

*Poisson Deformations and Birational Geometry**

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§1. Introduction

A complex normal variety X is a *symplectic variety* if there is a holomorphic symplectic 2-form ω on the regular locus X_{reg} of X and ω extends to a holomorphic 2-form on a resolution \tilde{X} of X . Notice that, if ω extends to a holomorphic 2-form on a particular resolution of X , then it also extends to a holomorphic 2-form on an arbitrary resolution of X .

In this article X is an affine symplectic variety with a coordinate ring R , where R is positively graded as $R = \bigoplus_{i \geq 0} R_i$ with $R_0 = \mathbf{C}$. Moreover we assume that ω is homogeneous with respect to the natural \mathbf{C}^* -action on X . The weight l of ω is automatically positive because X has only canonical singularities ([Na 1], Lemma 2.2). These varieties are constructed in various ways such as nilpotent orbit closures of a complex semisimple Lie algebra, Slodowy slices to nilpotent orbits, symplectic quotient singularities, Nakajima quiver varieties and so on.

In the remainder we assume that X has a (projective) crepant resolution¹ $\pi : Y \rightarrow X$. Then the symplectic 2-form ω extends to a symplectic 2-form ω_Y on Y . Moreover the \mathbf{C}^* -action uniquely extends to a \mathbf{C}^* -action on Y ([Na 2], Proposition A.7). Notice that the symplectic structures ω and ω_Y determine Poisson structures respectively on X and Y . A Poisson deformation of X (resp. Y) is just a deformation of the pair of X (resp. Y) and its Poisson structure. These have been extensively studied in [Ka], [K-V], [G-K], [Na 3] and [Na 4].

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¹More generally we may take a \mathbf{Q} -factorial terminalization as Y , which always exists by a main result of [BCHM].

Here let us consider the second cohomology space $H^2(Y, \mathbf{C})$. The first interpretation of this space is the Picard group $\text{Pic}(Y) \otimes_{\mathbf{Z}} \mathbf{C}$ tensorised with \mathbf{C} . One can consider various cones inside $\text{Pic}(Y) \otimes_{\mathbf{Z}} \mathbf{R}$. In particular, the π -movable cone $\text{Mov}(\pi)$ is decomposed into the ample cones $\text{Amp}(\pi')$ for different crepant resolutions $\pi' : Y' \rightarrow X$ of X . The codimension one faces of each ample cone generate hyperplanes of $\text{Pic}(Y) \otimes_{\mathbf{Z}} \mathbf{R}$ and these hyperplanes divide $\text{Mov}(\pi)$ into various chambers.

The second interpretation of $H^2(Y, \mathbf{C})$ is the *base space* of the universal Poisson deformation of Y . By using the \mathbf{C}^* -action on Y one may construct the universal Poisson deformation $\mathcal{Y} \rightarrow H^2(Y, \mathbf{C})$. The central fibre \mathcal{Y}_0 over $0 \in H^2(Y, \mathbf{C})$ is isomorphic to Y ; hence \mathcal{Y}_0 usually contains proper subvarieties as the exceptional locus of π . But it turns out that a general fibre \mathcal{Y}_t is an affine variety. We define the subset $\mathcal{D} \subset H^2(Y, \mathbf{C})$ as the locus where \mathcal{Y}_t is not affine.

The purpose of this article is to compare these two structures on $H^2(Y, \mathbf{C})$. We explain the main result in a more precise form. The universal Poisson deformation of X is constructed as a family of Poisson varieties over a quotient space $H^2(Y, \mathbf{C})/W$ by a finite group W . The universal Poisson deformations of X and Y fit into the commutative diagram

$$(1) \quad \begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ H^2(Y, \mathbf{C}) & \xrightarrow{q} & H^2(Y, \mathbf{C})/W \end{array}$$

Here the central fibres \mathcal{Y}_0 and $\mathcal{X}_{q(0)}$ are respectively Y and X and the induced map $\mathcal{Y}_0 \rightarrow \mathcal{X}_{q(0)}$ coincides with π .

Put $\mathcal{X}' := \mathcal{X} \times_{H^2(Y, \mathbf{C})/W} H^2(Y, \mathbf{C})$ and consider the induced commutative diagram

$$(2) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{\Pi} & \mathcal{X}' \\ \downarrow & & \downarrow \\ H^2(Y, \mathbf{C}) & \xrightarrow{id} & H^2(Y, \mathbf{C}) \end{array}$$

The maps $\Pi_t : \mathcal{Y}_t \rightarrow \mathcal{X}'_t$ are birational for all $t \in H^2(Y, \mathbf{C})$ and they are isomorphisms for general $t \in H^2(Y, \mathbf{C})$. Let $\mathcal{D} \subset H^2(Y, \mathbf{C})$ be the locus where Π_t is not an isomorphism. Notice that \mathcal{D} coincides with the locus

where \mathcal{X}'_t is singular. It can be also characterized as the locus where \mathcal{Y}_t is not an affine variety. Then our main theorem asserts that

MAIN THEOREM.

(i) *there are finitely many linear spaces $\{H_i\}_{i \in I}$ of $H^2(Y, \mathbf{Q})$ with $\text{Codim}(H_i) = 1$ such that $\mathcal{D} = \bigcup_{i \in I} (H_i)_{\mathbf{C}}$,*

(ii) *there are only finitely many crepant projective resolutions $\{\pi_k : Y_k \rightarrow X\}_{k \in K}$ of X , and*

(iii) *the set of open chambers determined by the real hyperplanes $\{(H_i)_{\mathbf{R}}\}_{i \in I}$ coincides with the set $\{w(\text{Amp}(\pi_k))\}$ where $w \in W$ and π_k are crepant projective resolutions of X .*

Notice that a crepant resolution $\pi : Y \rightarrow X$ is a relative Mori dream space in the sense of [H-K] when, more generally, X is a rational Gorenstein singularity (cf. Corollary 1.3.2 of [BCHM]). But the chamber structure is determined by the set of *full* hyperplanes when X is a symplectic singularity. This property does not hold for a general rational Gorenstein singularity (Example 13).

The simplest nontrivial example of the main theorem would be the following.

Example. Let $X := \{(x, y, z) \in \mathbf{C}^3; x^2 + y^2 + z^n = 0\}$ be an A_{n-1} -surface singularity and let $\pi : Y \rightarrow X$ be the minimal resolution. Let E_1, \dots, E_{n-1} be the irreducible components of $\text{Exc}(\pi)$. The 2-nd cohomology $H^2(Y, \mathbf{R})$ has dimension $n - 1$ and is generated by $[E_i]$'s. We introduce a negative definite symmetric form $(\ , \)$ on $H^2(Y, \mathbf{R})$ by the usual intersection pairings of E_i 's. The subset $\Phi := \{E := \sum a_i E_i; E^2 = -2, a_i \in \mathbf{Z}\}$ of $H^2(Y, \mathbf{R})$ determines a root system of A_{n-1} -type. The Weyl group of Φ is isomorphic to the symmetric group S_n . Thus S_n acts on $H^2(Y, \mathbf{R})$ as the Weyl group. Moreover $H^2(Y, \mathbf{R})$ is divided into $n!$ Weyl chambers. In particular, $\text{Amp}(\pi)$ coincides with one of them:

$$\text{Amp}(\pi) = \{x \in H^2(Y, \mathbf{R}); (x, [E_j]) > 0 \text{ for all } j\}.$$

