Mori Fibre Spaces for Kähler Threefolds

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In memory of Professor Kunihiko Kodaira

Abstract. Let $X$ be a compact Kähler threefold such that the base of the MRC-fibration has dimension two. We prove that $X$ is bimeromorphic to a Mori fibre space. Together with our earlier result [HP13] this completes the MMP for compact Kähler threefolds: let $X$ be a non-projective compact Kähler threefold. Then $X$ has a minimal model or $X$ is bimeromorphic to a Mori fibre space over a non-projective Kähler surface.

1. Introduction

This paper continues our study of the minimal model program (MMP) for compact Kähler threefolds. In [HP13] we established the existence of minimal models for compact Kähler threefolds such that $K_X$ is pseudoeffective. More precisely, minimal models are obtained, as in the projective setting, by a sequence of contractions of extremal rays (in a suitable cone) and flips. By a theorem of Brunella [Bru06] a smooth compact Kähler threefold has pseudoeffective $K_X$ if and only if $X$ is not uniruled. In the present work we deal with the remaining case where $X$ is uniruled. The general fibre of the MRC-fibration $X \to Z$ is rationally connected, so carries no holomorphic forms [Deb01, Cor.4.18]. Thus if the base $Z$ has dimension at most one, then we obtain $H^2(X, \mathcal{O}_X) = H^0(X, \Omega^2_X) = 0$. In particular the Kähler manifold $X$ is projective by Kodaira’s criterion. Since our main interest is the study of non-projective Kähler threefolds, we focus on the case where $Z$ has dimension two:

1.1. Theorem. Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \to Z$ has dimension two.

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Then $X$ is bimeromorphic to a Mori fibre space, i.e. there exists a MMP

$$X \dasharrow X',$$

consisting of contractions of extremal rays and flips, such that $X'$ admits a fibration $\varphi : X' \to S$ onto a normal compact $\mathbb{Q}$–factorial Kähler surface with at most klt singularities such that $-K_{X'}$ is $\varphi$-ample and $p(X'/S) = 1$.

It will be important to work with a special type of Kähler classes:

1.2. Definition. Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dasharrow Z$ has dimension two, and let $F \simeq \mathbb{P}^1$ be a general fibre. A Kähler class $\omega$ on $X$ is normalised if $\omega \cdot F = 2$.

Since the canonical class $K_X$ has degree $-2$ on $F$, the adjoint class $K_X + \omega$ is trivial on $F$. Using a recent result of Păun [Pău12] we first prove that $K_X + \omega$ is pseudoeffective. The proof of Theorem 1.1 then proceeds in two steps, the first being the existence of a MMP for the adjoint class $K_X + \omega$:

1.3. Theorem. Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dasharrow Z$ has dimension two. Then there exists a MMP

$$X \dasharrow X'$$

such that for every normalised Kähler class $\omega'$ on $X'$ the adjoint class $K_{X'} + \omega'$ is nef.

Once we have a normalised Kähler class $\omega$ such that $K_X + \omega$ is nef, the adjoint class $K_X + \omega$ is a natural candidate for the “nef supporting class” that defines a Mori fibre space structure.

The second step is to prove an analogue of the base-point free theorem for the adjoint class $K_X + \omega$.

1.4. Theorem. Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dasharrow Z$ has dimension two. Let $\omega$ be a normalised Kähler class on $X$ such that $K_X + \omega$ is nef.
Then there exists a holomorphic fibration $\varphi : X \to S$ onto a normal compact Kähler surface $S$ such that $K_X + \omega$ is $\varphi$-trivial.

By construction, the anticanonical class $-K_X$ is ample with respect to the fibration $X \to S$, so we can use the cone and contraction theorem for projective morphisms ([Nak87], [KM98]) to run a relative MMP. This MMP terminates with the Mori fibre space we are looking for.

In the situation of Theorem 1.4 one can prove that $S$ is $\mathbb{Q}$-factorial with at most rational singularities, but it is not quite clear whether $S$ is klt. However we can prove this property for an elementary contraction of fibre type, cf. Lemma 4.1.

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2. Notation

We use the same notation as in [HP13]. For the convenience of the reader we recall the most important definitions and basic results.

