{Spec}\, k[t]$ be the smooth morphism defined by $t \mapsto y + xy^n$. We have $g \equiv f \bmod \mathfrak{m}_u^{n+1}$. The restriction $g|_D\colon D \to C$ is also an isomorphism. For $c \in \mathbf{A}^1(k)$, the Swan conductor $\mathrm{Sw}_{(0,c)}(j_!\mathcal{F}|_{g^{-1}(c)})$ of the restriction to the fiber is 0 for $c = 0$ and $p - 1 > 1$ for $c \neq 0$. Hence by [19], we have $\dim \phi^1_u(j_!\mathcal{F}, f) = p - 1 > 1$.

For closed subschemes C and C' and a closed subscheme Z of X defined by the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$, we say $C \equiv C' \bmod \mathcal{I}_Z$ if $C \times_X Z = C' \times_X Z$. If $f\colon X \to S$ and $g\colon X \to S$ satisfy $f \equiv g \bmod \mathcal{I}_Z$ and $T \subset S$ is closed subscheme, we have $(X, f) \times_S T \equiv (X, g) \times_S T \bmod \mathcal{I}_Z$.

Let C be a reduced excellent noetherian scheme of dimension 1 and u be a closed point of C with perfect residue field. Let $C' \to C$ be the normalization. Let Λ be a finite field of characteristic ℓ invertible at u and $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on $U = C - \{u\}$. Then, the total dimension $\dim \mathrm{tot}_u\mathcal{F}$ is defined as the sum $\sum_{u' \in C' \times_C u} \dim\, \mathrm{tot}_{u'}\mathcal{F}$.

Proposition 2.4. *Let X be a normal excellent noetherian scheme of dimension 2 and u be a closed point of X such that $\mathcal{O}_{X,u}$ is of dimension 2 and that the residue field is perfect. Let $U \subset X$ be a dense open subscheme. Let Λ be a finite field of characteristic ℓ invertible on X and let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on U.*

Let C be a reduced Cartier divisor of X containing u such that u is in the closure of $C\cap U$. Then, there exists an integer $N\geqq 1$ such that for a reduced Cartier divisor C_1 of X satisfying $C\equiv C_1 \bmod \mathfrak{m}_u^N$, the point u is in the closure of $C_1\cap U$ and we have

$$\dim\,\mathrm{tot}_u(\mathcal{F}|_{C\cap U})=\dim\,\mathrm{tot}_u(\mathcal{F}|_{C_1\cap U}). \tag{2.2}$$

PROOF. Let Z be a closed subscheme of X such that $U=X-Z$ and let $N\geqq 2$ be an integer such that $\mathcal{O}_{Z\cap C,u}$ is annihilated by $\mathfrak{m}_u^{N-1}$. Then, for a reduced Cartier divisor C_1 of X of dimension 1 satisfying $C\equiv C_1 \bmod \mathfrak{m}_u^N$, the ring $\mathcal{O}_{Z\cap C_1,u}$ is also annihilated by $\mathfrak{m}_u^{N-1}$ by Lemma 2.2 and the point u is contained in the closure of $C_1\cap U$.

Let $V\to U$ be a G-torsor for a finite group G such that the pull-back $\mathcal{F}_V$ of $\mathcal{F}$ is constant and let $f\colon Y\to X$ be the normalization of X in V. Let $\bar D$ and $\bar D_1$ be the normalizations of $D=C\times_X Y$ and $D_1=C_1\times_X Y$. For $\sigma\in G,\neq 1$ and a point v of $\bar D$ above u, let $\bar{\mathcal{I}}_{\sigma,v}$ denote the ideal of $\mathcal{O}_{\bar D,v}$ defining the intersection $\Delta_{\bar D}\cap\Gamma_\sigma\subset\Delta_{\bar D}=\bar D$ of the diagonal and the graph of σ in $\bar D\times_C\bar D$ and similarly $\bar{\mathcal{I}}_{\sigma,v_1}$ for v_1 of $\bar D_1$ above u. By the definition of the Swan conductor, it suffices to show the existence of $N\geqq 1$ such that the congruence $C\equiv C_1 \bmod \mathfrak{m}_u^N$ implies a bijection $\bar D\times_C\{u\}$ and $\bar D_1\times_C\{u\}$ satisfying the equalities length $\mathcal{O}_{\bar D,v}/\bar{\mathcal{I}}_{\sigma,v}=$ length $\mathcal{O}_{\bar D_1,v_1}/\bar{\mathcal{I}}_{\sigma,v_1}$ for the corresponding points and for $\sigma\in G,\neq 1$.

First, we prove the case where X,C and $D=C\times_X Y$ are regular. For $\sigma\in G,\neq 1$, let $\mathcal{I}_\sigma$ denote the ideal of $\mathcal{O}_Y$ defining the intersection $Y_\sigma=\Delta_Y\cap\Gamma_\sigma\subset\Delta_Y=Y$ of the diagonal and the graph of σ in $Y\times_X Y$. We have $\bar{\mathcal{I}}_{\sigma,v}=\mathcal{I}_\sigma\mathcal{O}_{D,v}$. Let $N\geqq 2$ be an integer such that $\mathcal{O}_{D,v}/\mathcal{I}_\sigma\mathcal{O}_{D,v}$ is annihilated by $\mathfrak{m}_u^{N-1}$ for every $\sigma\neq 1$ and $v\in f^{-1}(u)$. Let C_1 be an integral closed subscheme of dimension 1 satisfying $C\equiv C_1 \bmod \mathfrak{m}_u^N$. Then, since $D_1=C_1\times_X Y\equiv D \bmod \mathfrak{m}_v^2$, the scheme D_1 is also regular at every $v\in f^{-1}(u)$. Further, $\mathcal{O}_{D_1,v}/\mathcal{I}_\sigma\mathcal{O}_{D_1,v}$ is annihilated by $\mathfrak{m}_u^{N-1}$ by Lemma 2.2 and is isomorphic to $\mathcal{O}_{D,v}/\mathcal{I}_\sigma\mathcal{O}_{D,v}$ for every $\sigma\neq 1$ and $v\in f^{-1}(u)$. Thus the assertion is proved in this case.

We show the general case by reducing to the case proved above by using the following embedded resolution.

LEMMA 2.5 (cf. [22, Theorems 8.3.4, 9.2.26]). *Let X be a normal excellent noetherian scheme of dimension 2 and $C\subset X$ be a reduced closed* *subscheme of dimension* 1*. Let* $U \subset X$ *be the complement of finitely many closed points of codimension* 2 *of* X *contained in* C *such that* U *and* $C \cap U$ *are regular.*

Then, there exist a regular excellent noetherian scheme X' *of dimension* 2 *and a proper morphism* $g\colon X' \to X$ *such that* $g^{-1}(U) \to U$ *is an isomorphism and that the reduced part of the inverse image* $g^{-1}(C)$ *is a divisor of* X' *with simple normal crossings.*

In particular, the closure $C' \subset X'$ *of* $g^{-1}(C \cap U)$ *with the reduced scheme structure is regular and meets transversely the reduced part* E *of the effective Cartier divisor* $C \times_X X' - C'$*.*

By shrinking X if necessary, we may assume that $X - \{u\}$ and $C - \{u\}$ are regular. We apply Lemma 2.5 to $X - \{u\}$ to obtain $g\colon X' \to X$ and further to the inverse image $D' = C' \times_{X'} Y'$ in the normalization $f'\colon Y' \to X'$ in V to obtain $Y'' \to Y'$. Further applying Lemma 2.5 and replacing X' by a resolution of the quotient Y''/G, we may assume that D' is regular.

We set $C \times_X X' = C' + F$ where C' is the normalization of C and F is a divisor supported on the inverse image E of u. We regard E as a reduced divisor of X' and let $M \geqq 1$ denote an integer such that $(M-1)E \geqq F$. For a reduced Cartier divisor C_1 satisfying $C \equiv C_1 \bmod \mathfrak{m}_u^M$, there exists a Cartier divisor C_1' of X' such that $C_1 \times_X X' = C_1' + F$ by Lemma 2.2. Since $C \times_X X' \equiv C_1 \times_X X' \bmod \mathcal{I}_E^M$, we have $C' \times_{X'} E = C_1' \times_{X'} E$ and the divisor C_1' is reduced and meets E transversely. Hence, it is a normalization of C_1.

As we have shown above, there exists an integer $N' \geqq 1$ such that the congruence $C' \equiv C_2' \bmod \mathfrak{m}_{u'}^{N'}$ for a reduced Cartier divisor C_2' of X' and for each point $u' \in C' \cap E$ imply the equality (2.2) holds. Set $N = M + N'$ and let C' be a closed reduced subscheme of X satisfying $C \equiv C' \bmod \mathfrak{m}_u^N$. Then, we have $C' \equiv C_1' \bmod \mathcal{I}_E^{N'}$. Hence, we obtain the equality (2.2). $\square$

We show a continuity of the total dimension of the space of vanishing cycles.

Proposition 2.6. *Let* C *be a smooth curve over an algebraically closed field* k *of characteristic* p *and let* $g\colon Y \to C$ *be a smooth morphism of schemes over* k *of relative dimension* 1*. Let* $f\colon X \to Y$ *be a proper morphism of schemes over* k*. Let* Λ *be a finite field of characteristic* $\ell \neq p$ *and* *let $\mathcal{F}$ be a constructible sheaf of Λ-modules on X locally acyclic relatively to $X \to C$. Let $Z \subset X$ be a closed subscheme such that the restriction of $\mathcal{F}$ on $X - Z$ is universally locally acyclic relatively to $X \to Y$. Let B be a linear combination of divisors on Y flat over C supported on the closed subset $E = f(Z)$.*

Let u be a closed point of Z and set $v = f(u) \in E \subset Y$ and $s = g(v) \in C$. Assume that u is the unique point of the inverse image $Z \times_Y v$, that v is an isolated point of $E \cap Y_s$ and that $Z \to C$ is quasi-finite outside s. Assume also that, for every closed point $t \in C, t \neq s$ and for every point $y \in E_t$, we have

$$\sum_{z \in Z \times_Y y} \dim \operatorname{tot} \phi_z(\mathcal{F}|_{X_t}, f|_{X_t}) = (B, Y_t)_y. \tag{2.3}$$

Then, the equality (2.3) *holds also for $t = s$ and $y = v$.*

PROOF. By the assumptions that v is an isolated point of $E \cap Y_s$ and that Z is quasi-finite over C outside s, the closed subset $E \subset Y$ is quasi-finite over C on a neighborhood of v. Hence, by replacing C by an étale neighborhood of s and Y by an étale neighborhood of v, we may assume that E is finite over C and that v is the unique point of E above s. Then, Z is also finite over C and u is the unique point of Z above s. We define functions a and b on C by

$$a(t) = \sum_{z \in Z_t} \dim \operatorname{tot}_z \phi(\mathcal{F}|_{X_t}, f|_{X_t}), \quad b(t) = (B, Y_t).$$

By the assumption, we have $a = b$ on $C - \{s\}$. By the assumption that B is flat over C, the function b is constant on a neighborhood of s. Hence, it suffices to show that the function a is also constant on a neighborhood of s.

Let $t \in C$ be a closed point. The complex of nearby cycles $\phi(\mathcal{F}|_{X_t}, f|_{X_t})$ is supported on Z_t by the assumption that the restriction of $\mathcal{F}$ to $X - Z$ is universally locally acyclic relatively to $f\colon X \to Y$. For $y \in E_t$, let $\bar{\eta}_y$ be a geometric generic point of the strict localization of Y_t at y. Then, the distinguished triangle of vanishing cycles gives us a distinguished triangle

$$\to (Rf_*\mathcal{F}|_{Y_t})_y \to (Rf_*\mathcal{F}|_{Y_t})_{\bar{\eta}_y} \to \bigoplus_{z \in Z \times_Y y} \phi_z(\mathcal{F}|_{X_t}, f|_{X_t}) \to . \tag{2.4}$$

Hence $a(t)$ equals the sum of the Artin conductors $\sum_{y\in E_t} a_y(Rf_*\mathcal{F}|_{Y_t})$ defined as (1.8).

We prove

$$a(s) - a(t) = -\dim \phi_v(Rf_*\mathcal{F}, g) = 0. \tag{2.5}$$

for $t \in C - \{s\}$ to complete the proof. The first equality is a consequence of the lemma below. We show the vanishing $\phi_v(Rf_*\mathcal{F}, g) = 0$. Since $\mathcal{F}$ is assumed locally acyclic relatively to $X \to C$, the canonical morphism $\mathcal{F}_s \to \psi(\mathcal{F}, g\circ f)$ is an isomorphism on X_s. Since the formation of the nearby cycle complex is compatible with proper push-forward, it implies that the canonical morphism $Rf_*\mathcal{F}_s \to \psi(Rf_*\mathcal{F}, g)$ is an isomorphism. Thus, we obtain the required vanishing $\phi_v(Rf_*\mathcal{F}, g) = 0$ and the equality (2.5) is proved. □

LEMMA 2.7. *Let $g\colon X \to C$ be a smooth morphism over an algebraically closed field k of characteristic p from a smooth surface X to a smooth curve C. Let D be a divisor of X finite flat over C and s be a closed point of C such that the closed fiber D_s consists of a unique point x. Let Λ be a finite field of characteristic $\ell \neq p$ and let $\mathcal{K}$ be a constructible complex of Λ-modules on X such that the restriction $\mathcal{H}^q\mathcal{K}|_U$ of the cohomology sheaf on the complement $U = X - D$ is locally constant for every integer q.*

Then, on a neighborhood of s in C, the sum of the Artin conductors $\sum_{z\in D_t} a_z(\mathcal{K}|_{X_t})$ is constant except possibly for $s = t$ and satisfies

$$a_x(\mathcal{K}|_{X_s}) - \sum_{z\in D_t} a_z(\mathcal{K}|_{X_t}) = -\dim \phi_x(\mathcal{K}, g). \tag{2.6}$$

PROOF. By devissage, it suffices to consider the case where $\mathcal{K} = j_!\mathcal{F}$ for a locally constant constructible sheaf $\mathcal{F}$ on U and the open immersion $j\colon U \to X$ and the case where $\mathcal{K} = i_*\mathcal{G}$ for a constructible sheaf $\mathcal{G}$ on D and the closed immersion $i\colon D \to X$. The first case is [19, Théorème 5.1.1]. The second case follows from the exact sequence $0 \to \mathcal{G}_x \to \bigoplus_{z\in D_{\bar\eta}} \mathcal{G}_z \to \phi_x(i_*\mathcal{G}, g) \to 0$ where $\bar\eta$ denotes a geometric generic point of the strict localization S of C at s. □

2.2. An application of Elkik's theorem

To prove the stability of nearby cycles in the next subsection, we recall the following generalization of Hensel's lemma due to Elkik [10, Section 2], with slight reformulation. Let $S = \operatorname{Spec} R$ be an affine noetherian scheme and $Y = \operatorname{Spec} B$ be an affine scheme of finite type over S. By taking a finite presentation $B = R[T]/(f)$ where T denotes a system of indeterminates and f denotes a system of polynomials, a closed subscheme Z of Y is defined by the ideal $H_B = \sum K_{(\alpha)}\Delta_{(\alpha)} \subset R[T]$ in the notation [10, 0.2]. As noted there, the complement of the support of Z is the largest open subscheme of Y smooth over S and the ideal can only get larger by base change. Although Z depends on presentation, let $Z_{Y/S}$ denote it by abuse of notation.

LEMMA 2.8 ([10, Théorème 2]). *Let S be an affine noetherian scheme, $X = \operatorname{Spec} A$ be an affine noetherian scheme over S and let $J \subset A$ be an ideal such that the pair (A, J) is henselian* ([23, Chapitre IX Définition 3]). *For an integer $n \geqq 1$, set $X_n = \operatorname{Spec} A/J^n \subset X$. Let $h \geqq 0$ be an integer.*

Then, there exist integers $m \geqq r \geqq 0, m \geqq h$ such that, for any affine scheme Y over S of finite type and any morphism of schemes $\bar{f}\colon X_n \to Y$ over S for $n \geqq m$ satisfying $Z_{Y/S} \times_Y X_n \subset X_h$, there exists a morphism $f\colon X \to Y$ over S that makes the diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X_{n-r} \arrow[u, "\cup"] \arrow[r, "\subset"] & X_n \arrow[u, "\bar{f}"'] \end{tikzcd} \tag{2.7}$$

commutative.

PROOF. Since the ideal defining $Z_{Y/S}$ only get larger by base change as remarked in [10, 0.2], we may assume $X = S$ by taking the base change by $X \to S$.