Note that X has a \mathbf{C}^* -action $(x, y, z) \rightarrow (t^n x, t^n y, t^2 z)$, $t \in \mathbf{C}^*$ and the symplectic 2-form

$$\omega := \text{Res}(dx \wedge dy \wedge dz/x^2 + y^2 + z^n)$$

is homogeneous of weight 2 with respect to this \mathbf{C}^* -action. The universal Poisson deformation of X is given by

$$\mathcal{X} := \{(x, y, z, u_1, \dots, u_{n-1}) \in \mathbf{C}^{n+2}; \\ x^2 + y^2 + z^n + u_1 z^{n-2} + \dots + u_{n-2} z + u_{n-1} = 0\}$$

where $(u_1, \dots, u_{n-1}) \in \mathbf{C}^{n-1}$ is the base space. Let us consider an $n - 1$ dimensional vector space $V := \{(s_1, \dots, s_n) \in \mathbf{C}^n; \sum s_i = 0\}$ and a finite Galois covering $V \rightarrow \mathbf{C}^{n-1}$ defined by $(s_1, \dots, s_n) \rightarrow (\sigma_2(s_1, \dots, s_n), \dots, \sigma_n(s_1, \dots, s_n))$. Here σ_i are defined by

$$(z - s_1)(z - s_2) \cdots (z - s_n) = z^n + \sigma_1(s_1, \dots, s_n)z^{n-1} + \dots + \sigma_n(s_1, \dots, s_n).$$

The Galois group of this map is the symmetric group S_n acting on V by the permutation of s_i 's. Thus $\mathbf{C}^{n-1} = V/S_n$. We put $\mathcal{X}' := \mathcal{X} \times_{V/S_n} V$. Then \mathcal{X}' can be written as

$$\mathcal{X}' := \{(x, y, z, s_1, \dots, s_n) \in \mathbf{C}^{n+3}; \\ x^2 + y^2 + (z - s_1)(z - s_2) \cdots (z - s_n) = 0, \sum s_i = 0\}.$$

The map $\mathcal{X}' \rightarrow V$ can be viewed as an $n - 1$ dimensional family of surfaces. Define the discriminant locus \mathcal{D} by

$$\mathcal{D} := \{s = (s_1, \dots, s_n) \in V; \mathcal{X}'_s \text{ is singular}\}.$$

Then one has

$$\mathcal{D} = \bigcup_{1 \leq i < j \leq n} L_{ij},$$

where

$$L_{ij} = \{(s_1, \dots, s_n) \in V; s_i = s_j\}.$$

The family can be resolved simultaneously. The simultaneous resolution is not unique. Let us take one of them, say \mathcal{Y} . Then we have a commutative diagram

$$(3) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{\Pi} & \mathcal{X}' \\ \downarrow & & \downarrow \\ V & \xrightarrow{id} & V \end{array}$$

Then $\mathcal{Y} \rightarrow V$ is the universal Poisson deformation of Y .

For a general point $t_{ij} \in L_{ij}$, the induced map $\Pi_{t_{ij}} : \mathcal{Y}_{t_{ij}} \rightarrow \mathcal{X}'_{t_{ij}}$ contracts one smooth rational curve C_{ij} to a point. By using the \mathbf{C}^* -action on \mathcal{Y} we put $E_{ij} := \lim_{\lambda \rightarrow 0} \lambda(C_{ij})$. We may choose a simultaneous resolution $\Pi : \mathcal{Y} \rightarrow \mathcal{X}'$ so that $E_{12} = E_1, E_{23} = E_2, \dots, E_{n-1,n} = E_{n-1}$.

We introduce a positive definite symmetric form on

$$V(\mathbf{R}) := \{(s_1, \dots, s_n) \in \mathbf{R}^n; \Sigma s_i = 0\}$$

as the restriction of the standard metric on \mathbf{R}^n . Define an \mathbf{R} -linear isomorphism $\alpha : H^2(Y, \mathbf{R}) \rightarrow V(\mathbf{R})$ by $\alpha([E_1]) = (1, -1, 0, \dots, 0)$, $\alpha([E_2]) = (0, 1, -1, 0, \dots, 0)$, ..., $\alpha([E_{n-1}]) = (0, 0, \dots, 1, -1)$. The map α preserves the symmetric forms up to a reversal of sign. Each Weyl chamber of $H^2(Y, \mathbf{R})$ is mapped to a chamber of $V(\mathbf{R})$ determined by $L_{ij} \cap V(\mathbf{R})$'s. In particular, $\text{Amp}(\pi)$ is mapped to a chamber surrounded by $n - 1$ rays

$$\mathbf{R}_{\geq 0}(-n + i, \dots, -n + i, i, \dots, i) \quad (1 \leq i \leq n - 1)$$

Here $-n + i$ occurs i times and i occurs $n - i$ times.

The map α induces a \mathbf{C} -linear isomorphism $\alpha_{\mathbf{C}} : H^2(Y, \mathbf{C}) \rightarrow V$, whose inverse coincides with the period map for $\mathcal{Y} \rightarrow V$ determined by a suitable multiple of ω . (END of EXAMPLE)

The example above can be generalized to the Slodowy slice to an arbitrary nilpotent orbit in a complex simple Lie algebra (see Example 11).

One can find in [Na 6] explicit descriptions of the chamber structures and the flops connecting them in the case where X is a nilpotent orbit closure of a complex simple Lie algebra. In particular, we can determine the explicit equations of the hyperplanes H_i . In Example 12 we illustrate this by using one typical example.

The birational geometry for crepant resolutions of 4-dimensional symplectic singularities is also treated in [A-W].

Recently Bellamy [Bel, Theorem 1.4] has identified the Calogero-Moser deformation of a symplectic quotient singularity X with the family $\mathcal{X}' \rightarrow H^2(Y, \mathbf{C})$. Moreover, he explicitly computed the discriminant locus $\mathcal{D} \subset H^2(Y, \mathbf{C})$ in some interesting cases. As an application of our main theorem, he finally counted the number of crepant resolutions of X ([ibid, 4.2, 4.3]).

§2. Proof of Main Theorem

Let $\pi : Y \rightarrow X$ be the same as in Introduction. Let $\text{Amp}(\pi) \subset H^2(Y, \mathbf{R})$ be the open convex cone generated by π -ample line bundles on Y . Its closure $\overline{\text{Amp}(\pi)}$ coincides with the cone generated by π -nef line bundles. Thus we often call $\overline{\text{Amp}(\pi)}$ the *nef cone* of π . A line bundle L on Y is called π -movable if the support of $\text{Coker}[\pi^*\pi_*L \rightarrow L]$ has codimension at least two. The closed cone in $H^2(Y, \mathbf{R})$ generated by the classes of π -movable line bundles is denoted by $\overline{\text{Mov}(\pi)}$ and its interior is called the *movable cone* for π . In the remainder we will write $\text{Mov}(\pi)$ for the movable cone. By the definition we have $\text{Amp}(\pi) \subset \text{Mov}(\pi)$.

The following is a fundamental fact on birational geometry proved almost thirty years ago.

