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Restriction of Most Degenerate Representations of $O(1, N)$ *with Respect to Symmetric Pairs*

By Jan MÖLLERS and Yoshiki OSHIMA

On the occasion of the centennial anniversary of Professor Kunihiko Kodaira's birthday

Abstract. We find the complete branching law for the restriction of certain unitary representations of $O(1, n+1)$ to the subgroups $O(1, m+1) \times O(n-m)$, $0 \leq m \leq n$. The unitary representations we consider are those induced from a character of a parabolic subgroup or its irreducible quotient. They belong either to the unitary spherical principal series, the spherical complementary series or discrete series for the hyperboloid.

In the crucial case $0 < m < n$ the decomposition consists of a continuous part and a discrete part. The continuous part is given by a direct integral of unitary principal series representations whereas the discrete part consists of finitely many representations which either belong to the complementary series or are discrete series for the hyperboloid. The explicit Plancherel formula is computed on the Fourier transformed side of the non-compact realization of the representations by using the spectral decomposition of a certain hypergeometric type ordinary differential operator. The main tool connecting this differential operator with the representations are second order Bessel operators which describe the Lie algebra action in this realization.

To derive the spectral decomposition of the ordinary differential operator we use Kodaira's formula for the spectral decomposition of Schrödinger type operators.

Contents

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Introduction

Among his various different mathematical contributions, the spectral theory of self-adjoint differential operators is one of the earlier works of Professor Kunihiko Kodaira. The so-called Weyl–Stone–Kodaira–Titchmarsh theory gives eigenfunction expansions for self-adjoint second order differential operators in one variable. It provides a uniform treatment of classical eigenfunction expansions such as spectral decompositions into Bessel functions, Hermite polynomials or Laguerre functions.

Kodaira [21] considered a differential operator $L = \frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)$ on a possibly unbounded interval $(a,b) \subset \mathbb{R}$ (see [21, 22] for the precise setting). Then L extends to a self-adjoint operator on the space of square integrable functions on (a,b) with domain given by functions satisfying certain boundary conditions, and we have an expansion into eigenfunctions of the form:

$$u(x) = \sum_{j=1}^{2}\sum_{k=1}^{2}\int_{-\infty}^{\infty} s_j(x,\lambda)\int_a^b s_k(y,\lambda)u(y)\,\mathrm{d}y\,\mathrm{d}\rho_{jk}(\lambda).$$

Here, the functions $s_1(\cdot,\lambda)$ and $s_2(\cdot,\lambda)$ are linearly independent solutions to the equation $Lu = \lambda u$.

The existence of the density measure $\mathrm{d}\rho_{jk}$ for which the above expansion holds was first proved by Weyl [39]. Later Stone gave in [34, Theorem 10.22] a different proof using the general theory of operators on Hilbert spaces. About forty years after Weyl's result, Kodaira [21, 22] found an explicit formula for $\mathrm{d}\rho_{jk}$ in terms of the characteristic functions, revealing the explicit relation between the density measures and the asymptotic behaviour of eigenfunctions. The same formula was also obtained independently by Titchmarsh [35] using a different method, and it is called the Kodaira–Titchmarsh formula. In the second half of [22] and in [23] Kodaira studied the eigenfunctions and the density matrix in detail for some particular cases, which are important for applications. Moreover, in a subsequent paper [24] he generalized this formula to differential operators of any even order.

In his Gibbs lecture [40] Weyl wrote about Kodaira: 'The formula [40, (12)] was rediscovered by Kunihiko Kodaira (who of course had been cut off from our Western mathematical literature since the end of 1941); his construction of ρ and his proofs for [40, (12)] and the expansion formula [40, (9)], still unpublished, seem to clinch the issue. It is remarkable that forty years had to pass before such a thoroughly satisfactory direct treatment emerged; the fact is a reflection on the degree to which mathematicians during this period got absorbed in abstract generalizations and lost sight of their task of finishing up some of the more concrete problems of undeniable importance.'

The Kodaira–Titchmarsh formula makes it possible to apply the spectral decomposition theorem to concrete settings and in particular it has a significant impact on the harmonic analysis on Lie groups. For a given variety X and a Lie group G acting on it, a fundamental problem in the harmonic analysis on X is to expand arbitrary function on X into joint eigenfunctions for the G-invariant differential operators on X. An explicit description of such an expansion is called Plancherel Theorem. When X is a symmetric space of rank one, it amounts to eigenfunction expansions for the Laplacian. In this case the problem can be reduced to an eigenfunction expansion for a self-adjoint second order differential operator in one variable and hence the Kodaira–Titchmarsh formula can be applied directly. In fact, a special case studied in the second half of [22] and in [23] is enough to deduce the Plancherel Theorem for all symmetric spaces of rank one. For Riemannian symmetric spaces G/K of arbitrary rank the Plancherel Theorem was established by Harish-Chandra (see [11]). In this case the Plancherel measure is given by the c-function which can be explicitly written in terms of the Gamma function by the work of Gindikin–Karpelevič [9]. As the c-function is defined in terms of the asymptotic behaviour of joint eigenfunctions, we see once more the spirit of the Kodaira–Titchmarsh formula in this setting. For pseudo-Riemannian symmetric spaces G/H, special cases like hyperboloids $O(p,q)/O(p,q-1)$ were studied in the sixties by Shintani [31] and Molčanov [26]. The Plancherel Theorem for general semisimple symmetric spaces of arbitrary rank was established by the works of T. Oshima, Delorme [4], and van den Ban–Schlichtkrull [36].

Plancherel Theorems for reductive homogeneous spaces G/H can be viewed as induction problems, decomposing the induced representation $L^2(G/H) = \mathrm{Ind}_H^G(\mathbf{1})$ into irreducible G-representations. As well as induction problems we may consider restriction problems as advocated in [16], namely, we may ask how a representation decomposes when restricted to a subgroup. The restriction problem was e.g. solved in [18, 29] for the most degenerate principal series representations of $G = GL(n,\mathbb{R})$ and $G = GL(n,\mathbb{C})$ with respect to any symmetric pair (G,H). Since (degenerate) principal series representations are realized on L^2-sections of line bundles on a flag variety G/P, Mackey theory relates these restriction problems to the Plancherel type problems for the open H-orbits in G/P. Our focus is on the indefinite orthogonal group $O(1,n+1)$, $n \geq 1$, for which we study the restriction of certain unitary representations using the Kodaira–Titchmarsh formula.

We now introduce some notation in order to describe our results. Let $G = O(1,n+1)$. It is known that on the level of $(\mathfrak{g},K)$-modules all irreducible unitary representations of G are obtained as subrepresentations of representations induced from a parabolic subgroup $P = MAN$. Up to conjugation P is unique and there are group isomorphisms $M \cong O(n)\times(\mathbb{Z}/2\mathbb{Z})$, $A \cong \mathbb{R}_+$ and $N \cong \mathbb{R}^n$. We restrict our attention to representations induced from characters of P. Denote by $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ the representation of G, which is induced from the character of P given by the character $\sigma \in \mathbb{C}$ of A and the character $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ of the second factor of $M \cong O(n)\times(\mathbb{Z}/2\mathbb{Z})$ (normalized parabolic induction).

In our parameterization $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ is irreducible and unitarizable if and only if $\sigma \in i\mathbb{R}\cup(-n,n)$. By abuse of notation we denote by $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ also the corresponding irreducible unitary representations. For $\sigma \in i\mathbb{R}$ these representations are called *unitary principal series representations* and for $\sigma \in (-n,0)\cup(0,n)$ they are called *complementary series representations.* We have natural isomorphisms $\pi_{-\sigma,\varepsilon}^{O(1,n+1)} \cong \pi_{\sigma,\varepsilon}^{O(1,n+1)}$ for $\sigma \in i\mathbb{R}\cup(-n,n)$.

Further, for $\sigma = n+2u$, $u \in \mathbb{N}$, the representation $\pi_{\sigma,\varepsilon}^{O(1,n+1)}$ has a unique non-trivial subrepresentation $\pi_{\sigma,\varepsilon,\mathrm{sub}}^{O(1,n+1)}$. This subrepresentation is irreducible and unitarizable and we use the same notation to also denote the corresponding irreducible unitary representation. Its underlying $(\mathfrak{g},K)$-module is isomorphic to Zuckerman's module $A_{\mathfrak{q}}(\lambda)$ for certain $\mathfrak{q}$

and λ and it occurs discretely in the Plancherel formula for the hyperboloid $O(1,n+1)/O(1,n)$. We refer to these representations as *discrete series representations for the hyperboloid.*

In this paper we study the restriction of $\pi^{O(1,n+1)}_{\sigma,\varepsilon}$, $\sigma\in i\mathbb{R}\cup(-n,n)$, and $\pi^{O(1,n+1)}_{\sigma,\varepsilon,\mathrm{sub}}$, $\sigma\in n+2\mathbb{N}$, $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$, with respect to any symmetric pair (G,H). By Berger's list [1] of symmetric pairs, the subgroup H is conjugate to

$$H=O(1,m+1)\times O(n-m),\qquad\qquad -1\le m<n.$$

Since H is a maximal compact subgroup of G if $m=-1$, the branching law for the restriction of $\pi^{O(1,n+1)}_{\sigma,\varepsilon}$ and $\pi^{O(1,n+1)}_{\sigma,\varepsilon,\mathrm{sub}}$ to $O(1)\times O(n+1)$ is simply the K-type decomposition (1.1) or (1.2) which is well-known. Moreover, for $H=O(1,1)\times O(n)$, i.e. the case $m=0$, the branching law can easily be derived using classical Fourier analysis, see Section 1.4. The most interesting case is the branching to H for $0<m<n$. In the formulation of the branching law we use the conventions $[0,\alpha)=\emptyset$ for $\alpha\le 0$ and $[0,\alpha]=\emptyset$ for $\alpha<0$.

Theorem (see Theorem 4.7). *The unitary representations $\pi^G_{\sigma,\varepsilon}$ and $\pi^G_{\sigma,\varepsilon,\mathrm{sub}}$ of $G=O(1,n+1)$ decompose into irreducible representations of $H=O(1,m+1)\times O(n-m)$, $0<m<n$, as follows: for $\sigma\in i\mathbb{R}\cup(-n,n)$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ we have*

$$\pi^G_{\sigma,\varepsilon}\big|_H\cong\sum_{k=0}^{\infty}{}^{\oplus}\left(\int_{i\mathbb{R}_+}^{\oplus}\pi^{O(1,m+1)}_{\tau,\varepsilon+k}\,\mathrm{d}\tau\right.$$
$$\left.\oplus\bigoplus_{j\in\mathbb{N}\cap\left[0,\frac{|\mathrm{Re}\,\sigma|-n+m-2k}{4}\right)}\pi^{O(1,m+1)}_{|\mathrm{Re}\,\sigma|-n+m-2k-4j,\varepsilon+k}\right)\boxtimes\mathcal{H}^k(\mathbb{R}^{n-m}),$$

and for $\sigma=n+2u$, $u\in\mathbb{N}$, and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ we have

$$\pi^G_{\sigma,\varepsilon,\mathrm{sub}}\big|_H\cong\sum_{k=0}^{\infty}{}^{\oplus}\left(\int_{i\mathbb{R}_+}^{\oplus}\pi^{O(1,m+1)}_{\tau,\varepsilon+k}\,\mathrm{d}\tau\oplus\bigoplus_{j\in\mathbb{N}\cap\left[0,\frac{u-k}{2}\right]}\pi^{O(1,m+1)}_{m+2u-2k-4j,\varepsilon+k,\mathrm{sub}}\right.$$
$$\left.\oplus\bigoplus_{j\in\mathbb{N}\cap\left(\frac{u-k}{2},\frac{m+2u-2k}{4}\right)}\pi^{O(1,m+1)}_{m+2u-2k-4j,\varepsilon+k}\right)\boxtimes\mathcal{H}^k(\mathbb{R}^{n-m}),$$

where $\mathcal{H}^k(\mathbb{R}^{n-m})$ denotes the irreducible representation of $O(n-m)$ on the space of solid spherical harmonics of degree k on $\mathbb{R}^{n-m}$.

The explicit Plancherel formula is given in Theorem 4.1. We remark that the branching laws are multiplicity-free, a fact which is not true for general irreducible representations of G restricted to H if $m < n-1$.

Let us explain the branching formula in more detail. First of all, the restriction $\pi^G_{\sigma,\varepsilon}|_H$ resp. $\pi^G_{\sigma,\varepsilon,\mathrm{sub}}|_H$ is decomposed with respect to the action of $O(n-m)$, the second factor of H. Then the decomposition of each $\mathcal{H}^k(\mathbb{R}^{n-m})$-isotypic component into irreducible representations of $O(1,m+1)$ contains continuous and discrete spectrum in general. The continuous spectrum is a direct integral of unitary principal series representations $\pi^{O(1,m+1)}_{\tau,\varepsilon+k}$ of $O(1,m+1)$. The discrete spectrum appears if and only if $k < \frac{|\operatorname{Re}\sigma|-n+m}{2}$ and is a direct sum of finitely many complementary series representations in the case $\pi^G_{\sigma,\varepsilon}|_H$ and additionally finitely many discrete series representations for the hyperboloid in the case $\pi^G_{\sigma,\varepsilon,\mathrm{sub}}|_H$. Therefore the whole branching law of $\pi^G_{\sigma,\varepsilon}|_H$ resp. $\pi^G_{\sigma,\varepsilon,\mathrm{sub}}|_H$ contains only finitely many discrete components and the discrete spectrum is non-trivial if and only if $|\operatorname{Re}\sigma| > n-m$. In particular for $m>0$ there is always at least one discrete component in the restriction of the discrete series representations for the hyperboloid $\pi^G_{\sigma,\varepsilon,\mathrm{sub}}|_H$ and also in the restriction of complementary series representations $\pi^G_{\sigma,\varepsilon}|_H$ if σ is sufficiently close to the first reduction point n or $-n$.

For $\sigma \in i\mathbb{R}$ the decomposition is purely continuous. In this case the branching law is actually equivalent to the Plancherel formula for the Riemannian symmetric space $O(1,m+1)/(O(1)\times O(m+1))$ (see Appendix A or [3]) and therefore well-known. However, neither for the complementary series representations nor the discrete series representations for the hyperboloid can the decomposition be obtained in the same way.

The proof of the Plancherel formula we present works uniformly for all $\sigma \in i\mathbb{R} \cup (-n,n) \cup (n+2\mathbb{N})$, i.e. for both unitary principal series representations, complementary series, and discrete series representations for the hyperboloid. It uses the "Fourier transformed realization" of $\pi^G_{\sigma,\varepsilon}$ resp. $\pi^G_{\sigma,\varepsilon,\mathrm{sub}}$ on $L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$. For this consider first the non-compact realization on the nilradical $\overline{N}$ of the parabolic subgroup $\overline{P}$ opposite to P. We then take the Euclidean Fourier transform on $\overline{N} \cong \mathbb{R}^n$ to obtain a realization of $\pi^G_{\sigma,\varepsilon}$

resp. $\pi^G_{\sigma,\varepsilon,\mathrm{sub}}$ on $L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$. The advantage of this realization is that the invariant form is simply the L^2-inner product. The Lie algebra action in the Fourier transformed picture is given by differential operators up to order two, the crucial operators being second order Bessel type operators. We remark that these operators are a special case of the Bessel operators on Jordan algebras introduced by Dib [5] (see also [6, 12]). In the context of indefinite orthogonal groups these operators first appeared explicitly in the study of the minimal representation of $O(p,q)$ by Kobayashi–Mano [17] where they are called fundamental differential operators (see also [25]). Using these operators in our case we reduce the branching law to the spectral decomposition of an ordinary differential operator of hypergeometric type on $L^2(\mathbb{R}_+)$ (see Section 2). The spectral decomposition of this operator is derived in Section 3 from Kodaira's result on the Schrödinger type operators and is used in Section 4 to obtain the branching law and the explicit Plancherel formula for the restriction of the representations. An interesting formula for the intertwining operators realizing the branching law in the non-compact picture on $\overline{N}$ is computed in Section 5. These intertwining operators will be subject of a subsequent paper.

Here are some related results on the branching laws studied by different methods:

- For $n=2$ and $m=1$ the full decomposition of the complementary series was given by Mukunda [30] using the non-compact picture. This case corresponds to the branching law $\mathrm{SL}(2,\mathbb{C}) \searrow \mathrm{SL}(2,\mathbb{R})$.

- For n arbitrary and $m=1$ the full decomposition of the complementary series was given by Boyer [2]. He obtained an expansion of matrix coefficients using analytic continuation in σ and a result for principal series [3].

- Boyer's result [2] was extended to the case $\mathrm{SO}(m,n+1) \searrow \mathrm{SO}(m,n)$ by Molčanov [27].

- Extending his study on branching laws for discretely decomposable restrictions [14], Kobayashi constructed discrete components for Zuckerman's modules of $O(p,q)$ when restricted to $O(p',q') \times O(p'',q'')$, which was announced in his talk [15]. The restriction of $\pi^{O(1,n+1)}_{\sigma,\varepsilon,\mathrm{sub}}$, $\sigma \in (n+2\mathbb{N})$ is a special case of his result. By our Theorem it turns out that his construction gives all the discrete components of the restriction $\pi^{O(1,n+1)}_{\sigma,\varepsilon,\mathrm{sub}}|_{O(1,m+1)\times O(n-m)}$ for $\sigma \in (n+2\mathbb{N})$.

- After the announcement of Kobayashi's result [15], Speh–Venkataramana [32, Theorem 1] proved the existence of the discrete component $\pi^{O(1,n)}_{\sigma-1,0}$ in $\pi^{O(1,n+1)}_{\sigma,0}|_{O(1,n)}$ for $n \geq 2$, $m = n-1$ and $\sigma \in (1,n)$ as well as the existence of the discrete component $\pi^{O(1,n)}_{n-1,0,\mathrm{sub}}$ in $\pi^{O(1,n+1)}_{n,0,\mathrm{sub}}|_{O(1,n)}$ (special case $j = k = 0$, $\sigma \in (1,n]$ in our Theorem). They also use the Fourier transformed picture for their proof. This is a special case of their more general result for complementary series representations of G on differential forms, i.e. induced from more general (possibly non-scalar) P-representations.

- The same special case was obtained by Zhang [41, Theorem 3.6]. He actually proved that for all rank one groups $G = \mathrm{SU}(1,n+1;\mathbb{F})$, $\mathbb{F} = \mathbb{R},\mathbb{C},\mathbb{H}$, resp. $G = F_{4(-20)}$ certain complementary series representations of $H = \mathrm{SU}(1,n;\mathbb{F})$ resp. $H = \mathrm{Spin}(8,1)$ occur discretely in some spherical complementary series representations of G. His proof uses the compact picture and explicit estimates for the restriction of K-finite vectors.

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Notation. $\mathbb{Z}_+ = \{1,2,3,\dots\}$, $\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$, $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$.

1. L^2-model of Some Representations of $O(1,n+1)$

In this section we recall the necessary geometry of the group $G = O(1,n+1)$ and some of its representation theory. The L^2-models discussed in Section 1.3 were for the complementary series previously constructed by Vershik–Graev [37] and are new for the discrete series representations for the hyperboloid.