In the notation of [10, Théorème 2], the condition $J(\mathbf{a}^0) \subset \mathcal{J}^n$ means that a morphism $X_n \to Y$ is defined. Further, under this condition and $h \leqq n$, the condition $H_B(\mathbf{a}^0) \supset \mathcal{J}^h$ means a closed immersion $Z \times_Y X_n \subset X_h$. Since the condition $J(\mathbf{a}^0) = 0$ means that a morphism $X \to Y$ is defined and since the congruence $\mathbf{a} \equiv \mathbf{a}^0 \bmod \mathcal{J}^{n-r}$ means the commutative diagram (2.7), the assertion follows by [10, Théorème 2]. $\square$

Proposition 2.9. *Let $f\colon X = \operatorname{Spec} A \to S$ be a morphism of finite type of* affine *noetherian schemes and X_1 be the closed subscheme defined by an ideal $I \subset A$. Assume that X is normal and that the complement $U = X - X_1$ is a dense open subscheme smooth over S. Let $\widetilde{X} = \operatorname{Spec} \tilde{A}$ be a henselization* ([23, Chapitre IX Définition 4]) *of X along X_1. Let $V \to U$ be a G-torsor for a finite group G and let Y be the normalization of X in V.*

Then, there exist integers $r \geqq 0$ and $N \geqq r+2$ such that for a morphism $g\colon X \to S$ satisfying $g \equiv f \bmod I^N$, there exist isomorphisms $\tilde{p}\colon \widetilde{X} \to \widetilde{X}$ and $\tilde{q}\colon \widetilde{Y} = Y \times_X \widetilde{X} \to \widetilde{Y}$ satisfying the following properties: The diagram

(2.8)
$$\begin{tikzcd}
\widetilde{Y} \arrow[rr, "\tilde{q}"] \arrow[d] & & \widetilde{Y} \arrow[d] \\
\widetilde{X} \arrow[rr, "\tilde{p}"] \arrow[dr, "\tilde{g}"'] & & \widetilde{X} \arrow[dl, "\tilde{f}"] \\
& S &
\end{tikzcd}$$

where $\tilde{f}$ and $\tilde{g}$ denote the composition with f and g is commutative and compatible with the G-actions. They are congruent to the identity modulo $I^{N-r}\mathcal{O}_{\widetilde{X}}$ and $I^{N-r}\mathcal{O}_{\widetilde{Y}}$ respectively.

Proof. For a scheme T over X, let T_f denote T regarded as a scheme over S with respect to the composition with $f\colon X \to S$ and similarly for T_g for a morphism $g\colon X \to S$. For an integer $n \geqq 1$ and for a scheme T over X, let $T_n \subset T$ denote the closed subscheme $T \times_X \operatorname{Spec} A/I^n$. The canonical morphism $\widetilde{X} \to X$ induces an isomorphism $\widetilde{X}_n \to X_n$ for $n \geqq 1$. If $g\colon X \to S$ satisfies $g \equiv f \bmod I^N$ and if $n \leqq N$, we have $T_{n,g} = T_{n,f}$ for a scheme T over X and we will drop the subscripts f and g in this case.

Let Z be the closed subscheme $Z_{X_f/S}$ of X. By the assumption that U is smooth over S, the intersection $Z \cap U$ is empty. Hence, there exists an integer $h \geqq 1$ such that $\mathcal{O}_Z$ is annihilated by I^h. Let $m \geqq r \geqq 0, m \geqq h$ be integers as in Lemma 2.8 for the henselian pair $(\tilde{A}, I\tilde{A})$.

Let N be an integer satisfying $N \geqq m$ and $N \geqq 2+r$. Let $g\colon X \to S$ be a morphism of schemes satisfying $g \equiv f \bmod I^N$. We apply Lemma 2.8 to the canonical immersion $\widetilde{X}_N = X_N \to X_f$ over S. For $N \geqq h$, the assumption

$Z \subset X_h$ on h implies that the assumption $Z \times_{X_f} X_N \subset X_h$ of Lemma 2.8 is satisfied. Hence applying Lemma 2.8 we obtain a commutative diagram

$$
\begin{array}{ccc}
\widetilde{X}_g & \xrightarrow{p} & X_f \\
\cup\uparrow & & \uparrow \\
\widetilde{X}_{N-r} & \xrightarrow{\subset} & \widetilde{X}_N
\end{array}
\tag{2.9}
$$

of schemes over S.

We show that the induced morphism $\tilde{p}\colon \widetilde{X}_g \to \widetilde{X}_f$ on the henselizations is an isomorphism. Let W be an étale neighborhood of $\widetilde{X}_1 \to X_g$ such that the composition $\widetilde{X}_g \xrightarrow{\tilde{p}} \widetilde{X}_f \to X_f$ is induced by $W \to X_f$. Since $\tilde{p}\colon \widetilde{X}_g \to \widetilde{X}_f$ induces the identity on $X_2 = \widetilde{X}_2 \subset \widetilde{X}_{N-r}$, the endomorphisms induced by $W \to X_f$ on the completions of the local rings of $\widetilde{X}_g$ and $\widetilde{X}_f$ at all points of $\widetilde{X}_1$ are surjections and hence automorphisms. Thus the induced endomorphisms on the henselizations of the local rings of $\widetilde{X}_g$ and $\widetilde{X}_f$ at all points of $\widetilde{X}_1$ are also automorphisms and the morphism $W \to X_f$ is étale on a neighborhood $W' \subset \widetilde{X}_g$ of $\widetilde{X}_1$. Hence, by replacing W by W', we may assume that $W \to \widetilde{X}_f$ itself is étale. Since $(\widetilde{X}, \widetilde{X}_1)$ is a henselian pair, the morphism $\tilde{p}\colon \widetilde{X}_g \to \widetilde{X}_f$ is an isomorphism.

The normalization Y in the étale covering $V \to U$ is finite over X. Let Z' be the closed subscheme $Z_{Y/X}$ of Y. The intersection $Z' \cap V$ is empty. Define an integer h' similarly as h above and let $m' \geqq r' \geqq 0, m' \geqq h'$ be integers defined for $\widetilde{Y}_f \to \widetilde{X}_f$ in Lemma 2.8. Then, by a similar argument as above for $Y_f \to X_f$, there exists an integer $N' \geqq N$ such that if $g \equiv f \bmod I^{N'}$, there exists an isomorphism $\tilde{q}\colon \widetilde{Y}_g \to \widetilde{Y}_f$ such that the diagram (2.8) is commutative.

We show that the morphism $\tilde{q}\colon \widetilde{Y}_g \to \widetilde{Y}_f$ is compatible with the action of G, after replacing N' by a larger integer if necessary. Since the set $\mathrm{Hom}_{\widetilde{X}}(\widetilde{Y}, \widetilde{Y})$ is finite and since the canonical morphism from the henselization to the completion is an injection, there exists an integer $n \geqq 1$ such that the restriction map $\mathrm{Hom}_{\widetilde{X}}(\widetilde{Y}, \widetilde{Y}) \to \mathrm{Hom}_{\widetilde{X}}(\widetilde{Y}_n, \widetilde{Y})$ is injective. Then, if $N \geqq m, N \geqq r+n$, both compositions in the diagram

$$\begin{tikzcd} \widetilde{Y}_g \arrow[d, "\sigma"'] & \widetilde{Y}_f \arrow[l, "\tilde{q}^{-1}"'] \arrow[d, "\sigma"] \\ \widetilde{Y}_g \arrow[r, "\tilde{q}"] & \widetilde{Y}_f \end{tikzcd} \tag{2.10}$$

for $\sigma \in G$ are the same after restricted to $\widetilde{Y}_n \subset \widetilde{Y}_{N-r}$. Hence the diagram (2.10) itself is commutative for $\sigma \in G$ and $\tilde{q}\colon \widetilde{Y}_g \to \widetilde{Y}_f$ is compatible with the action of G. □

2.3. Stability of nearby cycles

The stability of nearby cycles at an isolated singularity is an immediate consequence of Proposition 2.9.

PROPOSITION 2.10. *Let X be a scheme of finite type over a perfect field k of characteristic p, C be a smooth curve over k, and let $f\colon X \to C$ be a flat morphism over k. Let u be a closed point of X such that $U = X - \{u\}$ is smooth over C and $j\colon U \to X$ be the open immersion. Let Λ be a finite field of characteristic $\ell \neq p$ and $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on U.*

Then, there exists an integer $N \geqq 1$ such that, for a morphism $g\colon X \to C$ satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, there exists an isomorphism

$$R\psi_u(j_!\mathcal{F}, f) \to R\psi_u(j_!\mathcal{F}, g). \tag{2.11}$$

PROOF. Replacing X by the normalization, we may assume X is normal. Let $\widetilde{X}$ be the henselization of X at u and $\widetilde{\mathcal{F}}$ be the pull-back of $\mathcal{F}$ on $\widetilde{U} = U \times_X \widetilde{X}$. Then, by Proposition 2.9 applied to $X \to S$ and a finite Galois covering $V \to U$ trivializing $\mathcal{F}$, there exists an integer $N \geqq 1$ such that, for a morphism $g\colon X \to C$ satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, there exists an isomorphism $\tilde{p}\colon \widetilde{X}_g \to \widetilde{X}_f$ over C and an isomorphism $\tilde{p}^*\colon \widetilde{\mathcal{F}} \to \widetilde{\mathcal{F}}$. They induce an isomorphism (2.11). □

To prove the main result in this section, we show the vanishing of a certain limit of the spaces of vanishing cycles. We begin with the study of the limit of the local rings with respect to a sequence of blow-up.

LEMMA 2.11 ([1, Proposition 1.9.4], [12, 5.4]). *Let A be a local ring, $\mathfrak{p}$ be a prime ideal and let $f \in A$ be a non-zero divisor. Assume that $\bar{A} = A/\mathfrak{p}$ is a discrete valuation ring and that $\bar{f} \in \bar{A}$ is a uniformizer.*

1. *Let A' denote the subring $A[\mathfrak{p}/f^n; n \geqq 1] \subset A[1/f]$. Then, $\mathfrak{p}' = \mathfrak{p}A[1/f]$ is a prime ideal of A' and the canonical morphism $A/\mathfrak{p} \to A'/\mathfrak{p}'$ is an isomorphism. We have $f\mathfrak{p}' = \mathfrak{p}'$ and the ideal fA' is a maximal ideal of A'. The canonical morphism $A[1/f] \to A'[1/f]$ is an isomorphism.*

2. *Assume $f\mathfrak{p} = \mathfrak{p}$. Then the ring $A[1/f]$ equals the local ring $A_{\mathfrak{p}}$ and the canonical morphism $\mathfrak{p} \to \mathfrak{p}A_{\mathfrak{p}}$ is an isomorphism.*

3. *Assume that A is henselian and that $f\mathfrak{p} = \mathfrak{p}$. Then, the local ring $A_{\mathfrak{p}}$ is also henselian.*

We record a proof of 1. and 2. for the convenience of the reader.

PROOF. 1. By the commutative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{p} & \longrightarrow & A & \longrightarrow & \bar{A} & \longrightarrow & 0 \\
 & & \cap\big\downarrow & & \big\downarrow\cap & & \big\downarrow\cap & & \\
0 & \longrightarrow & \mathfrak{p}A[1/f] & \longrightarrow & A[1/f] & \longrightarrow & \bar{A}[1/\bar{f}] & \longrightarrow & 0,
\end{array}$$

the subring $A' = A + \mathfrak{p}A[1/f] \subset A[1/f]$ is the inverse image of $\bar{A} \subset \bar{A}[1/\bar{f}]$ by the surjection $A[1/f] \to \bar{A}[1/\bar{f}]$. Hence, we obtain an isomorphism $A'/\mathfrak{p}' \to \bar{A}$ and $\mathfrak{p}' = \mathfrak{p}A[1/f]$ is a prime ideal of A'. We have $f\mathfrak{p}' = f\mathfrak{p}A[1/f] = \mathfrak{p}A[1/f] = \mathfrak{p}'$. Since $A'/fA' = (A'/\mathfrak{p}')/(f) = \bar{A}/\bar{f}\bar{A}$ is the residue field of A, the ideal fA' is a maximal ideal of A'. The inclusions $A \to A' \to A[1/f]$ imply an isomorphism $A[1/f] \to A'[1/f]$.

2. We show that $A[1/f]$ is a local ring and that its maximal ideal is $\mathfrak{p}A[1/f]$. Let $g \in A$ and $n \geqq 0$ be such that $g/f^n \in A[1/f]$ is not in $\mathfrak{p}A[1/f] = \operatorname{Ker}(A[1/f] \to \bar{A}[1/\bar{f}])$. Since g is not contained in $\mathfrak{p}$ and $\bar{f}$ is a uniformizer of $\bar{A}$, it is of the form $g = uf^m + b$ for $u \in A^\times, m \geqq 0, b \in \mathfrak{p} = f\mathfrak{p}$. Writing $b = f^m c$ for $c \in \mathfrak{p}$, we obtain $g = f^m(u + c)$ and $u + c \in A$ is invertible. Hence $A[1/f]$ is a local ring and is equal to $A_{\mathfrak{p}}$.

Since $f\mathfrak{p} = \mathfrak{p}$ and $A[1/f] = A_{\mathfrak{p}}$, we have $\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}A[1/f] = \mathfrak{p}$.

3. Let $\tilde{B}$ be the local ring of an étale algebra over $A_{\mathfrak{p}}$ at a maximal ideal above $\mathfrak{p}$ such that the residue field is isomorphic to the residue field $\kappa(\mathfrak{p})$ of the local ring $A_{\mathfrak{p}}$. We show that the canonical morphism $A_{\mathfrak{p}} \to \tilde{B}$ is an isomorphism.

By Zariski's main theorem, there exist a finite A-algebra B and an isomorphism $B_{\mathfrak{q}} \to \tilde{B}$ from the localization at a prime ideal $\mathfrak{q}$ of B above $\mathfrak{p}$. By replacing B by the quotient by the $\mathfrak{p}$-torsion part, we may assume that the canonical morphism $B \to B \otimes_A A_{\mathfrak{p}}$ is an injection. The finite $\kappa(\mathfrak{p})$-algebra $B \otimes_A \kappa(\mathfrak{p})$ is decomposed as $\kappa(\mathfrak{q}) \times C$. We identify the residue field $\kappa(\mathfrak{q})$ with $\kappa(\mathfrak{p})$ by the canonical isomorphism.

The image of B in $\kappa(\mathfrak{q}) = \kappa(\mathfrak{p})$ equals $A/\mathfrak{p}$ since B is finite over A and $A/\mathfrak{p}$ is normal. Let $\bar{B}_1$ be the image of B in C. Define a subring $B' \subset B \otimes_A A_{\mathfrak{p}}$ containing B as a subring to be the inverse image by the canonical surjection $B \otimes_A A_{\mathfrak{p}} \to B \otimes_A \kappa(\mathfrak{p})$ of $\bar{B}' = A/\mathfrak{p} \times \bar{B}_1 \subset \kappa(\mathfrak{q}) \times C = B \otimes_A \kappa(\mathfrak{p})$. Since the kernel of the surjection $B \otimes_A A_{\mathfrak{p}} \to B \otimes_A \kappa(\mathfrak{p})$ is the image of $B \otimes_A \mathfrak{p}A_{\mathfrak{p}}$ and is contained in B by 2, the cokernel of the injection $B \to B'$ is isomorphic to the cokernel of $B/\mathfrak{p}B \to \bar{B}' = A/\mathfrak{p} \times \bar{B}_1$ and is of finite length as an $A/\mathfrak{p}$-module. Hence B' is also finite over A and the canonical morphism $B \otimes_A A_{\mathfrak{p}} \to B' \otimes_A A_{\mathfrak{p}}$ is an isomorphism.

Since A is henselian, the finite A-algebra B' is the product of local rings. Thus, replacing B by the factor of B' whose spectrum contains $\mathfrak{q}$, we may assume that $\kappa(\mathfrak{p}) \to B \otimes_A \kappa(\mathfrak{p})$ is an isomorphism. Then, the morphism $A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ of finite étale $A_{\mathfrak{p}}$-algebras is an isomorphism. □

Proposition 2.12. *Let S be the spectrum of a discrete valuation ring and X be a scheme of finite type over S. Let D be a closed regular integral subscheme of X finite and flat over S and let E be a Cartier divisor of X meeting D transversely. Let x be a closed point of $D \cap E$. For $n \geqq 1$, let $X_n \to X$ denote the blow-up at $D \cap nE$, let x_n be the closed point above x of the proper transform of D and let $\bar{x}_n$ be a geometric point of X_n above x_n.*

Let Λ be a finite field of characteristic ℓ invertible on S and $\mathcal{F}$ be a constructible sheaf of Λ-modules on X. Then, for an integer $q > 0$, the inductive limit $\varinjlim_n R^q\psi_{\bar{x}_n}\mathcal{F}$ is zero.

Proof. Replacing S by a strict localization, we may assume S is strictly local. Let A denote the local ring $\mathcal{O}_{X,x}$, $\mathfrak{p} \subset A$ be the prime ideal defined by D and $f \in A$ be a function defining the divisor E on a neighborhood of x. Then, $\varprojlim_n X_{n,\bar{x}_n}$ is Spec A'^h of the strict localization A'^h of $A' = A[\mathfrak{p}/f^n; n \geqq 1]$ at the maximal ideal fA' in the notation of Lemma 2.11.1. By Lemma 2.11.1, the ideal $\mathfrak{p}'^h = \mathfrak{p}A'^h$ satisfies $f\mathfrak{p}'^h = \mathfrak{p}'^h$.