LEMMA 1. *The nef cone $\overline{\text{Amp}(\pi)}$ is a rational polyhedral cone.*

PROOF. Take an effective divisor D of Y in such a way that $-D$ is π -ample. If we take a rational number $\epsilon > 0$ very small and put $\Delta := \epsilon D$. Then (Y, Δ) has klt (Kawamata log terminal) singularities. Since $\overline{K_Y}$ is trivial, $-(K_Y + \Delta)$ is π -ample. The cone of effective 1-cycle cone $\overline{NE}(\pi)$ is then a rational polyhedral cone (cf. [KMM, Theorem 4-2-1, Proposition 4-2-4]). Its dual cone $\overline{\text{Amp}(\pi)}$ is hence a rational polyhedral cone. \square

The symplectic 2-form ω on X_{reg} identifies the sheaf $\Theta_{X_{reg}}$ of holomorphic vector fields with the sheaf $\Omega_{X_{reg}}^1$ of holomorphic 1-forms. Moreover it induces an isomorphism $\wedge^2 \Theta_{X_{reg}} \cong \Omega_{X_{reg}}^2$. By this isomorphism ω is identified with a 2-vector θ on X_{reg} , which we call the *Poisson 2-vector*. The Poisson 2-vector defines a bracket on the structure sheaf $\mathcal{O}_{X_{reg}}$ by $\{f, g\} := \theta(df \wedge dg)$ with $f \in \mathcal{O}_{X_{reg}}$ and $g \in \mathcal{O}_{X_{reg}}$. By the definition this bracket is a skew-symmetric \mathbf{C} -bilinear form and is a biderivation with respect to each factor. One can prove that this bracket satisfies the Jacobi identity by using the fact that ω is d-closed. Thus X_{reg} admits a *Poisson structure*. By the normality of X , this Poisson structure uniquely extends to a Poisson structure on X .

A T -scheme $\mathcal{X} \rightarrow T$ is called a Poisson T -scheme if there is an \mathcal{O}_T -bilinear Poisson bracket $\{, \}_\mathcal{X} : \mathcal{O}_\mathcal{X} \times \mathcal{O}_\mathcal{X} \rightarrow \mathcal{O}_\mathcal{X}$. Let T be a scheme over \mathbf{C} and let $0 \in T$ be a closed point.

A Poisson deformation of the Poisson variety X is a Poisson T -scheme $f : \mathcal{X} \rightarrow T$ together with an isomorphism $\phi : \mathcal{X}_0 \cong X$ which satisfies the following conditions

- (i) f is a flat surjective morphism, and
- (ii) $\{ , \}_{\mathcal{X}}$ restricts to the original Poisson structure $\{ , \}_X$ on X via the identification ϕ .

Two Poisson deformations $(\mathcal{X}/T, \phi)$ and $(\mathcal{X}'/T, \phi')$ with the same base are *equivalent* if there is a T -isomorphism $\mathcal{X} \cong \mathcal{X}'$ of Poisson schemes such that it induces the identity map on the central fibre.

For a local Artinian \mathbf{C} -algebra A with residue field \mathbf{C} , denote by $PD_X(A)$ the set of equivalence classes of Poisson deformations of X over $\text{Spec}(A)$. Then it defines a functor

$$PD_X : (Art)_{\mathbf{C}} \rightarrow (Set)$$

from the category of local Artinian \mathbf{C} -algebra with residue field \mathbf{C} to the category of sets.

Assume that a crepant projective resolution $\pi : Y \rightarrow X$ is given. Then the symplectic 2-form ω extends to a symplectic 2-form ω_Y on Y . The symplectic 2-form ω_Y determines a Poisson structure $\{ , \}_Y$ on Y . Notice that $\pi : (Y, \{ , \}_Y) \rightarrow (X, \{ , \}_X)$ is a morphism of Poisson varieties, i.e. $\pi^*\{f, g\} = \{\pi^*f, \pi^*g\}_Y$ for $f, g \in \mathcal{O}_X$.

Assume that $A \in (Art)_{\mathbf{C}}$ and $T := \text{Spec}A$. If $(\mathcal{Y}/T, \phi)$ is a Poisson deformation of Y , then $\text{Spec}\Gamma(Y, \mathcal{O}_{\mathcal{Y}}) \rightarrow T$ is a flat deformation of $X = \text{Spec}\Gamma(Y, \mathcal{O}_Y)$ because $H^1(Y, \mathcal{O}_Y) = 0$ (cf. [Wa]). Moreover, the Poisson structure on \mathcal{Y} determines a Poisson bracket on $\Gamma(Y, \mathcal{O}_{\mathcal{Y}})$. Thus we have a morphism of functors

$$\pi_* : PD_{\mathcal{Y}} \rightarrow PD_X.$$

We can apply Schlessinger's theory [Sch] to these functors to get the prorepresentable hulls R_X and R_Y . By definition R_X and R_Y are both complete local \mathbf{C} -algebras with residue field \mathbf{C} . Let m_X (resp. m_Y) be the maximal ideal of R_X (resp. R_Y) and put $R_{X,n} := R_X/m_X^{n+1}$ and $R_{Y,n} := R_Y/m_Y^{n+1}$. There are formal semiuniversal Poisson deformations $\{X_n \rightarrow \text{Spec}R_{X,n}\}_{n \geq 0}$ and $\{Y_n \rightarrow \text{Spec}R_{Y,n}\}_{n \geq 0}$. Moreover, for each $n \geq 0$, there is a commutative

diagram

$$(4) \quad \begin{array}{ccc} Y_n & \longrightarrow & X_n \\ \downarrow & & \downarrow \\ \text{Spec}R_{Y,n} & \longrightarrow & \text{Spec}R_{X,n} \end{array}$$

By Corollary 15 of [Na 2] the Poisson deformation functor PD_Y is unobstructed and its tangent space $PD_Y(\mathbf{C}[\epsilon])$ is naturally isomorphic to $H^2(Y, \mathbf{C})$ (cf. [Na 2], Proposition 8, Corollary 10). By Theorem 5.1 of [Na 3], PD_X is also unobstructed and R_Y is a finite R_X -module. Moreover, by Corollary 2.5 of [Na 3], both functors PD_X and PD_Y are prorepresentable, in other words, $\{X_n \rightarrow \text{Spec}R_{X,n}\}_{n \geq 0}$ and $\{Y_n \rightarrow \text{Spec}R_{Y,n}\}_{n \geq 0}$ are respectively formal universal Poisson deformations of X and Y .

The main result of [Na 4] asserts that $\pi_* : PD_Y \rightarrow PD_X$ is a finite *Galois* covering with a finite Galois group W . By the \mathbf{C}^* -actions we can algebraize the commutative diagram above ([Na 3, Section 5])

$$(5) \quad \begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ H^2(Y, \mathbf{C}) & \xrightarrow{q} & H^2(Y, \mathbf{C})/W \end{array}$$

Here W acts linearly on $H^2(Y, \mathbf{Q})$ and the diagram is \mathbf{C}^* -equivariant. Each linear function on $H^2(Y, \mathbf{C})$ has weight $l = wt(\omega)$. For each $t \in H^2(Y, \mathbf{C})$, the induced map $\mathcal{Y}_t \rightarrow \mathcal{X}_{q(t)}$ is a birational morphism and it is an isomorphism for general t . Notice that $\mathcal{Y}_0 \rightarrow \mathcal{X}_{q(0)}$ coincides with $\pi : Y \rightarrow X$. The map $\mathcal{Y} \rightarrow H^2(Y, \mathbf{C})$ is a C^∞ -fibre bundle and every fibre \mathcal{Y}_t is diffeomorphic to Y . In particular, there is a natural identification of $H^2(\mathcal{Y}_t, \mathbf{Z})$ with $H^2(Y, \mathbf{Z})$.