1.1. Subgroups and decompositions

Let $G = O(1, n+1)$, $n \geq 1$, realized as the subgroup of $\mathrm{GL}(n+2,\mathbb{R})$ leaving the quadratic form

$$\mathbb{R}^{n+2} \to \mathbb{R}, \quad x = (x_1, \ldots, x_{n+2})^t \mapsto x_1^2 - (x_2^2 + \cdots + x_{n+2}^2),$$

invariant. We fix the Cartan involution θ of G given by $\theta(g) = g^{-t} = (g^t)^{-1}$, $g \in G$, which corresponds to the maximal compact subgroup $K := G^\theta = O(1) \times O(n+1)$. On the Lie algebra level the Lie algebra $\mathfrak{g}$ of G has the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into the ± 1 eigenspaces $\mathfrak{k}$ and $\mathfrak{p}$ of θ where $\mathfrak{k}$ is the Lie algebra of K. Choose the maximal abelian subalgebra $\mathfrak{a} := \mathbb{R}H \subseteq \mathfrak{p}$ spanned by the element

$$H := 2(E_{1,n+2} + E_{n+2,1}),$$

where E_{ij} denotes the $(n+2)\times(n+2)$ matrix with 1 in the (i,j)-entry and 0 elsewhere. The root system of the pair $(\mathfrak{g},\mathfrak{a})$ consists only of the roots $\pm 2\gamma$ where $\gamma \in \mathfrak{a}_{\mathbb{C}}^*$ is defined by $\gamma(H) := 1$. Put

$$\mathfrak{n} := \mathfrak{g}_{2\gamma}, \qquad \overline{\mathfrak{n}} := \mathfrak{g}_{-2\gamma} = \theta\mathfrak{n}$$

and let

$$N := \exp_G(\mathfrak{n}), \qquad \overline{N} := \exp_G(\overline{\mathfrak{n}}) = \theta N$$

be the corresponding analytic subgroups of G. Since $\dim(\mathfrak{n}) = \dim(\overline{\mathfrak{n}}) = n$ the half sum of all positive roots is given by $\rho = n\gamma$. We introduce the following coordinates on N and $\overline{N}$: For $1 \leq j \leq n$ let

$$\begin{aligned} N_j &:= E_{1,j+1} + E_{j+1,1} - E_{j+1,n+2} + E_{n+2,j+1}, \\ \overline{N}_j &:= E_{1,j+1} + E_{j+1,1} + E_{j+1,n+2} - E_{n+2,j+1}. \end{aligned}$$

For $x \in \mathbb{R}^n$ let

$$n_x := \exp\left(\sum_{j=1}^n x_j N_j\right) \in N, \qquad \overline{n}_x := \exp\left(\sum_{j=1}^n x_j \overline{N}_j\right) \in \overline{N}.$$

Further put $M := Z_K(\mathfrak{a})$ and $A := \exp(\mathfrak{a})$ and denote by $\mathfrak{m}$ the Lie algebra of M. We write $M = M^+ \cup m_0 M^+$ where

$$M^+ := \{\mathrm{diag}(1, k, 1) : k \in O(n)\} \cong O(n) \qquad \text{and}$$
$$m_0 := \mathrm{diag}(-1, 1, \ldots, 1, -1).$$

Via conjugation the element m_0 acts on N and $\overline{N}$ by

$$m_0 n_x m_0^{-1} = n_{-x} \quad \text{and} \quad m_0 \overline{n}_x m_0^{-1} = \overline{n}_{-x}$$

and the action of $m \in M^+ \cong O(n)$ on N and $\overline{N}$ by conjugation is given by

$$m n_x m^{-1} = n_{mx} \quad \text{and} \quad m \overline{n}_x m^{-1} = \overline{n}_{mx}$$

for $x \in \mathbb{R}^n$, where mx is the usual action of $O(n)$ on $\mathbb{R}^n$. Further A acts on N and $\overline{N}$ by

$$e^{tH} n_x e^{-tH} = n_{e^{2t}x} \quad \text{and} \quad e^{tH} \overline{n}_x e^{-tH} = \overline{n}_{e^{-2t}x}$$

for $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. The following decomposition holds

$$\mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n} \qquad \text{(Gelfand–Naimark decomposition)}.$$

The groups

$$P := MAN \quad \text{and} \quad \overline{P} := MA\overline{N} = \theta(P)$$

are opposite parabolic subgroups in G and $\overline{N}P \subseteq G$ is an open dense subset. Let $W := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ be the Weyl group corresponding to $\mathfrak{a}$. Then $W = \{\mathbf{1}, [w_0]\}$ where the non-trivial element is represented by the matrix

$$w_0 = \mathrm{diag}(-1, 1, \ldots, 1) \in K.$$

The element w_0 has the property that $w_0 N w_0^{-1} = \overline{N}$ and hence $w_0 P w_0^{-1} = \overline{P}$. More precisely,

$$w_0 n_x w_0^{-1} = \overline{n}_{-x} \quad \text{and} \quad w_0 e^{tH} w_0^{-1} = e^{-tH}.$$

We have the disjoint union

$$G = \overline{P} \cup \overline{P} w_0 \overline{P} \qquad \text{(Bruhat decomposition)}.$$

The following lemma is a straightforward calculation:

Lemma 1.1. *For $x \in \mathbb{R}^n$, $x \neq 0$, we have $w_0^{-1}\overline{n}_x = \overline{n}_y m e^{tH} n_z \in \overline{N}P$ with*

$$y = -|x|^{-2}x, \qquad z = |x|^{-2}x, \qquad t = \log|x|, \qquad m = \begin{pmatrix} -1 & & \\ & \mathbf{1}_n - 2|x|^{-2}xx^t & \\ & & -1 \end{pmatrix}.$$

Let τ be the involution of G given by conjugation with the matrix

$$\mathrm{diag}(\mathbf{1}_m, -\mathbf{1}_{n-m}, 1).$$

Then the subgroup $H := G^\tau$ is isomorphic to $O(1,m+1)\times O(n-m)$. The subgroup H is generated by the subgroups N_H, $\overline{N}_H$, M_H and A, where (viewing $\mathbb{R}^m$ as the subspace $\mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$)

$$N_H := \{n_x : x \in \mathbb{R}^m\} \quad \text{and} \quad \overline{N}_H := \{\overline{n}_x : x \in \mathbb{R}^m\}$$

and $M_H := M_H^+ \cup m_0 M_H^+$ with

$$\begin{aligned} M_H^+ &:= \{\mathrm{diag}(1,k_1,k_2,1) : k_1 \in O(m), k_2 \in O(n-m)\} \\ &\cong O(m)\times O(n-m). \end{aligned}$$

Also denote by

$$P_H := M_H A N_H \quad \text{and} \quad \overline{P}_H := M_H A \overline{N}_H$$

the corresponding parabolic subgroups. We write $\mathfrak{h}$ for the Lie algebra of H.

1.2. Principal series representations – non-compact picture and standard intertwining operators

We identify $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$ by $\lambda \mapsto \lambda(H)$, i.e. $\sigma \in \mathbb{C}$ corresponds to $\sigma\gamma \in \mathfrak{a}_\mathbb{C}^*$. Under this identification ρ corresponds to n. For $\sigma \in \mathbb{C}$ let e^σ be the character of A given by $e^\sigma(e^{tH}) = e^{\sigma t}$, $t \in \mathbb{R}$. Further, for $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ denote by ξ_ε the character of $M = M^+ \cup m_0 M^+$ with $\xi_\varepsilon(m_0) = (-1)^\varepsilon$ and

$\xi_\varepsilon(m)=1$ for $m\in M^+$. For $\sigma\in\mathbb{C}$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ we consider the character $\chi_{\sigma,\varepsilon}:=\xi_\varepsilon\otimes e^\sigma\otimes\mathbf{1}$ on $P=MAN$ and induce it to a representation of G:

$$\begin{aligned}\widetilde{I}^G_{\sigma,\varepsilon}&:=\operatorname{Ind}_P^G(\chi_{\sigma,\varepsilon})\\&=\{f\in C^\infty(G):f(gman)=\xi_\varepsilon(m)^{-1}a^{-\sigma-\rho}f(g)\,\forall\,g\in G,\\&\qquad\qquad\qquad\qquad\qquad\qquad man\in P=MAN\}.\end{aligned}$$

The group G acts on $\widetilde{I}^G_{\sigma,\varepsilon}$ by left-translations and this action will be denoted by $\widetilde{\pi}^G_{\sigma,\varepsilon}$. Restricting to K it is easy to see that the K-type decomposition of the representations $\widetilde{\pi}^G_{\sigma,\varepsilon}$ is given by

$$\widetilde{\pi}^G_{\sigma,\varepsilon}\big|_K\cong\sum_{k=0}^{\infty}{}^{\oplus}\operatorname{sign}^{\varepsilon+k}\boxtimes\mathcal{H}^k(\mathbb{R}^{n+1}),\tag{1.1}$$

where sign denotes the non-trivial character of $O(1)$, and $O(n+1)$ acts as usual on the space $\mathcal{H}^k(\mathbb{R}^{n+1})$ of spherical harmonics of degree k on $\mathbb{R}^n$, giving combined the action of $K\cong O(1)\times O(n+1)$.

The following fact on the structure of $\widetilde{\pi}^G_{\sigma,\varepsilon}$ is known (see [13]).

FACT 1.2.

(i) The representation $(\widetilde{\pi}^G_{\sigma,\varepsilon},\widetilde{I}^G_{\sigma,\varepsilon})$ is irreducible if and only if $\sigma\notin\pm(n+2\mathbb{N})$. It is unitarizable if and only if $\sigma\in(-n,n)\cup i\mathbb{R}$.

(ii) For $\sigma=n+2u$, $u\in\mathbb{N}$, the representation $(\widetilde{\pi}^G_{\sigma,\varepsilon},\widetilde{I}^G_{\sigma,\varepsilon})$ has a unique non-trivial subrepresentation $(\widetilde{\pi}^G_{\sigma,\varepsilon,\mathrm{sub}},\widetilde{I}^G_{\sigma,\varepsilon,\mathrm{sub}})$. This subrepresentation is irreducible and unitarizable and its K-type decomposition is given by

$$\widetilde{\pi}^G_{\sigma,\varepsilon,\mathrm{sub}}\big|_K\cong\sum_{k=u+1}^{\infty}{}^{\oplus}\operatorname{sign}^{\varepsilon+k}\boxtimes\mathcal{H}^k(\mathbb{R}^{n+1}),\tag{1.2}$$

(iii) For $\sigma=-n-2u$, $u\in\mathbb{N}$, the representation $(\widetilde{\pi}^G_{\sigma,\varepsilon},\widetilde{I}^G_{\sigma,\varepsilon})$ has a unique non-trivial subrepresentation $(\widetilde{\pi}^G_{\sigma,\varepsilon,\mathrm{sub}},\widetilde{I}^G_{\sigma,\varepsilon,\mathrm{sub}})$. This subrepresentation is finite-dimensional and irreducible and the quotient $\widetilde{I}^G_{\sigma,\varepsilon}/\widetilde{I}^G_{\sigma,\varepsilon,\mathrm{sub}}$ is unitarizable and isomorphic to $\widetilde{I}^G_{-\sigma,\varepsilon,\mathrm{sub}}$.

Since $\overline{N}P \subseteq G$ is dense, a function in $\widetilde{I}^G_{\sigma,\varepsilon}$ is already uniquely determined by its values on $\overline{N}$ and for $f \in \widetilde{I}^G_{\sigma,\varepsilon}$ we put

$$f_{\overline{N}}(x) := f(\overline{n}_x), \qquad x \in \mathbb{R}^n.$$

Let $I^G_{\sigma,\varepsilon} := \left\{ f_{\overline{N}} : f \in \widetilde{I}^G_{\sigma,\varepsilon} \right\}$ and denote by $\pi^G_{\sigma,\varepsilon}$ the corresponding induced action, i.e.

$$\pi^G_{\sigma,\varepsilon}(g) f_{\overline{N}} := (\widetilde{\pi}^G_{\sigma,\varepsilon}(g) f)_{\overline{N}}, \qquad f \in \widetilde{I}^G_{\sigma,\varepsilon}.$$

Further, denote by $(\pi^G_{\sigma,\varepsilon,\mathrm{sub}}, I^G_{\sigma,\varepsilon,\mathrm{sub}})$ the corresponding subrepresentations of $(\pi^G_{\sigma,\varepsilon}, I^G_{\sigma,\varepsilon})$ for $\sigma \in \pm(n+2\mathbb{N})$. Note that if $f \in \widetilde{I}^G_{\sigma,\varepsilon}$ is a $\mathfrak{k}$-fixed vector, namely, in the $k=0$ term on the right hand side of (1.1), then $f_{\overline{N}}$ is equal to $(1+|x|^2)^{-\frac{\sigma+n}{2}}$ up to a constant multiple.

In view of the Bruhat decomposition $G = \overline{P} \cup \overline{P} w_0 \overline{P}$ the action $\pi^G_{\sigma,\varepsilon}$ can be completely described by the action of $\overline{P}$ and w_0. Using Lemma 1.1 we find

$$\begin{aligned}
\pi^G_{\sigma,\varepsilon}(\overline{n}_a) f(x) &= f(x-a), && \overline{n}_a \in \overline{N},\\
\pi^G_{\sigma,\varepsilon}(m) f(x) &= f(m^{-1}x), && m \in M^+ \cong O(n),\\
\pi^G_{\sigma,\varepsilon}(m_0) f(x) &= (-1)^\varepsilon f(-x),\\
\pi^G_{\sigma,\varepsilon}(e^{tH}) f(x) &= e^{(\sigma+n)t} f(e^{2t}x), && e^{tH} \in A,\\
\pi^G_{\sigma,\varepsilon}(w_0) f(x) &= (-1)^\varepsilon |x|^{-\sigma-n} f(-|x|^{-2} x).
\end{aligned}$$

This also gives the following expressions for the differential action $\mathrm{d}\pi^G_\sigma = \mathrm{d}\pi^G_{\sigma,\varepsilon}$ of the Lie algebra $\mathfrak{g}$, which is independent of ε:

$$\begin{aligned}
\mathrm{d}\pi^G_\sigma(\overline{N}_j) f(x) &= -\frac{\partial f}{\partial x_j}(x), && j = 1, \dots, n,\\
\mathrm{d}\pi^G_\sigma(T) f(x) &= -D_{Tx} f(x), && T \in \mathfrak{m} \cong \mathfrak{so}(n),\\
\mathrm{d}\pi^G_\sigma(H) f(x) &= (2E + \sigma + n) f(x),\\
\mathrm{d}\pi^G_\sigma(N_j) f(x) &= -|x|^2 \frac{\partial f}{\partial x_j}(x) + x_j (2E + \sigma + n) f(x), && j = 1, \dots, n,
\end{aligned}$$

where D_a denotes the directional derivative in direction $a \in \mathbb{R}^n$ and $E = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$ is the Euler operator on $\mathbb{R}^n$. For the action of $\mathfrak{n}$ we have used the identity $\mathrm{d}\pi^G_\sigma(N_a) = \pi^G_{\sigma,\varepsilon}(w_0) \mathrm{d}\pi^G_\sigma(\overline{N}_{-a}) \pi^G_{\sigma,\varepsilon}(w_0^{-1})$.

For $\sigma \in i\mathbb{R}$ the usual L^2-inner product on $\mathbb{R}^n$ provides unitarizations $(\mathcal{H}^G_{\sigma,\varepsilon}, \pi^G_{\sigma,\varepsilon})$ on $\mathcal{H}^G_{\sigma,\varepsilon} = L^2(\mathbb{R}^n)$ and these representations form the *unitary principal series.*

Now consider the normalized Knapp–Stein intertwining operators $\widetilde{J}(\sigma,\varepsilon) : \widetilde{I}^G_{\sigma,\varepsilon} \to \widetilde{I}^G_{-\sigma,\varepsilon}$ which are for $\operatorname{Re}\sigma > 0$ given by

$$\widetilde{J}(\sigma,\varepsilon)f(g) := \frac{(-1)^\varepsilon}{\Gamma(\frac{\sigma}{2})} \int_{\overline{N}} f(gw_0\overline{n})\,\mathrm{d}\overline{n}, \qquad g \in G,\ f \in \widetilde{I}^G_{\sigma,\varepsilon},$$

where $\mathrm{d}\overline{n}$ is the Haar measure on $\overline{N}$ given by the push-forward of the Lebesgue measure on $\mathbb{R}^n$ by the map $\mathbb{R}^n \to \overline{N}$, $x \mapsto \overline{n}_x$, and extended analytically to all $\sigma \in \mathbb{C}$. This intertwining operator induces an intertwining operator $J(\sigma,\varepsilon) : I^G_{\sigma,\varepsilon} \to I^G_{-\sigma,\varepsilon}$ by $J(\sigma,\varepsilon)f_{\overline{N}} := (\widetilde{J}(\sigma,\varepsilon)f)_{\overline{N}}$, $f \in \widetilde{I}^G_{\sigma,\varepsilon}$. Using Lemma 1.1 we obtain

$$\begin{aligned} J(\sigma,\varepsilon)f_{\overline{N}}(x) &= \frac{(-1)^\varepsilon}{\Gamma(\frac{\sigma}{2})} \int_{\mathbb{R}^n} f(\overline{n}_x w_0 \overline{n}_z)\,\mathrm{d}z \\ &= \frac{1}{\Gamma(\frac{\sigma}{2})} \int_{\mathbb{R}^n} |z|^{-\sigma-n} f_{\overline{N}}(x - |z|^{-2} z)\,\mathrm{d}z. \end{aligned}$$

Consider the coordinate change $y := x - |z|^{-2} z$. Its Jacobian $\left|\det(\frac{\partial y}{\partial z})\right|$ is homogeneous of degree $-2n$, $O(n)$-invariant and has value 1 for $z = e_1$. Hence it is equal to $|z|^{-2n}$. This finally gives

$$J(\sigma,\varepsilon)f(x) = \frac{1}{\Gamma(\frac{\sigma}{2})} \int_{\mathbb{R}^n} |x-y|^{\sigma-n} f(y)\,\mathrm{d}y = \frac{1}{\Gamma(\frac{\sigma}{2})}(|-|^{\sigma-n} * f)(x), \tag{1.3}$$

so $J(\sigma,\varepsilon)$ is up to a constant given by convolution with the distribution $|-|^{\sigma-n}$ and its residues. We define a G-invariant Hermitian form $(- \mid -)_{\sigma,\varepsilon}$ on $I^G_{\sigma,\varepsilon}$ by

$$\begin{aligned} (f \mid g)_{\sigma,\varepsilon} &:= (f \mid J(\sigma,\varepsilon)g)_{L^2(\mathbb{R}^n)} \\ &= \frac{1}{\Gamma(\frac{\sigma}{2})} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |x-y|^{\sigma-n} f(x)\overline{g(y)}\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{1.4}$$

For $\sigma \in (-n,n)$ this form is in fact positive definite and in this case the completion $\mathcal{H}^G_{\sigma,\varepsilon}$ of $I^G_{\sigma,\varepsilon}$ with respect to the inner product $(- \mid -)_{\sigma,\varepsilon}$ gives an irreducible unitary representation $(\mathcal{H}^G_{\sigma,\varepsilon}, \pi^G_{\sigma,\varepsilon})$ of G. The intertwining operator extends to an (up to scalar) unitary isomorphism $J(\sigma,\varepsilon):\mathcal{H}^G_{\sigma,\varepsilon}\to\mathcal{H}^G_{-\sigma,\varepsilon}$. These representations comprise the *complementary series*.

For $\sigma=n+2u$, $u\in\mathbb{N}$ the operator $J(-\sigma,\varepsilon)$ vanishes on the finite-dimensional subrepresentation $I_{-\sigma,\varepsilon,\mathrm{sub}}$ and maps onto the infinite-dimensional subrepresentation $I_{\sigma,\varepsilon,\mathrm{sub}}$. Therefore the Hermitian form $(-\mid-)_{-\sigma,\varepsilon}$ vanishes on the finite-dimensional subrepresentation $I^G_{-\sigma,\varepsilon,\mathrm{sub}}$ and induces a G-invariant positive definite Hermitian form on the unitarizable quotient $I^G_{-\sigma,\varepsilon}/I^G_{-\sigma,\varepsilon,\mathrm{sub}}$. Since this quotient is isomorphic to the subrepresentation $I^G_{\sigma,\varepsilon,\mathrm{sub}}$ via the intertwining operator $J(-\sigma,\varepsilon)$ we also obtain an irreducible unitary representation $(\mathcal{H}^G_{\sigma,\varepsilon,\mathrm{sub}},\pi^G_{\sigma,\varepsilon,\mathrm{sub}})$. These representations are isomorphic to the unitarizations of certain Zuckerman's modules $A_{\mathfrak{q}}(\lambda)$ of G and occur discretely in the Plancherel formula for the hyperboloids $O(1,n+1)/O(1,n)$. We call them *discrete series representations for the hyperboloid*.

Note that for any $\sigma\in i\mathbb{R}\cup(-n,n)$ the intertwining operator $J(\sigma,\varepsilon)$ extends to an isometry between the irreducible unitary representations $(\mathcal{H}^G_{\sigma,\varepsilon},\pi^G_{\sigma,\varepsilon})$ and $(\mathcal{H}^G_{-\sigma,\varepsilon},\pi^G_{-\sigma,\varepsilon})$.

1.3. The Fourier transformed picture

Consider the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^n}:\mathcal{S}'(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ given by

$$\mathcal{F}_{\mathbb{R}^n}u(x)=(2\pi)^{-\frac{n}{2}}\int_{\mathbb{R}^n}e^{-i(x\,|\,y)}u(y)\,\mathrm{d}y. \tag{1.5}$$

For $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$ and $\sigma\in i\mathbb{R}\cup(-n,n)$ resp. $\sigma\in(n+2\mathbb{N})$ we define a representation $\rho^G_{\sigma,\varepsilon}$ of G on $\mathcal{F}^{-1}_{\mathbb{R}^n}I^G_{\sigma,\varepsilon}$ resp. $\mathcal{F}^{-1}_{\mathbb{R}^n}I^G_{\sigma,\varepsilon,\mathrm{sub}}$ by

$$\pi^G_{\sigma,\varepsilon}(g)\circ\mathcal{F}_{\mathbb{R}^n}=\mathcal{F}_{\mathbb{R}^n}\circ\rho^G_{\sigma,\varepsilon}(g),\qquad g\in G.$$

resp.

$$\pi^G_{\sigma,\varepsilon,\mathrm{sub}}(g)\circ\mathcal{F}_{\mathbb{R}^n}=\mathcal{F}_{\mathbb{R}^n}\circ\rho^G_{\sigma,\varepsilon}(g),\qquad g\in G.$$

It is easy to calculate the group action of $\overline{P}=MA\overline{N}$:

$$\rho_{\sigma,\varepsilon}(\overline{n}_a)f(x)=e^{i(x\,|\,a)}f(x),\qquad \overline{n}_a\in\overline{N}, \tag{1.6}$$

$$\rho_{\sigma,\varepsilon}(m)f(x)=f(m^{-1}x),\qquad m\in M^+\cong O(n), \tag{1.7}$$

$$\rho_{\sigma,\varepsilon}(m_0)f(x)=(-1)^{\varepsilon}f(-x), \tag{1.8}$$

$$\rho_{\sigma,\varepsilon}(e^{tH})f(x) = e^{(\sigma-n)t}f(e^{-2t}x), \qquad t \in \mathbb{R}. \tag{1.9}$$

The action of w_0 in the Fourier transformed picture is more involved (see e.g. [37, Proposition 2.3]). Note that by these formulas the restriction $\rho_{\sigma,\varepsilon}|_{\overline{P}}$ also acts on $C^\infty(\mathbb{R}^m \setminus \{0\})$. Using the classical intertwining relations

$$x_j \circ \mathcal{F}_{\mathbb{R}^n} = \mathcal{F}_{\mathbb{R}^n} \circ (-i\tfrac{\partial}{\partial x_j}),$$
$$\tfrac{\partial}{\partial x_j} \circ \mathcal{F}_{\mathbb{R}^n} = \mathcal{F}_{\mathbb{R}^n} \circ (-ix_j)$$

it is easy to compute the differential action $\mathrm{d}\rho^G_\sigma$ of $\rho^G_{\sigma,\varepsilon}$:

$$\mathrm{d}\rho^G_\sigma(\overline{N}_j)f(x) = ix_jf(x), \qquad j = 1, \ldots, n, \tag{1.10}$$
$$\mathrm{d}\rho^G_\sigma(T)f(x) = -D_{Tx}f(x), \qquad T \in \mathfrak{m} \cong \mathfrak{so}(n), \tag{1.11}$$
$$\mathrm{d}\rho^G_\sigma(H)f(x) = -(2E - \sigma + n)\, f(x), \tag{1.12}$$
$$\mathrm{d}\rho^G_\sigma(N_j)f(x) = -i\mathcal{B}^{n,\sigma}_j f(x), \qquad j = 1, \ldots, n, \tag{1.13}$$

where we abbreviate

$$\mathcal{B}^{n,\sigma}_j := x_j\Delta - (2E - \sigma + n)\,\frac{\partial}{\partial x_j}.$$

The operators $\mathcal{B}^{n,\sigma}_j$ are called *Bessel operators* and are polynomial differential operators on $\mathbb{R}^n$. Therefore the action $\mathrm{d}\rho^G_\sigma$ defines a representation of $\mathfrak{g}$ on $C^\infty(\Omega)$ for every open subset $\Omega \subseteq \mathbb{R}^n$.