Let η and $\bar{\eta}$ denote the generic point of S and the point defined by a separable closure. By Lemma 2.11.1 and 2., the complement $\widetilde{X}\times_X(X-E)=\operatorname{Spec} A'^h[1/f]\subset\widetilde{X}=\operatorname{Spec} A'^h$ of the inverse image of E equals the spectrum $\operatorname{Spec} A'^h_{\mathfrak{p}'^h}$ of the localization of A'^h at the prime ideal $\mathfrak{p}'^h$ and hence is contained in the generic fiber $\widetilde{X}\times_S\eta$. Further, since the underlying set of the inverse image $\widetilde{X}\times_X E$ consists of the unique closed point and is contained in the closed fiber, the generic fiber $\widetilde{X}\times_S\eta$ equals $\operatorname{Spec} A'^h_{\mathfrak{p}'^h}$ and is henselian by Lemma 2.11.3. The residue field of $\mathfrak{p}'^h$ is a finite extension of the function field of S. Hence $\widetilde{X}\times_S\bar{\eta}$ is a finite disjoint union of strictly local schemes and $\varinjlim_n R^q\psi_{\bar{x}_n}\mathcal{F}=H^q(\widetilde{X}\times_S\bar{\eta},\mathcal{F})$ is zero for $q>0$. $\square$

LEMMA 2.13. *Let S be the spectrum of an excellent discrete valuation ring, X be a normal flat scheme of finite type over S of relative dimension 1. Let $D\subset X$ be a reduced closed subscheme of X finite and flat over S and let $j\colon U=X-D\to X$ denote the open immersion.*

Let $X'\to X$ be a proper birational morphism as in Lemma 2.5 such that the proper transform $D'\subset X'$ of D is regular and meets the reduced part E of the closed fiber X'_s transversely.

For $n\geqq 1$, let $X_n\to X'$ be the blow-up at $nE\cap D'$ and let $D_n\subset X_n$ denote the proper transform of D'. Then for a geometric point $\bar{x}$ of $D_{\bar{s}}$ and for a locally constant sheaf $\mathcal{F}$ of Λ-modules on U_K, the canonical mapping

$$H^1_c((X_n-D_n)\times_X\bar{x},R\psi j_!\mathcal{F})\to R^1\psi_{\bar{x}}j_!\mathcal{F}\tag{2.12}$$

is injective. Further, there exists an integer $m\geqq 1$ such that, for every $n\geqq m$, the canonical mapping (2.12) is an isomorphism.

PROOF. The canonical morphism $H^1(X_n\times_X\bar{x},R\psi j_!\mathcal{F})\to R^1\psi_{\bar{x}}j_!\mathcal{F}$ is an isomorphism by the proper base change theorem. Hence, the injectivity follows from the exact sequence

$$\begin{aligned}\bigoplus_{\bar{x}'\in D_n\times_X\bar{x}}R^0\psi_{\bar{x}'}j_!\mathcal{F}&\to H^1_c((X_n-D_n)\times_X\bar{x},R\psi j_!\mathcal{F})\\&\to R^1\psi_{\bar{x}}j_!\mathcal{F}\to\bigoplus_{\bar{x}'\in D_n\times_X\bar{x}}R^1\psi_{\bar{x}'}j_!\mathcal{F}\end{aligned}\tag{2.13}$$

and $R^0\psi_{\bar{x}'}j_!\mathcal{F}=0$ for $\bar{x}'\in D_n\times_X\bar{x}$. By Proposition 2.12, the inductive limit $\varinjlim_n$ of the last term in (2.13) is zero. Since $R^1\psi_{\bar{x}}j_!\mathcal{F}$ is of finite dimension, there exists an integer $m \geqq 1$ such that the last map in (2.13) is the zero-map for $n \geqq m$. Hence, for $n \geqq m$, the second arrow in the exact sequence (2.13) is an isomorphism. $\square$

We prove the following stability of nearby cycles for a fibration from a surface to a curve. A similar stability is proved by Laumon in [21, Théorème 6.1.4] in arbitrary dimension, under the assumption that the normalization of a covering trivializing the sheaf has an isolated singularity.

Theorem 2.14. *Let X be a normal surface and C be a smooth curve over a perfect field k of characteristic p, and let $f\colon X \to C$ be a flat morphism over k. Let D be a closed subscheme of X and $j\colon U = X - D \to X$ be the open immersion. Let Λ be a finite field of characteristic $\ell \neq p$ and $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on U.*

Let u be a closed point of X such that u is an isolated characteristic point *of $f\colon X \to C$ with respect to $j_!\mathcal{F}$ and that $D - \{u\}$ is* étale *over C.*

1. *There exists an integer $N \geqq 1$ such that, for a morphism $g\colon X \to C$ satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, we have an equality*

$$\dim R^1\psi_u(j_!\mathcal{F}, f) = \dim R^1\psi_u(j_!\mathcal{F}, g). \tag{2.14}$$

2. *There exists an integer $N \geqq 1$ such that, for a morphism $g\colon X \to C$ satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, there exists an isomorphism*

$$R^1\psi_u(j_!\mathcal{F}, f) \to R^1\psi_u(j_!\mathcal{F}, g). \tag{2.15}$$

Proof. By Proposition 2.10, it suffices to prove the case where u is in the closure of $D - \{u\}$. By shrinking X, we may assume D is flat over C and u is the unique point of the fiber of $D \to C$.

1. First, we deduce the case where $\mathcal{F} = \Lambda_U$ from Proposition 2.10. Let $i\colon D \to X$ denote the closed immersion. By the exact sequence $0 \to j_!\Lambda_U \to \Lambda_X \to i_*\Lambda_D \to 0$ and $R^q\psi_u(j_!\Lambda_U, f) = 0$ for $q \neq 1$, we have an equality

$$-\dim R^1\psi_u(j_!\Lambda_U, f) = \dim R\psi_u(\Lambda_X, f) - \dim R\psi_u(\Lambda_D, f|_D). \tag{2.16}$$

By the assumption, $X - \{u\} \to C$ is smooth and $D - \{u\} \to C$ is étale. Hence, by Proposition 2.10, there exists an integer $N \geqq 1$ such that, for a morphism $g\colon X \to C$ satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, we have isomorphisms $R\psi_u(\Lambda_X, f) \to R\psi_u(\Lambda_X, g)$ and $R\psi_u(\Lambda_D, f|_D) \to R\psi_u(\Lambda_D, g|_D)$. Further, we have an equality (2.16) with f replaced by g by Proposition 2.1 and by the same argument as above. Hence the equality (2.14) holds for $\mathcal{F} = \Lambda_U$.

We prove the general case. Let $\mathcal{F}_0 = \mathcal{F} - \operatorname{rank} \mathcal{F} \cdot \Lambda_U$ denote the virtual difference. Let S denote the strict localization of C at a geometric point $\bar{s}$ above $f(u)$ and let $\bar{\eta}$ be a geometric point of S defined by an algebraic closure of the fraction field. Then, we have

$$\begin{aligned}(2.17)\qquad &\dim R^1\psi_u(j_!\mathcal{F}, f) - \operatorname{rank} \mathcal{F} \cdot \dim R^1\psi_u(j_!\Lambda_U, f)\\ &= \sum_{x \in (D,f)\times_S \bar{\eta}} \dim \operatorname{tot}_x(j_!\mathcal{F}_0|_{(U,f)\times_S \bar{\eta}}) - \dim \operatorname{tot}_u(j_!\mathcal{F}_0|_{(U,f)\times_S \bar{s}})\end{aligned}$$

and similarly for $\dim R^1\psi_u(j_!\mathcal{F}, g)$ by [19, Théorème 5.1.1], [17, Theorem (6.7)], [14, Theorem 11.9].

Let $N \geqq 1$ be an integer satisfying the conditions in Propositions 2.1 and 2.4 and in the first part of this proof and let g be a morphism satisfying $g \equiv f \bmod \mathfrak{m}_u^N$. Then, since u is an isolated characteristic point of g, the fiber of g containing u is a reduced Cartier divisor on a neighborhood of u. Hence by Propositions 2.1 and 2.4 and by what we have proved above, we have

$$\begin{aligned}(2.18)\qquad &\sum_{x \in (D,f)\times_S \bar{\eta}} \dim \operatorname{tot}_x(j_!\mathcal{F}|_{(U,f)\times_S \bar{\eta}})\\ &\qquad = \sum_{x \in (D,g)\times_S \bar{\eta}} \dim \operatorname{tot}_x(j_!\mathcal{F}|_{(U,g)\times_S \bar{\eta}}),\end{aligned}$$

$$(2.19)\qquad \dim \operatorname{tot}_u(j_!\mathcal{F}|_{(U,f)\times_S \bar{s}}) = \dim \operatorname{tot}_u(j_!\mathcal{F}|_{(U,g)\times_S \bar{s}})$$

respectively and the equality (2.14) for $\mathcal{F} = \Lambda_U$. By (2.17), they imply the equality (2.14).

2. By Lemma 2.13, there exists an integer $n \geqq 0$ such that the morphism

$$(2.20)\qquad H^1_c((X_n - D_n) \times_X \bar{u}, R\psi(j_!\mathcal{F}, f)) \to R^1\psi_u(j_!\mathcal{F}, f)$$

is an isomorphism in the notation loc. cit. Recall that X' is regular and an elementary computation show that the blow-up X_n is normal. Changing the notation, let X' and D' denote X_n and D_n. Further by Lemma 2.13, the canonical morphism

$$H^1_c((X' - D') \times_X \bar{u}, R\psi(j_!\mathcal{F}, g)) \to R^1\psi_u(j_!\mathcal{F}, g) \tag{2.21}$$

is an injection. Thus, by 1., it suffices to show that there exists an integer $N \geqq 1$ such that for a morphism $g\colon X \to C$ satisfying $g \equiv f \bmod \mathfrak{m}_u^N$, there exists an isomorphism

$$\begin{aligned} &H^1_c((X' - D') \times_X \bar{u}, R\psi(j_!\mathcal{F}, f)) \\ &\qquad \to H^1_c((X' - D') \times_X \bar{u}, R\psi(j_!\mathcal{F}, g)). \end{aligned} \tag{2.22}$$

Since $X' - D'$ may not be affine, we cannot immediately apply Proposition 2.9. By shrinking C and X, we may assume that C and X are affine. To apply Proposition 2.9, we use a contraction $X' \to X'' \to X$ satisfying the following property, constructed in the proof of [11, Lemme A]: The morphism $\varphi\colon X'' \to X$ is proper, its restriction $X'' - \varphi^{-1}(u) \to X - \{u\}$ is an isomorphism and the morphism $X' \to X''$ contracts exactly those components of $\pi^{-1}(u)$ not meeting D'. Referring to loc. cit. for the detail of the construction, we briefly sketch it here. Let $\pi\colon X' \to X$ be the canonical morphism. Take an integer $m \geqq 1$ such that $\pi^*\pi_*\mathcal{O}_{X'}(mD') \to \mathcal{O}_{X'}(mD')$ is surjective. Then $X' \to X''$ is defined as the Stein factorization of $X' \to \mathrm{Proj}_X \bigoplus_{m \geqq 0} \pi_*\mathcal{O}_{X'}(mD')$. Since X' is normal, its contraction X'' is also normal.

Since $X' \to X''$ is an isomorphism on a neighborhood of D', we identify D' as a divisor of X''. Then, by the proper base change theorem, the canonical morphism $H^1_c((X'' - D') \times_X \bar{u}, R\psi(j_!\mathcal{F}, f)) \to H^1_c((X' - D') \times_X \bar{u}, R\psi(j_!\mathcal{F}, f))$ is an isomorphism and the same for g. Thus to define an isomorphism (2.22), we may replace X' by X''.

By the assumption that $f\colon X - \{u\} \to C$ is smooth, the restriction of f to U is smooth. Hence, the restriction of f to the complement $(X'' - D') - \varphi^{-1}(u)$ is also smooth. The divisor D' is φ-ample and the complement $X'' - D'$ is a scheme affine over X and hence is an affine scheme. Let $V \to U$ be a G-torsor for a finite group G such that the pull-back of $\mathcal{F}$ on V is constant.

We apply Proposition 2.9 to the composition $X'' - D' \to X \to C$ of morphisms of affine schemes and to the pull-back of the G-torsor $V \to U$. Let $\widetilde{X}''$ be the henselization of $X'' - D'$ along the inverse image $\varphi^{-1}(u)$, let $\tilde{j}\colon U \times_X \widetilde{X}'' \to \widetilde{X}''$ be the open immersion and let $\widetilde{\mathcal{F}}$ be the pull-back of $\mathcal{F}$. Let $\widetilde{X}''_f$ and $\widetilde{X}''_g$ denote the scheme $\widetilde{X}''$ regarded as schemes over C with respect to the compositions of $\widetilde{X}'' \to X$ with $f\colon X \to C$ and $g\colon X \to C$ respectively, as in the proof of Proposition 2.9. Then, we obtain an isomorphism $h\colon \widetilde{X}''_g \to \widetilde{X}''_f$ over C together with an isomorphism $h^*(\tilde{j}_!\widetilde{\mathcal{F}}) \to \tilde{j}_!\widetilde{\mathcal{F}}$ on $\widetilde{X}''_g$.

Since the henselization $\widetilde{X}'' \to X'' - D'$ is defined as the limit of étale neighborhoods of $\varphi^{-1}(u)$, the restriction to $\varphi^{-1}(u)$ of the nearby cycle complex of $\tilde{j}_!\widetilde{\mathcal{F}}$ with respect to the morphism $\widetilde{X}'' \to C$ equals the restriction of the nearby cycle complex of $j_!\mathcal{F}$ with respect to $X'' - D' \to C$. Hence the isomorphisms $h\colon \widetilde{X}''_g \to \widetilde{X}''_f$ over C and $h^*(\tilde{j}_!\widetilde{\mathcal{F}}) \to \tilde{j}_!\widetilde{\mathcal{F}}$ on $\widetilde{X}''_g$ induce an isomorphism (2.22) with X' replaced by X'' as required. $\square$

3. Radon Transform and the Characteristic Cycle

3.1. Preliminaries on the universal family of hyperplane sections

For the formalism of dual variety, we refer to [18]. Let X be a normal projective irreducible scheme over an algebraically closed field k of characteristic $p > 0$ and let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module. Let

$$X \to \mathbf{P} = \mathbf{P}(E^\vee) = \mathrm{Proj}_k S^\bullet E$$

be the closed immersion defined by $\mathcal{L}$ to the projective space associated to the dual $E^\vee$ of the k-vector space $E = \Gamma(X, \mathcal{L})$. We use an anti-Grothendieck notation to denote a projective space $\mathbf{P}(E)(k) = (E - \{0\})/k^\times$.

Let $\mathbf{P}^\vee = \mathbf{P}(E)$ be the dual of $\mathbf{P}$. The universal hyperplane $\mathbf{H} = \{(x, H) \mid x \in H\} \subset \mathbf{P} \times \mathbf{P}^\vee$ is defined by the identity $\mathrm{id} \in \mathrm{End}(E)$ regarded as a section $F \in \Gamma(\mathbf{P} \times \mathbf{P}^\vee, \mathcal{O}(1,1)) = E \otimes E^\vee$. By the canonical injection $\Omega^1_{\mathbf{P}/k}(1) \to E \otimes \mathcal{O}_{\mathbf{P}}$, the universal hyperplane $\mathbf{H}$ is identified with the covariant projective space bundle $\mathbf{P}(T^*\mathbf{P})$ associated to the cotangent bundle $T^*\mathbf{P}$. Further, the identity of $\mathbf{H}$ is the same as the map $\mathbf{H} = \mathbf{P}(T^*_{\mathbf{H}}(\mathbf{P} \times \mathbf{P}^\vee)) \to \mathbf{H} = \mathbf{P}(T^*\mathbf{P})$ induced by the locally splitting injection $\mathcal{N}_{\mathbf{H}/\mathbf{P}\times\mathbf{P}^\vee} \to \mathrm{pr}_1^*\Omega^1_{\mathbf{P}/k}$.

The fibered product $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is the intersection of $X \times \mathbf{P}^\vee$ with $\mathbf{H}$ in $\mathbf{P} \times \mathbf{P}^\vee$ and is the universal family of hyperplane sections. We consider the universal family of hyperplane sections $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$.

Assume that X is smooth of dimension d and $X \subsetneqq \mathbf{P}$. We say that a reduced closed subscheme $T \subset T^*X$ is a conic subscheme if it is stable under the multiplication by scalars. For a reduced closed conic subscheme $T \subset T^*X$, we define a reduced subscheme

$$P(T) \subset X \times_{\mathbf{P}} \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P}) \tag{3.1}$$

as follows. First, we consider the inverse image $\widetilde{T}$ by the canonical surjection $X \times_{\mathbf{P}} T^*\mathbf{P} \to T^*X$ and its restriction to the complement $X \times_{\mathbf{P}} (T^*\mathbf{P} - T^*_{\mathbf{P}}\mathbf{P}) \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ of the 0-section. Then, $P(T)$ is defined to be the unique reduced closed subscheme of $X \times_{\mathbf{P}} \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ such that its pull-back by the canonical projection $X \times_{\mathbf{P}} (T^*\mathbf{P} - T^*_{\mathbf{P}}\mathbf{P}) \to \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ is equal to the restriction to the complement of the 0-section. Alternatively, the conic closed subscheme $\widetilde{T} \subset X \times_{\mathbf{P}} T^*\mathbf{P}$ is defined by a graded ideal of the graded $\mathcal{O}_X$-algebra $\mathcal{O}_{X \times_{\mathbf{P}} T^*\mathbf{P}}$ and $P(T) \subset \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ is defined as Proj of the quotient graded algebra.