Put $\mathcal{X}' := \mathcal{X} \times_{H^2(Y, \mathbf{C})/W} H^2(Y, \mathbf{C})$ and consider the induced commutative diagram

$$(6) \quad \begin{array}{ccc} \mathcal{Y} & \xrightarrow{\Pi} & \mathcal{X}' \\ f \downarrow & & g \downarrow \\ H^2(Y, \mathbf{C}) & \xrightarrow{id} & H^2(Y, \mathbf{C}) \end{array}$$

Here $\Pi : \mathcal{Y} \rightarrow \mathcal{X}'$ is a crepant projective resolution. Let $\Pi' : \mathcal{Y}' \rightarrow \mathcal{X}'$ be another crepant projective resolution. Since the birational map $\mathcal{Y}' \dashrightarrow \mathcal{Y}$

\mathcal{Y} is an isomorphism in codimension one, we have a natural identification $H^2(\mathcal{Y}', \mathbf{R}) \cong H^2(\mathcal{Y}, \mathbf{R})$. By this identification the nef cone $\overline{\text{Amp}}(\Pi')$ can be seen as a cone in $H^2(\mathcal{Y}, \mathbf{R})$. Furthermore, as the restriction map $H^2(\mathcal{Y}, \mathbf{R}) \rightarrow H^2(Y, \mathbf{R})$ is an isomorphism, we can regard $\overline{\text{Amp}}(\Pi')$ as a cone in $H^2(Y, \mathbf{R})$. By the same reasoning as in Lemma 1 $\overline{\text{Amp}}(\Pi')$ is a rational polyhedral cone.

Take a nonzero element $\alpha \in H^2(Y, \mathbf{C})$ and consider the complex line $\mathbf{A}_\alpha^1 := \{t\alpha\}_{t \in \mathbf{C}}$ inside $H^2(Y, \mathbf{C})$. We put

$$\mathcal{Y}^\alpha := \mathcal{Y} \times_{H^2(Y, \mathbf{C})} \mathbf{A}_\alpha^1,$$

and

$$\mathcal{X}^\alpha := \mathcal{X}' \times_{H^2(Y, \mathbf{C})} \mathbf{A}_\alpha^1.$$

We then get a commutative diagram

$$(7) \quad \begin{array}{ccc} \mathcal{Y}^\alpha & \xrightarrow{\Pi^\alpha} & \mathcal{X}^\alpha \\ \downarrow & & \downarrow \\ \mathbf{A}_\alpha^1 & \xrightarrow{id} & \mathbf{A}_\alpha^1 \end{array}$$

According to Kaledin [Ka] we call $\mathcal{Y}^\alpha \rightarrow \mathbf{A}_\alpha^1$ the *twistor deformation* of Y determined by α . There is a relative symplectic 2-form $\omega_{\mathcal{Y}}$ on \mathcal{Y} extending the symplectic 2-form ω_Y on Y . Define $\omega_{\mathcal{Y}^\alpha}$ to be the restriction of $\omega_{\mathcal{Y}}$ to \mathcal{Y}^α . A remarkable property of the twistor deformation is that, for each $t \in \mathbf{A}_\alpha^1$, the 2-nd cohomology class $[\omega_{\mathcal{Y}_t^\alpha}] \in H^2(\mathcal{Y}_t^\alpha, \mathbf{C})$ coincides with $t \cdot \alpha \in H^2(Y, \mathbf{C})$ under the natural identification $H^2(\mathcal{Y}_t^\alpha, \mathbf{C}) \cong H^2(Y, \mathbf{C})$ (cf. [G-K]).

LEMMA 2. *If $(\alpha, C) \neq 0$ for any proper curve $C \subset Y$ such that $\pi(C)$ is a point, then Π_t^α is an isomorphism for all $t \in \mathbf{A}_\alpha^1 - \{0\}$.*

PROOF. Assume that a proper curve $C_t \subset \mathcal{Y}_t^\alpha$ is contracted to a point by Π_t^α for some $t \in \mathbf{A}_\alpha^1 - \{0\}$. Since $\omega_{\mathcal{Y}_t^\alpha}$ is a holomorphic 2-form, we must have $([\omega_{\mathcal{Y}_t^\alpha}], C_t) = 0$. By the \mathbf{C}^* -action, we have a proper (but, not necessarily irreducible or reduced) curve $C \subset Y$ as a limit of C_t . Then we have $(t\alpha, C) = 0$; hence $(\alpha, C) = 0$. \square

It is not true that if $(\alpha, C) = 0$ for some proper curve $C \subset Y$, then the birational map Π_t^α contracts some curves to points for each $t \in \mathbf{A}_\alpha^1$. But we have the following:

LEMMA 3. *Assume that π factorizes as $Y \xrightarrow{\mu} Z \rightarrow X$ with Z normal, μ birational, and $\text{Exc}(\mu) \neq \emptyset$. If $\alpha \in \mu^*(H^2(Z, \mathbf{C}))$, then $\text{Exc}(\Pi_t^\alpha) \neq \emptyset$ for all $t \in \mathbf{A}_\alpha^1$.*

PROOF. Since Z has only rational singularities, we have an exact sequence

$$0 \rightarrow H^2(Z, \mathbf{C}) \rightarrow H^2(Y, \mathbf{C}) \rightarrow H^0(Z, R^2\mu_*\mathbf{C}).$$

Recall that a 1-st order Poisson deformation $\mathcal{Y}_1 \rightarrow \text{Spec}\mathbf{C}[\epsilon]$ of Y corresponds to an element α of $H^2(Y, \mathbf{C})$ ([Na 2], Proposition 8, Corollary 10). For a point $p \in Z$ take a sufficiently small analytic neighborhood U of p . Then \mathcal{Y}_1 induces a 1-st order Poisson deformation of $\mu^{-1}(U)$ and it determines an element of $H^2(\mu^{-1}(U), \mathbf{C})$ ([ibid], Proposition 8, Corollary 10). If $\alpha \in H^2(Z, \mathbf{C})$, then the induced Poisson deformation of $\mu^{-1}(U)$ is trivial by the exact sequence above.

Now let us define a subfunctor PD_Y^μ of PD_Y as follows.

For $A \in (\text{Art})_{\mathbf{C}}$, define $\text{PD}_Y^\mu(A)$ to be the set of equivalence classes of the Poisson deformations $f_T : \mathcal{Y}_T \rightarrow T$ of Y ($T := \text{Spec}(A)$) with the following property:

(*) For each $p \in Z$, there is a sufficiently small (analytic) neighborhood U of p such that f_T induce trivial Poisson deformations of $\mu^{-1}(U)$.

By the argument above, the tangent space $PD_Y^\mu(\mathbf{C}[\epsilon])$ of the functor PD_Y^μ is isomorphic to $H^2(Z, \mathbf{C})$. The functor PD_Y^μ has a prorepresentable hull R_Y^μ in the sense of [Sch]; one can easily check all axioms of [Sch] in the same way as in PD_Y . Now one can apply the T^1 -lifting method to prove that PD_Y^μ is unobstructed. More precisely, let $A_n := \mathbf{C}[t]/(t^{n+1})$ and take an element $[\mathcal{Y}_n \rightarrow \text{Spec}(A_n)] \in \text{PD}_Y^\mu(A_n)$. Put $\mathcal{Y}_{n-1} := \mathcal{Y}_n \times_{\text{Spec}(A_n)} \text{Spec}(A_{n-1})$. The birational map $\mu : Y \rightarrow Z$ extends to a birational map $\mu_n : \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ over $\text{Spec}(A_n)$ by [Wa] because $R^1\mu_*O_Y = 0$. Put $\mathcal{Z}_{n-1} := \mathcal{Z}_n \times_{\text{Spec}(A_n)} \text{Spec}(A_{n-1})$. Notice that \mathcal{Z}_n and \mathcal{Z}_{n-1} are both Poisson deformations of Z . There is a natural restriction map $H^2(\mathcal{Z}_n, A_n) \rightarrow H^2(\mathcal{Z}_{n-1}, A_{n-1})$, which is a surjection. This implies that PD_Y^μ has the T^1 -lifting property and hence PD_Y^μ is unobstructed ([Kaw]).