REMARK 1.3. The operators $\mathcal{B}^{n,\sigma}_j$ are called Bessel operators since they resemble the classical Bessel operators on $\mathbb{R}$ given by

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\nu}{x}\frac{\mathrm{d}}{\mathrm{d}x}.$$

They are a special instance of the Bessel operators in the theory of Jordan algebras which were already studied by Dib [5] in the early 90s and investigated further in [6, 12]. In the special case of indefinite orthogonal groups these operators first occur in the work by Kobayashi–Mano [17] on the minimal representation of $O(p,q)$ where they are called fundamental differential operators and denoted by $P_j(b)$ (see also [25]). In fact, for $(p,q) = (n+1,1)$ we have the relation $P_j(b) = \mathcal{B}^{n,2b}_j$.

To describe the representation spaces $\mathcal{H}^G_{\sigma,\varepsilon}$ resp. $\mathcal{H}^G_{\sigma,\varepsilon,\mathrm{sub}}$ in the Fourier transformed picture we recall that the Fourier transform $\mathcal{F}_{\mathbb{R}^n}$ intertwines convolution and multiplication operators. Further, the Riesz distributions $R_\lambda \in \mathcal{S}'(\mathbb{R}^n)$ given by

$$\langle R_\lambda, \varphi\rangle = \frac{2^{-\frac{\lambda}{2}}}{\Gamma(\frac{\lambda+n}{2})}\int_{\mathbb{R}^n} \varphi(x)\,|x|^\lambda \,\mathrm{d}x, \qquad \varphi\in\mathcal{S}(\mathbb{R}^n),$$

for $\operatorname{Re}\lambda > -n$ and extended analytically to $\lambda\in\mathbb{C}$ satisfy the following classical functional equation (see [8, equation (2') in II.3.3])

$$\mathcal{F}_{\mathbb{R}^n} R_\lambda = R_{-\lambda-n}.$$

With this observation as well as (1.3) and (1.4) we see that in the Fourier transformed picture the G-invariant inner product is simply given by the inner product of $L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$ and hence the map $\mathcal{F}_{\mathbb{R}^n}^{-1} : I^G_{\sigma,\varepsilon} \to L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$ extends to $\mathcal{F}_{\mathbb{R}^n}^{-1} : \mathcal{H}^G_{\sigma,\varepsilon} \to L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$ for $\sigma \in i\mathbb{R}\cup(-n,n)$. Similarly, the map $\mathcal{F}_{\mathbb{R}^n}^{-1} : I^G_{\sigma,\varepsilon,\mathrm{sub}} \to L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$ extends to $\mathcal{F}_{\mathbb{R}^n}^{-1} : \mathcal{H}^G_{\sigma,\varepsilon,\mathrm{sub}} \to L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$ for $\sigma\in(n+2\mathbb{N})$. We note that already the action of the parabolic subgroup $\overline{P}$ given by (1.6)–(1.9) extends to an irreducible unitary representation of $\overline{P}$ on $L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$ by Mackey theory and therefore $\mathcal{F}_{\mathbb{R}^n}^{-1}\mathcal{H}^G_{\sigma,\varepsilon}$ resp. $\mathcal{F}_{\mathbb{R}^n}^{-1}\mathcal{H}^G_{\sigma,\varepsilon,\mathrm{sub}}$ is actually equal to $L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$. This yields the L^2-realizations

$$(\rho^G_{\sigma,\varepsilon}, L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x))$$

for $\sigma\in i\mathbb{R}\cup(-n,n)\cup(n+2\mathbb{N})$. The Fourier transform is a unitary (up to scalar multiples) isomorphism $\mathcal{F}_{\mathbb{R}^n} : L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x) \to \mathcal{H}^G_{\sigma,\varepsilon}$ resp. $\mathcal{F}_{\mathbb{R}^n} : L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x) \to \mathcal{H}^G_{\sigma,\varepsilon,\mathrm{sub}}$ intertwining the representations $\rho^G_{\sigma,\varepsilon}$ and $\pi^G_{\sigma,\varepsilon}$ resp. $\pi^G_{\sigma,\varepsilon,\mathrm{sub}}$. For $\sigma\in i\mathbb{R}\cup(-n,n)$ the standard intertwining operators $J(\sigma,\varepsilon)$ are in this picture (up to scalar multiples) given by multiplication

$$L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x) \to L^2(\mathbb{R}^n, |x|^{\operatorname{Re}\sigma}\,\mathrm{d}x), \qquad f(x)\mapsto |x|^{-\sigma} f(x).$$

The explicit K-type decompositions (1.1) and (1.2) are difficult to see in the Fourier transformed picture. However, one can still describe the spaces of K-finite vectors. For this consider the renormalized K-Bessel function $\widetilde{K}_\alpha(z)$ from Appendix B.1. For $\sigma\in i\mathbb{R}\cup(-n,n)\cup(n+2\mathbb{N})$ put

$$\psi^G_\sigma(x) := \widetilde{K}_{-\frac{\sigma}{2}}(|x|), \qquad x\in\mathbb{R}^n\setminus\{0\}. \tag{1.14}$$

By Appendix B.4 and the integral formula (B.11), $\mathcal{F}_{\mathbb{R}^n}\psi_\sigma^G$ is equal to $(1+|x|^2)^{-\frac{\sigma+n}{2}}$ up to a constant multiple and hence ψ_σ^G is a $\mathfrak{k}$-fixed vector in $\mathcal{F}_{\mathbb{R}^n}^{-1}I_{\sigma,\varepsilon}^G$. If $\sigma \in i\mathbb{R}\cup(-n,n)$, it constitutes the minimal $\mathfrak{k}$-type in $L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$. Note that as K-representation the minimal K-type is for $\varepsilon\neq 0$ not the trivial representation since $m_0\in K$ acts on it by $(-1)^\varepsilon$. To describe the underlying $(\mathfrak{g},K)$-module we denote for $f\in C^\infty(\mathbb{R}_+)$ and $k\in\mathbb{N}$ by $f\otimes|x|^{2k}$ the function $f(|x|)\,|x|^{2k}$ and by $f\otimes|x|^{2k}\,\mathbb{C}[x]$ the space of all functions of the form $f(|x|)\,|x|^{2k}\,p(x)$ for some polynomial $p\in\mathbb{C}[x]$.

Lemma 1.4. *For $\sigma\in i\mathbb{R}\cup(-n,n)\cup(n+2\mathbb{N})$ the underlying $(\mathfrak{g},K)$-module of $\mathcal{F}_{\mathbb{R}^n}^{-1}I_{\sigma,\varepsilon}^G$ is given by*

$$(\mathcal{F}_{\mathbb{R}^n}^{-1}I_{\sigma,\varepsilon}^G)_K=\sum_{k=0}^\infty \widetilde{K}_{-\frac{\sigma}{2}+k}\otimes|x|^{2k}\,\mathbb{C}[x]=\sum_{k=0,1}\widetilde{K}_{-\frac{\sigma}{2}+k}\otimes|x|^{2k}\,\mathbb{C}[x]. \tag{1.15}$$

If $\sigma\in i\mathbb{R}\cup(-n,n)$ the underlying $(\mathfrak{g},K)$-module of the representation $(\rho_{\sigma,\varepsilon},L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x))$ is given by

$$L^2(\mathbb{R}^n,|x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)_K=\sum_{k=0}^\infty \widetilde{K}_{-\frac{\sigma}{2}+k}\otimes|x|^{2k}\,\mathbb{C}[x]. \tag{1.16}$$

Proof. Since $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\overline{\mathfrak{n}}$ the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}$ decomposes by the Poincaré–Birkhoff–Witt Theorem into $\mathcal{U}(\mathfrak{g})=\mathcal{U}(\overline{\mathfrak{n}})\mathcal{U}(\mathfrak{a})\mathcal{U}(\mathfrak{k})$. The $(\mathfrak{g},K)$-module $(\mathcal{F}_{\mathbb{R}^n}^{-1}I_{\sigma,\varepsilon}^G)_K$ is generated by the $\mathfrak{k}$-fixed vector ψ_σ^G and hence

$$(\mathcal{F}_{\mathbb{R}^n}^{-1}I_{\sigma,\varepsilon}^G)_K=\mathcal{U}(\mathfrak{g})\psi_\sigma^G=\mathcal{U}(\overline{\mathfrak{n}})\mathcal{U}(\mathfrak{a})\psi_\sigma^G.$$

By (1.10) and (1.12) we have $\mathcal{U}(\overline{\mathfrak{n}})=\mathbb{C}[x]$ and $\mathcal{U}(\mathfrak{a})=\mathbb{C}[E]$. Using (B.4) we further find that the Euler operator E acts on functions of the form $\widetilde{K}_\alpha(|x|)\,|x|^{2k}$, $\alpha\in\mathbb{R}$, $k\in\mathbb{N}$, by

$$E\left(\widetilde{K}_\alpha(|x|)\,|x|^{2k}\right)=-\tfrac{1}{2}\widetilde{K}_{\alpha+1}(|x|)\,|x|^{2k+2}+2k\widetilde{K}_\alpha(|x|)\,|x|^{2k}$$

Hence

$$\mathcal{U}(\overline{\mathfrak{n}})\mathcal{U}(\mathfrak{a})\psi_\sigma^G=\mathcal{U}(\overline{\mathfrak{n}})\sum_{k=0}^\infty\mathbb{C}\left(\widetilde{K}_{-\frac{\sigma}{2}+k}\otimes|x|^{2k}\right)=\sum_{k=0}^\infty\widetilde{K}_{-\frac{\sigma}{2}+k}\otimes|x|^{2k}\,\mathbb{C}[x]$$

which proves the first equality in (1.15). The second equality in (1.15) follows immediately from (B.5). Since the K-finite vectors do not depend on the globalization, (1.16) follows. $\square$

Now let $\sigma = n + 2u$, $u \in \mathbb{N}$. For $f \in C^\infty(\mathbb{R}_+)$ and $j, k \in \mathbb{N}$ we denote by $f \otimes |x|^{2k} \mathcal{H}^j(\mathbb{R}^n)$ the space of functions of the form $f(|x|)|x|^{2k}p(x)$ with $p \in \mathcal{H}^j(\mathbb{R}^n)$. Further, let $\mathbb{C}[x]_{>j}$ be the space of all polynomials which are sums of homogeneous polynomials of degree $> j$. Then $f \otimes |x|^{2k} \mathbb{C}[x]_{>j}$ denotes the space of functions of the form $f(|x|)|x|^{2k}p(x)$ with $p \in \mathbb{C}[x]_{>j}$.

LEMMA 1.5. *Let* $\sigma = n + 2u$, $u \in \mathbb{N}$. *The lowest* K*-type* $\mathrm{sign}^{\varepsilon+u+1} \boxtimes \mathcal{H}^{u+1}(\mathbb{R}^{n+1})$ *in the representation* $(\rho_{\sigma,\varepsilon}, L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\, \mathrm{d}x))$ *is given by*

$$\bigoplus_{k=0}^{u+1} \widetilde{K}_{-\frac{\sigma}{2}+k} \otimes |x|^{2k} \mathcal{H}^{u-k+1}(\mathbb{R}^n). \tag{1.17}$$

and for the underlying $(\mathfrak{g}, K)$*-module the following inclusion holds:*

$$L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\, \mathrm{d}x)_K \subseteq \sum_{k=0}^{\infty} \widetilde{K}_{-\frac{\sigma}{2}+k} \otimes |x|^{2k} \mathbb{C}[x]_{>u-k}. \tag{1.18}$$

PROOF. By (1.15), we see that (1.17) is contained in $(\mathcal{F}_{\mathbb{R}^n}^{-1} I_{\sigma,\varepsilon}^G)_K$. We now show that (1.17) is a K-subrepresentation of $(\mathcal{F}_{\mathbb{R}^n}^{-1} I_{\sigma,\varepsilon}^G)_K$. The group $O(n)$ leaves $\mathcal{H}^j(\mathbb{R}^n)$ invariant for every $j \in \mathbb{N}$ and hence M leaves each summand in (1.17) invariant. It remains to show that $\mathfrak{k} \cap (\mathfrak{n} + \overline{\mathfrak{n}}) = \operatorname{span}\{\overline{N}_j - N_j : j = 1, \ldots, n\}$ leaves (1.17) invariant. An easy calculation using (B.4) and (B.5) shows that for $p \in \mathcal{H}^{u-k+1}(\mathbb{R}^n)$ we have

$$\begin{aligned}
&\mathrm{d}\rho_\sigma^G(\overline{N}_j - N_j) \left[\widetilde{K}_{-\frac{\sigma}{2}+k}(|x|)\, |x|^{2k}\, p(x)\right] \\
&= i(\mathcal{B}_j^{n,\sigma} + x_j) \left[\widetilde{K}_{-\frac{\sigma}{2}+k}(|x|)\, |x|^{2k}\, p(x)\right] \\
&= 4ik\widetilde{K}_{-\frac{\sigma}{2}+k-1}(|x|)\, |x|^{2k-2}\, p_j^+(x) \\
&\quad + i(n + 2u - k)\widetilde{K}_{-\frac{\sigma}{2}+k+1}(|x|)\, |x|^{2k+2}\, p_j^-(x),
\end{aligned}$$

where $x_j p = p_j^+ + |x|^2 p_j^-$ with $p_j^\pm \in \mathcal{H}^{u-k+1\pm1}(\mathbb{R}^n)$ given by

$$p_j^+ = x_j p - \frac{|x|^2}{2u - 2k + n} \frac{\partial p}{\partial x_j}, \qquad p_j^- = \frac{1}{2u - 2k + n} \frac{\partial p}{\partial x_j}.$$

As in [28, Lemma B.1.2], we can construct an explicit isomorphism between (1.17) and $\mathrm{sign}^{\varepsilon+u+1} \boxtimes \mathcal{H}^{u+1}(\mathbb{R}^{n+1})$ which respects K-actions. In view of (1.1) the K-type $\mathrm{sign}^{\varepsilon+u+1} \boxtimes \mathcal{H}^{u+1}(\mathbb{R}^{n+1})$ occurs only once in $\mathcal{F}_{\mathbb{R}^n}^{-1} I_{\sigma,\varepsilon}^G|_K$ and hence (1.17) must coincide with the lowest K-type in $\mathcal{F}_{\mathbb{R}^n}^{-1} I_{\sigma,\varepsilon,\mathrm{sub}}^G$ and in $L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x)$. This shows the first part of the claim. Now an argument similar to the proof of Lemma 1.4 shows the inclusion (1.18). □

1.4. Branching rule for $m = 0$

Using the L^2-model $(\rho_{\sigma,\varepsilon}^G, L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x))$ we can now easily derive the branching rule for the restriction of $\rho_{\sigma,\varepsilon}^G$ to $H = O(1,1) \times O(n)$ (the case $m = 0$). Taking conjugation if necessary we may and do assume that $H = MA \cup w_0 MA$. Note that $MA = SO(1,1) \times O(n)$ and the action of MA is given by (1.7), (1.8) and (1.9). Therefore the isometric isomorphism

$$L^2(\mathbb{R}^n, |x|^{-\operatorname{Re}\sigma}\,\mathrm{d}x) \to L^2(\mathbb{R}^n), \quad f(x) \mapsto |x|^{-\frac{\sigma}{2}} f(x)$$

is an intertwining operator $\rho_{\sigma,\varepsilon}^G|_{MA} \to \rho_{0,\varepsilon}^G|_{MA}$ and the restrictions $\rho_{\sigma,\varepsilon}^G|_{MA}$ are pairwise equivalent. The decomposition of $\rho_{0,\varepsilon}^G|_{MA}$ can be done by the Mellin transform with respect to the variable $|x|$ giving

$$L^2(\mathbb{R}^n) = \sum_{k=0}^{\infty}{}^{\oplus} \left(\int_{i\mathbb{R}}^{\oplus} {\pi'}_{\tau,\varepsilon+k}^{SO(1,1)}\,\mathrm{d}\tau \right) \boxtimes \mathcal{H}^k(\mathbb{R}^n),$$

where ${\pi'}_{i\lambda,\delta}^{SO(1,1)}$ ($\lambda \in \mathbb{R}$, $\delta \in \mathbb{Z}/2\mathbb{Z}$) denotes the unitary character of $SO(1,1) = A \cup m_0 A$ given by

$${\pi'}_{i\lambda,\delta}^{SO(1,1)}(e^{tH}) = e^{i\lambda t}, \qquad {\pi'}_{i\lambda,\delta}^{SO(1,1)}(m_0) = (-1)^{\delta}.$$

Let $\pi_{i\lambda,\delta}^{O(1,1)}$ ($\lambda \in \mathbb{R}_+$, $\delta \in \mathbb{Z}/2\mathbb{Z}$) denote the two-dimensional irreducible unitary representation of $O(1,1) = A \cup m_0 A \cup w_0 A \cup m_0 w_0 A$ given by

$$\pi_{i\lambda,\delta}^{O(1,1)}(e^{tH}) = \begin{pmatrix} e^{i\lambda t} & 0 \\ 0 & e^{-i\lambda t} \end{pmatrix},$$

$$\pi_{i\lambda,\delta}^{O(1,1)}(m_0) = (-1)^{\delta}, \quad \pi_{i\lambda,\delta}^{O(1,1)}(w_0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Any irreducible unitary representation of $O(1,1)$ is either isomorphic to $\pi_{i\lambda,\delta}^{O(1,1)}$ or a character which factors through $O(1,1)/A \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Therefore, the only possibility for the branching law to H is

$$L^2(\mathbb{R}^n) = \sum_{k=0}^{\infty}{}^{\oplus} \left(\int_{i\mathbb{R}_+}^{\oplus} \pi_{\tau,\varepsilon+k}^{O(1,1)} \, \mathrm{d}\tau \right) \boxtimes \mathcal{H}^k(\mathbb{R}^n).$$

2. Reduction to an Ordinary Differential Operator

This section deals with the reduction of the branching problem for $\rho_{\sigma,\varepsilon}^G|_H$ to an ordinary differential equation on $\mathbb{R}_+$. For this we assume $0 < m < n$ throughout the rest of this paper.

Consider the L^2-realization $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma} \, \mathrm{d}x \, \mathrm{d}y)$ of the representation $\rho_{\sigma,\varepsilon}^G$ where we split variables $(x,y) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. We realize the unitary representations $\rho_{\tau,\delta}^{O(1,m+1)}$ of the first factor $O(1,m+1)$ of $H = O(1,m+1) \times O(n-m)$ in the same way on $L^2(\mathbb{R}^m, |x|^{-\operatorname{Re}\tau} \, \mathrm{d}x)$. For the second factor $O(n-m)$ denote by $\mathcal{H}^k(\mathbb{R}^{n-m})$ its representation on solid spherical harmonics on $\mathbb{R}^{n-m}$ of degree $k \in \mathbb{N}$ by left-translation.

In the following proposition we construct $\mathfrak{h}$-isotypic components in $C^\infty(\mathbb{R}^n \setminus \{x = 0\})$ using a hypergeometric differential equation. This is motivated by the work of Speh–Venkataramana [32] who studied the special case $m = n-1$, $k = 0$ and $\tau = \sigma - 1$ (i.e. $u \equiv 1$) of the following result:

Proposition 2.1. *Let $\sigma \in i\mathbb{R} \cup (-n,n) \cup (n+2\mathbb{N})$ and $\tau \in i\mathbb{R} \cup (-m,m) \cup (m+2\mathbb{N})$. For every solution $F \in C^\infty(\mathbb{R}_+)$ to the second order ordinary differential equation*

$$\begin{aligned} t(1+t)u''(t) + \left(\tfrac{-\sigma+2k+n-m+2}{2}t + \tfrac{2k+n-m}{2}\right) u'(t) \\ + \tfrac{1}{4} \left(\left(\tfrac{-\sigma+2k+n-m}{2}\right)^2 - \left(\tfrac{\tau}{2}\right)^2 \right) u(t) = 0 \end{aligned}$$

which is regular at $t = 0$ the map

$$\begin{aligned} \Psi : C^\infty(\mathbb{R}^m \setminus \{0\}) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}) \to C^\infty(\mathbb{R}^n \setminus \{x = 0\}), \\ \Psi(f \otimes \phi)(x,y) := |x|^{\frac{\sigma-\tau-2k-n+m}{2}} F\left(\tfrac{|y|^2}{|x|^2}\right) f(x) \phi(y), \end{aligned}$$

is $\overline{P}_H$- and $\mathfrak{h}$-equivariant. Here, $\overline{P}_H$- and $\mathfrak{h}$-actions on $C^\infty(\mathbb{R}^m \setminus \{0\})$ are given by (1.6)–(1.13) and actions on $C^\infty(\mathbb{R}^n \setminus \{x = 0\})$ are given by the restriction of (1.6)–(1.13) for $\overline{P}$ and $\mathfrak{g}$.