Assume that a reduced closed conic subscheme T of T^*X is of codimension $d = \dim X$. Then, since $X \times_{\mathbf{P}} T^*\mathbf{P} \to T^*X$ is a surjection of vector bundles, the projectivization $P(T) \subset X \times_{\mathbf{P}} \mathbf{H}$ of the inverse image is also of codimension d. Since $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is of relative dimension $d-1$, the image $p(P(T)) \subset \mathbf{P}^\vee$ is of codimension at least 1.

LEMMA 3.1. *Let X be a projective smooth scheme of dimension d over k and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module.*

1. *Assume that $\mathcal{L}$ is very ample and satisfies the following condition:*

(L) *For every pair of distinct closed points $u \neq v$ of X, the canonical mapping*

$$E = \Gamma(X, \mathcal{L}) \to \mathcal{L}_u/\mathfrak{m}_u^2\mathcal{L}_u \oplus \mathcal{L}_v/\mathfrak{m}_v^2\mathcal{L}_v \tag{3.2}$$

is a surjection.

Then, for an irreducible closed conic subscheme $T \subset T^*X$ of codimension $d = \dim X$, either the morphism $P(T) \to p(P(T))$ induced by $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is generically radicial or $p(P(T)) \subset \mathbf{P}^\vee$ is of codimension $\geqq 2$. For another irreducible closed conic subscheme $T' \subset T^*X$ of codimension d different from T, the intersection $p(P(T)) \cap p(P(T')) \subset \mathbf{P}^\vee$ is of codimension $\geqq 2$.

2. *There exists an integer m such that $\mathcal{L}^{\otimes n}$ is very ample and satisfies the condition* (L) *for every $n \geqq m$.*

Proof. 1. Let $\mathcal{I}_\Delta \subset \mathcal{O}_{X\times X}$ denote the ideal sheaf defining the diagonal immersion $\Delta\colon X \to X \times X$. Let $Z \subset X \times X$ be the closed subscheme defined by $\mathcal{I}_\Delta^2$ and $p_1, p_2\colon Z \to X$ be the restriction of the projections. Define a vector bundle V over X and a line bundle L associated to a locally free $\mathcal{O}_X$-module $\tilde{\mathcal{L}} = p_{1*}p_2^*\mathcal{L}$ of rank $d+1$ and the invertible $\mathcal{O}_X$-module $\mathcal{L}$ respectively. The canonical isomorphism $\mathcal{L} \otimes \Omega^1_X \to \mathcal{L} \otimes (\mathcal{I}_\Delta/\mathcal{I}_\Delta^2) \subset \tilde{\mathcal{L}}$ induces an injection

$$T^*X \otimes L \to V \tag{3.3}$$

of vector bundles. The cokernel of (3.3) is the line bundle L.

The linear morphism

$$E \times (X \times X)^\circ \to V \times_X (X \times X)^\circ \times_X V \tag{3.4}$$

of vector bundles on $(X \times X)^\circ = X \times X - \Delta_X$ defined by $\Gamma(X, \mathcal{L}) \otimes \mathcal{O}_{(X\times X)^\circ} \to \mathrm{pr}_1^*\tilde{\mathcal{L}} \oplus \mathrm{pr}_2^*\tilde{\mathcal{L}}$ sends a closed point (l, u, v) to $(l \bmod \mathfrak{m}_u^2\mathcal{L}_u, l \bmod \mathfrak{m}_v^2\mathcal{L}_v)$ in the fiber of (u, v). The condition (L) means that (3.4) is a surjection.

The images of twists by L of $T, T' \subset T^*X$ by (3.3) define closed subschemes $T \otimes L$ and $T' \otimes L$ of V. Then, the intersection of the pull-backs of $T \otimes L$ and $T' \otimes L$ by the projections defines a closed subscheme of $V \times_X (X \times X)^\circ \times_X V$. Pulling it back by (3.4) and applying the construction similar to the definition of $P(T)$ in (3.1), we define a closed subscheme $R(T, T')$ of $\mathbf{P}^\vee \times (X \times X)^\circ$. It consists of triples (H, u, v) of a hyperplane containing points $u \neq v$ such that $(u, H), (v, H) \in \mathbf{H}$ are contained in $P(T)$ and $P(T')$ respectively. Since $T, T' \subset T^*X$ are of codimension d and $T^*X \otimes L \subset V$ is of codimension 1, the intersection of their pull-backs in $V \times_X (X \times X)^\circ \times_X V$ is of codimension $2(d+1)$. Hence the codimension of $R(T, T') \subset \mathbf{P}^\vee \times (X \times X)^\circ$ is $2(d+1)$. Hence, the codimension of the closure of its image $S(T, T') \subset \mathbf{P}^\vee$ by the projection is at least $2(d+1) - 2d = 2$.

Assume $T = T'$ and let $H \in \mathbf{P}^\vee$ be a hyperplane not contained in $S(T, T)$. Then, there exists no two distinct points $u \neq v$ in X such that both (u, H) and (v, H) are contained in $P(T)$. In other words, the restriction $P(T) \to \mathbf{P}^\vee$ of $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is radicial outside $S(T, T)$.

Assume $T \neq T'$ and let $H \in \mathbf{P}^\vee$ be a hyperplane not contained in $S(T,T')$. Then, there exists no two distinct points $u \neq v$ such that (u,H) is in $P(T)$ and (v,H) is in $P(T')$. In other words, the intersection $p(P(T)) \cap p(P(T')) \subset \mathbf{P}^\vee$ is contained in the union of $S(T,T')$ and the image $p(P(T) \cap P(T'))$. By the assumption that T and T' have no common irreducible components, the intersection $P(T) \cap P(T') \subset X \times_{\mathbf{P}} \mathbf{H}$ is of codimension $\geqq d+1$. Since $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is smooth of relative dimension $d-1$, the image $p(P(T) \cap P(T')) \subset \mathbf{P}^\vee$ is of codimension $\geqq (d+1)-(d-1) = 2$. Hence the assertion follows.

2. Define $Z \subset X \times X$ as in the proof of 1 and let $S = X \times X - \Delta_X$ be the complement of the diagonal. Then, it suffices to apply the following Lemma to the proper flat scheme $X \times S$ over S and to the closed subscheme $\mathrm{pr}_{12}^* Z \amalg \mathrm{pr}_{13}^* Z \subset X \times S$ where $\mathrm{pr}_{1j}\colon X \times S \to X \times X$ denote the restrictions of the projections $\mathrm{pr}_{1j}\colon X \times X \times X \to X \times X$. □

Lemma 3.2. *Let S be a noetherian scheme, $f\colon X \to S$ be a proper flat scheme over S and $\mathcal{L}$ be an f-ample invertible $\mathcal{O}_X$-module. For a closed subscheme Z of X flat over S, there exists an integer m such that for every $n \geqq m$ and for every point $s \in S$, the restriction*

$$\Gamma(X_s, \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{X_s}) \to \Gamma(Z_s, \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{Z_s}) \tag{3.5}$$

is a surjection.

Proof. Let $\mathcal{I}_Z \subset \mathcal{O}_X$ be the ideal sheaf defining Z. Since $\mathcal{L}$ is f-ample, there exists an integer m such that for every $n \geqq m$ and for every $q > 0$, we have $R^q f_* \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n} = 0$. For $n \geqq m$, the spectral sequence $E_2^{p,q} = Tor_{-p}^{\mathcal{O}_X}(R^q f_* \mathcal{I}_Z \otimes \mathcal{L}^{\otimes n}, k(s)) \Rightarrow H^{p+q}(X_s, \mathcal{I}_{Z_s} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{X_s})$ implies the vanishing $H^1(X_s, \mathcal{I}_{Z_s} \otimes \mathcal{L}^{\otimes n} \otimes \mathcal{O}_{X_s}) = 0$ and (3.5) is a surjection. □

The following lemma will be used to show the existence of a pencil defining a fibration close to f.

Lemma 3.3. *Let X be a projective normal scheme of dimension d over k and $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let u be a closed point of X such that $X^\circ = X - \{u\}$ is smooth and let $N \geqq 1$ be an integer.*

1. *Assume that $\mathcal{L}$ is very ample and satisfies the condition:*

(N) *For every closed point $x \in X, \neq u$, the canonical morphism*

$$E = \Gamma(X, \mathcal{L}) \to \mathcal{L}_u/\mathfrak{m}_u^N \mathcal{L}_u \oplus \mathcal{L}_x/\mathfrak{m}_x^2 \mathcal{L}_x \tag{3.6}$$

is a surjection.

Let $l_\infty \in E = \Gamma(X, \mathcal{L})$ be a non-zero section such that the hyperplane section X_∞ defined by l_∞ does not contain u. Let $T_1, \ldots, T_m$ be a finite family of closed conic irreducible subschemes of T^*X° of codimension d.

Then, for $f \in \mathfrak{m}_u/\mathfrak{m}_u^N$, there exists a non-zero section $l \in E = \Gamma(X, \mathcal{L})$ such that $l \neq l_\infty$,

$$l/l_\infty \equiv f \bmod \mathfrak{m}_u^N \tag{3.7}$$

and that the hyperplane section X_0 of X defined by $l = 0$ satisfies the following condition: The complement $X_0^\circ = X_0 - \{u\} \subset X^\circ$ is smooth and, for every $i = 1, \ldots, m$, the intersection $T^*_{X_0^\circ}X^\circ \cap T_i$ with the conormal bundle is contained in the 0-section.

2. *There exists an integer $m \geqq 0$ such that $\mathcal{L}^{\otimes n}$ is very ample and satisfies the condition* (N) *for every $n \geqq m$.*

PROOF. 1. We regard the k-vector space $W = \mathcal{O}_{X,u}/\mathfrak{m}_u^N$ as an affine space over k and let $E_f \subset E = \Gamma(X, \mathcal{L})$ denote the inverse image of f mod $\mathfrak{m}_u^N$ by the surjection $E = \Gamma(X, \mathcal{L}) \to W = \mathcal{O}_{X,u}/\mathfrak{m}_u^N$ sending l to l/l_∞ mod $\mathfrak{m}_u^N$.

We define a closed subscheme $Z \subset X^\circ \times X^\circ$, a vector bundle V of rank $d+1$ over X° and an injection $T^*X^\circ \otimes L \to V$ (3.3) of vector bundles of codimension 1 on X° similarly as in the proof of Lemma 3.1.1. We consider the pull-back $E \otimes \mathcal{O}_Z \to p_2^*\mathcal{L}$ of the canonical morphism $E \otimes \mathcal{O}_{X^\circ} \to \mathcal{L}$. Since $E \otimes \mathcal{O}_Z = p_1^*(E \otimes \mathcal{O}_{X^\circ})$, it induces $E \otimes \mathcal{O}_{X^\circ} \to p_{1*}p_2^*\mathcal{L}$ by adjunction and hence $E \times X^\circ \to V$. We define a linear morphism

$$E \times X^\circ \to V \times W \tag{3.8}$$

of vector bundles on X° to be its product with the canonical morphism $E \to W$. It maps a closed point (l, x) to $(l \bmod \mathfrak{m}_x^2\mathcal{L}_x, l \bmod \mathfrak{m}_u^N\mathcal{L}_u)$ in the fiber of x. The condition (N) means that (3.8) is a surjection. For a closed point $(l, x) \in E \times X^\circ$ and $l \neq 0$, its image in $V \times W$ is contained in

$(T^*X^\circ \otimes L) \times W$ if and only if x is a point of the hyperplane section X_0° defined by $l = 0$.

We put $T_0 = T^*_{X^\circ}X^\circ$ and for each T_i, let $T_i \otimes L \subset V$ denote the image of the twist of T_i by L by (3.3) and define

$$E_{f,i} \subset E_f \times X^\circ$$

to be the inverse image of $(T_i \otimes L) \times \{f\}$ by (3.8). For $(l, x) \in E_f \times X^\circ$ such that $l \neq 0$, the condition $(l, x) \in E_{f,0}$ is equivalent to $l \equiv 0 \bmod \mathfrak{m}_x^2 \mathcal{L}_x$, that means that $x \neq u$ is a singular point of the hyperplane section X_0° defined by $l = 0$. Further, for $(l, x) \in E_f \times X^\circ$ not contained in $E_{f,0}$ and for $i \neq 0$, the condition $(l, x) \in E_{f,i}$ is equivalent to that $x \neq u$ is a smooth point of the hyperplane section X_0° and the fiber of the conormal bundle $T^*_{X_0^\circ}X^\circ$ at x, that is spanned by the twist of l, is contained in T_i.

Consequently, $l \in E_f, \neq 0$ is not in the image of $E_{f,0}$ by the projection $E_f \times X^\circ \to E_f$ if and only if X_0° is smooth. Further, for such l, it is not in the image of $E_{f,i}$ if and only if the intersection $T^*_{X_0^\circ}X^\circ \cap T_i$ is contained in the 0-section. Thus, the hyperplane section X_0 satisfies the condition if and only if $l \in E_f, \neq 0$ is not in the union of the images of $E_{f,0}, \cdots, E_{f,m}$ by the projection $E_f \times X^\circ \to E_f$.

The conic subscheme $T_0 = T^*_{X^\circ}X^\circ \subset T^*X^\circ$ is of codimension $d = \dim X$ and $T_i \subset T^*X^\circ$ for $i = 1, \ldots, m$ are assumed to be of codimension d. Since the morphism (3.8) is surjective and the injection (3.3) is of codimension 1, the subvariety $E_{f,i} \subset E_f \times X^\circ$ is of codimension $d + 1$. The images of $E_{f,i}$ by the projection $E_f \times X^\circ \to E_f$ are of codimension at least 1 in E_f and the assertion is proved.

2. Let $P \subset X \times X$ be the closed subscheme defined by $\mathcal{I}_X^2$ and let $T \subset X$ be the closed subscheme defined by $\mathfrak{m}_u^N$. Then, it suffices to apply Lemma 3.2 to $S = X - \{u\}$ and the closed subscheme $Z = (T \times S) \amalg (P \cap (X \times S))$ of $X \times S$. □

Combining Lemmas 3.1 and 3.3, we obtain the following.

Proposition 3.4. *Let X be a projective irreducible smooth scheme of dimension d over an algebraically closed field k and $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let T be a conic irreducible closed subscheme of T^*X of codimension d. Then, there exists an integer $m \geqq 0$ such that for every $n \geqq m$,* *the invertible $\mathcal{O}_X$-module $\mathcal{L}^{\otimes n}$ is very ample and satisfies the condition* (L) *in Lemma* 3.1 *and the morphism $P(T) \to p(P(T))$ is generically radicial.*

Proof. For a hyperplane $H_0 \in \mathbf{P}^\vee$, the inverse image of $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is identified with the hyperplane section $X \cap H_0$. Hence, it is contained in the open set, a priori can be empty, of $p(P(T))$ where $P(T) \to p(P(T))$ is radicial if and only if the intersection of $X \cap H_0$, regarded as a subset of $X \times_{\mathbf{P}} \mathbf{H}$, with $P(T)$ consists of a unique point. Since the condition (L) has been already studied in Lemma 3.1.2, it suffices to show the existence of an integer m such that for $n \geqq m$, there exists a hyperplane $H_0 \in \mathbf{P}^\vee$ such that the intersection of $X \cap H_0$ with $P(T)$ for $\mathcal{L}^{\otimes n}$ consists of a unique point.

Let u be a closed point of X in the image of T by the canonical map $T^*X \to X$. Take a function f defined on a neighborhood of u such that $f(u) = 0$ and that T and the section df of T^*X meet on the fiber of T^*X above u.

By Lemma 3.3.2, there exists an integer m such that for $n \geqq m$, the invertible $\mathcal{O}_X$-module $\mathcal{L}^{\otimes n}$ satisfies the condition (N) in Lemma 3.3 for u and $N = 2$. Then by Lemma 3.3.1, for an integer $n \geqq m$, there exist non-zero sections $l_\infty, l \in \Gamma(X, \mathcal{L}^{\otimes n}), l_\infty \neq l$ such that the hyperplane section X_∞ defined by l_∞ does not contain u and $l/l_\infty \equiv f \bmod \mathfrak{m}_u^2$, that the hyperplane section X_0 of X defined by $l = 0$ is smooth outside u and that the intersection $T^*_{X_0^\circ}X^\circ \cap T$ with the conormal bundle of $X_0^\circ = X_0 - \{u\} \subset X^\circ = X - \{u\}$ is contained in the 0-section.