Recall that the twistor deformation \mathcal{Y}^α is constructed inductively as the objects of PD_Y by using the T^1 -lifting property of PD_Y (cf. [Na 2], p. 281). But, if $\alpha \in H^2(Z, \mathbf{C})$, then it is already constructed as the objects of PD_Y^μ by the T^1 -lifting property of PD_Y^μ . This means that our twistor deformation induces a trivial Poisson deformation of $\mu^{-1}(U)$. In particular, $\text{Exc}(\Pi_t^\alpha) \neq \emptyset$ for all $t \in \mathbf{A}_\alpha^1$. \square

Let $\mathcal{C} \subset \mathcal{X}'$ be the locus on which Π is not an isomorphism. The locus \mathcal{C} can be also defined as the subset where g is not a smooth morphism. We define $\mathcal{D} \subset H^2(Y, \mathbf{C})$ to be the closure of $g(\mathcal{C})$. By Lemma 2, \mathcal{D} is a closed algebraic subset of $H^2(Y, \mathbf{C})$ with $\mathcal{D} \neq H^2(Y, \mathbf{C})$. We call \mathcal{D} the *discriminant locus* for Π .

LEMMA 4. *Assume that $\alpha \in H^2(Y, \mathbf{C})$ is contained in a complex hyperplane of $H^2(Y, \mathbf{C})$ generated by a codimension one face F of $\text{Amp}(\pi')$ for a crepant resolution $\pi' : Y' \rightarrow X$. Then $\alpha \in g(\mathcal{C})$.*

PROOF. The codimension one face F defines a birational contraction map $\mu : Y' \rightarrow Z$ and it factorizes π' as $Y' \rightarrow Z \rightarrow X$. Consider the universal Poisson deformation $\mathcal{Y}' \rightarrow H^2(Y', \mathbf{C})$ of Y' . We identify the base space $H^2(Y', \mathbf{C})$ with $H^2(Y, \mathbf{C})$. Then we have two families \mathcal{Y}' and \mathcal{X}' of Poisson varieties over $H^2(Y, \mathbf{C})$ and there is a birational projective morphism $\Pi' : \mathcal{Y}' \rightarrow \mathcal{X}'$. By the assumption $\alpha \in \mu^*(H^2(Z, \mathbf{C}))$. By applying Lemma 3 to $\pi' : Y' \rightarrow X$, we see that $\text{Exc}(\Pi'_\alpha) \neq \emptyset$. Hence $\alpha \in g(\mathcal{C})$. \square

COROLLARY 5. *There are only finitely many crepant projective resolutions of X .*

PROOF. Suppose that there are infinitely many crepant projective resolutions of X . Then we have infinitely many rational polyhedral cones inside $H^2(Y, \mathbf{R})$ corresponding to the nef cones of crepant resolutions. Note that the interior part of these cones do not intersect. The codimension one faces of these cones generates infinitely many complex hyperplanes of $H^2(Y, \mathbf{C})$. Since $\mathcal{D} \neq H^2(Y, \mathbf{C})$, these complex hyperplanes are all irreducible components of \mathcal{D} by Lemma 4. This is absurd. \square

THEOREM ([BCHM]). *Let $f : V \rightarrow W$ be a \mathbf{Q} -factorial terminalization of a normal variety W with rational Gorenstein singularities. Assume that*

f is an isomorphism in codimension one and $\rho(V/W) = 1$. Then there is a flop $f' : V' \rightarrow W$, in other words, f' is another \mathbf{Q} -factorial terminalization of W such that the proper transform D' of an f -negative divisor D is f' -ample.

In fact, $\bigoplus_{m \geq 0} f_* \mathcal{O}_V(mD)$ is a finitely generated \mathcal{O}_W -algebra. The flop V' is nothing but $\mathbf{Proj}_W \bigoplus_{m \geq 0} f_* \mathcal{O}_V(mD)$.

LEMMA 6. $\overline{\text{Mov}(\pi)}$ is the union of the nef cones of crepant projective resolutions of X .

PROOF. Assume that D is a π -movable divisor. If $D \notin \overline{\text{Amp}(\pi)}$, there is a codimension one face of $\overline{\text{Amp}(\pi)}$ such that the corresponding birational X -morphism $\phi_1 : Y \rightarrow Z_1$ contracts D -negative curves. Since D is π -movable, ϕ_1 is an isomorphism in codimension one. By the previous theorem we can take its flop $\phi'_1 : Y_1 \rightarrow Z_1$. Let $\pi_1 : Y_1 \rightarrow X$ be the composition of ϕ'_1 and $Z_1 \rightarrow X$. Notice that π_1 is a crepant resolution by [Na 2, Corollary 31]. Let $D_1 \subset Y_1$ be the proper transform of D . If D_1 is π_1 -nef, then $D \in \overline{\text{Amp}(\pi_1)}$. If D_1 is not π_1 -nef, we repeat the same process. Since there are only finitely many different crepant projective resolutions of X , if this process does not terminate, there appear the same crepant resolution twice in the process, say Y_i and Y_j ($i < j$). Take a discrete valuation v of the function field K of Y_i in such a way that its center is contained in $\text{Exc}(\phi_{i-1})$. Let $D_i \subset Y_i$ be the proper transform of D . Then we have inequalities for discrepancies (cf. [KMM], Proposition 5-1-11):

$$a(v, D_i) < a(v, D_{i+1}) \leq a(v, D_{i+2}) \leq \dots \leq a(v, D_j).$$

Here the first inequality is a strict one by the choice of v . On the other hand, we have $Y_i = Y_j$ and $D_i = D_j$ by the assumption. This is absurd. Thus we finally reach $\pi_n : Y_n \rightarrow X$ such that D_n is π_n -nef. Then $D \in \overline{\text{Amp}(\pi_n)}$. \square

The following fact is pointed out by Braden, Proudfoot and Webster.

PROPOSITION ([BPW, Proposition 2.19]). *The movable cone $\text{Mov}(\pi) \subset H^2(Y, \mathbf{R})$ is a fundamental domain for the W -action on $H^2(Y, \mathbf{R})$.*

The group W also acts on \mathcal{X}' and the map $\mathcal{X}' \rightarrow H^2(Y, \mathbf{C})$ is W -equivariant. Let $\Pi' : \mathcal{Y}' \rightarrow \mathcal{X}'$ be another crepant projective resolution of \mathcal{X}' . For $w \in W$, we take the fibre product

$$(8) \quad \begin{array}{ccc} \mathcal{Y}'_w & \longrightarrow & \mathcal{Y}' \\ \Pi'_w \downarrow & & \Pi' \downarrow \\ \mathcal{X}' & \xrightarrow{w} & \mathcal{X}' \end{array}$$

We call \mathcal{Y}'_w the w -twist of \mathcal{Y}' .

Let $\{\pi_k : Y_k \rightarrow X\}_{k \in K}$ be the set of all crepant projective resolutions of X . For each $k \in K$, we can construct the universal Poisson deformation $\mathcal{Y}_k \rightarrow H^2(Y, \mathbf{C})$ of Y_k . Notice that there is a $H^2(Y, \mathbf{C})$ -birational projective morphism $\Pi_k : \mathcal{Y}_k \rightarrow \mathcal{X}'$ and Π_k is a crepant resolution of \mathcal{X}' .