2.1. Definition. An irreducible and reduced complex space $X$ is Kähler if there exists a Kähler form $\omega$, i.e. a positive closed real $(1,1)$-form $\omega \in \mathcal{A}^{1,1}_R(X)$, such that the following holds: for every point $x \in X_{\text{sing}}$ there exists an open neighbourhood $x \in U \subset X$ and a closed embedding $i_U : U \subset V$ into an open set $V \subset \mathbb{C}^N$, and a strictly plurisubharmonic $C^\infty$-function $f : V \to \mathbb{C}$ with $\omega|_{U \cap X_{\text{nons}}} = (i\partial \bar{\partial} f)|_{U \cap X_{\text{nons}}}$. In the same manner one can define $(p,q)$–forms on an irreducible reduced complex space [Dem85], by duality we obtain the usual notions of currents.

We will next define the appropriate analogue of the Néron-Severi space $N^1(X)$ for a normal compact Kähler space, as well as the cones $\overline{\text{NE}}(X)$ and $\overline{\text{NA}}(X)$ contained in its dual $N_1(X)$. For any details we refer to [HP13, Sect.3].

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2.2. Definition. [BPEG13, Defn. 4.6.2] [HP13, Defn. 3.6] Let $X$ be an irreducible reduced complex space. Let $\mathcal{H}_X$ be the sheaf of real parts of holomorphic functions multiplied with $i$. A $(1,1)$-form with local potentials on $X$ is a global section of the quotient sheaf $\mathcal{A}^0_X/\mathcal{H}_X$. We define the Bott-Chern cohomology

$$N^1(X) := H^1(X, \mathcal{H}_X).$$

2.3. Remark. Using the exact sequence

$$0 \to \mathcal{H}_X \to \mathcal{A}^0_X \to \mathcal{A}^0_X/\mathcal{H}_X \to 0,$$

and the fact that $\mathcal{A}^0_X$ is acyclic, we obtain a surjective map

$$H^0(X, \mathcal{A}^0_X/\mathcal{H}_X) \to H^1(X, \mathcal{H}_X).$$

Thus we can see an element of the Bott-Chern cohomology group as a closed $(1,1)$-form with local potentials modulo all the forms that are globally of the form $dd^c u$.

Let $\mathcal{D}_X$ be the sheaf of distributions. Using the exact sequence

$$0 \to \mathcal{H}_X \to \mathcal{D}_X \to \mathcal{D}_X/\mathcal{H}_X \to 0,$$

we see that one obtains the same Bott-Chern group, considering $(1,1)$-currents $T$ with local potentials, which is to say that locally $T = dd^c u$ with $u$ a distribution.

Dually we define

2.4. Definition. Let $X$ be a normal compact complex space. Then $N_1(X)$ is the vector space of real closed currents of bidimension $(1,1)$ modulo the following equivalence relation: $T_1 \sim T_2$ if and only if

$$T_1(\eta) = T_2(\eta)$$

for all real closed $(1,1)$-forms $\eta$.

In [HP13, Prop. 3.9] we established a canonical isomorphism

$$\Phi : N^1(X) \to N_1(X)^*$$

(1)
for any normal compact complex space $X$ in the Fujiki class $C$, i.e., for those $X$ which are bimeromorphic to a Kähler space.

2.5. Definition. Let $X$ be a normal compact complex space in class $C$. We define $\overline{NA}(X) \subset N_1(X)$ as the cone generated by the positive closed currents of bidimension $(1, 1)$.

Given an irreducible curve $C \subset X$, we associate to $C$ the current of integration $T_C$. In the case of isolated singularities, which is the only case relevant in our setting, we define

$$T_C(\omega) = \int_C \omega = \int_{\hat{C}} \pi^*(\omega),$$

where $\pi : \hat{X} \to X$ is a resolution of singularities, the curve $\hat{C}$ is the strict transform of $C$, and $\omega$ a $d$-closed $(1, 1)$-form on $X$. We define the Mori cone $\overline{NE}(X) \subset N_1(X)$ as the closure of the cone generated by the currents $T_C$ and clearly have an inclusion

$$\overline{NE}(X) \subset \overline{NA}(X).$$

2.6. Definition. Let $X$ be an irreducible reduced compact complex space in class $C$. We denote by $\text{Nef}(X) \subset N^1(X)$ the cone generated by cohomology classes which are nef in the sense of [Pâu98, Defn.3]: let $u \in N^1(X)$ be a class represented by a form $\alpha$ with local potentials. Then $u$ is nef if for some positive $(1, 1)$-form $\omega$ on $X$ and for every $\epsilon > 0$ there exists $f_\epsilon \in A^0(X)$ such that

$$\alpha + i\partial\bar{\partial}f_\epsilon \geq -\epsilon \omega.$$

The class $u$ is pseudo-effective, if it can be represented by a current $T$ which is locally of the form $T = \partial\bar{\partial}\varphi$ with $\varphi$ a plurisubharmonic function.