Let H_0 be the hyperplane defined by $l = 0$ and g be the function l/l_∞ defined on $X - X_\infty$. Then the congruence $l/l_\infty \equiv f \bmod \mathfrak{m}_u^2$ implies that $dg(u) = df(u)$ in T^*_uX. Hence, the pair $(u, H_0) \in X \times_{\mathbf{P}} \mathbf{H}$ is a point of $P(T)$. Further, the conditions that X_0° is smooth and that $T^*_{X_0^\circ}X^\circ \cap T$ is contained in the 0-section imply that the intersection of the fiber $X_0 \times \{H_0\}$ of $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ at H_0 with $P(T)$ is a subset of $\{u\}$. Thus, u is the unique point of the fiber $P(T) \to p(P(T))$. $\square$

Let $\mathbf{G} = \mathrm{Gr}(1, \mathbf{P}^\vee)$ be the Grassmannian variety parametrizing lines in $\mathbf{P}^\vee$. The universal line $\mathbf{D} \subset \mathbf{G} \times \mathbf{P}^\vee$ is canonically identified with the flag variety parametrizing pairs (L, H) of points H of $\mathbf{P}^\vee$ and lines L passing through H. We also identify $\mathbf{D}$ with the projective space bundle $\mathbf{P}(T\mathbf{P}^\vee)$ associated to the tangent bundle of $\mathbf{P}^\vee$. We define $X_{\mathbf{G}}$ by the cartesian diagram

$$(3.9)\qquad \begin{array}{ccc} X_{\mathbf{G}} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{H} \\ \Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ \mathbf{D} & \longrightarrow & \mathbf{P}^{\vee} \\ \Big\downarrow & & \\ \mathbf{G}. & & \end{array}$$

The Grassmannian variety $\mathbf{G}$ is also regarded as $\mathrm{Gr}(2, E)$ parametrizing subspace of E of dimension 2. If $\mathbf{V} \subset E \times \mathbf{G}$ denotes the universal sub vector bundle of rank 2, the universal axis $\mathbf{A} \subset \mathbf{P} \times \mathbf{G} \to \mathbf{G}$ is the projective space bundle $\mathbf{P}(\mathbf{V}^{\perp})$ associated to the annihilators $\mathbf{V}^{\perp} \subset E^{\vee} \times \mathbf{G}$. The intersection $X \times_{\mathbf{P}} \mathbf{A} = \mathbf{A} \cap (X \times \mathbf{G})$ is identified with the Grassmannian bundle $\mathrm{Gr}(2, X \times_{\mathbf{P}} T^*\mathbf{P})$ parametrizing sub vector bundles of rank 2 of $X \times_{\mathbf{P}} T^*\mathbf{P}$. Hence $X \times_{\mathbf{P}} \mathbf{A}$ is proper smooth over X and the immersion $X \times_{\mathbf{P}} \mathbf{A} = \mathrm{Gr}(2, X \times_{\mathbf{P}} T^*\mathbf{P}) \to X \times \mathbf{G} = \mathrm{Gr}(2, X \times E)$ is a regular immersion of codimension 2. The intersection of the twist of the pull-back of the universal sub bundle $\mathbf{V} \subset E \times \mathbf{G}$ to $X \times \mathbf{G}$ with the pull-back of $X \times_{\mathbf{P}} T^*\mathbf{P}$ defines a sub line bundle on the complement $X \times \mathbf{G} - X \times_{\mathbf{P}} \mathbf{A}$ and hence morphisms to $X \times_{\mathbf{P}} \mathbf{H}$ and to $\mathbf{D}$. They define a morphism $(X \times \mathbf{G}) - (X \times_{\mathbf{P}} \mathbf{A}) \to X_{\mathbf{G}}$ to the fiber product and further induce an isomorphism from the blow-up of $X \times \mathbf{G}$ at $X \times_{\mathbf{P}} \mathbf{A}$ to $X_{\mathbf{G}}$.

For a line $L \subset \mathbf{P}^{\vee}$, we define X_L by the cartesian diagram

$$(3.10)\qquad \begin{array}{ccc} X_L & \longrightarrow & X \times_{\mathbf{P}} \mathbf{H} \\ {\scriptstyle p_L}\Big\downarrow & & \Big\downarrow{\scriptstyle p} \\ L & \longrightarrow & \mathbf{P}^{\vee}. \end{array}$$

It is equal to $\{(x, H) \in X \times L \mid x \in X \cap H\}$. If the axis $A_L = \bigcap_{H \in L} H$ of L meets X transversely, then X_L is the blow up of X at the intersection $X \cap A_L$ and is smooth over k.

Let $T \subset T^*X$ be a conic reduced closed subscheme of codimension d and u be a closed point of X. Let f be a morphism to a smooth curve C over k defined on a neighborhood of u and assume that the intersection of T with the image of $df\colon X \times_C T^*C \to T^*X$ is contained in the union of the fiber of u and the 0-section on a neighborhood of u. Then, for a basis ω of $X \times_C T^*C$

on a neighborhood of u, the intersection number $(T,[\omega])_{T^*X,u}$ with the image $[\omega]$ of the section of T^*X defined on a neighborhood of u is defined. It is independent of the choice of ω since T is assumed conic. More intrinsically, it is the intersection multiplicity of the twist $Hom(X\times_C T^*C, T)$ with the image of the section df of the twisted vector bundle $Hom(X\times_C T^*C, T^*X)$ at the inverse image of u and we will write it as

$$(T,[df])_{T^*X,u} \tag{3.11}$$

by abuse of notation.

LEMMA 3.5. *Let $T\subset T^*X$ be a conic closed subscheme of dimension $d=\dim X$ and L be a line in $\mathbf{P}^\vee$. Assume that the axis A_L meets X transversely and that T is contained in the* 0*-section T^*_XX on a neighborhood of $X\cap A_L$.*

Let u be a closed point of X not in $X\cap A_L$ and set $v=p_L(u)$. Assume that, on a neighborhood of $T^*X\times_X(p_L^{-1}(v)-(X\cap A_L))$, the intersection of T with the image of $(X-(X\cap A_L))\times_L T^*L\to T^*(X-(X\cap A_L))$ is contained in the fiber T^*_uX of u.

Then, v is an isolated point of the intersection $p_*(P(T))\cap L\subset\mathbf{P}^\vee$ if v is contained in it and we have

$$(p_*(P(T)),L)_{\mathbf{P}^\vee,v}=(T,[dp_L])_{T^*X,u}. \tag{3.12}$$

PROOF. We claim that the intersection of $p_L^{-1}(v)\subset X_L\subset X\times_{\mathbf{P}}\mathbf{H}=\mathbf{P}(X\times_{\mathbf{P}}T^*\mathbf{P})$ with $P(T)$ is contained in the fiber of u. On the complement of $X\cap A_L$, the immersion $X-(X\cap A_L)\to X_L\to X\times_{\mathbf{P}}\mathbf{H}=\mathbf{P}(X\times_{\mathbf{P}}T^*\mathbf{P})$ corresponds to the restriction on $X-(X\cap A_L)$ of the injection $T^*L\times_L(\mathbf{P}-A_L)\to T^*\mathbf{P}\times_{\mathbf{P}}(\mathbf{P}-A_L)$ induced by the morphism $\mathbf{P}-A_L\to L$. Hence on a neighborhood of $p_L^{-1}(v)-(X\cap A_L)$, the claim follows from the assumption on the intersection of T with the image of $X\times_L T^*L\to T^*X$. Since X meets A_L transversely, $p_L\colon X_L\to L$ is smooth on a neighborhood of $p_L^{-1}(v)\cap(X\cap A_L)$. Hence, on a neighborhood of $p_L^{-1}(v)\cap(X\cap A_L)$, the claim follows from the assumption that T is contained in the 0-section T^*_XX on a neighborhood of $X\cap A_L$. Applying the projection formula to (3.10), we have $(p_*(P(T)),L)_{\mathbf{P}^\vee,v}=(P(T),X_L)_{X\times_{\mathbf{P}}\mathbf{H},u}$.

Let $\tilde{T}$ be the inverse image of T by the surjection $X \times_{\mathbf{P}} T^*\mathbf{P} \to T^*X$ appeared in the definition (3.1) of $P(T)$. For a basis ω of $X \times_L T^*L$ on a neighborhood of u, we have $(P(T), X_L)_{X\times_{\mathbf{P}}\mathbf{H},u} = (\tilde{T}, [\omega])_{X\times_{\mathbf{P}}T^*\mathbf{P},u}$ by the definition of $P(T)$. Further, the right hand side is equal to $(T, [\omega])_{T^*X,u}$. □

3.2. Radon transform and vanishing cycles

Let X be a smooth projective connected surface over an algebraically closed field k of characteristic $p > 0$. Let $D \subsetneqq X$ be a reduced closed subscheme of X and $j\colon U = X - D \to X$ be the open immersion of the complement. Let Λ be a finite field of characteristic $\ell \neq p$ and $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on $U = X - D$.

Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module. We set $E = \Gamma(X, \mathcal{L})$ and let $X \to \mathbf{P} = \mathbf{P}(E^\vee)$ be the closed immersion as in the previous section. The Radon transform $\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}$ is defined to be $Rp_*q^*j_!\mathcal{F}$ using the universal family of hyperplane sections

$$X \xleftarrow{\ q\ } X \times_{\mathbf{P}} \mathbf{H} \xrightarrow{\ p\ } \mathbf{P}^\vee = \mathbf{P}(E). \tag{3.13}$$

We study the ramification of the cohomology sheaves $\mathcal{R}^s_{\mathcal{L}} j_!\mathcal{F} = R^sp_*q^*j_!\mathcal{F}$ in Lemma 3.7 below. We define several closed subsets of $\mathbf{P}^\vee$. Let $D_i, i \in I$ be the irreducible components of dimension 1 of D. For each $i \in I$, let $D_i^\circ \subset D_i$ be a dense open smooth subscheme not meeting $D_{i'}$ for $i' \neq i$ along which the ramification of $\mathcal{F}$ is non-degenerate. We define a finite set Σ of closed points of D by

$$\Sigma = D - \bigcup_{i\in I} D_i^\circ \tag{3.14}$$

and let $j^\circ\colon U \to X - \Sigma$ denote the open immersion. For an irreducible component $D_i, i \in I$ of codimension 1, let $T_{ij}^\circ, j \in J_i$ be the irreducible components of the singular support $SS(j^\circ_!\mathcal{F}) \subset T^*(X - \Sigma)$ dominating D_i° and $T_{ij} \subset T^*X$ be the closure. They are irreducible conic closed subschemes of T^*X of dimension 2. We set

$$J = \coprod_{i\in I} J_i, \tag{3.15}$$

let $ij \in J$ denote $j \in J_i \subset J$ and

$$SS(j_!\mathcal{F}) = T^*_X X \cup \bigcup_{ij\in J} T_{ij} \cup \bigcup_{x\in\Sigma} T^*_x X. \tag{3.16}$$

Applying the construction of $P(T)$ (3.1) for conic closed subscheme $T \subset T^*X$, we define closed subvarieties $P(T^*_X X)$, $P(T_{ij})$ for $ij \in J$ and $P(T^*_x X)$ for $x \in \Sigma$ of $X \times_{\mathbf{P}} \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$. They are irreducible subschemes of $X \times_{\mathbf{P}} \mathbf{H} = \mathbf{P}(X \times_{\mathbf{P}} T^*\mathbf{P})$ of codimension 2. We define a closed subset $P(j_!\mathcal{F}) \subset X \times_{\mathbf{P}} \mathbf{H}$ to be the union

$$P(j_!\mathcal{F}) = P(T^*_X X) \cup \bigcup_{ij \in J} P(T_{ij}) \cup \bigcup_{x \in \Sigma} P(T^*_x X). \tag{3.17}$$

The image of $P(T^*_X X)$ by the projection $p\colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is the dual variety $X^\vee$. Let $T^\vee_{ij} \subset \mathbf{P}^\vee$ denote the image $p(P(T_{ij}))$ for $ij \in J$. The image $H_x = p(P(T^*_x X)) \subset \mathbf{P}^\vee$ is the dual hyperplane $\mathbf{P}(T^*_x \mathbf{P}) = \{H \mid x \in H\}$ for $x \in \Sigma$. Since $P(T^*_X X), P(T_{ij}), P(T^*_x X) \subset X \times_{\mathbf{P}} \mathbf{H}$ are of codimension 2 and $\dim X \times_{\mathbf{P}} \mathbf{H} = \dim \mathbf{P} - 1 + 2 = \dim \mathbf{P}^\vee + 1$, their images in $\mathbf{P}^\vee$ are of codimension $\geqq 1$. For $x \in \Sigma$, the canonical morphism $P(T^*_x X) \to H_x$ is an isomorphism.

Lemma 3.6. *Let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module.*

1. *Assume that $\mathcal{L}$ is very ample and satisfies the condition* (L) *in Lemma* 3.1 *and the following condition:*

(R) *The closed subset $X^\vee$ and $T^\vee_{ij} \subset \mathbf{P}^\vee$ for $ij \in J$ are of codimension* 1.

Then, $X^\vee, T^\vee_{ij}$ for $ij \in J$ and H_x for $x \in \Sigma$ are distinct to each other and the morphisms $P(T^*_X X) \to X^\vee$ and $P(T_{ij}) \to T^\vee_{ij}$ for $ij \in J$ are generically radicial.

2. *There exists an integer m such that for every $n \geqq m$, the invertible $\mathcal{O}_X$-module $\mathcal{L}^{\otimes n}$ satisfies the condition* (L) *in Lemma* 3.1 *and the condition* (R).

Proof. 1. It follows from Lemma 3.1.1.

2. It follows from Lemma 3.1.2 and Proposition 3.4. □

We define a closed subset $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}) \subset \mathbf{P}^\vee$ to be the union

$$D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}) = X^\vee \cup \bigcup_{ij \in J} T^\vee_{ij} \cup \bigcup_{x \in \Sigma} H_x. \tag{3.18}$$

Under the condition (R), the closed subset $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}) \subset \mathbf{P}^\vee$ is the underlying subset of a Cartier divisor. For an irreducible component D_i of D

of dimension 1, the $\mathcal{O}_{D_i}$-module $\mathcal{L}_i = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{D_i}$ is very ample and the linear subspace $\mathbf{P}_i^\vee = \mathbf{P}(E_i) \subset \mathbf{P}^\vee = \mathbf{P}(E)$ associated to $E_i = \operatorname{Ker}(E \to \Gamma(D_i, \mathcal{L}_i))$ is of codimension $\geqq 2$. The product $D_i \times \mathbf{P}_i^\vee$ is a subset of $X \times_{\mathbf{P}} \mathbf{H}$ for $i \in I$ and $x \times H_x$ is also a subset of $X \times_{\mathbf{P}} \mathbf{H}$ for $x \in \Sigma$.

Lemma 3.7. 1. *The morphism $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is flat of relative dimension 1. It is smooth outside $P(T_X^* X)$. Outside the union $\bigcup_{i \in I}(D_i \times \mathbf{P}_i^\vee) \cup \bigcup_{x \in \Sigma}(x \times H_x)$, the closed subscheme $D \times_{\mathbf{P}} \mathbf{H} \subset X \times_{\mathbf{P}} \mathbf{H}$ is a divisor flat over $\mathbf{P}^\vee$.*

2. *Outside the union $P(j_! \mathcal{F}) \cup \bigcup_{i \in I}(D_i \times \mathbf{P}_i^\vee)$, the morphism $p \colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is universally locally acyclic relatively to the pull-back $q^* j_! \mathcal{F}$.*

3. *The cohomology sheaf $\mathcal{R}^s_{\mathcal{L}} j_! \mathcal{F} = R^s p_* q^* j_! \mathcal{F}$ is 0 except for $s = 0, 1, 2$. The restrictions of $\mathcal{R}^s_{\mathcal{L}} j_! \mathcal{F}$ on the complement $V = \mathbf{P}^\vee - (D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}) \cup \bigcup_{i \in I} \mathbf{P}_i^\vee)$ is locally constant for every s.*

Proof. 1. Since a hyperplane $H \subset \mathbf{P}$ is defined by a non-zero section $l \in \Gamma(X, \mathcal{L})$, the intersection $X \cap H$ is a Cartier divisor of X. Further $X \times_{\mathbf{P}} \mathbf{H}$ is a smooth divisor of a scheme $X \times \mathbf{P}^\vee$ flat of relative dimension 2 over $\mathbf{P}^\vee$. Hence the morphism $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is flat of relative dimension 1. The smoothness outside $P(T_X^* X)$ follows from $P(T_X^* X) = \mathbf{P}(T_X^* \mathbf{P}) \subset \mathbf{P}(X \times_{\mathbf{P}} T^* \mathbf{P}) = X \times_{\mathbf{P}} \mathbf{H}$.

Outside Σ, $D - \Sigma$ is a smooth divisor of $X - \Sigma$. If $H \notin \mathbf{P}_i^\vee$, we have $D_i \cap H \subsetneqq D_i$. The condition $H \notin H_x$ means $x \notin H$. Hence if $(u, H) \in D \times_{\mathbf{P}} \mathbf{H}$ is not a point of the union $\bigcup_{i \in I}(D_i \times \mathbf{P}_i^\vee) \cup \bigcup_{x \in \Sigma}(x \times H_x)$, the intersection $D \cap H$ is a Cartier divisor of $X \cap H$ at x. Thus, outside the union $\bigcup_{i \in I}(D_i \times \mathbf{P}_i^\vee) \cup \bigcup_{x \in \Sigma}(x \times H_x)$, the closed subscheme $D \times_{\mathbf{P}} \mathbf{H} \subset X \times_{\mathbf{P}} \mathbf{H}$ is a divisor flat over $\mathbf{P}^\vee$.