COROLLARY 7. *Every \mathbf{Q} -factorial terminalization of \mathcal{X}' is obtained as the w -twist $\mathcal{Y}_{k,w}$ of \mathcal{Y}_k for $w \in W$. In particular, they are all smooth.*

PROOF. Denote by $\Pi_{k,w}$ the crepant resolution $\mathcal{Y}_{k,w} \rightarrow \mathcal{X}'$. The nef cone $\overline{\text{Amp}(\Pi_{k,w})} \subset H^2(Y, \mathbf{R})$ is $w(\overline{\text{Amp}(\pi_k)})$. By the previous proposition and Lemma 6, $H^2(Y, \mathbf{R})$ is covered by the cones $\{w(\overline{\text{Amp}(\pi_k)})\}$. \square

LEMMA 8. *Assume that $\alpha \in \mathcal{D}$. Then there exists a crepant projective resolution $\Pi' : \mathcal{Y}' \rightarrow \mathcal{X}'$ and a codimension one face of F of $\overline{\text{Amp}(\Pi')} \subset H^2(Y, \mathbf{R})$ such that the complex hyperplane H of $H^2(Y, \mathbf{C})$ generated by F contains α .*

PROOF. We may assume that $\alpha \in g(\mathcal{C})$. Then $\Pi_\alpha : \mathcal{Y}_\alpha \rightarrow \mathcal{X}'_\alpha$ has non-empty exceptional locus². Let H be a Π -ample divisor. Since $\text{Codim}_Y \text{Exc}(\Pi) \geq 2$, we have $D := -H \in \text{Mov}(\Pi)$. Notice that $(D, C) < 0$ for any proper curve C in \mathcal{Y}_α contracted by Π_α . Choose a codimension one face of $\overline{\text{Amp}(\Pi)}$ and let $\phi_1 : \mathcal{Y} \rightarrow \mathcal{Z}_1$ be the corresponding birational morphism. By the previous theorem there is a flop $\phi'_1 : \mathcal{Y}_1 \rightarrow \mathcal{Z}_1$. We denote by Π_1 the composition of ϕ'_1 and the map $\mathcal{Z}_1 \rightarrow \mathcal{X}'$. The resulting variety \mathcal{Y}_1 is again a crepant projective resolution of \mathcal{X}' by Corollary 7. Let $D_1 \subset \mathcal{Y}_1$ be the proper transform of D . If D_1 is not Π_1 -nef, one can find a codimension one face F_1 of $\overline{\text{Amp}(\Pi_1)}$ in such a way that the corresponding

²Caution: \mathcal{Y}_α is not the twistor deformation, but the fibre over α .

birational map $\phi_2 : \mathcal{Y}_1 \rightarrow \mathcal{Z}_2$ contracts D_1 -negative curves. By taking the flop of ϕ_2 we get a new crepant resolution $\Pi_2 : \mathcal{Y}_2 \rightarrow \mathcal{X}'$. Since there are only finitely many crepant resolutions of \mathcal{X}' by Corollary 7, this process eventually terminates by the same argument as in Lemma 6 and the proper transform $D_n \subset \mathcal{Y}_n$ of D is Π_n -nef for some n . Each map ϕ_i induces a birational map $\phi_{i,\alpha} : \mathcal{Y}_{i-1,\alpha} \rightarrow \mathcal{Z}_{i,\alpha}$. Suppose that all $\phi_{i,\alpha}$ are isomorphisms; then the birational map $\mathcal{Y} \dashrightarrow \mathcal{Y}_n$ is an isomorphism on a neighborhood of \mathcal{Y}_α . On the other hand, D is Π -negative around \mathcal{Y}_α and D_n is Π_n -nef around $\mathcal{Y}_{n,\alpha}$. This is absurd. Thus $\text{Exc}(\phi_{i_0,\alpha}) \neq \emptyset$ for some i_0 .

By Corollary 7, \mathcal{Y}_{i_0-1} can be written as $\mathcal{Y}_{k,w}$ with a crepant resolution Y_k of X and $w \in W$. Let C_α be a proper curve on $\mathcal{Y}_{i_0-1,\alpha}$ contracted by $\phi_{i_0,\alpha}$. Then $([\omega_{\mathcal{Y}_{i_0-1,\alpha}}], [C_\alpha]) = 0$. By the \mathbf{C}^* -action on \mathcal{Y}_{i_0-1} we obtain a proper curve C on Y_k as $\lim_{t \rightarrow 0} t(C_\alpha)$. As $\mathcal{Y}_{i_0-1,\alpha}$ is diffeomorphic to Y_k , there is a natural identification $H^2(\mathcal{Y}_{i_0-1,\alpha}, \mathbf{C}) \cong H^2(Y_k, \mathbf{C}) (\cong H^2(Y, \mathbf{C}))$. Under this identification we have $[\omega_{\mathcal{Y}_{i_0-1,\alpha}}] = \alpha$; thus $(\alpha, [C]) = 0$.

Let F be the codimension one face of $\overline{\text{Amp}(\Pi_{i_0-1})}$ which determines ϕ_i . Let H be the complex hyperplane of $H^2(Y, \mathbf{C})$ generated by F . Note that $H = \{x \in H^2(Y, \mathbf{C}); (x, [C]) = 0\}$. As $(\alpha, [C]) = 0$, we have $\alpha \in H$. \square

REMARK 9. The locus $g(\mathcal{C})$ is invariant by W . By Lemma 4 and Corollary 7, if $\alpha \in H^2(Y, \mathbf{C})$ is contained in the complex hyperplane generated by a codimension one face F of $\overline{\text{Amp}(\Pi')}$, then $\alpha \in g(\mathcal{C})$. Lemma 8 together with this fact implies that $\mathcal{D} = g(\mathcal{C})$ and that \mathcal{D} coincides with the union of complex hyperplanes generated by $F \subset \overline{\text{Amp}(\Pi')}$ for all crepant projective resolutions $\Pi' : \mathcal{Y}' \rightarrow \mathcal{X}'$ and for all codimension one faces F of their nef cones.

Now our results can be summarized as follows:

MAIN THEOREM.

(i) *There are finitely many codimension one linear subspaces $\{H_i\}_{i \in I}$ of $H^2(Y, \mathbf{Q})$ such that $\mathcal{D} = \cup_i (H_i)_{\mathbf{C}}$. Moreover $\mathcal{D} = g(\mathcal{C})$.*

(ii) *There are only finitely many crepant projective resolutions of X , say $\{\pi_k : Y_k \rightarrow X\}_{k \in K}$. The closed movable cone $\overline{\text{Mov}(\pi)}$ is the union of nef cones: $\overline{\text{Mov}(\pi)} = \bigcup_{k \in K} \overline{\text{Amp}(\pi_k)}$. Moreover $H^2(Y, \mathbf{R}) = \bigcup_{k \in K, w \in W} w(\overline{\text{Amp}(\pi_k)})$.*

(iii) The chambers $\{w(\text{Amp}(\pi_k))\}_{k \in K}$ coincide with the open chambers determined by $\{(H_i)_{\mathbf{R}}\}_{i \in I}$.