If $X$ is a normal compact Kähler space, we can also consider the open cone $K$ generated by the classes of Kähler forms. In this case we know by [Dem92, Prop.6.1] that

$$\text{Nef}(X) = \overline{K}.$$

\[^2\]The statement in [Dem92, Prop.6.1.iii]] is for compact manifolds, but the proof works in the singular setting, cf. also [HP13, Rem.3.5] for regularisation arguments in the singular setting.
As in the projective setting, we have a duality statement:

2.7. **Proposition.** [HP13, Prop.3.15] Let $X$ be a normal compact threefold in class $\mathcal{C}$. Then the cones $\text{Nef}(X)$ and $\overline{\text{NA}}(X)$ are dual via the canonical isomorphism $\Phi: N^1(X) \to N_1(X)^*$ given by (1).

Finally we define the notion of the contraction of an extremal ray $R$. It is very important to consider extremal rays in the dual Kähler cone $\overline{\text{NA}}(X)$ rather than in the Mori cone $\overline{\text{NE}}(X)$.

2.8. **Definition.** Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler space with at most terminal singularities, and let $\omega$ be a Kähler class on $X$. Let $R$ be a $(K_X + \omega)$-negative extremal ray in $\overline{\text{NA}}(X)$. A contraction of the extremal ray $R$ is a morphism $\varphi: X \to Y$ onto a normal compact Kähler space $Y$, such that $-(K_X + \omega)$ is a Kähler class on every fibre and a curve $C \subset X$ is contracted if and only if $[C] \in R$.

3. **MMP for the Adjoint Class**

In order to simplify the statements we will work under the following

3.1. **Assumption.** Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dasharrow Z$ has dimension two, and let $\omega$ be a normalised Kähler class on $X$.

3.2. **Remark.** Observe that the surface $Z$ is not uniruled: since this is a bimeromorphic statement we can suppose that $X$ and $Z$ are smooth and the MRC-fibration is a morphism $\varphi: X \to Z$. If $(C_t)_{t \in T} \subset Z$ is a dominant family of rational curves, the surface $\varphi^{-1}(C_t)$ is uniruled by the fibres of $\varphi^{-1}(C_t) \to C_t$. Thus it carries no holomorphic 2-form, in particular $\varphi^{-1}(C_t)$ is projective by Kodaira’s criterion. Thus Tsen’s theorem applies and we obtain that $\varphi^{-1}(C_t)$ is rationally connected. Now we conclude as in the algebraic case that $\varphi$ is not the MRC-fibration. The same line of arguments also shows that the theorem of Graber-Harris-Starr [GHS03] is also true in Kähler category.
3.A. Remarks on adjunction

Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Let $S \subset X$ be a prime divisor, i.e. an irreducible and reduced compact surface. Let $m \in \mathbb{N}$ be the smallest positive integer such that both $mK_X$ and $mS$ are Cartier divisors on $X$. Then the canonical class $K_S \in \text{Pic}(S) \otimes \mathbb{Q}$ is defined by

$$K_S := \frac{1}{m}(mK_X + mS)|_S.$$ 

Since $X$ is smooth in codimension two, there exist at most finitely many points $\{p_1, \ldots, p_q\}$ where $K_X$ and $S$ are not Cartier. Thus by the adjunction formula $K_S$ is isomorphic to the dualising sheaf $\omega_S$ on $S \setminus \{p_1, \ldots, p_q\}$.

Let now $\nu : \tilde{S} \to S$ be the normalisation. Then we have

$$K_{\tilde{S}} \sim_{\mathbb{Q}} \nu^*K_S - N,$$

where $N$ is an effective Weil divisor defined by the conductor ideal. Indeed this formula holds by [Rei94] for the dualising sheaves. Since $O_{\tilde{S}}(\nu^*K_S)$ is isomorphic to $\nu^*\omega_S$ on the complement of $\nu^{-1}(p_1, \ldots, p_q)$, the formula holds for the canonical classes.