2. We apply [19, Théorème 2.1.1 (ii)] to the flat morphism $p \colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$. Let $(x, H) \in X \times_{\mathbf{P}} \mathbf{H}$ be a point outside the union $P(j_! \mathcal{F}) \cup \bigcup_{i \in I}(D_i \times \mathbf{P}_i^\vee)$. Then, by 1., $p \colon X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is smooth at (x, H). Further, if x is a point of D, $D \times_{\mathbf{P}} \mathbf{H}$ is a Cartier divisor of $X \times_{\mathbf{P}} \mathbf{H}$ flat over $\mathbf{P}^\vee$ at (x, H). By the condition that (x, H) is not in $P(j_! \mathcal{F})$, the immersion $X \cap H \to X$ is non-characteristic with respect to $j_! \mathcal{F}$. Hence the total dimension of the restriction of $j_! \mathcal{F}$ to $X \cap H$ is computed as the intersection number $(X \cap H, DT(j_! \mathcal{F}))$ by (1.9).

In other words, it is the intersection number of the fiber of $X \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ at the point $[H]$ with the pull-back of $DT(j_! \mathcal{F})$ flat over $\mathbf{P}^\vee$. Hence the assumption on the function φ in [19, Théorème 2.1.1 (ii)] is satisfied and the assertion follows by [19, Théorème 2.1.1 (ii)].

3. By 1., we have $R^s p_* q^* j_! \mathcal{F} = 0$ except for $s = 0, 1, 2$. By 2. and [15, 2.4], $R^s p_* q^* j_! \mathcal{F}$ is locally constant on V for every s. $\square$

We define the characteristic cycle of $j_!\mathcal{F}$ as a cycle in the cotangent bundle T^*X using the ramification of the Radon transform.

DEFINITION 3.8. Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module satisfying the conditions (L) in Lemma 3.1 and (R) in Lemma 3.6 and let

$$a(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}) = a_X^{\mathcal{L}}(j_!\mathcal{F}) \cdot X^\vee + \sum_{ij \in J} a_{ij}^{\mathcal{L}}(j_!\mathcal{F}) \cdot T_{ij}^\vee + \sum_{x \in \Sigma} a_x^{\mathcal{L}}(j_!\mathcal{F}) \cdot H_x \tag{3.19}$$

denote the Artin divisor (1.8) of the Radon transform $\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}$. We define the *characteristic cycle* of $j_!\mathcal{F}$ relative to $\mathcal{L}$ by

$$\begin{aligned} \mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F}) = -\Big(& \frac{a_X^{\mathcal{L}}(j_!\mathcal{F})}{[P(T_X^*X) : X^\vee]} \cdot [T_X^*X] \\ & + \sum_{ij \in J} \frac{a_{ij}^{\mathcal{L}}(j_!\mathcal{F})}{[P(T_{ij}) : T_{ij}^\vee]} \cdot [T_{ij}] + \sum_{x \in \Sigma} a_x^{\mathcal{L}}(j_!\mathcal{F}) \cdot [T_x^*X] \Big) \end{aligned} \tag{3.20}$$

as a cycle of dimension 2 in the cotangent bundle T^*X.

We have

$$p_* P(\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F})) = -a(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}) \tag{3.21}$$

by the definition. We study the coefficients in more detail in Proposition 3.11.

We prove an analogue Theorem 3.17 of the Milnor formula [6] in several steps. In the following, we assume that $\mathcal{L}$ is a very ample invertible $\mathcal{O}_X$-module satisfying the conditions (L) in Lemma 3.1 and (R) in Lemma 3.6. The following immediate consequence of Lemma 3.7 is fundamental in the study of the morphism defined by a pencil.

LEMMA 3.9. *Let L be a line in $\mathbf{P}^\vee$ such that the axis A_L meets X transversely and does not meet D. We identify the blow-up X_L of X at*

$X \cap A_L$ as a closed subscheme of $X \times_{\mathbf{P}} \mathbf{H}$ by the cartesian diagram (3.10)*. Then, outside the intersection $X_L \cap \big(P(j_!\mathcal{F}) \cup \bigcup_{i\in I}(D_i \times \mathbf{P}_i^\vee)\big)$, the morphism $p_L \colon X_L \to L$ is locally acyclic relatively to the pull-back of $j_!\mathcal{F}$ and is non-characteristic with respect to the pull-back of $j_!\mathcal{F}$. Further, outside the same locus, the restriction $D \times_X X_L \to L$ of p_L is flat.*

PROOF. The local acyclicity and the flatness of $D \times_X X_L \to L$ is clear from Lemma 3.7. The non-characteristicity is clear from the definition of $P(j_!\mathcal{F})$. $\square$

Let $X^{\vee\circ}$ be a smooth dense open subscheme of $X^\vee$ satisfying the following conditions: The intersections with other components of $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})$ (3.18) and with $\mathbf{P}_i^\vee$ for $i \in I$ are empty. The inverse image of $P(T^*_X X) \to X^\vee$ consists of one point for every point of $X^{\vee\circ}$. The ramification of $(\mathcal{R}^s_{\mathcal{L}} j_!\mathcal{F})|_V$ along $X^{\vee\circ}$ is non-degenerate. The restriction $(\mathcal{R}^s_{\mathcal{L}} j_!\mathcal{F})|_{X^{\vee\circ}}$ is locally constant for $s = 0, 1, 2$.

Similarly, we define smooth dense open subschemes $T_{ij}^{\vee\circ} \subset T_{ij}^\vee$ for $ij \in J$ and $H_x^{\vee\circ} \subset H_x$ for $x \in \Sigma$. Let $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ$ denote the disjoint union

$$D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ = X^{\vee\circ} \cup \bigcup_{ij\in J} T_{ij}^{\vee\circ} \cup \bigcup_{x\in\Sigma} H_x^{\vee\circ} \tag{3.22}$$

as in (3.18). It is a dense open subscheme of $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})$ and is smooth of codimension 1 in $\mathbf{P}^\vee$.

LEMMA 3.10. *Let L be a line in $\mathbf{P}^\vee$ such that the axis A_L meets X transversely and does not meet D. Let y be a closed point of L corresponding to a hyperplane $H \subset \mathbf{P}$ and suppose that L meets $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})$ at y properly.*

1. *Let $z \in X$ be a closed point not contained in A_L satisfying $y = p_L(z)$. Assume that $p_L \colon X_L \to L$ is non-characteristic with respect to (the pull-back of) $j_!\mathcal{F}$ on a neighborhood of $p_L^{-1}(y)$ except at z. Then, we have*

$$-(a(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}), L)_y = (\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F}), [dp_L])_z. \tag{3.23}$$

2. *Assume that L meets $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ$ transversely at y and that the immersion $L \to \mathbf{P}^\vee$ is non-characteristic with respect to $(\mathcal{R}^s_{\mathcal{L}} j_!\mathcal{F})|_V$ at y for $s = 0, 1, 2$. Then, the intersection $((X \cap H) \times \{y\}) \cap P(j_!\mathcal{F}) \subset X \times_{\mathbf{P}} \mathbf{H}$ consists of one point z and we have*

$$\dim \mathrm{tot}\phi_z(j_!\mathcal{F}, p_L) = (a(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}), L)_y. \tag{3.24}$$

PROOF. 1. We verify that the assumptions of Lemma 3.5 is satisfied. By the assumption $X \cap A_L \subset U$, the singular support is contained in the 0-section on a neighborhood of $X \cap A_L$. By the assumption of non-characteristicity, the intersection of T with the image of $X \times_L T^*L \to T^*X$ is satisfied. By $p_*P(\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F})) = -a(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})$ (3.21) and Lemma 3.5, we obtain (3.23)

2. Let $\bar{\eta}_y$ denote a geometric generic point of the strict localization of L at y. By the definition of $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ$, the point $y \in L$ is not contained in $\mathbf{P}_i^\vee$ for $i \in I$ and z is the unique point of $p_L^{-1}(y)$ contained in $P(j_!\mathcal{F}) \subset X \times_{\mathbf{P}} \mathbf{H}$. Hence, by Lemma 3.9, the morphism $p_L \colon X_L \to L$ is locally acyclic relatively to the pull-back of $j_!\mathcal{F}$ on $p_L^{-1}(y)$ except at z. Hence the distinguished triangle of vanishing cycles gives a distinguished triangle

$$\to (\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})_y \to (\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})_{\bar{\eta}_y} \to \phi_z(j_!\mathcal{F}, p_L) \to . \tag{3.25}$$

This implies $\dim \mathrm{tot}\phi_z(j_!\mathcal{F}, p_L) = a_y\big((\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})|_L\big)$.

By the assumption that the immersion $L \to \mathbf{P}^\vee$ is non-characteristic at y and by (1.9), the total dimension of the restriction $\dim \mathrm{tot}_{\bar{\eta}_y}(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})|_L$ equals the intersection number $(DT(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}), L)_y$. By the definition of $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ$, the dimension $\dim(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})_y$ is the rank of the restriction of $\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}$ on the component of $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ$ containing y. Hence we have $a_y\big((\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})|_L\big) = (a(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}), L)_y$. $\square$

PROPOSITION 3.11 (cf. [9, p.7 Question]). *The coefficients of $[T^*_X X]$ in $\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F})$ is the rank of $\mathcal{F}$. The coefficient of $[T_{ij}]$ for $ij \in J$ is a rational number at least 0 and its denominator is a power of p. The coefficient of $[T^*_x X]$ for $x \in \Sigma$ is an integer at least 0, if x is not an isolated point of D. If $x \in \Sigma$ is an isolated point of D, it is $-\mathrm{rank}\ \mathcal{F}$.*

PROOF. Let L be a line as in Lemma 3.10.2 and y be a point of intersection $L \cap D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ$. By Lemma 3.10.2, we have $\dim \mathrm{tot}\phi_z(j_!\mathcal{F}, p_L) = (a(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}), L)_y$. If z is not an isolated point of D, we have $\phi^q_z(j_!\mathcal{F}, p_L) = 0$ except for $q \neq 1$ and the coefficient of the component containg y in $a(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})$ are integers at most 0. Hence, the coefficients in $\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F})$ are rational numbers at least 0 except for the coefficients of $[T^*_x X]$ for an isolated point $x \in \Sigma$ of D. If z is an isolated point of D, we have $\phi^0_z(j_!\mathcal{F}, p_L) = (j_*\mathcal{F})_z$ and $\phi^q_z(j_!\mathcal{F}, p_L) = 0$ except for $q \neq 0$. Hence, the coefficient of $[T^*_z X]$ is $-\mathrm{rank}\ \mathcal{F}$.

Assume that y is in $X^{\vee\circ}$. Since $\mathcal{F}$ is locally constant on a neighborhood of z, by the Milnor formula [6], we have $-\dim \operatorname{tot}\phi_z(j_!\mathcal{F}, p_L) = -\operatorname{rank} \mathcal{F} \cdot \dim \operatorname{tot}\phi_z(j_!\Lambda_U, p_L) = \operatorname{rank} \mathcal{F} \cdot (T^*_X X, [dp_L])_{T^*X,z}$. Hence the coefficient of $[T^*_X X]$ is rank $\mathcal{F}$ by Lemma 3.10.2.

Since $P(T_{ij}) \to T^\vee_{ij}$ is generically purely inseparable by Lemma 3.6, the degree $[P(T_{ij}) : T^\vee_{ij}]$ is a power of p. □

We show the existence of a good pencil for an invertible sheaf satisfying the conditions (L) and (R).

Lemma 3.12. *Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module satisfying the conditions* (L) *and* (R)*. Then, the open subscheme of the Grassmannian variety* $\mathbf{G}$ *consisting of lines $L \subset \mathbf{P}^\vee$ satisfying the following conditions* (P1)–(P3) *is non-empty:*

(P1) *The axis A_L meets X transversely and does not meet D. The morphism $p_L|_D \colon D \to L$ is generically étale.*

(P2) *The intersection $L \cap D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})$ is finite and is contained in $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ$. Further L meets $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})^\circ$ transversely. The intersection $L \cap \bigcup_{i \in I} \mathbf{P}^\vee_i$ is empty.*

(P3) *The immersion $L \to \mathbf{P}^\vee$ is non-characteristic with respect to $j_{V!}(\mathcal{R}^s_{\mathcal{L}} j_!\mathcal{F})_V$ for $s = 0, 1, 2$ for the open immersion $j_V \colon V \to \mathbf{P}^\vee$ for V in Lemma* 3.7.

The condition (P2) implies that the inverse image of the intersection $X \cap A_L$ in X_L does not meet the intersection $Z_L = P(j_!\mathcal{F}) \cap X_L$. The condition (P2) further implies that the intersection Z_L consists of finitely many closed points and that the restriction $p_L|_{Z_L} \colon Z_L \to L$ is an injection.

Proof. Since each condition is an open condition on $\mathbf{G}$, it suffices to show that there exists a line $L \subset \mathbf{P}^\vee$ satisfying each condition (P1)–(P3), separately.

By Bertini's theorem, there exists a hyperplane $H \in \mathbf{P}^\vee$ meeting X transversely and another hyperplane $H' \in \mathbf{P}^\vee$ meeting $X \cap H$ transversely. Then, for the line $L \subset \mathbf{P}^\vee$ spanned by H and H', the axis $A_L = H \cap H' \subset \mathbf{P}$ meets X transversely. Similarly, there exists a hyperplane $H \in \mathbf{P}^\vee$ meeting

D transversely and another hyperplane $H' \in \mathbf{P}^\vee$ not meeting $D \cap H$. For the line $L \subset \mathbf{P}^\vee$ spanned by H and H', the intersection $A_L \cap D$ is empty. A line L satisfying these conditions satisfies (P1).

Since $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})^\circ$ is a dense open subscheme of a divisor $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})$ of $\mathbf{P}^\vee$, there exists a line $L \subset \mathbf{P}^\vee$ such that the intersection with $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F}) - D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})^\circ$ is empty. Since $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})^\circ$ is smooth, there exists a line L meeting $D(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})^\circ$ transversely. Since $\mathbf{P}_i^\vee$ is of codimension $\geqq 2$ for every $i \in I$, there exists a line $L \subset \mathbf{P}^\vee$ such that the intersection $L \cap \mathbf{P}_i^\vee$ is empty for every $i \in I$. A line L satisfying these conditions satisfies (P2).

For $ij \in J$, let $\Sigma_{ij}^\circ \subset \mathbf{D} \times_{\mathbf{P}^\vee} T_{ij}^{\vee\circ}$ be the subset consisting of pairs (L, H) of a hyperplane $H \in T_{ij}^{\vee\circ} \subset \mathbf{P}^\vee$ and a line $L \subset \mathbf{P}^\vee$ passing through it such that the immersion $L \to \mathbf{P}^\vee$ is *not* non-characteristic at H. Since the closure $\Sigma_{ij} \subset \mathbf{D}$ of Σ_{ij}° is of codimension 2 and since $\mathbf{D}$ is a $\mathbf{P}^1$-bundle over $\mathbf{G}$, its image $Q_{ij} \subset \mathbf{G}$ is of codimension $\geqq 1$. We define $\Sigma_X \subset \mathbf{D}$ and $\Sigma_x \subset \mathbf{D}$ for $x \in \Sigma$ similarly. By the same argument, their images $Q_X \subset \mathbf{G}$ and $Q_x \subset \mathbf{G}$ for $x \in \Sigma$ are of codimension at least 1. A line $L \in \mathbf{G}$ not contained in the union of Q_X, Q_{ij} and $Q_x \subset \mathbf{G}$ for $x \in \Sigma$ satisfies (P3). $\square$

PROPOSITION 3.13. *Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module satisfying the conditions* (L) *and* (R). *Let $L \subset \mathbf{P}^\vee$ be a line such that the axis A_L meets X transversely and does not meet D and set $Z_L = X_L \cap P(j_! \mathcal{F}) \subset X_L$.*

Let u be a closed point of $X - (X \cap A_L)$ satisfying the following condition:

(u) *$v = p_L(u)$ is an isolated point of $p_L(Z_L)$ and that u is the unique point in the intersection $Z_L \cap p_L^{-1}(v)$. Further, for $i \in I$, if the restriction $p_L|_{D_i} : D_i \to L$ is not flat, v is not in the image $p_L(D_i)$.*

Then, we have

$$-\dim \operatorname{tot}\phi_u(j_! \mathcal{F}, p_L) = (\operatorname{Char}_{\mathcal{L}}(j_! \mathcal{F}), [dp_L])_{T^*X,u}. \tag{3.26}$$

PROOF. By Lemma 3.12, the open subscheme $V \subset \mathbf{G}$ consisting of lines satisfying the conditions (P1)–(P3) is non-empty. We take a line C in $\mathbf{G}$ passing the point $s \in \mathbf{G}$ defined by L and meeting V. By replacing C by a neighborhood C of s, we may assume that for every point $t \in C - \{s\}$, the corresponding line L_t satisfies the conditions (P1)–(P3).