Proof. Put $\mu := 2k+n-m$ and $\alpha := \frac{\sigma-\tau-\mu}{2}$ so that

$$\Psi(f\otimes\phi)(x,y) = |x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(x)\phi(y).$$

Since $\mathfrak{h} = \mathfrak{n}_H + \mathfrak{m}_H + \mathfrak{a} + \overline{\mathfrak{n}}_H$ it suffices to check the intertwining property for $\overline{N}_H$, M_H, A and $\mathfrak{n}_H$.

(i) For $\overline{n}_a \in \overline{N}_H$ both $\rho^G_{\sigma,\varepsilon}(\overline{n}_a)$ and $\rho^{O(1,m+1)}_{\tau,\varepsilon+k}(\overline{n}_a)$ are by (1.6) the multiplication operators $e^{i(x\,|\,a)}$ and hence the intertwining property is clear.

(ii) Let $m = \mathrm{diag}(1,k_1,k_2,1) \in M_H^+$, $k_1 \in O(m)$, $k_2 \in O(n-m)$. Then with $m' = \mathrm{diag}(1,k_1,\mathbf{1}_{n-m+1})$ we have by (1.7)

$$\begin{aligned}\rho^G_{\sigma,\varepsilon}(m)\Psi(f\otimes\phi)(x,y) &= \Psi(f\otimes\phi)(k_1^{-1}x,k_2^{-1}y)\\ &= \left|k_1^{-1}x\right|^\alpha F(\frac{\left|k_2^{-1}y\right|^2}{\left|k_1^{-1}x\right|^2})f(k_1^{-1}x)\phi(k_2^{-1}y)\\ &= |x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(k_1^{-1}x)\phi(k_2^{-1}y)\\ &= \Psi(\rho^{O(1,m+1)}_{\tau,\varepsilon+k}(m')f\otimes(k_2\cdot\phi))(x,y).\end{aligned}$$

Further, for m_0 we have with (1.8)

$$\begin{aligned}\rho^G_{\sigma,\varepsilon}(m_0)\Psi(f\otimes\phi)(x,y) &= (-1)^\varepsilon\Psi(f\otimes\phi)(-x,-y)\\ &= (-1)^\varepsilon\,|(-x)|^\alpha F(\tfrac{|(-y)|^2}{|(-x)|^2})f(-x)\phi(-y)\\ &= (-1)^{\varepsilon+k}\,|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(-x)\phi(y)\\ &= \Psi(\rho^{O(1,m+1)}_{\tau,\varepsilon+k}(m_0)f\otimes\phi)(x,y).\end{aligned}$$

(iii) For $a = e^{tH} \in A$ we obtain with (1.9)

$$\begin{aligned}\rho^G_{\sigma,\varepsilon}(a)\Psi(f\otimes\phi)(x,y) &= e^{(\sigma-n)t}\Psi(f\otimes\phi)(e^{-2t}x,e^{-2t}y)\\ &= e^{(\sigma-n)t}\left|e^{-2t}x\right|^\alpha F(\frac{\left|e^{-2t}y\right|^2}{|e^{-2t}x|^2})f(e^{-2t}x)\phi(e^{-2t}y)\\ &= e^{(\sigma-n-2\alpha-2k)t}\,|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(e^{-2t}x)\phi(y)\\ &= e^{(\tau-m)t}\,|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(e^{-2t}x)\phi(y)\\ &= \Psi(\rho^{O(1,m+1)}_{\tau,\varepsilon+k}(a)f\otimes\phi)(x,y).\end{aligned}$$

(iv) To show the intertwining property for $\mathfrak{n}_H$ it suffices by (1.13) to show the identity

$$\mathcal{B}_j^{n,\sigma}\Psi(f\otimes\phi) = \Psi(\mathcal{B}_j^{m,\tau}f\otimes\phi)$$

for $j=1,\dots,m$ which follows from the next lemma. □

For $\sigma,\mu\in\mathbb{C}$ we introduce the ordinary differential operator

$$\mathcal{D}_{\sigma,\mu} := t(1+t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left(\frac{\mu-\sigma+2}{2}t+\frac{\mu}{2}\right)\frac{\mathrm{d}}{\mathrm{d}t}. \tag{2.1}$$

Lemma 2.2. *Let $\sigma,\tau,\alpha\in\mathbb{C}$, $k\in\mathbb{N}$, $\mu=2k+n-m$, $F\in C^\infty([0,\infty))$, $f\in C^\infty(\mathbb{R}^m\setminus\{0\})$ and $\phi\in\mathcal{H}^k(\mathbb{R}^{n-m})$. Then for every $j=1,\dots,m$ we have*

$$\begin{aligned}\mathcal{B}_j^{n,\sigma}&\left[|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(x)\phi(y)\right]\\ &= |x|^\alpha F(\tfrac{|y|^2}{|x|^2})\mathcal{B}_j^{m,\tau}f(x)\phi(y)\\ &\quad + x_j\,|x|^{\alpha-2} f(x)\phi(y)\left(4\mathcal{D}_{\sigma,\mu}+\alpha(\sigma-\mu-\alpha)\right)F(\tfrac{|y|^2}{|x|^2}).\end{aligned}$$

Proof. We first note the following basic identities, where $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are the gradients in $x\in\mathbb{R}^m$ and $y\in\mathbb{R}^{n-m}$ respectively, and Δ_x and Δ_y the Laplacians on $\mathbb{R}^m$ and $\mathbb{R}^{n-m}$ respectively:

$$\frac{\partial}{\partial x}|x|^\alpha = \alpha\,|x|^{\alpha-2}x, \qquad \Delta_x|x|^\alpha = \alpha(\alpha+m-2)\,|x|^{\alpha-2},$$

$$\frac{\partial}{\partial x}F(\tfrac{|y|^2}{|x|^2}) = -\frac{2\,|y|^2}{|x|^4}F'(\tfrac{|y|^2}{|x|^2})x, \qquad \Delta_xF(\tfrac{|y|^2}{|x|^2}) = 4\frac{|y|^4}{|x|^6}F''(\tfrac{|y|^2}{|x|^2}) - 2(m-4)\frac{|y|^2}{|x|^4}F'(\tfrac{|y|^2}{|x|^2}),$$

$$\frac{\partial}{\partial y}F(\tfrac{|y|^2}{|x|^2}) = \frac{2}{|x|^2}F'(\tfrac{|y|^2}{|x|^2})y, \qquad \Delta_yF(\tfrac{|y|^2}{|x|^2}) = \frac{4\,|y|^2}{|x|^4}F''(\tfrac{|y|^2}{|x|^2}) + \frac{2(n-m)}{|x|^2}F'(\tfrac{|y|^2}{|x|^2}).$$

The calculation is split into several parts. In what follows we abbreviate $t:=\frac{|y|^2}{|x|^2}$.

(i) We begin with calculating $x_j\Delta_x\Psi(f\otimes\phi)$:

$$
\begin{aligned}
&x_j\Delta_x\Psi(f\otimes\phi)(x,y)\\
&\quad=\Psi(x_j\Delta_x f\otimes\phi)(x,y)+x_j\Delta_x|x|^\alpha\cdot F(\tfrac{|y|^2}{|x|^2})f(x)\phi(y)\\
&\qquad+x_j\Delta_x F(\tfrac{|y|^2}{|x|^2})\cdot|x|^\alpha f(x)\phi(y)+2x_j\frac{\partial|x|^\alpha}{\partial x}\cdot\frac{\partial f}{\partial x}(x)\cdot F(\tfrac{|y|^2}{|x|^2})\phi(y)\\
&\qquad+2x_j\frac{\partial|x|^\alpha}{\partial x}\cdot\frac{\partial F(\frac{|y|^2}{|x|^2})}{\partial x}\cdot f(x)\phi(y)\\
&\qquad+2x_j\frac{\partial F(\frac{|y|^2}{|x|^2})}{\partial x}\cdot\frac{\partial f}{\partial x}(x)\cdot|x|^\alpha\phi(y)\\
&\quad=\Psi(x_j\Delta_x f\otimes\phi)(x,y)+x_j|x|^{\alpha-2}Ef(x)\phi(y)\left(-4tF'(t)+2\alpha F(t)\right)\\
&\qquad+x_j|x|^{\alpha-2}f(x)\phi(y)\left(4t^2F''(t)-2(2\alpha+m-4)tF'(t)\right.\\
&\qquad\qquad\left.+\alpha(\alpha+m-2)F(t)\right).
\end{aligned}
$$

(ii) Next we calculate $x_j\Delta_y\Psi(f\otimes\phi)$:

$$
\begin{aligned}
&x_j\Delta_y\Psi(f\otimes\phi)(x,y)\\
&\quad=x_j\Delta_y F(\tfrac{|y|^2}{|x|^2})\cdot|x|^\alpha f(x)\phi(y)+x_j\Delta_y\phi(y)\cdot|x|^\alpha F(\tfrac{|y|^2}{|x|^2})f(x)\\
&\qquad+2x_j\frac{\partial F(\frac{|y|^2}{|x|^2})}{\partial y}\cdot\frac{\partial\phi}{\partial y}\cdot|x|^\alpha f(x)\\
&\quad=x_j|x|^{\alpha-2}f(x)\phi(y)\left(4tF''(t)+2(2k+n-m)F'(t)\right)
\end{aligned}
$$

since $E\phi=k\phi$ and $\Delta_y\phi=0$.

(iii) We now calculate $\frac{\partial}{\partial x_j}\Psi(f\otimes\phi)$:

$$
\begin{aligned}
&\frac{\partial}{\partial x_j}\Psi(f\otimes\phi)(x,y)\\
&\quad=\frac{\partial|x|^\alpha}{\partial x_j}\cdot F(\tfrac{|y|^2}{|x|^2})f(x)\phi(y)+\frac{\partial F(\frac{|y|^2}{|x|^2})}{\partial x_j}\cdot|x|^\alpha f(x)\phi(y)\\
&\qquad+\frac{\partial f}{\partial x_j}(x)\cdot|x|^\alpha F(\tfrac{|y|^2}{|x|^2})\phi(y)
\end{aligned}
$$

$$= \frac{\partial f}{\partial x_j}(x) \cdot |x|^{\alpha} F(\tfrac{|y|^2}{|x|^2})\phi(y) + x_j\,|x|^{\alpha-2} f(x)\phi(y)\left(-2tF'(t) + \alpha F(t)\right).$$

(iv) Next we find $(2E - \sigma + n)\frac{\partial}{\partial x_j}\Psi(f \otimes \phi)$ by using (iii):

$$\begin{aligned}
&(2E - \sigma + n)\frac{\partial}{\partial x_j}\Psi(f \otimes \phi)(x, y) \\
&\quad = (2E - \sigma + n + 2(\alpha + k))\frac{\partial f}{\partial x_j}(x) \cdot |x|^{\alpha} F(\tfrac{|y|^2}{|x|^2})\phi(y) \\
&\qquad + 2x_j\,|x|^{\alpha-2} Ef(x)\phi(y)\left(-2tF'(t) + \alpha F(t)\right) \\
&\qquad + (2(\alpha + k - 1) - \sigma + n)x_j\,|x|^{\alpha-2} f(x)\phi(y)\left(-2tF'(t) + \alpha F(t)\right)
\end{aligned}$$

since $E\,|x|^{\beta} = \beta\,|x|^{\beta}$, $EF(\frac{|y|^2}{|x|^2}) = 0$ and $E\phi = k\phi$.

Now, putting (i), (ii) and (iv) together gives the claimed identity. $\square$

3. Spectral Decomposition of an Ordinary Second Order Differential Operator

Proposition 2.1 and Lemma 2.2 suggest that the decomposition of the $O(n-m)$-isotypic component of $\mathcal{H}^k(\mathbb{R}^{n-m})$ in $\rho^G_{\sigma,\varepsilon}$ into irreducible $O(1, m+1)$-representations is given by the spectral decomposition of the second order differential operator $\mathcal{D}_{\sigma,\mu}$ defined in (2.1) where $\mu = 2k + n - m$. In this section we find the spectral decomposition of $\mathcal{D}_{\sigma,\mu}$ acting on $L^2(\mathbb{R}_+, t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}t)$ using the theory developed by Weyl–Stone–Kodaira–Titchmarsh.

3.1. Kodaira's result

The spectral decomposition formula for general self-adjoint ordinary differential operators of the second order was established by Kodaira [21, 22] and Titchmarsh [35]. In [22] and [23], Kodaira studied Schrödinger type operators in detail and deduced a simpler formula for the spectral measure of these operators, which also laid a mathematical foundation for Heisenberg's S-matrix theory. We can apply this simpler formula to our setting, because $\mathcal{D}_{\sigma,\mu}$ turns out to be a Schrödinger type operator after a suitable change of variables.

We first recall Kodaira's spectral decomposition theorem for Schrödinger type operators (see the original papers [22] or [23] for the proof). Let

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\nu(\nu+1)}{x^2} + V(x) \quad (0 < x < \infty),$$

where $\nu \geq -\frac{1}{2}$ and $V(x)$ is a real-valued continuous function such that

$$V(x) = O(x^{-2+\epsilon}) \text{ as } x \to 0, \quad V(x) = \frac{\alpha + O(x^{-\epsilon})}{x} \text{ as } x \to \infty$$

for some $\alpha \in \mathbb{R}$ and $\epsilon > 0$. A system of two solutions $s_1(x,\lambda)$, $s_2(x,\lambda)$ to $Lu = \lambda u\,(\lambda \in \mathbb{C})$ is called *a system of fundamental solutions* if it has the following three properties:

- $W(s_2, s_1) = 1$, where $W(u,v) = u\dfrac{\mathrm{d}v}{\mathrm{d}x} - v\dfrac{\mathrm{d}u}{\mathrm{d}x}$ denotes the Wronskian,
- $s_j(x,\overline{\lambda}) = \overline{s_j(x,\lambda)}$ for $j = 1,2$,
- $s_j(x,\lambda)$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}s_j(x,\lambda)$ are holomorphic in $\lambda \in \mathbb{C}$ for $j = 1,2$.

For Schrödinger type operators there exists a system of fundamental solutions s_1, s_2 with the following asymptotic behaviour as $x \to 0$:

$$\begin{aligned} &s_1(x,\lambda) \sim x^{\nu+1}, && s_2(x,\lambda) \sim \frac{1}{2\nu+1}x^{-\nu} && \text{if } \nu > -\frac{1}{2}, \\ &s_1(x,\lambda) \sim x^{\frac{1}{2}}, && s_2(x,\lambda) \sim -x^{\frac{1}{2}}\log x && \text{if } \nu = -\frac{1}{2}. \end{aligned}$$

We note that the function s_1 is uniquely determined because a solution to $Lu = \lambda u$ with $u(x) \sim x^{\nu+1}$ is unique. Since s_1 is L^2 near $x = 0$ for any $\nu \geq -\frac{1}{2}$ and s_2 is L^2 near $x = 0$ if and only if $\nu < \frac{1}{2}$, we conclude that $x = 0$ is of limit point type (LPT) if $\nu \geq \frac{1}{2}$ and of limit circle type (LCT) if $-\frac{1}{2} \leq \nu < \frac{1}{2}$. In the case of (LCT) at $x = 0$ we impose the following additional boundary condition (which is automatic in the case of (LPT)):

$$\text{(BC)} \qquad \lim_{x\to 0} W(s_1(-,0),u)(x) = 0.$$

Then in both the (LPT) and the (LCT) case $s_1(x,\lambda)$ is the unique solution to $Lu = \lambda u$ which is L^2 near $x = 0$ and satisfies the boundary condition (BC).

On the other hand, the point $x=\infty$ is always of (LPT) and we have:

THEOREM 3.1 ([22, Theorem 5.1], [23, Theorem 26]). *If* $\operatorname{Im}\kappa \geq 0$ *and* $\kappa \neq 0$, *the equation* $Lu=\kappa^2 u$ *has one and only one solution* $u_0(-,\kappa)$ *such that*

$$u_0(x,\kappa) \sim \exp\left(i\kappa x - \frac{i\alpha}{2\kappa}\log x\right) \qquad \textit{as } x \to \infty.$$

As functions of the two variables x *and* κ, $u_0(x,\kappa)$ *and* $\frac{\mathrm{d}}{\mathrm{d}x}u_0(x,\kappa)$ *are continuous in* $0<x<\infty$, $\operatorname{Im}\kappa \geq 0$ *and* $\kappa \neq 0$. *As functions of* κ, *they are holomorphic in* $\operatorname{Im}\kappa > 0$.

The differential operator L defines a self-adjoint operator on $L^2(\mathbb{R}_+)$ with domain the space of functions u satisfying the following five conditions:

- $u \in L^2(\mathbb{R}_+)$,
- u is differentiable,
- $\dfrac{\mathrm{d}u}{\mathrm{d}x}$ is absolutely continuous in every closed interval $[a,b]$ $(0<a<b<\infty)$,
- $Lu \in L^2(\mathbb{R}_+)$,
- u satisfies the boundary condition (BC).

The spectral decomposition of L is given in terms of the functions $A(\kappa)$ and $B(\kappa)$ defined by

$$u_0(x,\kappa) = A(\kappa)s_2(x,\kappa^2) - B(\kappa)s_1(x,\kappa^2).$$

This equation implies

$$A(\kappa) = W(u_0(-,\kappa), s_1(-,\kappa^2)) \qquad \text{and} \qquad B(\kappa) = W(u_0(-,\kappa), s_2(-,\kappa^2)).$$

The functions $A(\kappa)$ and $B(\kappa)$ are holomorphic in $\operatorname{Im}\kappa>0$ and continuous in $\operatorname{Im}\kappa \geq 0$ and $\kappa \neq 0$. In $\operatorname{Im}\kappa > 0$, all zeros of $A(\kappa)$ lie on the imaginary axis and are of order one. Denote these zero points by $\kappa_j = i|\kappa_j|$ $(j \in J)$. Then it can be proved that the discrete spectrum of L is $\lambda = \kappa_j^2$ and possibly $\lambda = 0$.

The continuous spectrum of L is the interval $[0,\infty)$. The eigenfunction expansion formula for L is

Theorem 3.2 ([22, 23]). *In the setting and the notation above, we have an expansion of any L^2-function $u(x)$ of the following form*

$$u(x)=\sum_{j\in J}s_1(x,\kappa_j^2)\rho_j\int_0^\infty s_1(y,\kappa_j^2)u(y)\,\mathrm{d}y \\ +s_1(x,0)\rho^0\int_0^\infty s_1(y,0)u(y)\,\mathrm{d}y \\ +\frac{2}{\pi}\int_0^\infty s_1(x,\kappa^2)\frac{\kappa^2}{|A(\kappa)|^2}\int_0^\infty s_1(y,\kappa^2)u(y)\,\mathrm{d}y\,\mathrm{d}\kappa, \tag{3.1}$$

where

$$\rho_j=\frac{1}{\pi}|\kappa_j|B(\kappa_j)\oint_{\kappa_j}\frac{\mathrm{d}\kappa}{A(\kappa)},\ \textit{and}\ \rho^0=\lim_{\epsilon\to+0}\frac{1}{\pi}\int_0^\pi\frac{B(\epsilon e^{i\theta})}{A(\epsilon e^{i\theta})}\epsilon^2e^{2i\theta}\,\mathrm{d}\theta.$$

We remark that $\rho^0=0$ in many cases.

To reformulate it as an isomorphism between Hilbert spaces put

$$S:=\{\kappa_j^2:j\in J\}(\cup\{0\})\cup\mathbb{R}_+,$$

where $\{0\}$ is included if $\rho^0>0$. Define a measure on S by

$$\int_S g(\lambda)\,\mathrm{d}\rho(\lambda):=\sum_{j\in J}\rho_jg(\kappa_j^2)(+\rho^0g(0))+\frac{1}{\pi}\int_0^\infty\frac{\sqrt{\lambda}}{|A(\sqrt{\lambda})|^2}g(\lambda)\,\mathrm{d}\lambda.$$

Then by [22, Theorem 4.2] or [23, Theorem 19]:

Theorem 3.3. *The map*

$$L^2(\mathbb{R}_+)\xrightarrow{\sim}L^2(S,\mathrm{d}\rho),\qquad u\mapsto g(\lambda)=\int_0^\infty s_1(x,\lambda)u(x)\,\mathrm{d}x,$$

is a unitary isomorphism with inverse

$$L^2(S,\mathrm{d}\rho)\xrightarrow{\sim}L^2(\mathbb{R}_+),\qquad g\mapsto u(x)=\int_S s_1(x,\lambda)g(\lambda)\,\mathrm{d}\rho(\lambda).$$

3.2. Simplifications

In the rest of this section we apply the above result to find the spectral decomposition of $\mathcal{D}_{\sigma,\mu}$. We fix $\sigma \in i\mathbb{R} \cup (0,\infty)$. (In the case $\sigma \in (-\infty,0)$ only the derivation of the discrete spectrum is slightly different. However, since $\pi_{\sigma,\varepsilon} \cong \pi_{-\sigma,\varepsilon}$ for $\sigma \in (-n,n)$ the decomposition of the representations is again the same and it suffices to consider $\sigma \in i\mathbb{R}\cup(0,\infty)$ for our purpose.) Further fix $k \in \mathbb{N}$ and put $\mu := 2k+n-m$. We assume $m < n$ so that $\mu > 0$. Writing

$$\mathcal{D}_{\sigma,\mu} = t(1+t)\frac{\mathrm{d}^2}{\mathrm{d}t^2} + ((a+b+1)t+c)\frac{\mathrm{d}}{\mathrm{d}t}$$

with

$$a = -\frac{\sigma-\mu}{4} + \frac{\tau}{4}, \qquad b = -\frac{\sigma-\mu}{4} - \frac{\tau}{4}, \qquad c = \frac{\mu}{2},$$

it is easy to see from (B.6) that the hypergeometric function

$$F(t,\tau) := {}_2F_1(a,b;c;-t) \tag{3.2}$$

solves the equation

$$\mathcal{D}_{\sigma,\mu}f + \lambda^* f = 0, \qquad \lambda^* = ab = \left(\frac{\sigma-\mu}{4}\right)^2 - \left(\frac{\tau}{4}\right)^2.$$

We find a spectral decomposition of $\mathcal{D}_{\sigma,\mu}$ in terms of $F(t,\tau)$.