Let $\mu : \hat{S} \to \tilde{S}$ be the minimal resolution of the normal surface $\tilde{S}$, then we have

$$K_{\hat{S}} \sim_{\mathbb{Q}} \mu^*K_{\tilde{S}} - N',$$

where $N'$ is an effective $\mu$-exceptional $\mathbb{Q}$-divisor [Sak84, 4.1]. Thus if $\pi : \hat{S} \to S$ is the composition $\nu \circ \mu$, there exists an effective, canonically defined $\mathbb{Q}$-divisor $E \subset \hat{S}$ such that

$$K_{\hat{S}} \sim_{\mathbb{Q}} \pi^*K_S - E.$$ 

Let $C \subset S$ be a curve such that $C \not\subset S_{\text{sing}}$. Then the morphism $\pi$ is an isomorphism at the general point of $C$, and we can define the strict transform $\hat{C} \subset \hat{S}$ as the closure of $C \setminus S_{\text{sing}}$. Since $\hat{C}$ is an (irreducible) curve that is not contained in the divisor $N$ defined by the conductor, we have $\hat{C} \not\subset E$.

By the projection formula and (3) we obtain

$$K_{\hat{S}} \cdot \hat{C} \leq K_S \cdot C.$$
3.B. Divisorial Zariski decomposition for $K_X + \omega$

The starting point of our investigation is the following observation:

3.3. Lemma. Under the Assumption 3.1 the adjoint class $K_X + \omega$ is pseudoeffective.

Proof. Being pseudoeffective is a closed property in $N^1(X)$, so it is sufficient to prove that for every $\varepsilon > 0$, the class $K_X + (1 + \varepsilon)\omega$ is pseudoeffective. Let $\mu : X' \to X$ be a bimeromorphic morphism from a smooth Kähler threefold $X'$ such that the MRC-fibration is a morphism $\phi' : X' \to Z'$ onto a smooth surface $Z'$. The projection formula yields

$$\mu^* (K_{X'} + (1 + \varepsilon)\mu^* \omega) = K_X + (1 + \varepsilon)\omega,$$

so it is sufficient to prove that $K_{X'} + (1 + \varepsilon)\mu^* \omega$ is pseudoeffective. However by a recent result of Păun [Pău12, Thm.1.1], the class $K_{X'/Z'} + (1 + \varepsilon)\mu^* \omega$ is pseudoeffective. Since the surface $Z'$ is not uniruled (cf. Remark 3.2) and Kähler by [Var86, Thm.3], the canonical class $K_{Z'}$ is pseudoeffective. Thus $K_{X'} + (1 + \varepsilon)\mu^* \omega$ is pseudoeffective. \square

Since $K_X + \omega$ is pseudoeffective, we may apply [Bou04, Thm.3.12] to obtain a divisorial Zariski decomposition

$$(5) \quad K_X + \omega = \sum_{j=1}^{r} \lambda_j S_j + P_{\omega},$$

where the $S_j$ are integral surfaces in $X$, the coefficients $\lambda_j \in \mathbb{R}^+$ and $P_{\omega}$ is a pseudoeffective class which is nef in codimension one [Bou04, Prop.2.4], that is for every surface $S \subset X$ the restriction $P_{\omega}|_S$ is pseudoeffective.

3.4. Lemma. Under the Assumption 3.1, let $S$ be a surface such that $(K_X + \omega)|_S$ is not pseudoeffective. Then $S$ is one of the surfaces $S_j$ in

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3 The statements in [Bou04] are for complex compact manifolds, but generalise immediately to our situation: take $\mu : X' \to X$ a desingularisation, and let $m \in \mathbb{N}$ be the Cartier index of $K_X$. Then $\mu^* (m(K_X + \omega))$ is a pseudoeffective class with divisorial Zariski decomposition $\mu^* (m(K_X + \omega)) = \sum \eta_j S'_j + P'_\omega$. The decomposition of $K_X + \omega$ is defined by the push-forwards $\mu_* (\frac{1}{m} \sum \eta_j S'_j)$ and $\mu_* (\frac{1}{m} P'_\omega)$. Since a prime divisor $D \subset X$ is not contained in the singular locus of $X$, the decomposition has the stated properties.
the divisorial Zariski decomposition (5) of $K_X + \omega$. Moreover $S = S_j$ is Moishezon and any desingularisation $\hat{S}_j$ is a uniruled projective surface.