We apply Proposition 2.6 to the cartesian diagram

$$\begin{array}{ccccccc}
X_L & \longrightarrow & X_C & \longrightarrow & X_{\mathbf{G}} & \longrightarrow & X \times_{\mathbf{P}} \mathbf{H} \\
{\scriptstyle p_L}\downarrow & & \downarrow & & \downarrow & & \downarrow \\
L & \longrightarrow & L_C & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{P}^\vee \\
\downarrow & & \downarrow & & \downarrow & & \\
s & \longrightarrow & C & \longrightarrow & \mathbf{G} & &
\end{array}$$

and to the pull-back B of the Artin divisor $a(\mathcal{R}_{\mathcal{L}} j_! \mathcal{F})$ to $Y = L_C$. We show that the pull-back of $j_!\mathcal{F}$ to X_C is locally acyclic relatively to $X_C \to C$. By the assumption (P1), the axis A_{L_C} meets $X \times C$ transversely and does not meet $D \times C$. Hence the pull-back of $j_!\mathcal{F}$ is locally constant on a smooth scheme $X_C - (D \times C)$ over C and is locally acyclic relatively to $X_C - (D \times C) \to C$ by the local acyclicity of smooth morphism. Further, on a neighborhood of $D \times C$, it is the pull-back by the projection and is also locally acyclic relatively to $X_C \to C$ by [7, Théorème 2.13]. Hence the pull-back of $j_!\mathcal{F}$ to X_C is locally acyclic relatively to $X_C \to C$.

The pull-back of $j_!\mathcal{F}$ is universally locally acyclic relatively to $X_C \to L_C$ outside the inverse image Z' of the union of $P(j_!\mathcal{F}) \cup \bigcup_{i \in I}(D_i \times \mathbf{P}_i^\vee)$ by Lemma 3.7. By condition (P2), the morphism $Z' \to C$ is quasi-finite on $C - \{s\}$. By the assumption (u), v is not contained in $L \cap \bigcup_{i \in I} \mathbf{P}_i^\vee$ and is an isolated point of $p_L(Z_L) = p_L(Z' \cap X_L)$. Further, u is the unique point of $Z_L \cap p_L^{-1}(v) = Z' \cap p_L^{-1}(v)$ and the assumptions of Proposition 2.6 are satisfied. Hence by Lemma 3.10.2 and Proposition 2.6, we obtain $\dim \mathrm{tot}\phi_u(j_!\mathcal{F}, p_L) = (a(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}), L)_v$. Hence (3.26) follows from (3.23). □

Proposition 3.14. *Let*

$$C \xleftarrow{\ f\ } X' \xrightarrow{\ \varphi\ } X$$

be an étale morphism $\varphi\colon X' \to X$ of smooth surfaces over k and a flat morphism $f\colon X' \to C$ to a smooth curve C over k. Assume that X is projective and let $\mathcal{F}$ be a locally constant constructible sheaf on the complement $U = X - D$ of a reduced closed subscheme $D \subsetneqq X$. Let $j'\colon U' \to X'$ be the pull-back of $j\colon U \to X$ and $\mathcal{F}'$ be the pull-back of $\mathcal{F}$ on U'. Let u be a closed point of X' such that u is an isolated characteristic point of $f\colon X' \to C$ with *respect to* $j'_!\mathcal{F}'$ *and assume that the restriction* $f|_{D'}\colon D'=D\times_X X'\to C$ *is étale on a neighborhood of* u *except at* u*.*

Then, for an ample invertible $\mathcal{O}_X$*-module* $\mathcal{L}$*, there exists an integer* m *such that for every integer* $n\geqq m$*, the invertible* $\mathcal{O}_X$*-module* $\mathcal{L}^{\otimes n}$ *is very ample and satisfies the conditions* (L) *and* (R) *and we have*

$$-\dim \operatorname{tot}\phi_u(j'_!\mathcal{F}',f)=((T^*\varphi)^*\mathrm{Char}_{\mathcal{L}^{\otimes n}}(j_!\mathcal{F}),[df])_{T^*X',u}. \tag{3.27}$$

PROOF. We prove Proposition by reducing it to Proposition 3.13 using the stability of nearby cycles Theorem 2.14. By taking an étale morphism $C\to\mathbf{P}^1$ on a neighborhood of $v=f(c)$, we may assume $C=\mathbf{P}^1$ and $v=0$. By Theorem 2.14, there exists an integer $N\geqq 1$ such that for a morphism $g\colon X'\to C$ congruent to f mod $\mathfrak{m}_u^N$, we have an isomorphism $\phi_u(j'_!\mathcal{F}',f)\simeq\phi_u(j'_!\mathcal{F}',g)$.

Similarly as Proposition 2.1, there exists an integer $N\geqq 1$ such that for a morphism $g\colon X'\to C$ congruent to f mod $\mathfrak{m}_u^N$, we have an equality $(T,[df])_{T^*X',u}=(T,[dg])_{T^*X',u}$ for every irreducible component T of the singular support $SS(j_!\mathcal{F})$.

By Lemmas 3.1.2 and 3.3.2, there exists an integer $m\geqq 1$ such that for every integer $n\geqq m$, the invertible $\mathcal{O}_X$-module $\mathcal{L}^{\otimes n}$ is very ample and satisfies the conditions (L) and (N) for the integer $N\geqq 1$ above. We show the equality (3.27) for $n\geqq m$ above. By changing the notation, we write $\mathcal{L}$ for $\mathcal{L}^{\otimes n}$. We show the existence of a pencil L such that p_L satisfies the conditions in Proposition 3.13 and that the composition $g=p_L\circ\varphi$ is congruent to f mod $\mathfrak{m}_u^N$.

Take a hyperplane $H_\infty\in\mathbf{P}^\vee$ not contained in $D(\mathcal{R}_{\mathcal{L}}j_!\mathcal{F})\cup\bigcup_{i\in I}\mathbf{P}_i^\vee$ and a section $l_\infty\in E=\Gamma(X,\mathcal{L})$ defining H_∞. Then, the hyperplane section $X_\infty=X\cap H_\infty$ is smooth and does not contain $x\in\Sigma$. Further, we may assume that u is not contained in H_∞ and that $D_\infty=D\cap H_\infty$ is étale. We apply Lemma 3.3 to the family of subschemes T^*_XX, T_{ij} for $ij\in J$, T^*_xX for $x\in\Sigma\cup D_\infty$ and $T^*_{X_\infty}X$. Then, there exists $l\in E_f$ satisfying the conditions loc. cit. for this family. By the rational function l/l_∞, we also identify $L=\mathbf{P}^1$.

The complement $X_0^\circ=X_0-\{u\}$ is a smooth divisor of $X-\{u\}$. The condition that the intersection $T^*_{X_0^\circ}X\cap T^*_{X_\infty}X$ is contained in the 0-section means that X_∞ and X_0° meet transversely and hence the axis A_L of the pencil L spanned by l and l_∞ meets X transversely. The condition that the intersection $T^*_{X_0^\circ}X \cap T^*_x X$ is contained in the 0-section for $x \in D_\infty$ means that the axis A_L does not meet D.

Let $p_L\colon X_L \to L$ be the morphism defined by the pencil L. By the assumption that $D_\infty = D \cap H_\infty$ is étale, the restrictions $p_L|_{D_i}\colon D_i \to L$ are flat and $L \cap \mathbf{P}^\vee_i = \varnothing$ for $i \in I$. Let $Z_L \subset X_L$ be the intersection with $P(j_!\mathcal{F}) \subset X \times_{\mathbf{P}} \mathbf{H}$. By the conditions that H_∞ is not contained in $L \cap D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F}) = p_L(Z_L)$ and that $X \cap A_L$ is contained in U, the morphism $p_L\colon X_L \to L$ is smooth outside the finite set $p_L(Z_L)$. Further, $p_L\colon X_L \to L$ is locally acyclic relatively to the pull-back of $j_!\mathcal{F}$ and is non-characteristic with respect to the pull-back of $j_!\mathcal{F}$ outside Z_L by Lemma 3.9.

The condition that the intersection $T^*_{X_0^\circ}X \cap T^*_X X$ is contained in the 0-section means that the morphism $p_L\colon X_L \to L$ is smooth on a neighborhood of $p_L^{-1}(v) - \{u\}$. The condition that the intersection of $T^*_{X_0^\circ}X$ with $T^*_X X, T_{ij}$ for $ij \in J$, $T^*_x X$ for $x \in \Sigma$ is contained in the 0-section means that the intersection $X_0^\circ \cap Z_L$ is empty. Thus, the condition (u) in Proposition 3.13 is satisfied and we have an equality $-\dim \operatorname{tot}\phi_{\varphi(u)}(j_!\mathcal{F}, p_L) = (\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F}), [dp_L])_{T^*X,\varphi(u)}$.

The congruence $l/l_\infty \equiv f \bmod \mathfrak{m}_u^N$ means that the composition $p_L \circ \varphi\colon X \to L$ is congruent to $f\colon X \to C \bmod \mathfrak{m}_u^N$. Thus, by Theorem 2.14, we have an isomorphism $\phi_u(\varphi^* j_!\mathcal{F}, f) \to \phi_{\varphi(u)}(j_!\mathcal{F}, p_L)$. Since we also have $((T^*\varphi)^*\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F}), [df])_{T^*X',u} = (\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F}), [dp_L])_{T^*X,\varphi(u)}$, the assertion follows. $\square$

COROLLARY 3.15. *Let $f\colon X' \to X$ be an étale morphism of smooth surfaces over k. Let $\overline{X} \supset X$ and $\overline{X}' \supset X'$ be projective smooth surfaces containing X and X' as dense open subschemes and let $\mathcal{L}$ and $\mathcal{L}'$ be a very ample invertible $\mathcal{O}_{\bar{X}}$-module and $\mathcal{O}_{\bar{X}'}$-module satisfying the conditions* (L) *and* (R).

Let U be a dense open subscheme of X and $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on U. Let $\bar{j}\colon U \to \bar{X}$ and $\bar{j}'\colon U' = U \times_X X' \to \bar{X}'$ be the open immersions and let $\mathcal{F}'$ be the pull-back of $\mathcal{F}$ on U'. Then, we have

$$(T^*f)^*(\mathrm{Char}_{\mathcal{L}}(\bar{j}_!\mathcal{F})|_X) = \mathrm{Char}_{\mathcal{L}'}(\bar{j}'_!\mathcal{F}')|_{X'}. \tag{3.28}$$

Proof. It is sufficient to show that the coefficient of an irreducible component T_0 of $SS(j_!\mathcal{F}) \subset T^*X$ in $\mathrm{Char}_{\mathcal{L}}(\bar{j}_!\mathcal{F})|_X = \sum_i a_i[T_i]$ equals that of an irreducible component T_0' of the pull-back of T_0 in $\mathrm{Char}_{\mathcal{L}'}(\bar{j}_!'\mathcal{F}')|_{X'} = \sum_{i'} a'_{i'}[T'_{i'}]$. Let $\bar{T}_0 \subset T^*\bar{X}$ be the closure. By Lemma 3.6.1, the restriction $P(\bar{T}_0) \to \bar{T}_0^\vee$ of the morphism $\bar{X} \times_{\mathbf{P}} \mathbf{H} \to \mathbf{P}^\vee$ is generically radicial. By Lemma 3.12, we may take a pencil $L \subset \mathbf{P}^\vee$ satisfying the conditions (P1)–(P3) there. Let v be a point $L \cap \bar{T}_0^\vee$ and $u \in P(\bar{T}_0)$ be the unique point of the inverse image $p_L^{-1}(v) \subset X_L$. By the condition (P2), $u \in X \times_{\mathbf{P}} \mathbf{H}$ is not contained in $P(\bar{T}_i)$ for $i \neq 0$. Similarly, we take a pencil $L' \subset \mathbf{P}'^\vee$ satisfying the conditions (P1)–(P3) and a unique point $u' \in P(\bar{T}'_0)$ of the inverse image $p_{L'}^{-1}(p_{L'}(u')) \subset X'_{L'}$. Let $j\colon U \to X$ and $j'\colon U' \to X'$ be the open immersions. Then, by Proposition 3.13, we have

$$\begin{aligned}
-\dim \mathrm{tot}_u\phi(j_!\mathcal{F}, p_L) &= (\mathrm{Char}_{\mathcal{L}}(\bar{j}_!\mathcal{F}), [dp_L])_{T^*X,u} = a_0 \cdot (T_0, [dp_L])_{T^*X,u},\\
-\dim \mathrm{tot}_{u'}\phi(j'_!\mathcal{F}', p_{L'}) &= (\mathrm{Char}_{\mathcal{L}'}(\bar{j}'_!\mathcal{F}'), [dp_{L'}])_{T^*X',u'}\\
&= a'_0 \cdot (T'_0, [dp_{L'}])_{T^*X',u'}.
\end{aligned}$$

By Proposition 3.14, there exists an integer n such that the invertible $\mathcal{O}_{\bar{X}}$-module $\mathcal{M} = \mathcal{L}^{\otimes n}$ is very ample, satisfying the condition (L) and (R) and

$$\begin{aligned}
-\dim \mathrm{tot}_u\phi(j_!\mathcal{F}, p_L) &= (\mathrm{Char}_{\mathcal{M}}(\bar{j}_!\mathcal{F}), [dp_L])_{T^*X,u},\\
-\dim \mathrm{tot}_{u'}\phi(j'_!\mathcal{F}', p_{L'}) &= ((T^*f)^*\mathrm{Char}_{\mathcal{M}}(\bar{j}_!\mathcal{F}), [dp_{L'}])_{T^*X',u'}.
\end{aligned}$$

Thus, if we set $\mathrm{Char}_{\mathcal{M}}(\bar{j}_!\mathcal{F}) = \sum_i b_i[T_i]$, we obtain

$$\begin{aligned}
a_0 \cdot (T_0, [dp_L])_{T^*X,u} &= b_0 \cdot (T_0, [dp_L])_{T^*X,u}\\
a'_0 \cdot (T'_0, [dp_{L'}])_{T^*X',u'} &= b_0 \cdot (T'_0, [dp_{L'}])_{T^*X',u'}.
\end{aligned}$$

Since $(T_0, [dp_L])_{T^*X,u} > 0$ and $(T'_0, [dp_{L'}])_{T^*X',u'} > 0$, we obtain $a_0 = b_0 = a'_0$. Thus the equality (3.28) is proved. □

Corollary 3.15 means that the characteristic cycle $\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F})$ is independent of the choice of a very ample $\mathcal{O}_X$-module $\mathcal{L}$ satisfying (L) and (R) and that the construction of $\mathrm{Char}_{\mathcal{L}}(j_!\mathcal{F})$ is étale local. Thus, we can make the following definition.

Definition 3.16. Let X be a smooth surface over k and U be the complement of a Cartier divisor. Let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on U. Then, we define $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F})$ to be the restriction to T^*X of $\mathrm{Char}_{\mathcal{L}}(\bar{j}_!\mathcal{F})$ for a smooth compactification $X \to \bar{X}$, the composition $\bar{j}\colon U \to \bar{X}$ and a very ample invertible $\mathcal{O}_{\bar{X}}$-module $\mathcal{L}$ satisfying (L) and (R).

The construction of $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F})$ is additive in the sense that we have

$$\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}) = \mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}') + \mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}'')$$

for an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of locally constant constructible sheaves on $U = X - D$. We record the equality (3.26) for the convenience of the reference.

THEOREM 3.17 (cf. [9, p.7 Principe]). *Let X be a smooth surface over k and let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on a dense open subscheme U. Let $f\colon X \to C$ be a flat morphism to a smooth curve and u be a closed point of X. Assume that u is an isolated characteristic point of f with respect to $j_!\mathcal{F}$ and that D is étale over C on a neighborhood of u except at u. Then, we have*

$$-\dim \mathrm{tot}\phi_u(j_!\mathcal{F}, f) = (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), [df])_{T^*X,u}. \tag{3.29}$$

PROOF. Clear from Proposition 3.14. □

We prove a variant of Theorem 3.17 for a normal surface later at Proposition 3.22.

3.3. Euler characteristic and the characteristic cycle

We compute the Euler characteristic. Let X be a smooth connected surface over a perfect field k, let $D \subsetneqq X$ be a reduced closed subscheme and $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on $U = X - D$. Let $Y \to X$ be a closed immersion of a smooth curve such that the immersion $Y \to X$ is non-characteristic with respect to $j_!\mathcal{F}$. Then let $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y)$ denote -1-*times* the cycle of T^*Y defined as the image of the fiber $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}) \times_X Y$ by the surjection $T^*X \times_X Y \to T^*Y$.

LEMMA 3.18. *Let X be a projective smooth connected surface, $D \subset X$ be a reduced closed subscheme and C be a proper smooth connected curve of* *genus g over an algebraically closed field k. Let $f\colon X \to C$ be a proper flat morphism over k such that the restriction $f|_D\colon D \to C$ is finite. Let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on $U = X - D$.*

Assume that $f\colon X \to C$ is non-characteristic with respect to $j_!\mathcal{F}$ on the complement of a finite set Z of closed points of X. Let $c \in C$ be a closed point such that on a neighborhood $V \subset C$ of c, the morphism $X \times_C V \to V$ is smooth and non-characteristic with respect to $j_!\mathcal{F}$ and set $Y = X \times_C c$.