REMARK 10. Even if X has no crepant resolution, Main theorem equally holds for a \mathbf{Q} -factorial terminalization $\pi : Y \rightarrow X$. In fact, we can construct a commutative diagram of universal Poisson deformations of Y and X (cf. [Na 3], Theorem 5.5) over affine spaces

$$(9) \quad \begin{array}{ccc} \mathcal{Y} & \longrightarrow & \mathcal{X} \\ f \downarrow & & \downarrow \\ \mathbf{A}^n & \xrightarrow{q} & \mathbf{A}^n \end{array}$$

Notice that each fibre of $\mathcal{Y} \xrightarrow{f} \mathbf{A}^n$ has singularities, but f is a locally trivial deformation of Y ([Na 3], Theorem 5.5, (b)), i.e. for each point $p \in Y$, there is an open subset \mathcal{U} of \mathcal{Y} in the classical topology such that $p \in \mathcal{U}$ and $f|_{\mathcal{U}}$ is identified with the projection $(\mathcal{U} \cap Y) \times \Delta^n \rightarrow \Delta^n$. By the period map $p : \mathbf{A}^n \rightarrow H^2(Y, \mathbf{C})$ ([Na 5], (4.2)), the first affine space \mathbf{A}^n is naturally identified with $H^2(Y, \mathbf{C})$. Moreover Theorem 1.1 of [Na 4] still holds in our case and q is a finite Galois covering. Thus the second affine space \mathbf{A}^n is identified with $H^2(Y, \mathbf{C})/W$ for a finite Galois group W .

§3. Examples

Example 11. Let \mathfrak{g} be a complex simple Lie algebra and G its adjoint group. Let $\chi : \mathfrak{g} \rightarrow \mathfrak{g}/G$ be the adjoint quotient map. If \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} and W is the Weyl group of \mathfrak{g} with respect to \mathfrak{h} , then \mathfrak{g}/G can be identified with \mathfrak{h}/W . Choose a Borel subalgebra \mathfrak{b} of \mathfrak{g} containing \mathfrak{h} . Define a map $\mu_B : G \times^B \mathfrak{b} \rightarrow \mathfrak{g}$ by $(g, x) \rightarrow Ad_g(x)$. Let $O \subset \mathfrak{g}$ be a nilpotent orbit and $S \subset \mathfrak{g}$ the Slodowy slice to O . Consider the map $\chi|_S : S \rightarrow \mathfrak{h}/W$ and denote by S_0 its null fibre. By the Kostant-Kirillov form, S_0 is a symplectic variety and it admits a good \mathbf{C}^* -action. On the other hand, we define $S_B := \mu_B^{-1}(S)$ and consider the composition map $S_B \subset G \times^B \mathfrak{b} \rightarrow \mathfrak{h}$. We then denote by $S_{B,0}$ its null fibre. The map $S_B \rightarrow S$ induces a map $\pi : S_{B,0} \rightarrow S_0$, which is a crepant resolution of S_0 . We have $\text{Mov}(\pi) = \text{Amp}(\pi)$. In other words, S_0 has a unique crepant resolution. In

[LNS] we proved that the commutative diagram

$$(10) \quad \begin{array}{ccc} S_B & \longrightarrow & S \\ \downarrow & & \downarrow \\ \mathfrak{h} & \longrightarrow & \mathfrak{h}/W \end{array}$$

gives the universal Poisson deformations of $S_{B,0}$ and S_0 with some exceptions. The exceptional cases are when

- (i) O is the subregular orbit in B_n, C_n, G_2 or F_4 ,
- (ii) $O_{[n,n]}, O_{[2n-2i,2i]}$ ($1 < i \leq n/2$) in C_n or
- (iii) 8 dimensional nilpotent orbit in G_2 .

When O is not item (i) on the list, we have $H^2(S_B, \mathbf{C}) \cong \mathfrak{h}$ and our chamber structure for π is nothing but the Weyl chamber structure of $\mathfrak{h}(\mathbf{R})$.

When O is item (i) on the list, we know that S_0 is isomorphic to the null fibre of the Slodowy slice to the subregular nilpotent orbit in another simply-laced Lie algebra (as symplectic \mathbf{C}^* -varieties). For B_n , the simply-laced Lie algebra is A_{2n-1} and other cases are: $C_n \rightarrow D_{n+1}, G_2 \rightarrow D_4, F_4 \rightarrow E_6$.

For (ii), the nullfibres of the Slodowy slices for the following pairs are mutually isomorphic as symplectic \mathbf{C}^* -varieties ([H-L], Corollary 5.2) :

$$\begin{aligned} (O_{[n,n]} \subset C_n, O_{[n+1,n+1]} \subset D_{n+1}) \\ (O_{[2n-2i,2i]} \subset C_n, O_{[2n-2i+1,2i+1]} \subset D_{n+1}). \end{aligned}$$

Finally, the nullfibre of the Slodowy slice in the case (iii) is isomorphic to the null fibre of the Slodowy slice for $O_{[4,1^2]} \subset C_3$ as symplectic \mathbf{C}^* -varieties.

In any of these cases, we have $H^2(S_B, \mathbf{C}) \cong \mathfrak{h}'$ for a Cartan subalgebra \mathfrak{h}' of a suitable simple Lie algebra \mathfrak{g}' . Then the chamber structure for π again coincides with the Weyl chamber structure of $\mathfrak{h}'(\mathbf{R})$.

Example 12. Let G be a complex simple Lie group and let P_0 be its parabolic subgroup. Let $s : T^*(G/P_0) \rightarrow \mathfrak{g}$ be the Springer map from the cotangent bundle of G/P_0 to the Lie algebra \mathfrak{g} of G . The Springer map s is a generically finite projective morphism. As s is G -equivariant, the image of s coincides with the closure \bar{O} of a nilpotent orbit O of \mathfrak{g} . We take the Stein factorization of $s: T^*(G/P_0) \xrightarrow{\mu} \tilde{O} \rightarrow \bar{O}$. Then μ is a crepant resolution of \tilde{O} . In [Na 6] we have studied the birational geometry related with μ . According to [Na 6] one can describe the chamber structures of

$H^2(T^*(G/P_0), \mathbf{R})$ quite explicitly³. Here we shall illustrate it by using one typical example.

We put $G = SL(5, \mathbf{C})$. Fix a maximal torus T and consider the corresponding root system Φ of \mathfrak{g} . Choose a base $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of Φ . Denote by Φ^+ (resp. Φ^-) the set of positive roots in Φ . Let us consider the marked Dynkin diagram

$$\circ_{\alpha_1} \text{---} \bullet_{\alpha_2} \text{---} \bullet_{\alpha_3} \text{---} \circ_{\alpha_4}$$

Let I be the set of simple roots corresponding to the white vertexes and denote by Φ_I the root subsystem of Φ generated by I . We set $\Phi_I^- := \Phi_I \cap \Phi^+$. We define a parabolic subalgebra \mathfrak{p}_0 of \mathfrak{g} by

$$\mathfrak{p}_0 := \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \Phi_I^-} \mathfrak{g}_\alpha.$$

Here $\mathfrak{h} := \text{Lie}(T)$. Let P_0 be the corresponding parabolic subgroup of G . Let $L \subset P_0$ be the Levi part of P_0 containing T . By Theorem 1.3 and Proposition 2.3 of [Na 6], there is a one-to-one correspondence between the chambers (inside $H^2(T^*(G/P_0), \mathbf{R})$) and the parabolic subgroups of G with the Levi part L . Moreover, all such parabolic subgroups are constructed from P_0 by performing the operations “twist” successively.

If we twist P_0 by the vertex α_2 , we have a parabolic subgroup P_1 :

$$\bullet_{-\alpha_2} \text{---} \circ_{-\alpha_1} \text{---} \bullet_{\alpha_1 + \alpha_2 + \alpha_3} \text{---} \circ_{\alpha_4}$$

We can continue twisting one by one. Then we get

P_2 (as the twist of P_1 by $\alpha_1 + \alpha_2 + \alpha_3$):

$$\bullet_{\alpha_3 + \alpha_4} \text{---} \circ_{-\alpha_4} \text{---} \bullet_{-\alpha_1 - \alpha_2 - \alpha_3} \text{---} \circ_{\alpha_1}$$

P_3 (as the twist of P_2 by $\alpha_3 + \alpha_4$):

$$\circ_{\alpha_4} \text{---} \bullet_{-\alpha_3 - \alpha_4} \text{---} \bullet_{-\alpha_1 - \alpha_2} \text{---} \circ_{\alpha_1}$$

³In [Na 6] the base space $H^2(T^*(G/P_0), \mathbf{C})$ is denoted by $\mathfrak{k}(\mathfrak{p}_0)$.