First make the transformation $t = \sinh^2(\frac{x}{2})$. Using $t\frac{\mathrm{d}}{\mathrm{d}t} = \tanh(\frac{x}{2})\frac{\mathrm{d}}{\mathrm{d}x}$ we write the operator $\mathcal{D}_{\sigma,\mu}$ as

$$\begin{aligned}\mathcal{D}_{\sigma,\mu} &= \frac{1}{t}\left((1+t)\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^2 + \left(\frac{\mu-\sigma}{2}t + \frac{\mu-2}{2}\right)t\frac{\mathrm{d}}{\mathrm{d}t}\right)\\ &= \frac{\mathrm{d}^2}{\mathrm{d}x^2} + \beta(x)\frac{\mathrm{d}}{\mathrm{d}x}\end{aligned}$$

with

$$\beta(x) = \frac{\mu-1}{2}\tanh\left(\frac{x}{2}\right)^{-1} - \frac{\sigma-1}{2}\tanh\left(\frac{x}{2}\right).$$

Putting

$$u(x) = r(x)^{-1} f\left(\sinh^2\left(\frac{x}{2}\right)\right) \quad \text{with} \quad r(x) = \sinh\left(\frac{x}{2}\right)^{-\frac{\mu-1}{2}} \cosh\left(\frac{x}{2}\right)^{\frac{\sigma-1}{2}}$$

we finally see that the differential equation $\mathcal{D}_{\sigma,\mu}f + \lambda^* f = 0$ is equivalent to

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + q^*(x)u = \lambda^* u$$

with

$$\begin{aligned} q^*(x) &= \tfrac{1}{4}\beta(x)^2 + \tfrac{1}{2}\beta'(x) \\ &= \frac{(\mu-1)(\mu-3)}{16}\tanh\left(\frac{x}{2}\right)^{-2} \\ &\quad - \frac{\mu(\sigma-2)+1}{8} + \frac{(\sigma+1)(\sigma-1)}{16}\tanh\left(\frac{x}{2}\right)^2. \end{aligned}$$

To stay in line with the setting in Section 3.1 we shift the eigenvalues by putting $q(x) := q^*(x) - \left(\frac{\sigma-\mu}{4}\right)^2$ and $\lambda := \lambda^* - \left(\frac{\sigma-\mu}{4}\right)^2$ and obtain

$$-\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + q(x)u = \lambda u. \tag{3.3}$$

Note that $q(x)$ is real-valued for $\sigma \in i\mathbb{R} \cup \mathbb{R}$ and hence the operator $-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x)$ is formally self-adjoint on $L^2(\mathbb{R}_+)$. Moreover by putting $\nu = \frac{\mu-3}{2}$ and $\alpha = 0$, the differential operator $L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q(x)$ is of Schrödinger type if $\mu \geq 2$. This is also true for $\mu = 1$ if we put $\nu = 0$. However, we should rather put $\nu = -1$ in order to impose an appropriate boundary condition. Thus we cannot use the general result in Section 3.1 directly for $\mu = 1$, but one can see that the proof of Theorem 3.2 in [22] or [23] is still valid in this case and thus (3.1) gives the correct formula.

3.3. Singularities and the boundary condition

We put $\nu = \frac{\mu-3}{2}$ for $\mu \geq 1$ and $\kappa = \sqrt{\lambda}$. The differential equation (3.3) has regular singular points at $x = 0$ and $x = \infty$. The corresponding asymptotic behaviour of solutions at $x = 0$ is given by $x^{\frac{\mu-1}{2}}$ and $x^{-\frac{\mu-3}{2}}$ for $\mu \neq 2$ and by $x^{\frac{1}{2}}$ and $\log(x)x^{\frac{1}{2}}$ for $\mu = 2$. Hence $x = 0$ is of limit point type (LPT) if $\mu \geq 4$ and of limit circle type (LCT) if $\mu = 1, 2, 3$. The solution

$$s_1(x,\lambda) = 2^{\frac{\mu-1}{2}} r(x)^{-1} {}_2F_1(a,b;c;-\sinh^2(\tfrac{x}{2}))$$

has asymptotic behaviour $x^{\frac{\mu-1}{2}} (= x^{\nu+1})$ near $x = 0$, where

$$a = -\frac{\sigma-\mu}{4} + i\kappa, \qquad b = -\frac{\sigma-\mu}{4} - i\kappa, \qquad c = \frac{\mu}{2}.$$

Note that $s_1(x, \lambda)$ is holomorphic in $\lambda \in \mathbb{C}$ and $\overline{s_1(x, \lambda)} = s_1(x, \overline{\lambda})$ if $\sigma \in \mathbb{R} \cup i\mathbb{R}$ by Kummer's transformation formula (B.9). For $\mu \geq 2$, s_1 is the unique solution which has asymptotic behaviour $x^{\frac{\nu+1}{2}}$ near $x = 0$. Hence we can find s_2 such that s_1, s_2 is a system of fundamental solutions. For $\mu = 1$, put

$$s_2(x, \lambda) = -2r(x)^{-1} \sinh(\tfrac{x}{2})_2F_1(1 + a - c, 1 + b - c; 2 - c; -\sinh^2(\tfrac{x}{2})).$$

Then s_2 has asymptotic behaviour $-x$ near $x = 0$ and s_1, s_2 is a system of fundamental solutions.

In the case of (LCT) at $x = 0$ we impose the additional boundary condition (BC). Then in both the (LPT) and the (LCT) case (i.e. for every $\mu \geq 1$) $s_1(x, \lambda)$ is the unique solution to (3.3) which is L^2 near $x = 0$ and satisfies the boundary condition (BC).

In view of Theorem 3.1 we consider another solution

$$u_0(x, \kappa) = 2^{2i\kappa} r(x)^{-1} \sinh^{-2b}(\tfrac{x}{2})_2F_1(b, b - c + 1; b - a + 1; -\sinh^{-2}(\tfrac{x}{2})),$$

which has asymptotic behaviour $e^{ix\kappa}$ as $x \to \infty$ and hence is L^2 near $x = \infty$ for $\operatorname{Im} \kappa > 0$. Note that a linearly independent solution is obtained by interchanging a and b and has asymptotics $e^{-ix\kappa}$ whence $x = \infty$ is always of (LPT).

Altogether the operator in (3.3) extends to a self-adjoint operator on $L^2(\mathbb{R}_+)$ under the boundary condition (BC) and its spectral decomposition is given by Theorem 3.2. We now make this spectral decomposition explicit.

3.4. The function $A(\kappa)$

We calculate the Wronskian

$$\begin{aligned}
A(\kappa) &= W(u_0(-, \kappa), s_1(-, \kappa^2)) \\
&= 2^{\frac{\mu-1}{2} + 2i\kappa} r(x)^{-2} \\
&\quad \times W(\sinh^{-2b}(\tfrac{-}{2})_2F_1(b, b - c + 1; b - a + 1; -\sinh^{-2}(\tfrac{-}{2})), \\
&\qquad\qquad {}_2F_1(a, b; c; -\sinh^2(\tfrac{-}{2})))(x) \\
&= 2^{\frac{\mu-1}{2} + 2i\kappa} r(x)^{-2} \sinh(\tfrac{x}{2}) \cosh(\tfrac{x}{2}) \\
&\quad \times W(z^{-b}{}_2F_1(b, b - c + 1; b - a + 1; -\tfrac{1}{z}), {}_2F_1(a, b; c; -z))(\sinh^2(\tfrac{x}{2}))
\end{aligned}$$

$$
\begin{aligned}
&= 2^{\frac{\mu-1}{2}+2i\kappa} r(x)^{-2}\sinh(\tfrac{x}{2})\cosh(\tfrac{x}{2})\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}\\
&\quad\times W(z^{-b}{}_2F_1(b,b-c+1;b-a+1;-\tfrac{1}{z}),\\
&\qquad\qquad z^{-a}{}_2F_1(a,a-c+1;a-b+1;-\tfrac{1}{z}))(\sinh^2(\tfrac{x}{2}))\\
&= 2^{\frac{\mu-1}{2}+2i\kappa}(b-a)\frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}\\
&= 2^{\frac{\mu-1}{2}+2i\kappa}\frac{\Gamma(-2i\kappa+1)\Gamma(\frac{\mu}{2})}{\Gamma(-\frac{\sigma-\mu}{4}-i\kappa)\Gamma(\frac{\sigma+\mu}{4}-i\kappa)}.
\end{aligned}
$$

Then we see that all the zeros of $A(\kappa)$ in the upper half-plane $\operatorname{Im}\kappa > 0$ are of order one and exactly at the points where $-\frac{\sigma-\mu}{4} - i\kappa \in -\mathbb{N}$. This gives $i\kappa = -\frac{\sigma-\mu}{4} + j$ and $\lambda = -\left(\frac{\sigma-\mu}{4} - j\right)^2$ for $j \in \mathbb{N}$ with $j < \frac{\sigma-\mu}{4}$. Hence $A(\kappa)$ has zeros in $\operatorname{Im}\kappa > 0$ if and only if $\sigma \in \mathbb{R}$ and $\sigma > \mu$. If this is the case, we put $\kappa_j = i(\frac{\sigma-\mu}{4} - j)$ for $j \in [0, \frac{\sigma-\mu}{4}) \cap \mathbb{Z}$. Using $\operatorname{res}_{z=-n}\Gamma(z) = \frac{(-1)^n}{n!}$ we find

$$
\begin{aligned}
\operatorname{res}_{\kappa=\kappa_j}\frac{1}{A(\kappa)} &= 2^{-\frac{\mu-1}{2}-2i\kappa_j}\frac{\Gamma(\frac{\sigma}{2}-j)}{\Gamma(\frac{\sigma-\mu}{2}-2j+1)\Gamma(\frac{\mu}{2})}\\
&\quad\times \operatorname{res}_{\kappa=\kappa_j}\Gamma\left(-\frac{\sigma-\mu}{4}-i\kappa\right)\\
&= 2^{-\frac{\mu-1}{2}-2i\kappa_j}\frac{\Gamma(\frac{\sigma}{2}-j)}{\Gamma(\frac{\sigma-\mu}{2}-2j+1)\Gamma(\frac{\mu}{2})}\frac{(-1)^j}{(-i)j!}.
\end{aligned}
\tag{3.4}
$$

To calculate $B(\kappa)$ for $\kappa = \kappa_j$, we note that $b = -j$ and therefore, by (B.7)

$$
\begin{aligned}
2^{-\frac{\mu-1}{2}} s_1(x,\kappa_j^2) &= \frac{(a)_j\Gamma(c)}{\Gamma(c+j)} 2^{-2i\kappa_j} u_0(x,\kappa_j)\\
&= 2^{-2i\kappa_j}\frac{(-\frac{\sigma-\mu}{2}+j)_j\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu}{2}+j)} u_0(x,\kappa_j).
\end{aligned}
$$

Here, $(a)_j = a(a+1)\cdots(a+j-1)$ denotes the Pochhammer symbol. As a result,

$$
B(\kappa_j) = -2^{-\frac{\mu-1}{2}+2i\kappa_j}\frac{\Gamma(\frac{\mu}{2}+j)}{(-\frac{\sigma-\mu}{2}+j)_j\Gamma(\frac{\mu}{2})}.
\tag{3.5}
$$

3.5. The spectral theorem for $\mathcal{D}_{\sigma,\mu}$

For $\mu \geq 2$ Theorem 3.2 gives the spectral formula (3.1). Following the proof in [22] or [23] it is easy to see that (3.1) is still valid for $\mu = 1$. For

$\kappa > 0$ we calculate

$$\begin{aligned}\frac{\kappa^2}{|A(\kappa)|^2} &= 2^{-(\mu-1)}\kappa^2 \left|\frac{\Gamma(-\frac{\sigma-\mu}{4} - i\kappa)\Gamma(\frac{\sigma+\mu}{4} - i\kappa)}{\Gamma(-2i\kappa+1)\Gamma(\frac{\mu}{2})}\right|^2 \\ &= 2^{-(\mu+1)} \left|\frac{\Gamma(-\frac{\sigma-\mu}{4} + i\kappa)\Gamma(\frac{\sigma+\mu}{4} + i\kappa)}{\Gamma(2i\kappa)\Gamma(\frac{\mu}{2})}\right|^2.\end{aligned}$$

For $j \in [0, \frac{\operatorname{Re}\sigma-\mu}{4}) \cap \mathbb{Z}$ we have

$$\begin{aligned}\rho_j &= \frac{1}{\pi}|\kappa_j| B(\kappa_j) \oint_{\kappa_j} \frac{\mathrm{d}\kappa}{A(\kappa)} \\ &= \frac{1}{\pi}\left(\frac{\sigma-\mu}{4} - j\right) \times \frac{-2^{-\frac{\mu-1}{2}+2i\kappa_j}\Gamma(\frac{\mu}{2}+j)}{(-\frac{\sigma-\mu}{2}+j)_j\Gamma(\frac{\mu}{2})} \\ &\quad \times 2\pi i \frac{2^{-\frac{\mu-1}{2}-2i\kappa_j} i(-1)^j\Gamma(\frac{\sigma}{2}-j)}{j!\Gamma(\frac{\sigma-\mu}{2}-2j+1)\Gamma(\frac{\mu}{2})} \\ &= \frac{2^{-(\mu-1)}(\frac{\sigma-\mu}{2}-2j)\Gamma(\frac{\sigma}{2}-j)\Gamma(\frac{\mu}{2}+j)}{j!\Gamma(\frac{\mu}{2})^2\Gamma(\frac{\sigma-\mu}{2}-j+1)}\end{aligned}$$

by (3.4) and (3.5). Moreover, since $\frac{B(\kappa)}{A(\kappa)}$ has at most a pole of order one at $x = 0$,

$$\rho^0 = \lim_{\epsilon\to+0} \frac{1}{\pi}\int_0^\pi \frac{B(\epsilon e^{i\theta})}{A(\epsilon e^{i\theta})}\epsilon^2 e^{2i\theta}\,\mathrm{d}\theta = 0.$$

Consequently, (3.1) gives the expansion formula:

$$\begin{aligned}u(x) = &\sum_{j\in[0,\frac{\operatorname{Re}\sigma-\mu}{4})\cap\mathbb{Z}} s_1(x,\kappa_j^2)\frac{2^{-(\mu-1)}(\frac{\sigma-\mu}{2}-2j)\Gamma(\frac{\sigma}{2}-j)\Gamma(\frac{\mu}{2}+j)}{j!\Gamma(\frac{\mu}{2})^2\Gamma(\frac{\sigma-\mu}{2}-j+1)} \\ &\qquad\qquad \times \int_0^\infty s_1(y,\kappa_j^2)u(y)\,\mathrm{d}y \\ &+ \frac{1}{\pi}\int_0^\infty s_1(x,\kappa^2)2^{-\mu}\left|\frac{\Gamma(-\frac{\sigma-\mu}{4}+i\kappa)\Gamma(\frac{\sigma+\mu}{4}+i\kappa)}{\Gamma(2i\kappa)\Gamma(\frac{\mu}{2})}\right|^2 \\ &\qquad\qquad \times \int_0^\infty s_1(y,\kappa^2)u(y)\,\mathrm{d}y\,\mathrm{d}\kappa.\end{aligned}$$

Using the different normalization

$$\eta_1(x,\lambda) := r(x)^{-1}{}_2F_1(a,b;c;-\sinh^2(\tfrac{x}{2}))(= 2^{-\frac{\mu-1}{2}}s_1(x,\lambda)),$$

this can be rewritten as

$$\begin{aligned} u(x) = \sum_{j\in[0,\frac{\operatorname{Re}\sigma-\mu}{4})\cap\mathbb{Z}} &\eta_1(x,-\big(\tfrac{\sigma-\mu}{4}-j\big)^2)\frac{(\frac{\sigma-\mu}{2}-2j)\Gamma(\frac{\sigma}{2}-j)\Gamma(\frac{\mu}{2}+j)}{j!\Gamma(\frac{\mu}{2})^2\Gamma(\frac{\sigma-\mu}{2}-j+1)} \\ &\times\int_0^\infty \eta_1(y,-\big(\tfrac{\sigma-\mu}{4}-j\big)^2)u(y)\,\mathrm{d}y \\ +\frac{1}{4\pi}\int_0^\infty \eta_1(x,\lambda)&\left|\frac{\Gamma(-\frac{\sigma-\mu}{4}+i\sqrt{\lambda})\Gamma(\frac{\sigma+\mu}{4}+i\sqrt{\lambda})}{\Gamma(2i\sqrt{\lambda})\Gamma(\frac{\mu}{2})}\right|^2 \\ &\times\int_0^\infty \eta_1(y,\lambda)u(y)\,\mathrm{d}y\frac{\mathrm{d}\lambda}{\sqrt{\lambda}}. \end{aligned}$$

To obtain an isomorphism between Hilbert spaces let

$$S(\sigma,\mu) := (0,\infty)\cup\bigcup_{j\in[0,\frac{\operatorname{Re}\sigma-\mu}{4})\cap\mathbb{Z}}\left\{-\left(\frac{\sigma-\mu}{4}-j\right)^2\right\}.$$

Note that $S(\sigma,\mu) = (0,\infty)$ for $\sigma\in i\mathbb{R}$. On $S(\sigma,\mu)$ we define a measure $\mathrm{d}\nu_{\sigma,\mu}$ by

$$\begin{aligned} &\int_{S(\sigma,\mu)} g(\lambda)\,\mathrm{d}\nu_{\sigma,\mu}(\lambda) \\ &\quad := \frac{1}{4\pi}\int_0^\infty g(\lambda)\left|\frac{\Gamma(-\frac{\sigma-\mu}{4}+i\sqrt{\lambda})\Gamma(\frac{\sigma+\mu}{4}+i\sqrt{\lambda})}{\Gamma(2i\sqrt{\lambda})\Gamma(\frac{\mu}{2})}\right|^2\frac{\mathrm{d}\lambda}{\sqrt{\lambda}} \\ &\quad\quad + \sum_{j\in[0,\frac{\operatorname{Re}\sigma-\mu}{4})\cap\mathbb{Z}}\frac{(\frac{\sigma-\mu}{2}-2j)\Gamma(\frac{\sigma}{2}-j)\Gamma(\frac{\mu}{2}+j)}{j!\Gamma(\frac{\mu}{2})^2\Gamma(\frac{\sigma-\mu}{2}-j+1)}g(-\big(\tfrac{\sigma-\mu}{4}-j\big)^2). \end{aligned}$$

Then by Theorem 3.3:

Theorem 3.4. *For $\sigma\in i\mathbb{R}\cup(0,\infty)$ and $\mu\in\mathbb{Z}_+$ the map*

$$L^2(\mathbb{R}_+)\xrightarrow{\sim}L^2(S(\sigma,\mu),\,\mathrm{d}\nu_{\sigma,\mu}),\qquad u\mapsto g(\lambda)=\int_0^\infty\eta_1(x,\lambda)u(x)\,\mathrm{d}x,$$

is a unitary isomorphism with inverse

$$L^2(S(\sigma,\mu),\,\mathrm{d}\nu_{\sigma,\mu}) \xrightarrow{\sim} L^2(\mathbb{R}_+),$$
$$g \mapsto u(x) = \int_{S(\sigma,\mu)} \eta_1(x,\lambda)g(\lambda)\,\mathrm{d}\nu_{\sigma,\mu}(\lambda).$$

For our application we need the spectral decomposition of the operator $\mathcal{D}_{\sigma,\mu}$ which follows from Theorem 3.4 by the transformation $f(t) \mapsto r(x)^{-1}f(\sinh^2(\frac{x}{2}))$. To state this put

$$T(\sigma,\mu) := i\mathbb{R}_+ \cup \bigcup_{j\in[0,\frac{\mathrm{Re}\,\sigma-\mu}{4})\cap\mathbb{Z}} \{\sigma-\mu-4j\}$$

and define a measure $\mathrm{d}m_{\sigma,\mu}$ on $T(\sigma,\mu)$ by

$$\int_{T(\sigma,\mu)} g(\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau) := \frac{1}{8\pi}\int_{i\mathbb{R}_+} g(\tau)\left|\frac{\Gamma(\frac{-\sigma+\mu+\tau}{4})\Gamma(\frac{\sigma+\mu+\tau}{4})}{\Gamma(\frac{\tau}{2})\Gamma(\frac{\mu}{2})}\right|^2\mathrm{d}\tau$$
$$+\sum_{j\in[0,\frac{\mathrm{Re}\,\sigma-\mu}{4})\cap\mathbb{Z}} \frac{(\frac{\sigma-\mu}{2}-2j)\Gamma(\frac{\sigma}{2}-j)\Gamma(\frac{\mu}{2}+j)}{j!\Gamma(\frac{\mu}{2})^2\Gamma(\frac{\sigma-\mu}{2}-j+1)}g(\sigma-\mu-4j).$$

Corollary 3.5. *For $\sigma \in i\mathbb{R}\cup(0,\infty)$ and $\mu\in\mathbb{Z}_+$ the map*

$$L^2(\mathbb{R}_+, t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\mathrm{Re}\,\sigma}{2}}\,\mathrm{d}t) \xrightarrow{\sim} L^2(T(\sigma,\mu),\,\mathrm{d}m_{\sigma,\mu}),$$
$$f \mapsto g(\tau) = \int_0^\infty F(t,\tau)f(t)t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\sigma}{2}}\,\mathrm{d}t$$

is a unitary isomorphism with inverse

$$L^2(T(\sigma,\mu),\,\mathrm{d}m_{\sigma,\mu}) \xrightarrow{\sim} L^2(\mathbb{R}_+, t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\mathrm{Re}\,\sigma}{2}}\,\mathrm{d}t),$$
$$g \mapsto f(t) = \int_{T(\sigma,\mu)} F(t,\tau)g(\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau).$$

Remark 3.6. For the discrete part, namely for $\tau=\sigma-\mu-4j$, $j\in\mathbb{N}$, the Gauß hypergeometric function $F(t,\tau)$ degenerates to a polynomial in t of degree j. More precisely, we have (see (B.10))

$$F(t,\sigma-\mu-4j) = \frac{j!}{(\frac{\mu}{2})_n}P_j^{(\frac{\mu-2}{2},-\frac{\sigma}{2})}(1+2t),$$

where $P_n^{(\alpha,\beta)}(z)$ denote the Jacobi polynomials.