**Proof.** The proof that $S = S_j$ for some $j$ is analogous to the proof in [HP13, Lemma 4.1], thus (up to renumbering) we may suppose that $S = S_1$. We have

$$S = S_1 = \frac{1}{\lambda_1}(K_X + \omega) - \frac{1}{\lambda_1} \left( \sum_{j=2}^{r} \lambda_j S_j + P_\omega \right),$$

so by adjunction

$$K_S = (K_X + S)|_S = \left( \frac{1}{\lambda_1} K_X|_S + \frac{1}{\lambda_1} \omega|_S \right) - \frac{1}{\lambda_1} \left( \sum_{j=2}^{r} \lambda_j (S_j \cap S) + P_\omega|_S \right).$$

Note now that $\frac{1}{\lambda_1} K_X|_S + \frac{1}{\lambda_1} \omega|_S$ is not pseudoeffective: otherwise

$$\left( \frac{1}{\lambda_1} K_X|_S + \frac{1}{\lambda_1} \omega|_S \right) + \omega|_S = \frac{1}{\lambda_1} (K_X + \omega)|_S$$

would be pseudoeffective, in contradiction to our assumption. Since

$$\frac{1}{\lambda_1} \left( \sum_{j=2}^{r} \lambda_j (S_j \cap S) + P_\omega|_S \right)$$

is pseudoeffective, the class $K_S$ cannot be pseudoeffective.

Let now $\pi : \hat{S} \to S$ be the composition of the normalisation and the minimal resolution of the surface $S$, then by (3) there exists an effective divisor $E$ such that

$$K_{\hat{S}} \sim_{Q} \pi^* K_S - E.$$ 

Thus $K_{\hat{S}}$ is not pseudoeffective, in particular $\kappa(\hat{S}) = -\infty$. It follows from the Castelnuovo-Kodaira classification that $\hat{S}$ is covered by rational curves, in particular $\hat{S}$ is a projective surface [BHPVdV04]. Thus $S$ is Moishezon. $\square$

### 3.5. Corollary.

Under the Assumption 3.1, the adjoint class $K_X + \omega$ is nef if and only if

$$(K_X + \omega) \cdot C \geq 0$$

for every curve $C \subset X$. 
Proof. We prove the non-trivial implication by contradiction, so suppose that \( K_X + \omega \) is not nef, but \((K_X + \omega) \cdot C \geq 0\) for all curves \( C \subset X \). Since \( K_X + \omega \) is pseudoeffective by Lemma 3.3 and the restriction to every curve is nef, there exists by [Pău98], [Bou04, Prop.3.4] an irreducible surface \( S \subset X \) such that \((K_X + \omega)|_S\) is not pseudoeffective. Fix a desingularisation \( \pi : \hat{S} \to S \) of the surface \( S \). By Lemma 3.4 the surface \( \hat{S} \) is projective and uniruled. The class \( \pi^*(K_X + \omega)|_S \) is not pseudoeffective and, since \( H^2(\hat{S}, \mathcal{O}_\hat{S}) = 0 \), the class is represented by an \( \mathbb{R} \)-divisor. Thus there exists a covering family of curves \( C_t \subset S \) such that

\[
(K_X + \omega) \cdot C_t = \pi^*(K_X + \omega)|_S \cdot \hat{C}_t < 0,
\]

where \( \hat{C}_t \) denotes the strict transform of \( C_t \) in \( \hat{S} \). This contradicts our assumption that \((K_X + \omega) \cdot C \geq 0\) for all curves \( C \subset X \).□

3.C. The adjoint cone theorem

The goal of this subsection is to prove a cone theorem for the adjoint class \( K_X + \omega \):

3.6. Theorem. Under the Assumption 3.1 there exists a countable family \((\Gamma_i)_{i \in I}\) of rational curves on \( X \) such that

\[
0 < -(K_X + \omega) \cdot \Gamma_i \leq 4
\]

and

\[
\overline{NA}(X) = \overline{NA}(X)^{(K_X + \omega) \geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]
\]

The proof of Theorem 3.6 is quite similar to the proof of [HP13, Thm.1.2.]; for sakes of completeness we explain the main steps:

3.7. Lemma. Under the Assumption 3.1, let \( C \subset X \) be a curve such that \((K_X + \omega) \cdot C < 0\) and \( \dim_C \text{Chow}(X) > 0 \).