1. *We have*

$$\chi_c(U, \mathcal{F}) = (2-2g) \cdot \chi_c(U \cap Y, \mathcal{F}|_{U\cap Y}) - \sum_{x\in Z} \dim \operatorname{tot}\phi_x(j_!\mathcal{F}, f). \tag{3.30}$$

2. *We have*

$$\begin{aligned}(\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*_X X)_{T^*X} = (2-2g) \cdot {} & (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y), T^*_Y Y)_{T^*Y} \\ & + \sum_{x\in Z} (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), [df])_{T^*X,x}.\end{aligned} \tag{3.31}$$

PROOF. 1. By the assumption that $X \times_C V \to V$ is non-characteristic with respect to $j_!\mathcal{F}$, the cohomology sheaves of $Rf_*j_!\mathcal{F}$ are locally constant on V similarly as in the proof of Lemma 3.7 and we have rank $(Rf_*j_!\mathcal{F})_V = \chi_c(U \cap Y, \mathcal{F}|_{U\cap Y})$. Hence it suffices to apply the Grothendieck-Ogg-Shafarevich formula [13, Théorème 7.1] to compute $\chi_c(U, \mathcal{F}) = \chi(C, Rf_*j_!\mathcal{F})$.

2. By the cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & T^*C \times_C X \arrow[r] \arrow[d, "p"] & T^*X \\ C \arrow[r] & T^*C, \end{tikzcd}$$

we have

$$(\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*_X X)_{T^*X} = (p_*A, T^*_C C)_{T^*C}$$

where $A = (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*C \times_C X)_{T^*X}$ denotes the pull-back by the top right arrow.

Since $X \times_C V \to V$ is assumed non-characteristic, the push-forward p_*A is supported in the union of the 0-section and the inverse image of $C - V$.

Hence, it is the sum A_1+A_2 of a multiple A_1 of the zero-section and a linear combination A_2 of fibers. We have

$$\begin{aligned}(A_1, T^*_C C)_{T^*C} &= (p_*A, T^*_c C)_{T^*C} \cdot (T^*_C C, T^*_C C)_{T^*C} \\ &= (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*C \times_C Y)_{T^*X} \cdot (2g-2).\end{aligned}$$

By the exact sequence $0 \to T^*C \times_C Y \to T^*X \times_X Y \to T^*Y \to 0$ and the definition of $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y)$, we have $(\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*C \times_C Y)_{T^*X} = -(\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y), T^*_Y Y)_{T^*Y}$ and $(A_1, T^*_C C)_{T^*C}$ equals the first term in the right hand side of (3.31). Since $(A_2, T^*_C C)_{T^*C}$ is equal to the second term, the equality (3.31) is proved. □

THEOREM 3.19 (cf. [9, p.13 Corollaire]). *Let X be a projective smooth surface over an algebraically closed field k of characteristic $p > 0$, U be a dense open subscheme and $j\colon U \to X$ be the open immersion. Let Λ be a finite field of characteristic $\ell \neq p$ and $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on U. Then, we have an equality*

$$\chi_c(U, \mathcal{F}) = (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*_X X)_{T^*X}. \tag{3.32}$$

Theorem 3.19 is proved in [20, Théorème 1.2.1] under the following "non-feroce" assumption on $\mathcal{F}$: There exists a finite Galois covering V of U trivializing $\mathcal{F}$ such that for every point $\xi \in X$ of codimension 1, the pull-back $V \times_U \mathrm{Spec}\, K_\xi$ to the local field $K_\xi = \mathrm{Frac}(\hat{\mathcal{O}}_{X,\xi})$ at ξ is isomorphic to $\amalg\, \mathrm{Spec}\, L_i$ for finite extensions L_i of local fields such that the residue fields are separable over that of K_ξ.

PROOF. Let $\mathcal{L}$ be a very ample invertible $\mathcal{O}_X$-module satisfying the conditions (L) and (R). By Lemma 3.12, there exists a line $L \subset \mathbf{P}^\vee$ satisfying the conditions (P1)–(P3). Let H be the hyperplane corresponding to a closed point of L not contained in $D(\mathcal{R}_{\mathcal{L}} j_!\mathcal{F})$ and $Y = X \cap H$ be the hyperplane section. We compare (3.30) and (3.31) for the blow-up X_L and the pull-back $\mathcal{F}_L$ of $\mathcal{F}$ to $U_L = U \times_X X_L$. Then, since the axis A_L meets X transversely and does not meet D, we have

$$\chi_c(U_L, \mathcal{F}) = \chi_c(U, \mathcal{F}) + \mathrm{rank}\, \mathcal{F} \cdot \deg(X \cap A_L)$$

and

$$(\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}_L), T^*_{X_L}X_L)_{T^*X_L} = (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*_X X)_{T^*X} + \mathrm{rank}\ \mathcal{F}\cdot \deg(X\cap A_L).$$

By the Grothendieck-Ogg-Shafarevich formula, we have

$$\chi_c(U\cap Y, j_!\mathcal{F}|_{U\cap Y}) = (\mathrm{Char}(j_!\mathcal{F}|_Y), T^*_Y Y)_{T^*Y}. \tag{3.33}$$

Hence, by (3.30), (3.31) and Theorem 3.17, we have

$$\begin{aligned}\chi_c(U,\mathcal{F}) &- (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*_X X)_{T^*X} \\ &= 2(\mathrm{Char}(j_!\mathcal{F}|_Y) - \mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y), T^*_Y Y)_{T^*Y}.\end{aligned} \tag{3.34}$$

For $ij\in J$, let D°_{ij} be a finite scheme over D°_i such that $T_{ij}\times_{D_i} D^\circ_i$ is a line bundle over D°_{ij}. We put $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}) = \mathrm{rank}\ \mathcal{F}\cdot[T^*_X X] + \sum_{ij\in J} s^{\mathcal{R}}_{ij}(j_!\mathcal{F})[T_{ij}] + \sum_{x\in\Sigma} s^{\mathcal{R}}_x(j_!\mathcal{F})[T^*_x X]$ and define an effective Cartier divisor $DT^{\mathcal{R}}(j_!\mathcal{F})$ supported on D by

$$DT^{\mathcal{R}}(j_!\mathcal{F}) = \sum_{ij\in J} s^{\mathcal{R}}_{ij}(j_!\mathcal{F})\cdot[D^\circ_{ij}:D^\circ_i]\cdot D_i. \tag{3.35}$$

Since the coefficients of the 0-section $T^*_Y Y$ in $\mathrm{Char}(j_!\mathcal{F}|_Y)$ and $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y)$ are both $-\mathrm{rank}\ \mathcal{F}$ and the other coefficients are defined by the intersection number $-(DT(j_!\mathcal{F}), Y)_X$ and $-(DT^{\mathcal{R}}(j_!\mathcal{F}), Y)_X$, for the right hand side of (3.34) we have

$$\begin{aligned}\chi_c(U,\mathcal{F}) &- (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*_X X)_{T^*X} \\ &= -2(DT(j_!\mathcal{F}) - DT^{\mathcal{R}}(j_!\mathcal{F}), c_1(\mathcal{L}))_X.\end{aligned} \tag{3.36}$$

The left hand side of (3.36) is independent of the choice of an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ satisfying the conditions (L) and (R). Since the Néron-Severi group is generated by the classes of ample invertible sheaves, the difference $DT(j_!\mathcal{F}) - DT^{\mathcal{R}}(j_!\mathcal{F})$ is a divisor numerically equivalent to 0 by Lemma 3.1.2. Hence the right hand side of (3.36) is 0 and we obtain (3.32). □

Proposition 3.20. *The restriction of* $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F})$ *to the non-degenerate locus is equal to* $\mathrm{Char}(j_!\mathcal{F})$ *whose definition is recalled in Section* 1.

Proof. It suffices to show that the coefficients of $T^*_X X, T_{ij}$ for $ij \in J$ and $T^*_x X$ for $x \in \Sigma$ in $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F})$ are equal to the corresponding ones in $\mathrm{Char}(j_!\mathcal{F})$ as long as the latter is defined. By Proposition 3.11, the coefficient of $T^*_X X$ in $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F})$ is rank $\mathcal{F}$ and the assertion follows in this case.

We deduce the assertion on the coefficients of T_{ij} for $ij \in J$ from that $DT(j_!\mathcal{F}) - DT^{\mathcal{R}}(j_!\mathcal{F})$ is numerically equivalent to 0 proved at the end of the proof of Theorem 3.19. Let D_1 be an irreducible component of dimension 1 of D. The assertion is étale local by Corollary 3.15. By the additivity of the characteristic cycles, we may assume that J_1 consists of one element 1. Let $s_{11}(j_!\mathcal{F})$ and $s^{\mathcal{R}}_{11}(j_!\mathcal{F})$ be the coefficients of T_{11} in $\mathrm{Char}(j_!\mathcal{F})$ and in $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F})$.

By approximation, there exists a finite Galois extension L of the function field K of Galois group G of X such that the local field K_1 splits completely and that the inertia group at K_i for $i \in I, i \neq 1$ acts trivially on the stalk of $\mathcal{F}$. Let $Y \to X$ be the normalization in L and let $X' \to Y$ be a resolution of singularities.

Let H be the class of an ample line bundle on Y and let $D_{1,Y}$ be the inverse image of D_1 in Y. Then, since the divisor $DT(f^*j_!\mathcal{F}) - DT^{\mathcal{R}}(f^*j_!\mathcal{F})$ of X' is numerically equivalent to 0, we have $(DT(f^*j_!\mathcal{F}) - DT^{\mathcal{R}}(f^*j_!\mathcal{F}), H)_Y = 0$. Since the right hand side is $(s_{11}(j_!\mathcal{F}) - s^{\mathcal{R}}_{11}(j_!\mathcal{F})) \cdot [D^\circ_{11} : D^\circ_1] \times (D_{1,Y}, H)_Y$ and $(D_{1,Y}, H)_Y > 0$, we have $s_{11}(j_!\mathcal{F}) = s^{\mathcal{R}}_{11}(j_!\mathcal{F})$. Thus the assertion for the coefficient of T_{ij} for $ij \in J$ is proved.

Assume that D has simple normal crossing and that $\mathcal{F}$ is non-degenerate along D and let u be a closed point of D. We show that the coefficients of $T^*_u X$ in $\mathrm{Char}(j_!\mathcal{F})$ and $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F})$ are equal. Since we assume that $\mathrm{Char}(j_!\mathcal{F})$ is defined, it suffices to consider the cases where $\mathcal{F}$ is tame ramified along D and is totally wild ramified separately.

In the tamely ramified case, we deduce the assertion from the following Lemma.

Lemma 3.21. *Let X be a smooth surface over an algebraically closed field k and $f\colon X \to C$ be a smooth morphism to a smooth curve. Let D be a divisor with simple normal crossing and u be a closed point of D. Let $\mathcal{F}$ be a locally constant constructible sheaf $U = X - D$ tamely ramified along D.*

1. *If the restriction $D \to C$ of f is étale at u, we have $\phi_u(j_!\mathcal{F}, f) = 0$.*

2. *Assume that u is in the intersection of two components D_1 and D_2 of D and that the restriction $f|_{D_1}: D_1 \to C$ and $f|_{D_2}: D_2 \to C$ are étale. Then, we have $\psi_u^q(j_!\mathcal{F}, f) = 0$ for $q \neq 1$ and we have $\dim \psi_u^1(j_!\mathcal{F}, f) =$ rank $\mathcal{F}$. Further the action of the Galois group of the local field K_v at $v = f(u) \in C$ on $\psi_u^1(j_!\mathcal{F}, f)$ is tamely ramified.*

PROOF. 1. By [15], $f: X \to C$ is locally acyclic relatively to $j_!\mathcal{F}$ in this case.

2. We have $\psi_u^q(j_!\mathcal{F}, f) = 0$ for $q \neq 1$ and $\dim \psi_u^1(j_!\mathcal{F}, f) = \text{rank } \mathcal{F}$ by [19]. We show that the action of the Galois group of the local field K_v at $v = f(u) \in C$ on $\psi_u^1(j_!\mathcal{F}, f)$ is tamely ramified. Since the local tame monodromy is abelian, we may assume that $\mathcal{F}$ is of rank 1. Let $\pi: X' \to X$ be the blow-up at u and set $v = f(u)$. Let E be the exceptional divisor and let $w_1, w_2, w_3 \in E$ be the intersection with the proper transforms of D_1, D_2 and of the fiber $f^{-1}(f(u))$ respectively and set $E^\circ = E - \{w_1, w_2, w_3\}$.

An elementary computation as in [24] shows the following: $\psi^0(\pi^* j_!\mathcal{F}, f)|_{E^\circ}$ is a locally constant constructible sheaf of rank 1 tamely ramified at w_1, w_2, w_3 with a tame Galois action of the local field K_v of C at v and $\psi^q(\pi^* j_!\mathcal{F}, f)|_{E^\circ} = 0$ for $q \neq 0$. We have $\psi_{w_1}(\pi^* j_!\mathcal{F}, f) = \psi_{w_2}(\pi^* j_!\mathcal{F}, f) = 0$. We have $\psi_{w_3}^q(\pi^* j_!\mathcal{F}, f) = 0$ except for $q = 0, 1$ and $\psi_{w_3}^q(\pi^* j_!\mathcal{F}, f)$ for $q = 0, 1$ have the same dimension with a tame Galois action of the local field K_v. Thus, the Galois action of the local field K_v on $\psi(j_!\mathcal{F}, f) = R\Gamma(E, \psi(\pi^* j_!\mathcal{F}, f)|_E)$ is tamely ramified. □

In the case 1 (resp. 2) of Lemma 3.21, by Theorem 3.17 and Lemma 3.21, we have $(\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), [df]) = 0$ (resp. $=$ rank $\mathcal{F}$) and hence the coefficient of $T_u^* X$ in $\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F})$ is zero (resp. rank $\mathcal{F}$). Thus the assertion follows in the tamely ramified case.

Assume that $\mathcal{F}$ is totally wildly ramified along D. Let $f: X \to C$ be a morphism to a smooth curve defined on a neighborhood of u that is non-characteristic with respect to $j_!\mathcal{F}$. Then, $f: X \to C$ is locally acyclic relatively to $j_!\mathcal{F}$ and we have $\psi_u(j_!\mathcal{F}, f) = 0$ by [25, Proposition 3.15]. Hence the assertion follows as above. □

PROPOSITION 3.22. *Let X be a normal surface over k and let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on a dense open subscheme $U = X - D$. Let $f: X \to C$ be a flat morphism to a smooth curve and u be* *a closed point of X. Assume that u is an isolated characteristic point of f with respect to $j_!\mathcal{F}$ and that $D - \{u\}$ is étale over C on a neighborhood of u. Let $\pi\colon X' \to X$ be a resolution, $\mathcal{F}'$ be the pull-back of $\mathcal{F}$ and $E = \pi^{-1}(u)$ be the inverse image. Then, we have*

$$-\dim \operatorname{tot}\phi_u(j_!\mathcal{F}, f) = (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}'), [df])_{T^*X',E}. \tag{3.37}$$

Proof. By resolution, we may assume that X is projective and $X - \{u\}$ is smooth over k. Let $N \geqq 1$ be an integer such that and $g \equiv f \bmod \mathfrak{m}_u^N$ implies an isomorphism $\phi_u(j_!\mathcal{F}, f) \simeq \phi_u(j_!\mathcal{F}, g)$ by Theorem 2.14. We take an ample invertible $\mathcal{O}_X$-module $\mathcal{L}$ satisfying the conditions (L) and (R) on the complement of u and take a pencil L such that $p_L\colon X_L \to L$ satisfies the condition in Proposition 3.13 and $f \equiv p_L \bmod \mathfrak{m}_u^N$.

Let $U' \subset X'$ be the inverse image of $U \subset X$. Similarly as Lemma 3.18, we obtain equalities

$$\begin{aligned}&\chi_c(U', \mathcal{F}) + \mathrm{rank}\ \mathcal{F} \cdot \deg(X \cap A_L)\\ &= 2\chi_c(U' \times_X Y, \mathcal{F}|_{U' \times_X Y}) - \dim \operatorname{tot}\phi_u(j_!\mathcal{F}, p_L)\\ &\quad - \sum_{x \in X} \dim \operatorname{tot}\phi_x(j_!\mathcal{F}, p_L).\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}&(\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), T^*_{X'}X')_{T^*X'} + \mathrm{rank}\ \mathcal{F} \cdot \deg(X \cap A_L)\\ &= 2(\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}|_Y), T^*_Y Y)_{T^*Y} + (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), [df])_{T^*X',E}\\ &\quad + \sum_{x \in X} (\mathrm{Char}^{\mathcal{R}}(j_!\mathcal{F}), [dp_L])_{T^*X,x}.\end{aligned} \tag{3.39}$$

The corresponding terms in (3.38) and (3.39) are equal to each other except for the second terms in the right hand, by Theorems 3.19 and 3.17 and the Grothendieck-Ogg-Shafarevich formula. Hence we have an equality also for the second terms and the assertion follows. □

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