Remark 3.7. For $\sigma \in (0,\infty)$ the results of Corollary 3.5 can also be found in [7, formula (A.11)] where the hypergeometric transform appears (essentially) as the radial part of the spherical Fourier transform on $\mathrm{SU}(1,n)/\mathrm{SU}(n)$. Since our approach provides a unified treatment of both complementary series, discrete series representations for the hyperboloid and principal series, including the case $\sigma \in i\mathbb{R}$, we gave a detailed proof in this section for convenience.

4. Decomposition of Representations and the Plancherel Formula

Using the spectral decomposition of $\mathcal{D}_{\sigma,\mu}$ obtained in Corollary 3.5 we find in this section the explicit Plancherel formula for the decomposition of $\rho^G_{\sigma,\varepsilon}|_H$.

Let us first consider the action of $O(n-m)$ on $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)$ which gives the following decomposition as $O(n-m)$-representations:

$$\begin{aligned}(4.1)\qquad & L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)\\ &= \sum_{k=0}^{\infty}{}^{\oplus} L^2(\mathbb{R}^m\times\mathbb{R}_+, (|x|^2+r^2)^{-\frac{\operatorname{Re}\sigma}{2}} r^{2k+n-m-1}\,\mathrm{d}x\,\mathrm{d}r)\\ &\qquad\qquad\qquad\qquad \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}),\end{aligned}$$

where $r=|y|$. We fix a summand for some $k\in\mathbb{N}$ and put again $\mu = 2k+n-m$. The coordinate change $t := \frac{r^2}{|x|^2}$ gives

$$\begin{aligned}& L^2(\mathbb{R}^m\times\mathbb{R}_+, (|x|^2+r^2)^{-\frac{\operatorname{Re}\sigma}{2}} r^{\mu-1}\,\mathrm{d}x\,\mathrm{d}r)\\ &\quad = L^2(\mathbb{R}^m\times\mathbb{R}_+, \tfrac12 |x|^{-\operatorname{Re}\sigma+\mu}\, t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}x\,\mathrm{d}t).\end{aligned}$$

Since

$$\begin{aligned}& L^2(\mathbb{R}^m\times\mathbb{R}_+, \tfrac12 |x|^{-\operatorname{Re}\sigma+\mu}\, t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}x\,\mathrm{d}t)\\ &\quad \cong L^2(\mathbb{R}^m, \tfrac12 |x|^{-\operatorname{Re}\sigma+\mu}\,\mathrm{d}x)\widehat{\otimes} L^2(\mathbb{R}_+, t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}t)\end{aligned}$$

we can apply Theorem 3.4 to find that the map

$$L^2(\mathbb{R}^m \times \mathbb{R}_+, \tfrac{1}{2}\,|x|^{-\operatorname{Re}\sigma+\mu}\, t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\operatorname{Re}\sigma}{2}}\,\mathrm{d}x\,\mathrm{d}t) \\ \to \int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}\,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau)$$

given by

$$f(x,t) \mapsto \hat{f}(x,\tau) := |x|^{-\frac{\sigma-\tau-\mu}{2}} \int_0^\infty F(t,\tau) f(x,t) t^{\frac{\mu-2}{2}}(1+t)^{-\frac{\sigma}{2}}\,\mathrm{d}t$$

is a unitary isomorphism, where $F(t,\tau)$ is defined by (3.2) and the measure $\mathrm{d}m_{\sigma,\mu}$ is given by (3.5). Its inverse is given by

$$g(x,\tau) \mapsto \check{g}(x,t) := \int_{T(\sigma,\mu)} |x|^{\frac{\sigma-\tau-\mu}{2}}\, F(t,\tau) g(x,\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau).$$

Now we put these things together. For $\sigma \in i\mathbb{R} \cup (0,\infty)$ and $k \in \mathbb{N}$ we put $\mu := 2k+n-m$ and define an operator

$$\Psi(\sigma,k) : \left(\int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}\,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau)\right) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}) \\ \to L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)$$

by

$$\Psi(\sigma,k)\,(f \otimes \phi)\,(x,y) \\ := \phi(y) \int_{T(\sigma,\mu)} |x|^{\frac{\sigma-\tau-\mu}{2}}\, {}_2F_1\left(\tfrac{\mu-\sigma+\tau}{4}, \tfrac{\mu-\sigma-\tau}{4}; \tfrac{\mu}{2}; -\tfrac{|y|^2}{|x|^2}\right) f(x,\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau).$$

THEOREM 4.1. *For $\sigma \in i\mathbb{R} \cup (0,n) \cup (n+2\mathbb{N})$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ the map $\Psi(\sigma,k)$ is H-equivariant between the representations*

$$\int_{T(\sigma,\mu)}^{\oplus} \rho_{\tau,\varepsilon+k}^{O(1,m+1)}\,\mathrm{d}m_{\sigma,\mu}(\tau) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}) \to \rho_{\sigma,\varepsilon}^{G}\big|_H$$

and constructs the $\mathcal{H}^k(\mathbb{R}^{n-m})$-isotypic component in $\rho^G_{\sigma,\varepsilon}|_H$. The following Plancherel formula holds:

$$\begin{aligned}&\|\Psi(\sigma,k)(f\otimes\phi)\|^2_{L^2(\mathbb{R}^n,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)}\\&\quad=\int_{T(\sigma,\mu)}\|f(-,\tau)\|^2_{L^2(\mathbb{R}^m,\frac{1}{2}|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)}\,\mathrm{d}m_{\sigma,\mu}(\tau)\cdot\|\phi\|^2_{L^2(S^{n-m-1})}\,.\end{aligned}$$

PROOF. We have already seen that $\Psi(\sigma,k)$ gives a unitary isomorphism so that the Plancherel formula above holds. Further, by Proposition 2.1 the map $\Psi(\sigma,k)$ intertwines the actions of $M_HA\overline{N}_H$ on smooth vectors and hence on the Hilbert spaces. Since H is generated by $M_HA\overline{N}_H$ and N_H it remains to prove the intertwining property for N_H. For this we use the Lie algebra action.

LEMMA 4.2. *Let L be a connected Lie group with Lie algebra $\mathfrak{l}$ and let $(\rho_1,\mathcal{H}_1)$ and $(\rho_2,\mathcal{H}_2)$ be unitary representations of L. Suppose that a continuous linear map $\varphi:\mathcal{H}_1\to\mathcal{H}_2$ is given and there exist subspaces $V_1\subset\mathcal{H}_1$ and $V_2\subset\mathcal{H}_2$ such that*

- *(i) V_i is dense in $\mathcal{H}_i$ for $i=1,2$,*
- *(ii) V_i is contained in the space of analytic vectors $\mathcal{H}_i^\omega$ for $i=1,2$,*
- *(iii) V_i is $\mathrm{d}\rho_i$-stable for $i=1,2$,*
- *(iv) $(\varphi(\mathrm{d}\rho_1(X)v_1)\mid v_2)_{\mathcal{H}_2}=-(\varphi(v_1)\mid\mathrm{d}\rho_2(X)v_2)_{\mathcal{H}_2}$ for $v_1\in V_1$, $v_2\in V_2$ and $X\in\mathfrak{l}$.*

Then φ is L-equivariant.

PROOF. For $v_1\in V_1$ and $v_2\in V_2$ we put

$$\begin{aligned}f_{v_1,v_2}(g)&:=(\varphi(\rho_1(g)v_1)\mid v_2)_{\mathcal{H}_2}, && g\in L,\\ h_{v_1,v_2}(g)&:=(\rho_2(g)\varphi(v_1)\mid v_2)_{\mathcal{H}_2}=(\varphi(v_1)\mid\rho_2(g^{-1})v_2)_{\mathcal{H}_2}, && g\in L,\end{aligned}$$

which are analytic functions on L by (ii). For a smooth function f on L and $X\in\mathfrak{l}$ we define derivatives by

$$(R(X)f)(g):=\lim_{t\to0}\frac{f(ge^{tX})-f(g)}{t},$$
$$(L(X)f)(g):=\lim_{t\to0}\frac{f(e^{-tX}g)-f(g)}{t}.$$

We have $R(X)f(e) = -L(X)f(e)$ for the identity element $e \in L$ and $R(X)$ commutes with $L(X')$ for any $X, X' \in \mathfrak{l}$. Hence

$$\begin{aligned} R(X_1)R(X_2)\cdots R(X_k)f(e) &= -L(X_1)R(X_2)\cdots R(X_k)f(e) \\ &= -R(X_2)\cdots R(X_k)L(X_1)f(e) \\ &\;\;\vdots \\ &= (-1)^k L(X_k)\cdots L(X_2)L(X_1)f(e) \end{aligned}$$

for $X_1, \ldots, X_k \in \mathfrak{l}$. Then (iv) implies

$$\begin{aligned} R(X_1)\cdots R(X_k)f_{v_1,v_2}(e) &= f_{\mathrm{d}\rho_1(X_1)\cdots\mathrm{d}\rho_1(X_k)v_1,v_2}(e) \\ &= (-1)^k h_{v_1,\mathrm{d}\rho_2(X_k)\cdots\mathrm{d}\rho_2(X_1)v_2}(e) \\ &= (-1)^k L(X_k)\cdots L(X_1)h_{v_1,v_2}(e) \\ &= R(X_1)\cdots R(X_k)h_{v_1,v_2}(e). \end{aligned}$$

Since f_{v_1,v_2} and h_{v_1,v_2} are analytic functions, they coincide. Therefore $\varphi(\rho_1(g)v_1) = \rho_2(g)\varphi(v_1)$ for $v_1 \in V_1$ and hence $\varphi(\rho_1(g)v) = \rho_2(g)\varphi(v)$ for any $v \in \mathcal{H}_1$ by (i). $\square$

We apply the lemma to the map $\varphi = \Psi(\sigma, k) : \mathcal{H}_1 \to \mathcal{H}_2$ where

$$\begin{aligned} \mathcal{H}_1 &:= \left(\int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}\,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau) \right) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m}), \\ \mathcal{H}_2 &:= L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y). \end{aligned}$$

So let ρ_1 and ρ_2 be the restrictions of

$$\left(\int_{T(\sigma,\mu)}^{\oplus} \rho_{\tau,\varepsilon+k}^{O(1,m+1)}\,\mathrm{d}m_{\sigma,\mu}(\tau) \right) \boxtimes \mathbf{1} \quad \text{and} \quad \rho_{\sigma,\varepsilon}^{G}$$

to $L = N_H$, respectively. To define V_1, we regard an element

$$f \in \int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}\,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau)$$

as a function $f(x,\tau)$ on $(\mathbb{R}^m \setminus \{0\}) \times T(\sigma,\mu)$. Let $V_{1,c}$ be the space consisting of linear combinations of the functions on $(\mathbb{R}^m \setminus \{0\}) \times i\mathbb{R}_+ \times \mathbb{R}^{n-m}$ of the form

$$(x,\tau,y) \mapsto (\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x)\phi(y)\chi(\tau),$$

where $X \in \mathcal{U}(\mathfrak{h})$, $\psi_\tau^{O(1,m+1)}$ is the spherical vector of $\rho_{\tau,\varepsilon+k}^{O(1,m+1)}$ as defined in (1.14), $\phi \in \mathcal{H}^k(\mathbb{R}^{n-m})$ and $\chi \in C_c(i\mathbb{R}_+)$, i.e. χ is a continuous function on $i\mathbb{R}_+$ with compact support. Let $V_{1,d}$ be the space consisting of sum of functions on $(\mathbb{R}^m \setminus \{0\}) \times (T(\sigma,\mu) \cap (0,\infty)) \times \mathbb{R}^{n-m}$ of the form

$$(x,\tau,y) \mapsto f_\tau(x)\phi(y),$$

where $f_\tau \in L^2(\mathbb{R}^m, |x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)_{K\cap O(1,m+1)}$, a $(K\cap O(1,m+1))$-finite vector in $\rho_{\tau,\varepsilon+k}^{O(1,m+1)}$, and $\phi \in \mathcal{H}^k(\mathbb{R}^{n-m})$. Then we put $V_1 := V_{1,c} \oplus V_{1,d}$. Further let V_2 be the space of all K-finite vectors in $L^2(\mathbb{R}^n, |(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)$. We now check conditions (i)–(iv):

(i) V_1 is dense in $\mathcal{H}_1$ since $C_c(i\mathbb{R}_+)$ is dense in $L^2(i\mathbb{R}_+, \mathrm{d}m_{\sigma,\mu})$ and the space of $(K \cap O(1,m+1))$-finite vectors for $\rho_\tau^{O(1,m+1)}$ is generated by $\psi_\tau^{O(1,m+1)}(x)$ and dense in $L^2(\mathbb{R}^m, |x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)$ for $\tau \in i\mathbb{R}_+$. The space V_2 is dense in $\mathcal{H}_2$ since it is the space of K-finite vectors for $\rho_{\sigma,\varepsilon}^{O(1,n+1)}$.

(ii) K-finite vectors are analytic vectors for G and in particular for $N_H \subseteq G$, hence $V_2 \subseteq \mathcal{H}_2^\omega$. Similarly, $V_{1,d} \subseteq \mathcal{H}_1^\omega$. The inclusion $V_{1,c} \subseteq \mathcal{H}_1^\omega$ follows from the lemma below.

(iii) It is clear that V_2 is $\mathrm{d}\rho_2$-stable since the space of K-finite vectors is $\mathrm{d}\rho_\sigma^{O(1,n+1)}$-stable. That V_1 is $\mathrm{d}\rho_1$-stable follows from the definition of V_1.

Lemma 4.3. *Let*

$$(\rho_1', \mathcal{H}_1') := \left(\int_{T(\sigma,\mu)}^{\oplus} \rho_{\tau,\varepsilon+k}^{O(1,m+1)}\,\mathrm{d}m_{\sigma,\mu}(\tau), \right.$$
$$\left. \int_{T(\sigma,\mu)}^{\oplus} L^2(\mathbb{R}^m, \tfrac{1}{2}\,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)\,\mathrm{d}m_{\sigma,\mu}(\tau) \right).$$

A function $f(x,\tau)$ on $(\mathbb{R}^m \setminus \{0\}) \times i\mathbb{R}_+$ of the form

$$f(x,\tau) := (\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x)\chi(\tau)$$

for $X \in \mathcal{U}(\mathfrak{h})$ and $\chi \in C_c(i\mathbb{R}_+)$ is an analytic vector of ρ_1'.

PROOF. It is enough to prove that for any $g_0 \in O(1,m+1)$ there exists a neighborhood $0 \in U \subset \mathfrak{so}(1,m+1)$ such that

$$a_N := \left\| \rho_1'(\exp Y)\rho_1'(g_0)f(x,\tau) - \sum_{l=0}^{N} \frac{1}{l!}\mathrm{d}\rho_1'(Y)^l \rho_1'(g_0)f(x,\tau) \right\|_{\mathcal{H}_1'}^2 \to 0$$

as $N \to \infty$ for $Y \in U$. Consider the Euclidean Fourier transform $\mathcal{F}_{\mathbb{R}^m}$ with respect to the variable x (see (1.5)) which gives a unitary equivalence between

$$\rho_1' = \int_{T(\sigma,\mu)}^{\oplus} \rho_{\tau,\varepsilon+k}^{O(1,m+1)} \,\mathrm{d}m_{\sigma,\mu}(\tau) \quad \text{and} \quad \pi_1 := \int_{T(\sigma,\mu)}^{\oplus} \pi_{\tau,\varepsilon+k}^{O(1,m+1)} \,\mathrm{d}m_{\sigma,\mu}(\tau).$$

Put $h(x,\tau) := \mathcal{F}_{\mathbb{R}^m}(\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x)$ then

$$\begin{aligned} a_N = \int_{i\mathbb{R}_+} \Big\| &\pi_1(\exp Y)\pi_1(g_0)h(x,\tau) \\ &- \sum_{l=0}^{N} \frac{1}{l!}\mathrm{d}\pi_1(Y)^l \pi_1(g_0)h(x,\tau) \Big\|_{L^2(\mathbb{R}^m, \frac{1}{2}|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)}^2 |\chi(\tau)|^2 \,\mathrm{d}m_{\sigma,\mu}(\tau). \end{aligned}$$

As in Section 1.2 the function $h(x,\tau)$ corresponds to a function $\tilde{h}(g,\tau)$ on $O(1,m+1) \times i\mathbb{R}_+$ satisfying $\tilde{h}(gman,\tau) = \xi_{\varepsilon+k}(m)^{-1}a^{-\tau-\rho}\tilde{h}(g,\tau)$ for $m \in O(1,m+1) \cap M$, $a \in A$ and $n \in N_H$. Consequently, a_N is given as

$$\begin{aligned} \int_{i\mathbb{R}_+} \Big(\int_{O(1)\times O(m+1)} \Big| &\pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}(\exp(-Y)k,\tau) \\ &- \sum_{l=0}^{N} \frac{1}{l!}\mathrm{d}\pi_\tau^{O(1,m+1)}(Y)^l \pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}(k,\tau) \Big|^2 \mathrm{d}k \Big) |\chi(\tau)|^2 \,\mathrm{d}m_{\sigma,\mu}(\tau) \end{aligned}$$

up to a constant factor, where $\mathrm{d}k$ is the Haar measure on $O(1) \times O(m+1)$. Since $\pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}$ is analytic on $O(1,m+1) \times i\mathbb{R}_+$, the sequence

$$\sum_{l=0}^{N} \frac{1}{l!}\mathrm{d}\pi_\tau^{O(1,m+1)}(Y)^l \pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}(k,\tau)$$

converges uniformly to $\pi_{\tau,\varepsilon+k}^{O(1,m+1)}(g_0)\tilde{h}(\exp(-Y)k,\tau)$ on the compact set $(k,\tau) \in (O(1) \times O(m+1)) \times \operatorname{supp}\chi$, which proves $a_N \to 0$. $\square$

To verify the intertwining condition (iv) we first prove the intertwining property for each single space $L^2(\mathbb{R}^m,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)$ for fixed τ by embedding it into the $\mathbb{C}$-antilinear algebraic dual of the Harish-Chandra module $L^2(\mathbb{R}^n,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)_K$ of K-finite vectors. For $\tau\in i\mathbb{R}_+$ and $X\in\mathcal{U}(\mathfrak{h})$ let

$$f_{\tau,X}(x):=(\mathrm{d}\rho_\tau^{O(1,m+1)}(X)\psi_\tau^{O(1,m+1)})(x),\qquad x\in\mathbb{R}^m\setminus\{0\}.$$

PROPOSITION 4.4. *Let $\phi\in\mathcal{H}^k(\mathbb{R}^{n-m})$ and $g\in L^2(\mathbb{R}^n,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)_K$.*

(i) For every $X\in\mathcal{U}(\mathfrak{h})$ and $\tau\in i\mathbb{R}_+$ the integral

$$\int_{\mathbb{R}^n}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f_{\tau,X}(x)\phi(y)\overline{g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y$$

converges absolutely and defines a continuous function in τ.

(ii) For every $X\in\mathcal{U}(\mathfrak{h})$, $\tau\in i\mathbb{R}_+$ and $j=1,\dots,m$ we have

$$\begin{aligned}(4.2)\quad &\int_{\mathbb{R}^n}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau}f_{\tau,X})(x)\phi(y)\overline{g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y\\ &=\int_{\mathbb{R}^n}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f_{\tau,X}(x)\phi(y)\overline{(\mathcal{B}_j^{n,\sigma}g)(x,y)}\\ &\qquad\times|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

(iii) For $\tau\in T(\sigma,\mu)\cap(0,\infty)$ and $f_\tau\in L^2(\mathbb{R}^m,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)_{K\cap O(1,m+1)}$, the integral

$$\int_{\mathbb{R}^n}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f_\tau(x)\phi(y)\overline{g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y$$

converges absolutely.

(iv) For $\tau\in T(\sigma,\mu)\cap(0,\infty)$, $f_\tau\in L^2(\mathbb{R}^m,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)_{K\cap O(1,m+1)}$, and $j=1,\dots,m$, we have

$$(4.3)\quad\int_{\mathbb{R}^n}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau}f_\tau)(x)\phi(y)\overline{g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_{\mathbb{R}^n} |x|^{\frac{\sigma-\tau-\mu}{2}} F(\tfrac{|y|^2}{|x|^2},\tau) f_\tau(x)\phi(y)\overline{(\mathcal{B}_j^{n,\sigma} g)(x,y)} \times |(x,y)|^{-\operatorname{Re}\sigma} \, \mathrm{d}x \, \mathrm{d}y.$$

Proof. We first note that by (1.16) the function $f_{\tau,X}(x)$ is a linear combination of functions of the form

$$f(x) = \widetilde{K}_{-\frac{\tau}{2}+a}(|x|)\,|x|^{2a}\,p(x)$$

for $a \in \mathbb{N}$ and $p \in \mathbb{C}[x]$ with coefficients depending smoothly on τ. Similarly, by (1.16) and (1.18), $f_\tau(x)$ is a linear combination of the form $f(x)$ above. Note that in the case $\tau = m + 2v \in m + 2\mathbb{N}$ we additionally have $p \in \mathbb{C}[x]_{>v-a}$. We may replace $f_{\tau,X}(x)$ and $f_\tau(x)$ by one of these functions $f(x)$. For the same reason we may assume that

$$g(x,y) = \widetilde{K}_{-\frac{\sigma}{2}+b}(|(x,y)|)\,|(x,y)|^{2b}\,q(x,y)$$

for some $b \in \mathbb{N}$ and $q \in \mathbb{C}[x,y]$ where for $\sigma = n+2u \in n+2\mathbb{N}$ we additionally have $q \in \mathbb{C}[x,y]_{>u-b}$.