Then there exists a unique surface \( S_j \) from the divisorial Zariski decomposition (5) such that \( C \) and its deformations are contained in the surface \( S_j \). Moreover we have

\[
K_{S_j} \cdot C < K_X \cdot C.
\]
Proof. Identical to the proof of [HP13, Lemma 5.4], simply replace $K_X$ by $K_X + \omega$. □

3.8. Lemma. Under the Assumption 3.1, let $S_1, \ldots, S_r$ be the surfaces appearing in the divisorial Zariski decomposition (5). Set

$$b := \max\{1, -(K_X + \omega) \cdot Z \mid Z \text{ a curve s.t. } Z \subset S_{j, \text{sing}} \text{ or } Z \subset S_j \cap S_{j'}, j \neq j'\}.$$

If $C \subset X$ is a curve such that

$$-(K_X + \omega) \cdot C > b,$$

then we have $\dim_C \text{Chow}(X) > 0$.

In the proof we will use the following deformation property:

3.9. Definition. [HP13, Defn.4.3] Let $X$ be a normal $\mathbb{Q}$-factorial Kähler threefold with at most terminal singularities. We say that a curve $C$ is very rigid if

$$\dim_{mC} \text{Chow}(X) = 0$$

for all $m > 0$.

Proof of Lemma 3.8. Since $\omega$ is nef, we have $-K_X \cdot C > b$. The condition $b \geq 1$ implies that the curve $C$ is not very rigid (cf. [HP13, Thm.4.5]). We can now argue exactly as in [HP13, Lemma 5.6] to deduce

$$P_\omega \cdot C \geq 0.$$

Since $(K_X + \omega) \cdot C < 0$, the divisorial Zariski decomposition implies that there exists a number $j \in \{1, \ldots, r\}$ such that $S_j \cdot C < 0$. In particular we have $C \subset S_j$. The class $\omega$ being nef, we thus obtain

$$K_{S_j} \cdot C < K_X \cdot C < -b.$$

By definition of $b$, the curve $C$ is not contained in the singular locus of $S_j$. Let $\pi_j : \hat{S}_j \to S_j$ be the composition of normalisation and minimal resolution (cf. Subsection 3.A). Then the strict transform $\hat{C}$ of $C$ is well-defined and from (4) we deduce

$$K_{\hat{S}_j} \cdot \hat{C} \leq K_{S_j} \cdot C < -b.$$
Since \( b \geq 1 \), [Kol96, Thm.1.15] yields
\[
\dim_{\text{Chow}}(\hat{S}) > 0,
\]
so \( \hat{C} \) deforms. Thus its push-forward \( \pi_\ast \hat{C} = C \) deforms. \( \square \)

3.10. Corollary. Under the Assumption 3.1, let \( b \) be the constant from Lemma 3.8 and set
\[
d := \max\{3, b\}.
\]
If \( C \subset X \) is a curve such that \( -(K_X + \omega) \cdot C > d \), we have
\[
[C] = [C_1] + [C_2]
\]
with \( C_1 \) and \( C_2 \) effective 1-cycles (with integer coefficients) on \( X \).

Proof. Since \( \omega \) is nef, we have \( -K_X \cdot C > d \). Using the Lemmas 3.7 and 3.8, the proof of [HP13, Cor.5.7] applies without changes. \( \square \)

3.11. Lemma. Under the Assumption 3.1, let \( \mathbb{R}^+ [\Gamma_i] \) be a \((K_X + \omega)\)-negative extremal ray in \( \overline{\text{NE}}(X) \), where \( \Gamma_i \) is a curve that is not very rigid (cf. Definition 3.9). Then the following holds:

a) There exists a curve \( C \subset X \) such that \([C] \in \mathbb{R}^+ [\Gamma_i]\) and \( \dim_{\text{Chow}}(X) > 0 \).

b) There exists a rational curve \( C \subset X \) such that \([C] \in \mathbb{R}^+ [\Gamma_i]\).