P_4 (as the twist of P_3 by $-\alpha_1 - \alpha_2$):

$$\overset{\circ}{\alpha_4} \text{---} \overset{\bullet}{-\alpha_2 - \alpha_3 - \alpha_4} \text{---} \overset{\circ}{-\alpha_1} \text{---} \overset{\bullet}{\alpha_1 + \alpha_2}$$

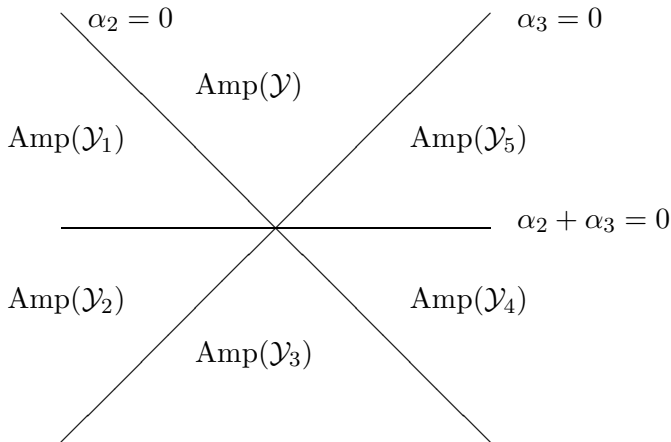
P_5 (as the twist of P_4 by $\alpha_2 + \alpha_3 + \alpha_4$):

$$\overset{\circ}{\alpha_1} \text{---} \overset{\bullet}{\alpha_2 + \alpha_3 + \alpha_4} \text{---} \overset{\circ}{-\alpha_4} \text{---} \overset{\bullet}{-\alpha_3}$$

These 6 parabolic subgroups are the complete lists of those with the Levi part L . In the remainder we put $Y := T^*(G/P_0)$, $\mathcal{Y} := G \times^{P_0} r(\mathfrak{p}_0)$ and $\mathcal{Y}_i := G \times^{P_i} r(\mathfrak{p}_i)$. Here $r(\mathfrak{p}_i)$ are solvable radicals of \mathfrak{p}_i . Since Y is a vector bundle over G/P , we have an isomorphism $H^2(Y, \mathbf{R}) \cong H^2(G/P, \mathbf{R})$. On the other hand, we have

$$H^2(G/P, \mathbf{R}) \cong \text{Hom}_{alg, gp}(L, \mathbf{C}^*) \otimes_{\mathbf{Z}} \mathbf{R} = \{x \in \mathfrak{h}(\mathbf{R})^*; (x, \alpha_1) = (x, \alpha_4) = 0\},$$

where $(,)$ is the Killing form on $\mathfrak{h}(\mathbf{R})^*$. Thus $H^2(Y, \mathbf{R})$ can be identified with the dual space of the 2-dimensional vector space $\mathbf{R}\alpha_2 \oplus \mathbf{R}\alpha_3$ by the Killing form. Now $H^2(Y, \mathbf{R})$ is decomposed into 6 chambers:



The movable cone $\text{Mov}(Y)$ is the upper half plane. The Weyl group W is isomorphic to $\mathbf{Z}/2\mathbf{Z}$, which acts on $H^2(Y, \mathbf{R})$ as the reflection with respect to the line $\{\alpha_2 + \alpha_3 = 0\}$.

Example 13. The following example is due to Ryo Yamagishi. It clarifies a difference between symplectic singularities and non-symplectic singularities.

Let us consider the line bundle $Z := p_1^* \mathcal{O}_{\mathbf{P}^1}(-1) \otimes p_2^* \mathcal{O}_{\mathbf{P}^1}(-1) \otimes p_3^* \mathcal{O}_{\mathbf{P}^1}(-1)$ on $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. Denote by $\Sigma \subset Z$ the zero section of the bundle map $Z \rightarrow \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. For $1 \leq i < j \leq 3$, let $p_{ij} : \Sigma \rightarrow \mathbf{P}^1 \times \mathbf{P}^1$ be the projection to the i -th and j -th factors. There is a birational projective morphism $f_{ij} : Z \rightarrow X_{ij}$ to a smooth variety X_{ij} such that $f_{ij}|_{\Sigma} = p_{ij}$ and f_{ij} is an isomorphism outside Σ . We put $\Sigma_{ij} := p_{ij}(\Sigma)$. Then $\Sigma_{ij} \cong \mathbf{P}^1 \times \mathbf{P}^1$ and f_{ij} is the blowing up of X_{ij} along Σ_{ij} . Moreover there is a birational projective morphism $f : Z \rightarrow X$ to an affine (singular) variety X , which contracts Σ to a point. By the construction f factors through f_{ij} for each pair (i, j) with $1 \leq i < j \leq 3$. Then X_{12} , X_{13} and X_{23} are three different crepant resolutions of X .

Recall that Σ_{ij} is naturally identified with the product $\mathbf{P}^1 \times \mathbf{P}^1$ of the i -th and the j -th factors of $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$. Let l_{ij}^i be a fibre of the projection map $q_{ij}^i : \Sigma_{ij} \rightarrow \mathbf{P}^1$ to the i -th factor. Then the normal bundle of such a curve (inside X_{ij}) is isomorphic to $\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbf{P}^1}$.

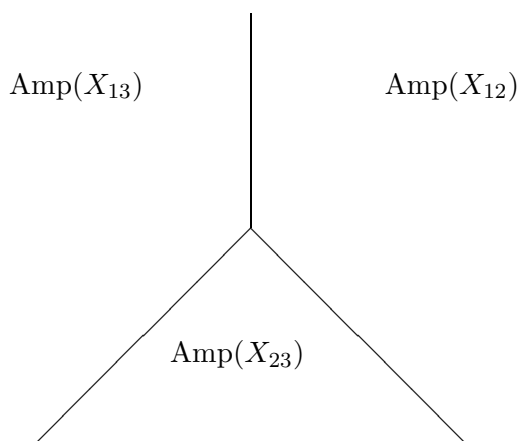
Now one has a birational projective morphism $g_{ij}^i : X_{ij} \rightarrow X_i^i$ such that $g_{ij}^i|_{\Sigma_{ij}} = q_{ij}^i$ and g_{ij}^i is an isomorphism outside Σ_{ij} . By the definition $X_{12}^1 = X_{13}^1$, $X_{12}^2 = X_{23}^2$ and $X_{13}^3 = X_{23}^3$. Thus we shall denote them by X_1 , X_2 and X_3 . Note that these 4-folds X_i have 3-dimensional ordinary double points along $\text{Sing}(X_i) (\cong \mathbf{P}^1)$. The 4-folds X_{12} , X_{13} and X_{23} are related by flops

$$X_{12} \rightarrow X_2 \leftarrow X_{23}$$

$$X_{23} \rightarrow X_3 \leftarrow X_{13}$$

$$X_{13} \rightarrow X_1 \leftarrow X_{12}.$$

Then $\text{Pic}(X_{12}) \otimes_{\mathbf{Z}} \mathbf{R}$ is divided by the ample cones of X_{12} , X_{23} and X_{13} . There are 3 open chambers and they are determined by half-lines. Thus X is not a symplectic singularity.



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