(i)&(iii) By (B.2) and (B.3) there exists a continuous function $C_1(\tau) > 0$ on $T(\sigma,\mu)$ and $N_1 > 0$ such that for $x \neq 0$:

$$\left|\widetilde{K}_{-\frac{\tau}{2}+a}(|x|)\,|x|^{2a}\,p(x)\right| \leq C_1(\tau)|x|^{-\delta_1}(1+|x|)^{N_1}e^{-|x|}\begin{cases} 1 & \text{for } 0 \leq \operatorname{Re}\tau < m, \\ |x|^{\frac{\operatorname{Re}\tau - m}{2}+1} & \text{for } \operatorname{Re}\tau \geq m, \end{cases}$$

for some arbitrarily small $\delta_1 > 0$ (covering the possible log-term for $\tau = 2a$). For the hypergeometric function we have by (B.7) and (B.10) (checking the cases $\tau \in i\mathbb{R}_+$ and $\tau \in (\operatorname{Re}\sigma - \mu - 4\mathbb{N}) \cap \mathbb{R}_+$ separately)

$$|F(t,\tau)| \leq C_2(\tau)(1+t)^{\frac{\operatorname{Re}\sigma - \operatorname{Re}\tau - \mu}{4}}, \qquad t > 0,$$

for some continuous function $C_2(\tau) > 0$ on $T(\sigma,\mu)$. We estimate

$$|\phi(y)| \leq C_3\,|y|^k \leq C_3\,|(x,y)|^k\,.$$

Further, for the K-Bessel function of parameter $-\frac{\sigma}{2}+b$ we again find by (B.2) and (B.3) that for $(x,y)\neq 0$:

$$\begin{aligned}\left|\widetilde{K}_{-\frac{\sigma}{2}+b}(|(x,y)|)\,|(x,y)|^{2b}\,q(x,y)\right|&\\ \leq C_4\,|(x,y)|^{-\delta_2}\,(1+|(x,y)|)^{N_2}e^{-|(x,y)|}&\\ \times\begin{cases}1 & \text{for } 0\leq\operatorname{Re}\sigma<n,\\ |(x,y)|^{\frac{\operatorname{Re}\sigma-n}{2}+1} & \text{for } \operatorname{Re}\sigma\geq n,\end{cases}&\end{aligned}$$

for some arbitrarily small $\delta_2>0$ (covering the possible log-term for $\sigma=2b$) and $C_4,N_2>0$. Now assume $0\leq\operatorname{Re}\tau<m$ and $0\leq\operatorname{Re}\sigma<n$ then we obtain

$$\begin{aligned}&\left||x|^{\frac{\sigma-\tau-\mu}{2}}\,F(\tfrac{|y|^2}{|x|^2},\tau)f(x)\phi(y)\overline{g(x,y)}\right|\\ &\leq C_1(\tau)C_2(\tau)C_3C_4\,|x|^{\frac{\operatorname{Re}\sigma-\operatorname{Re}\tau-\mu}{2}}\left(1+\tfrac{|y|^2}{|x|^2}\right)^{\frac{\operatorname{Re}\sigma-\operatorname{Re}\tau-\mu}{4}}\\ &\quad\times|x|^{-\delta_1}\,(1+|x|)^{N_1}e^{-|x|}\,|(x,y)|^{k-\delta_2}\,(1+|(x,y)|)^{N_2}e^{-|(x,y)|}\\ &\leq C(\tau)\,|x|^{-\delta_1}\,|(x,y)|^{\frac{\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}-\delta_2}\,(1+|(x,y)|)^{N}e^{-|(x,y)|}\end{aligned}$$

with $C(\tau)=C_1(\tau)C_2(\tau)C_3C_4$ and $N=N_1+N_2$. Since δ_1 and δ_2 can be chosen arbitrarily small the right hand side is integrable on $\mathbb{R}^n$ with respect to the measure $|(x,y)|^{-\operatorname{Re}\sigma}$ if and only if

$$\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}>-n.$$

This inequality holds by assumption and hence the integral converges absolutely. Moreover, we even have $n-\operatorname{Re}\sigma+m-\operatorname{Re}\tau>n-\operatorname{Re}\sigma>0$ for all τ and hence the convergence is uniformly in τ varying in a compact subset of $T(\sigma,\mu)$. The other two possibilities $0\leq\operatorname{Re}\tau<m$, $\operatorname{Re}\sigma\geq n$ and $\operatorname{Re}\tau\geq m$, $\operatorname{Re}\sigma\geq n$ are treated similarly which finishes the proof of (i) & (iii).

(ii)&(iv) First recall from Proposition 2.1 that

$$\begin{aligned}&|x|^{\frac{\sigma-\tau-\mu}{2}}\,F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau}f)(x)\phi(y)\\ &\quad=\mathcal{B}_j^{n,\sigma}\left[|x|^{\frac{\sigma-\tau-\mu}{2}}\,F(\tfrac{|y|^2}{|x|^2},\tau)f(x)\phi(y)\right].\end{aligned}$$

Therefore we have to show that

$$\begin{aligned}(4.4)\qquad &\int_{\mathbb{R}^n} \mathcal{B}_j^{n,\sigma}\Phi(x,y)\cdot\overline{g(x,y)}\cdot|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y\\ &\overset{!}{=}\int_{\mathbb{R}^n}\Phi(x,y)\cdot\overline{\mathcal{B}_j^{n,\sigma}g(x,y)}\cdot|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where we abbreviate

$$\Phi(x,y)=|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f(x)\phi(y).$$

The operator $\mathcal{B}_j^{n,\sigma}$ is formally self-adjoint with respect to $|(x,y)|^{-\operatorname{Re}\sigma}$ since $\mathrm{d}\rho_\sigma^G(N_j)=-i\mathcal{B}_j^{n,\sigma}$ is, as part of the Lie algebra action, formally skew-adjoint on $C_c^\infty(\mathbb{R}^n\setminus\{0\})\subseteq L^2(\mathbb{R}^n,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)^\infty$. Therefore it remains to show that we can integrate by parts without leaving any boundary terms. Fix $j\in\{1,\dots,m\}$ and consider the domain

$$\Omega_{j,\varepsilon}:=\{(x,y)\in\mathbb{R}^n:|x_j|>\varepsilon\}\subseteq\mathbb{R}^n$$

for $\varepsilon>0$. Clearly $\mathbb{R}^n\setminus\bigcup_{\varepsilon>0}\Omega_{j,\varepsilon}$ is of measure zero and hence (4.4) is equivalent to

$$\begin{aligned}(4.5)\qquad &\lim_{\varepsilon\to0}\int_{\Omega_{j,\varepsilon}}\mathcal{B}_j^{n,\sigma}\Phi(x,y)\cdot\overline{g(x,y)}\cdot|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y\\ &\overset{!}{=}\lim_{\varepsilon\to0}\int_{\Omega_{j,\varepsilon}}\Phi(x,y)\cdot\overline{\mathcal{B}_j^{n,\sigma}g(x,y)}\cdot|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

On $\Omega_{j,\varepsilon}$ both $|x|$ and $|(x,y)|$ are bounded from below by ε. Hence, by (B.4) and (B.8), all factors in the integrand

$$|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f(x)\phi(y)\overline{g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}$$

can be arbitrarily often differentiated in x and y and the result is a smooth function on $\overline{\Omega_{j,\varepsilon}}$. Since further the hypergeometric function grows at most polynomially and the K-Bessel functions decay exponentially near ∞, all such differentiated terms decay exponentially as $|(x,y)|\to\infty$ and are hence integrable on $\overline{\Omega_{j,\varepsilon}}$. Therefore we can arbitrarily integrate by parts and all intermediate integrals exist. It remains to show that for $\varepsilon\to0$ all boundary terms that occur while integrating by parts vanish. By the asymptotic behaviour of the K-Bessel functions at ∞ the boundary terms at ∞ always vanish. Hence, by the choice of $\Omega_{j,\varepsilon}$, the only boundary terms that occur are for derivatives in x_j at $x_j = \pm\varepsilon$. Therefore we only need to consider the parts

$$x_j \frac{\partial^2}{\partial x_j^2}, \frac{\partial}{\partial x_j} \quad \text{and} \quad E\frac{\partial}{\partial x_j}$$

of $\mathcal{B}_j^{n,\sigma}$. We treat these three parts separately. Here we start with the right hand side of (4.5) and then integrate by parts once or twice. We only carry out the details for the case $0 \le \operatorname{Re}\tau < m$ and $0 \le \operatorname{Re}\sigma < n$, the other cases are treated similarly with the corresponding estimates derived in part (i) & (iii).

(a) $\frac{\partial}{\partial x_j}$. The boundary terms that occur when integrating by parts are (up to multiplication with a constant) of the form

$$\int_{\mathbb{R}^{n-1}} \Big(\Phi(x', \varepsilon, y)\overline{g(x', \varepsilon, y)}\, \big|(x', \varepsilon, y)\big|^{-\operatorname{Re}\sigma} - \Phi(x', -\varepsilon, y)\overline{g(x', -\varepsilon, y)}\, \big|(x', -\varepsilon, y)\big|^{-\operatorname{Re}\sigma} \Big)\, \mathrm{d}x'\, \mathrm{d}y$$

where we write $x = (x', x_j)$ with $x' = (x_1, \dots, \widehat{x_j}, \dots, x_m) \in \mathbb{R}^{m-1}$. The integrand obviously converges pointwise almost everywhere to 0 as $\varepsilon \to 0$ and it suffices to find an integrable function independent of ε dominating the integrand to apply the Dominated Convergence Theorem. For this note that in both $\Phi(x, y)$ and $g(x, y)$ the only terms dependent on the sign of x_j are the polynomials $p(x)$ and $q(x, y)$, respectively. Using the same estimates as in the proof of part (i) & (iii) we find that

$$\begin{aligned} &\Big| \Phi(x', \varepsilon, y)\overline{g(x', \varepsilon, y)}\, \big|(x', \varepsilon, y)\big|^{-\operatorname{Re}\sigma} \\ &\quad - \Phi(x', -\varepsilon, y)\overline{g(x', -\varepsilon, y)}\, \big|(x', -\varepsilon, y)\big|^{-\operatorname{Re}\sigma} \Big| \\ &\le C \big|(x', \varepsilon, y)\big|^{\frac{-\operatorname{Re}\sigma - n + m - \operatorname{Re}\tau}{2} - \delta} \big(1 + \big|(x', \varepsilon, y)\big|\big)^N e^{-|(x', \varepsilon, y)|} \\ &\quad \times \Big| p(x', \varepsilon)\overline{q(x', \varepsilon, y)} - p(x', -\varepsilon)\overline{q(x', -\varepsilon, y)} \Big| \end{aligned}$$

for some $N > 0$ and an arbitrarily small $\delta > 0$. Now note that $p(x', \varepsilon)\overline{q(x', \varepsilon, y)} - p(x', -\varepsilon)\overline{q(x', -\varepsilon, y)}$ is an odd polynomial in

ε and hence of the form $\varepsilon \cdot r(x',\varepsilon,y)$. For the extra ε from this observation we use the estimate $|\varepsilon| \leq |(x',\varepsilon,y)|$. We further estimate $|r(x',\varepsilon,y)| \leq C'(1+|(x',\varepsilon,y)|)^{N'}$ for some $C',N'>0$ and find (assuming $\varepsilon \leq 1$)

$$\leq CC'\left|(x',\varepsilon,y)\right|^{\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}+1-\delta} \times (1+\left|(x',1,y)\right|)^{N+N'}e^{-|(x',y)|}.$$

Now suppose the exponent $\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}+1$ is ≤ 0. Then we can estimate

$$\leq CC'\left|(x',y)\right|^{\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}+1-\delta} \times (1+\left|(x',1,y)\right|)^{N+N'}e^{-|(x',y)|},$$

which is independent of $\varepsilon \in (0,1)$ and integrable on $\mathbb{R}^{n-1}$ for small $\delta>0$ since $\operatorname{Re}\sigma<n$ and $\operatorname{Re}\tau<m$. If the exponent $\frac{-\operatorname{Re}\sigma-n+m-\operatorname{Re}\tau}{2}+1-\delta$ is positive the estimate $\varepsilon \leq 1$ also yields a dominant integrable function independent of ε. Therefore, in both cases we can apply the Dominated Convergence Theorem and obtain that as $\varepsilon \to 0$ the boundary terms vanish.

(b) $x_j\frac{\partial^2}{\partial x_j^2}$. Integrating by part once gives (up to multiplication by a constant) the boundary terms

$$\begin{aligned}(4.6)\quad \int_{\mathbb{R}^{n-1}} \Bigg(& \Phi(x',\varepsilon,y)\left(x_j\overline{\frac{\partial g}{\partial x_j}(x,y)}\right)_{x_j=\varepsilon}\left|(x',\varepsilon,y)\right|^{-\operatorname{Re}\sigma} \\ & -\Phi(x',-\varepsilon,y)\left(x_j\overline{\frac{\partial g}{\partial x_j}(x,y)}\right)_{x_j=-\varepsilon}\left|(x',-\varepsilon,y)\right|^{-\operatorname{Re}\sigma}\Bigg)\,dx'\,dy.\end{aligned}$$

We have

$$g(x,y)=\widetilde{K}_{-\frac{\sigma}{2}+b}(|(x,y)|)\,|(x,y)|^{2b}\,q(x,y)$$

and use the product rule to find $x_j\frac{\partial g}{\partial x_j}(x,y)$. The first term is by (B.4)

$$\begin{aligned}&-\frac{x_j^2}{2}\widetilde{K}_{-\frac{\sigma}{2}+b+1}(|(x,y)|)\,|(x,y)|^{2b}\,q(x,y)\\ &=-\frac{x_j^2}{2\,|(x,y)|^2}\widetilde{K}_{-\frac{\sigma}{2}+b+1}(|(x,y)|)\,|(x,y)|^{2(b+1)}\,q(x,y)\end{aligned}$$

and putting $x_j = \pm\varepsilon$ gives

$$= -\frac{\varepsilon^2}{2\left|(x',\varepsilon,y)\right|^2}\widetilde{K}_{-\frac{\sigma}{2}+b+1}(\left|(x',\varepsilon,y)\right|) \times \left|(x',\varepsilon,y)\right|^{2(b+1)} q(x',\pm\varepsilon,y).$$

Again ε^2 can be estimated by $\left|(x',\varepsilon,y)\right|^2$ and we find that

$$\frac{\varepsilon^2}{2\left|(x',\varepsilon,y)\right|^2}\widetilde{K}_{-\frac{\sigma}{2}+b+1}(\left|(x',\varepsilon,y)\right|)\left|(x',\varepsilon,y)\right|^{2(b+1)} \quad \text{and}$$
$$\widetilde{K}_{-\frac{\sigma}{2}+b}(\left|(x,y)\right|)\left|(x,y)\right|^{2b}$$

satisfy the same estimates (see the proof of (i) & (iii)). The same argument applies to the other two terms in the product rule. Therefore the same argument as in (a) yields the vanishing of the boundary terms (4.6). Similar arguments yield the vanishing of the boundary terms that occur when integrating by parts for the second time. For this note that the formal adjoint of $\frac{\partial}{\partial x_j}$ on $L^2(\mathbb{R}^n, \left|(x,y)\right|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)$ is $-\frac{\partial}{\partial x_j} + (\operatorname{Re}\sigma)\frac{x_j}{\left|(x,y)\right|^2}$. Both summands are treated separately as above.

(c) $E\frac{\partial}{\partial x_j}$. We have

$$E\frac{\partial}{\partial x_j} = x_j\frac{\partial^2}{\partial x_j^2} + \sum_{k\neq j} x_k \frac{\partial}{\partial x_k}\frac{\partial}{\partial x_j} + \sum_k y_k \frac{\partial}{\partial y_k}\frac{\partial}{\partial x_j}.$$

The first term was already treated in part (b). For the other two terms note that we can first integrate by parts the derivatives with respect to x_k ($k \neq j$) and y_k without any boundary terms occurring. Secondly, integration by parts of the derivative with respect to x_j is dealt with as in part (b). This finishes the proof. □

REMARK 4.5. It is necessary in the proof of Proposition 4.4 (ii) & (iv) to restrict integration to the domain $\Omega_{j,\varepsilon}$. This is because the operator $\mathcal{B}_j^{n,\sigma}$ is of second order and we have to integrate by parts twice. The intermediate result, i.e. after integrating by parts once, may not be integrable on $\mathbb{R}^n$

and hence we need to restrict to a subdomain on which these intermediate results are integrable. The same problem occurs when one considers the two summands $x_j\Delta$ and $-(2E-\sigma+n)\frac{\partial}{\partial x_j}$ separately. Here the integral over $\mathbb{R}^n$ for each of the two summands may not converge while the integral for the sum $\mathcal{B}_j^{n,\sigma}$ does by Proposition 4.4 (i) & (iii).

Remark 4.6. The assertions (i) & (iii) of Proposition 4.4 construct an embedding of

$$L^2(\mathbb{R}^m,|x|^{-\operatorname{Re}\tau}\,\mathrm{d}x)_{K\cap O(1,m+1)}\boxtimes\mathcal{H}^k(\mathbb{R}^{n-m})$$

into the $\mathbb{C}$-antilinear algebraic dual of $L^2(\mathbb{R}^n,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)_K$ for every $\tau\in T(\sigma,\mu)$. By (ii) & (iv) this embedding is $\mathfrak{h}$-equivariant.

Let us now continue the proof of Theorem 4.1 by showing property (iv) in Lemma 4.2. Let $v_1\in V_{1,c}$ and $v_2\in V_2$. Suppose that

$$v_1(x,\tau,y)=f_{\tau,X}(x)\phi(y)\chi(\tau)\quad\text{and}\quad v_2(x,y)=g(x,y)$$

with $X\in\mathcal{U}(\mathfrak{h})$, $\chi\in C_c(T(\sigma,\mu))$, $\phi\in\mathcal{H}^k(\mathbb{R}^{n-m})$, and $g\in L^2(\mathbb{R}^n,|(x,y)|^{-\operatorname{Re}\sigma}\,\mathrm{d}x\,\mathrm{d}y)_K$. We have

$$\begin{aligned}
&(\varphi(\mathrm{d}\rho_1(N_j)v_1)\mid v_2)_{\mathcal{H}_2}\\
&\quad=-i\int_{\mathbb{R}^n}\int_{T(\sigma,\mu)}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau}f_{\tau,X})(x)\phi(y)\overline{g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}\\
&\qquad\qquad\chi(\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad=-i\int_{T(\sigma,\mu)}\int_{\mathbb{R}^n}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)(\mathcal{B}_j^{m,\tau}f_{\tau,X})(x)\phi(y)\overline{g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}\\
&\qquad\qquad\chi(\tau)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}m_{\sigma,\mu}(\tau),
\end{aligned}$$

where we were able to change the order of integration, because by Proposition 4.4 (i) the inner integral in the last line converges absolutely and is continuous in τ and the integration is only over the compact subset $\operatorname{supp}\chi\subseteq T(\sigma,\mu)$. Now, by Proposition 4.4 (ii) we find

$$\begin{aligned}
&=-i\int_{T(\sigma,\mu)}\int_{\mathbb{R}^n}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f_{\tau,X}(x)\phi(y)\overline{\mathcal{B}_j^{n,\sigma}g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}\\
&\qquad\qquad\chi(\tau)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}m_{\sigma,\mu}(\tau)
\end{aligned}$$

$$= -i\int_{\mathbb{R}^n}\int_{T(\sigma,\mu)}|x|^{\frac{\sigma-\tau-\mu}{2}}F(\tfrac{|y|^2}{|x|^2},\tau)f_{\tau,X}(x)\phi(y)\overline{\mathcal{B}_j^{n,\sigma}g(x,y)}\,|(x,y)|^{-\operatorname{Re}\sigma}$$
$$\chi(\tau)\,\mathrm{d}m_{\sigma,\mu}(\tau)\,\mathrm{d}x\,\mathrm{d}y$$
$$= (\varphi(v_1)\mid \mathrm{d}\rho_2(N_j)v_2)_{\mathcal{H}_2},$$

again using Proposition 4.4 (i) to change the order of integration. This shows property (iv) of Lemma 4.2 for $v_1 \in V_{1,c}$. A similar argument with Proposition 4.4 (iii) and (iv) shows Lemma 4.2 (iv) for $v_1 \in V_{1,d}$. We therefore obtain that $\varphi = \Psi(\sigma,k)$ intertwines the group action of N_H and hence of H. Thus the proof of Theorem 4.1 is complete. □

We obtain the whole spectral decomposition of $\rho^G_{\sigma,\varepsilon}|_H$ from (4.1) and Theorem 4.1.

THEOREM 4.7. *For $\sigma \in i\mathbb{R}\cup(-n,n)\cup(n+2\mathbb{N})$ the representation $\rho^G_{\sigma,\varepsilon}$ decomposes under the restriction to $H = O(1,m+1)\times O(n-m)$, $0<m<n$, as*

$$\rho^G_{\sigma,\varepsilon}\big|_H \cong \sum_{k=0}^{\infty}{}^{\oplus}\Big(\int^{\oplus}_{i\mathbb{R}_+}\rho^{O(1,m+1)}_{\tau,\varepsilon+k}\,\mathrm{d}\tau$$
$$\oplus \bigoplus_{j\in\mathbb{Z}\cap\left[0,\frac{|\operatorname{Re}\sigma|-n+m-2k}{4}\right)}\rho^{O(1,m+1)}_{|\operatorname{Re}\sigma|-n+m-2k-4j,\varepsilon+k}\Big)\boxtimes\mathcal{H}^k(\mathbb{R}^{n-m}).$$

5. Intertwining Operators in the Non-compact Picture

In Proposition 2.1 we explicitly found an intertwining operator $C^\infty(\mathbb{R}^m\setminus\{0\})\boxtimes\mathcal{H}^k(\mathbb{R}^{n-m})\to C^\infty(\mathbb{R}^n\setminus\{x=0\})$. In the Fourier transformed picture this operator is given by

$$A(\sigma,\tau)(f\otimes\phi)(x,y)$$
$$= |x|^{\frac{\sigma-\tau-\mu}{2}}\,{}_2F_1\left(\frac{\mu-\sigma+\tau}{4},\frac{\mu-\sigma-\tau}{4};\frac{\mu}{2};-\frac{|y|^2}{|x|^2}\right)f(x)\phi(y),$$

where again $\mu = 2k+n-m$. In Proposition 4.4 we even showed that for fixed $\sigma\in i\mathbb{R}\cup(-n,n)\cup(n+2\mathbb{N})$, $k\in\mathbb{N}$ and $\tau\in T(\sigma,2k-n+m)$

the operator $A(\sigma,\tau)$ is intertwining between the Harish-Chandra module of $\rho^{O(1,m+1)}_{\tau,\varepsilon+k} \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m})$ and the $\mathbb{C}$-antilinear algebraic dual of the Harish-Chandra module of $\rho^{O(1,n+1)}_{\sigma,\varepsilon}$. We now find a formal expression for this intertwiner in the non-compact picture.

Consider the following diagram

$$\begin{tikzcd} C_c^\infty(\mathbb{R}^m \setminus \{0\}) \otimes \mathcal{H}^k(\mathbb{R}^{n-m}) \arrow[r, "A(\sigma,\tau)"] \arrow[d, "\mathcal{F}_{\mathbb{R}^m}\otimes \mathrm{id}"'] & \mathcal{S}'(\mathbb{R}^n) \arrow[d, "\mathcal{F}_{\mathbb{R}^n}"] \\ \mathcal{F}_{\mathbb{R}^m} C_c^\infty(\mathbb{R}^m \setminus \{0\}) \otimes \mathcal{H}^k(\mathbb{R}^{n-m}) \arrow[r, "I(\sigma,\tau)"'] & \mathcal{S}'(\mathbb{R}^n). \end{tikzcd}$$

We extend the operator $A(\sigma,\tau)$ for all $\sigma,\tau \in \mathbb{C}$ and determine the operator $I(\sigma,\tau)$ for $\operatorname{Re}\sigma \ll \operatorname{Re}\tau \ll 0$. We have

$$\begin{aligned} &\mathcal{F}_{\mathbb{R}^n} A(\sigma,\tau)(f\otimes\phi)(\xi,\eta) \\ &\quad = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^m}\int_{\mathbb{R}^{n-m}} e^{-ix\cdot\xi - iy\cdot\eta} \, |x|^{\frac{\sigma-\tau-\mu}{2}} \\ &\qquad\qquad \times {}_2F_1\left(\frac{\mu-\sigma+\tau}{4}, \frac{\mu-\sigma-\tau}{4}; \frac{\mu}{2}; -\frac{|y|^2}{|x|^2}\right) f(x)\phi(y)\,\mathrm{d}y\,\mathrm{d}x. \end{aligned}$$

We first calculate the integral over $y \in \mathbb{R}^{n-m}$. Using Appendix B.4 and the integral formula (B.11) we find

$$\begin{aligned} &(2\pi)^{-\frac{n-m}{2}} \int_{\mathbb{R}^{n-m}} e^{-iy\cdot\eta} {}_2F_1\left(\frac{\mu-\sigma+\tau}{4}, \frac{\mu-\sigma-\tau}{4}; \frac{\mu}{2}; -\frac{|y|^2}{|x|^2}\right) \phi(y)\,\mathrm{d}y \\ &\quad = i^{-k}\phi(\eta)\,|\eta|^{-\frac{\mu-2}{2}} \int_0^\infty J_{\frac{\mu-2}{2}}(|\eta|\, s) \\ &\qquad \times {}_2F_1\left(\frac{\mu-\sigma+\tau}{4}, \frac{\mu-\sigma-\tau}{4}; \frac{\mu}{2}; -\frac{s^2}{|x|^2}\right) s^{\frac{\mu}{2}}\,\mathrm{d}s \\ &\quad = i^{-k}\phi(\eta)\,|\eta|^{-\frac{\mu-2}{2}} \frac{2^{\frac{\sigma+2}{2}}\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu-\sigma+\tau}{4})\Gamma(\frac{\mu-\sigma-\tau}{4})} \, |x|^{\frac{\mu-\sigma}{2}} \, |\eta|^{-\frac{\sigma+2}{2}} \, K_{\frac{\tau}{2}}(|x|\cdot|\eta|). \end{aligned}$$

If we let

$$\psi(x,\eta) := \frac{2^{\frac{\sigma+2}{2}} i^{-k}\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu-\sigma+\tau}{4})\Gamma(\frac{\mu-\sigma-\tau}{4})} \, |\eta|^{-\frac{\sigma+\mu}{2}} \, |x|^{-\frac{\tau}{2}} \, K_{\frac{\tau}{2}}(|x|\cdot|\eta|)$$

then we find that

$$\begin{aligned}\mathcal{F}_{\mathbb{R}^n} A(\sigma,\tau)(f\otimes\phi)(\xi,\eta) &= \mathcal{F}_{\mathbb{R}^m}(f\cdot\psi(-,\eta))(\xi)\cdot\phi(\eta)\\ &= (2\pi)^{-\frac{m}{2}}(\mathcal{F}_{\mathbb{R}^m}\psi(-,\eta)*\mathcal{F}_{\mathbb{R}^m}f)(\xi)\cdot\phi(\eta).\end{aligned}$$

Therefore we compute, using again Appendix B.4 and the integral formula (B.12) (noticing that $K_\nu(x) = K_{-\nu}(x)$)

$$\begin{aligned}&(\mathcal{F}_{\mathbb{R}^m}\psi(-,\eta))(\xi)\\ &= \frac{2^{\frac{\sigma+2}{2}} i^{-k}\Gamma(\frac{\mu}{2})}{\Gamma(\frac{\mu-\sigma+\tau}{4})\Gamma(\frac{\mu-\sigma-\tau}{4})}|\eta|^{-\frac{\sigma+\mu}{2}}|\xi|^{-\frac{m-2}{2}}\\ &\quad\times\int_0^\infty J_{\frac{m-2}{2}}(|\xi|\,s)s^{-\frac{\tau}{2}}K_{\frac{\tau}{2}}(|\eta|\,s)s^{\frac{m}{2}}\,\mathrm{d}s\\ &= \frac{2^{\frac{\sigma-\tau+m}{2}} i^{-k}\Gamma(\frac{\mu}{2})\Gamma(\frac{m-\tau}{2})}{\Gamma(\frac{\mu-\sigma+\tau}{4})\Gamma(\frac{\mu-\sigma-\tau}{4})}|\eta|^{-\frac{\sigma+\tau+\mu}{2}}(|\xi|^2+|\eta|^2)^{\frac{\tau-m}{2}}.\end{aligned}$$

Altogether we see that $I(\sigma,\mu)$ is a partial convolution operator combined with a multiplication operator

$$I(\sigma,\tau)(f\otimes\phi)(\xi,\eta) = \mathrm{const}\cdot|\eta|^{-\frac{\sigma+\tau+\mu}{2}}\phi(\eta)\int_{\mathbb{R}^m}(|\xi-\xi'|^2+|\eta|^2)^{\frac{\tau-m}{2}}f(\xi')\,\mathrm{d}\xi'.$$

For $m = n-1$ this operator appears in [19, 20] as a special case. This expression for $I(\sigma,\tau)$ is valid for $\operatorname{Re}\sigma \ll \operatorname{Re}\tau \ll 0$. It has a holomorphic extension to all $\sigma,\tau\in\mathbb{C}$ for $f\in\mathcal{F}_{\mathbb{R}^m}C_c^\infty(\mathbb{R}^m\setminus\{0\})$.

A. Decomposition of Principal Series

We give a short alternative proof for the decomposition of the principal series $\pi^G_{\sigma,\varepsilon}$, $\sigma\in i\mathbb{R}$, $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$, into irreducible H-representations. This decomposition turns out to be essentially equivalent to the Plancherel formula for $L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_\delta)$, where $\mathcal{L}'_\delta$ are the line bundles over the Riemannian symmetric space $O(1,m+1)/(O(1)\times O(m+1))$ induced by the characters $(a,g)\mapsto a^\delta$ of $O(1)\times O(m+1)$, $\delta\in\mathbb{Z}/2\mathbb{Z}$.

Consider the flag variety $X = G/P$. Since $G/P\cong K/M$ we can identify X with the unit sphere $S^n\subseteq\mathbb{R}^{n+1}$. For this we define a G-action on S^n by the formula

$$g\circ x := \frac{\mathrm{pr}_x(g(1,x))}{\mathrm{pr}_0(g(1,x))},\qquad x\in S^n,$$

where $\mathrm{pr}_0 : \mathbb{R}^{n+2} \to \mathbb{R}$ and $\mathrm{pr}_x : \mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ denote the projections onto the first coordinate and the last $n+1$ coordinates, respectively, and $g(1,x)$ is the usual action of g on $(1,x) \in \mathbb{R} \times \mathbb{R}^{n+1} \cong \mathbb{R}^{n+2}$. Then it is easy to prove the following:

LEMMA A.1. *The operation $\circ$ defines a transitive group action of G on S^n. The stabilizer of the point $e_{n+1} = (0,\dots,0,1) \in S^n$ is equal to the parabolic subgroup P. The maximal compact subgroup K also acts transitively on S^n and the stabilizer subgroup of the point x_0 is equal to M.*

Let us consider a slightly different embedding of $O(1,m+1)\times O(n-m)$ into $G = O(1,n+1)$. Let

$$H' := \{\mathrm{diag}(g,h) : g \in O(1,m+1), h \in O(n-m)\}.$$

Then clearly H and H' are conjugate and hence the branching to H is equivalent to the branching to H'. We shall therefore only deal with H' in this section.

LEMMA A.2. *Under the action $\circ$ of the group H' the sphere S^n decomposes into the two orbits*

$$\mathcal{O}_0 := H' \circ e_1 = \big\{(x',0) : x' \in S^m\big\},$$
$$\mathcal{O}_1 := H' \circ e_{n+1} = \big\{(x',x'') \in S^n : x' \in \mathbb{R}^{m+1}, x'' \in \mathbb{R}^{n-m}, x'' \neq 0\big\}.$$

The orbit $\mathcal{O}_1$ is open and dense in S^n. The isotropy group of e_{n+1} in H' is

$$S = \{(a,g,h,a) : a \in O(1), g \in O(m+1), h \in O(n-m-1)\}.$$

Now consider the realization of $\pi_{\sigma,\varepsilon}$ in the compact picture, i.e. on $L^2(G/P,\mathcal{L}_{\sigma,\varepsilon})$, where $\mathcal{L}_{\sigma,\varepsilon}$ denotes the line bundle over G/P associated to the character $man \mapsto \xi_\varepsilon(m)a^{\sigma+\rho}$ of P. Since the orbit $\mathcal{O}_1 \subseteq G/P$ is open and dense we have

$$L^2(G/P,\mathcal{L}_{\sigma,\varepsilon}) \cong L^2(\mathcal{O}_1,\mathcal{L}_{\sigma,\varepsilon}|_{\mathcal{O}_1}).$$

Now the stabilizer S of $eP \in G/P$ in H is contained in P and hence the restriction of the line bundle $\mathcal{L}_{\sigma,\varepsilon}$ to $\mathcal{O}_1 \cong H'/S$ is induced by the restriction of the corresponding character of P to S which is simply $\xi_\varepsilon|_S$. Therefore we find

$$L^2(G/P,\mathcal{L}_{\sigma,\varepsilon}) \cong L^2(\mathcal{O}_1,\mathcal{L}_\varepsilon),$$

where $\mathcal{L}_\varepsilon$ is the line bundle over $\mathcal{O}_1 \cong H'/S$ induced by the character $\xi_\varepsilon|_S$. Using the decomposition of $L^2(S^{n-m-1})$ into spherical harmonics we find

$$L^2(\mathcal{O}_1,\mathcal{L}_\varepsilon) \cong \sum_{k=0}^{\infty}{}^{\oplus} L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_{\varepsilon+k}) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m})$$

as H'-representations, where for $\delta \in (\mathbb{Z}/2\mathbb{Z})$ we denote by $\mathcal{L}'_\delta$ the line bundle over the symmetric space $O(1,m+1)/(O(1)\times O(m+1))$ induced by the character $(a,g)\mapsto a^\delta$ of $O(1)\times O(m+1)$. Together we obtain

$$\pi^G_{\sigma,\varepsilon}\big|_H \cong \sum_{k=0}^{\infty}{}^{\oplus} L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_{\varepsilon+k}) \boxtimes \mathcal{H}^k(\mathbb{R}^{n-m})$$

and hence the decomposition of $\pi^G_{\sigma,\varepsilon}|_H$ into irreducible H-representations is equivalent to the decomposition of $L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_\delta)$ into irreducible $O(1,m+1)$-representations, $\delta \in \mathbb{Z}/2\mathbb{Z}$. Since $O(1,m+1)/(O(1)\times O(m+1))$ is a Riemannian symmetric space of rank one the decomposition of $L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_\delta)$ is well-known and given by

$$L^2(O(1,m+1)/(O(1)\times O(m+1)),\mathcal{L}'_\delta) \cong \int_{i\mathbb{R}_+}^{\oplus} \pi^{O(1,m+1)}_{\tau,\delta}\,\mathrm{d}\tau,$$

the unitary isomorphism established by the spherical Fourier transform. This proves Theorem 4.7 for the special case $\sigma \in i\mathbb{R}$.

B. Special Functions

For the sake of completeness we collect here the necessary formulas for certain special functions needed in this paper.

B.1. The K-Bessel function

We renormalize the classical K-Bessel function $K_\alpha(z)$ by

$$\widetilde{K}_\alpha(z) := \left(\frac{z}{2}\right)^{-\alpha} K_\alpha(z).$$

Then $\widetilde{K}_\alpha(z)$ solves the differential equation

$$\frac{\mathrm{d}^2u}{\mathrm{d}z^2}+\frac{2\alpha+1}{z}\frac{\mathrm{d}u}{\mathrm{d}z}-u=0. \tag{B.1}$$

It has the following asymptotic behaviour as $x\to 0$ (see [38, Chapters III & VII]):

$$\widetilde{K}_\alpha(x)=\begin{cases}\frac{\Gamma(\alpha)}{2}\left(\frac{x}{2}\right)^{-2\alpha}+o(x^{-2\alpha}), & \text{for } \operatorname{Re}\alpha>0,\\ -\log\left(\frac{x}{2}\right)+o\left(\log\left(\frac{x}{2}\right)\right), & \text{for } \operatorname{Re}\alpha=0,\\ \frac{\Gamma(-\alpha)}{2}+o(1), & \text{for } \operatorname{Re}\alpha<0.\end{cases} \tag{B.2}$$

Further, as $x\to\infty$ we have

$$\widetilde{K}_\alpha(x)=\frac{\sqrt{\pi}}{2}\left(\frac{x}{2}\right)^{-\alpha-\frac{1}{2}}e^{-x}\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right). \tag{B.3}$$

For the derivative of $\widetilde{K}_\alpha(z)$ the following identity holds (see [38, equation III.71 (6)]):

$$\frac{\mathrm{d}}{\mathrm{d}z}\widetilde{K}_\alpha(z)=-\frac{z}{2}\widetilde{K}_{\alpha+1}(z). \tag{B.4}$$

This identity can be used to write the differential equation (B.1) as the three-term recurrence relation (see [38, equation III.71 (6)]):

$$z^2\widetilde{K}_{\alpha+1}(z)=4\alpha\widetilde{K}_\alpha(z)+4\widetilde{K}_{\alpha-1}(z). \tag{B.5}$$

B.2. The Gauß hypergeometric function

Consider the classical Gauß hypergeometric function

$$_2F_1(a,b;c;z)=\sum_{n=0}^\infty\frac{(a)_n(b)_n}{n!(c)_n}z^n,$$

where $(a)_n=a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. The function $_2F_1(a,b;c;z)$ is holomorphic in z for $z\notin[1,\infty)$ and meromorphic in the parameters $a,b,c\in\mathbb{C}$. It solves the differential equation

$$(1-z)z\frac{\mathrm{d}^2u}{\mathrm{d}z^2}+(c-(a+b+1)z)\frac{\mathrm{d}u}{\mathrm{d}z}-abu=0. \tag{B.6}$$

The following formula allows to study the asymptotic behaviour of the Gauß hypergeometric function near $z = -\infty$ (see [10, equation 9.132 (2)]):

$$\text{(B.7)}\quad {}_2F_1(a,b;c;z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,a-c+1;a-b+1;\tfrac{1}{z}) + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,b-c+1;b-a+1;\tfrac{1}{z}).$$

Both summands on the right hand side of (B.7) are generically linear independent solutions to (B.6). Their Wronskian is given by

$$W(z^{-a}{}_2F_1(a,a-c+1;a-b+1;-\tfrac{1}{z}), z^{-b}{}_2F_1(b,b-c+1;b-a+1;-\tfrac{1}{z})) = (a-b)(1+z)^{c-a-b-1}z^{-c}.$$

The following simple formula for the derivative of the hypergeometric function holds:

$$\text{(B.8)}\qquad \frac{\mathrm{d}}{\mathrm{d}z}{}_2F_1(a,b;c;z) = \frac{ab}{c}{}_2F_1(a+1,b+1;c+1;z).$$

We recall Kummer's transformation formula (see [10, equation 9.131 (1)]):

$$\text{(B.9)}\qquad {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a,c-b;c;z).$$

For $a \in -\mathbb{N}$ the hypergeometric function ${}_2F_1(a,b;c;z)$ degenerates to a polynomial which can be expressed in terms of the Jacobi polynomials $P_n^{(a,b)}(z)$ (see [10, equation 8.962 (1)]):

$$\text{(B.10)}\qquad {}_2F_1(-n,b;c;z) = \frac{n!}{(c)_n}P_n^{(c-1,b-c-n)}(1-2z), \qquad n \in \mathbb{N},$$

where

$$P_n^{(a,b)}(z) = \frac{1}{n!}\sum_{k=0}^{n}\frac{(-n)_k(a+b+n+1)_k(a+k+1)_{n-k}}{k!}\left(\frac{1-z}{2}\right)^k.$$

B.3. Integral formulas

We consider the J-Bessel function $J_\nu(z)$ and the K-Bessel function $K_\nu(z)$. For the J-Bessel function and the hypergeometric function the following integral formula holds for $y > 0$, $\operatorname{Re}\lambda > 0$ and $-1 < \operatorname{Re}\nu < 2\max(\operatorname{Re}\alpha, \operatorname{Re}\beta) - \frac{3}{2}$ (see [10, equation 7.542 (10)])

$$\text{(B.11)}\qquad \int_0^\infty {}_2F_1(\alpha,\beta;\nu+1;-\lambda^2x^2)J_\nu(xy)x^{\nu+1}\,\mathrm{d}x = \frac{2^{\nu-\alpha-\beta+2}\Gamma(\nu+1)}{\lambda^{\alpha+\beta}\Gamma(\alpha)\Gamma(\beta)} y^{\alpha+\beta-\nu-2}K_{\alpha-\beta}\left(\frac{y}{\lambda}\right).$$

For the J-Bessel function and the K-Bessel function we have the following integral formula for $\operatorname{Re}\mu > |\operatorname{Re}\nu| - 1$ and $\operatorname{Re} b > |\operatorname{Im} a|$ (see [10, equation 6.576 (7)])

$$\text{(B.12)}\qquad \int_0^\infty x^{\mu+\nu+1}J_\mu(ax)K_\nu(bx)\,\mathrm{d}x = 2^{\mu+\nu}a^\mu b^\nu \frac{\Gamma(\mu+\nu+1)}{(a^2+b^2)^{\mu+\nu+1}}.$$

B.4. Fourier and Hankel transform

Let $\mathcal{F}_{\mathbb{R}^n}$ denote the Euclidean Fourier transform on $\mathbb{R}^n$ as defined in (1.5). Let $k \in \mathbb{N}$ and $\phi \in \mathcal{H}^k(\mathbb{R}^n)$. For $f \in L^2(\mathbb{R}_+, r^{n+2k-1}\,\mathrm{d}r)$ denote by $f \otimes \phi \in L^2(\mathbb{R}^n)$ the function

$$(f\otimes\phi)(x) := f(|x|)\phi(x), \qquad x \in \mathbb{R}^n.$$

Then by [33, Chapter IV, Theorem 3.10]

$$\mathcal{F}_{\mathbb{R}^n}(f\otimes\phi) = i^{-k}(\mathcal{H}_{\frac{n+2k-2}{2}}f)\otimes\phi,$$

where $\mathcal{H}_\nu$ is the modified Hankel transform of parameter $\nu \geq -\frac{1}{2}$

$$\mathcal{H}_\nu f(r) = r^{-\nu}\int_0^\infty J_\nu(rs)f(s)s^{\nu+1}\,\mathrm{d}s,$$

which is a unitary isomorphism (up to a scalar multiple) on $L^2(\mathbb{R}_+, r^{2\nu+1}\,\mathrm{d}r)$.

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Jan Möllers
Department of Mathematics
The Ohio State University
231 West 18th Avenue
Columbus, OH 43210, USA
E-mail: mollers.1@osu.edu

Yoshiki Oshima
Kavli IPMU (WPI)
The University of Tokyo
5-1-5 Kashiwanoha
Kashiwa 277-8583, Japan
E-mail: yoshiki.oshima@ipmu.jp