Proof. This is completely analogous to [HP13, Lemma 5.8] since the existence of the rational curve \( C \subset X \) such that \([C] \in \mathbb{R}^+ [\Gamma_i]\) is a consequence of [HP13, Lemma 5.5 a)] which contains no assumption on \( K_X \). \( \square \)

Following the strategy of [HP13, Thm.6.2] we first establish the cone theorem for the Mori cone.

3.12. Theorem. Under the Assumption 3.1, there exists a number \( d \in \mathbb{N} \) and a countable family \((\Gamma_i)_{i \in I}\) of curves on \( X \) such that
\[
0 < -(K_X + \omega) \cdot \Gamma_i \leq d
\]
and

\[ \overline{\text{NE}}(X) = \overline{\text{NE}}(X)_{(K_X + \omega) \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i]. \]

If the ray \( \mathbb{R}^+ [\Gamma_i] \) is extremal in \( \overline{\text{NE}}(X) \), there exists a rational curve \( C_i \) on \( X \) such that \([C_i] \in \mathbb{R}^+ [\Gamma_i]\).

**Proof.** Let \( d \in \mathbb{N} \) be the bound from Corollary 3.10. There are only countably many curve classes \([C] \in \overline{\text{NE}}(X)\), such that

\[ 0 < -(K_X + \omega) \cdot C \leq d. \]

We choose a representative \( \Gamma_i \) for each such class \([C]\) and set

\[ V := \overline{\text{NE}}(X)_{(K_X + \omega) \geq 0} + \sum_{0 < -(K_X + \omega) \cdot \Gamma_i \leq d} \mathbb{R}^+ [\Gamma_i]. \]

Fix a Kähler class \( \eta \) on \( X \) such that \( \eta \cdot C \geq 1 \) for every curve \( C \subset X \).

**Step 1** We have \( \overline{\text{NE}}(X) = V \). By [HP13, Lemma 6.1] it is sufficient to prove that \( \overline{\text{NE}}(X) = V \), i.e. the class \([C]\) of every irreducible curve \( C \subset X \) is contained in \( V \). We will prove the statement by induction on the degree \( l := \eta \cdot C \). The start of the induction for \( l = 0 \) is trivial. Suppose now that we have shown the statement for all curves of degree at most \( l - 1 \) and let \( C \) be a curve such that \( l - 1 < \eta \cdot C \leq l \). If \(- (K_X + \omega) \cdot C \leq d\) we have \([C] \in V\) by definition. Otherwise there exists by Corollary 3.10 a decomposition

\[ [C] = [C_1] + [C_2] \]

with \( C_1 \) and \( C_2 \) effective 1-cycles (with integer coefficients) on \( X \). Since \( \eta \cdot C_i \geq 1 \) for \( i = 1, 2 \) we have \( \eta \cdot C_i \leq l - 1 \) for \( i = 1, 2 \). By induction both classes are in \( V \), so \([C]\) is in \( V \).

**Step 2** Every extremal ray contains the class of a rational curve. If the ray \( \mathbb{R}^+ [\Gamma_i] \) is extremal in \( \overline{\text{NE}}(X) \), we know by [HP13, Thm.4.5] and Lemma 3.11 that there exists a rational curve \( C_i \) such that \([C_i]\) is in the extremal ray. \( \square \)

We next pass from \( \overline{\text{NE}}(X) \) to \( \overline{\text{NA}}(X) \):
3.13. THEOREM. Under the Assumption 3.1 there exists a number \( d \in \mathbb{N} \) and a countable family \((\Gamma_i)_{i \in I}\) of curves on \( X \) such that

\[
0 < - (K_X + \omega) \cdot \Gamma_i \leq d
\]

and

\[
\overline{NA}(X) = \overline{NA}(X)_{(K_X + \omega) \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i].
\]

If the ray \( \mathbb{R}^+ [\Gamma_i] \) is extremal in \( \overline{NA}(X) \), there exists a rational curve \( C_i \) on \( X \) such that \( [C_i] \in \mathbb{R}^+ [\Gamma_i] \).

Theorem 3.13 is a consequence of Theorem 3.12 and the following proposition.

3.14. PROPOSITION. Under the Assumption 3.1, suppose that there exists a \( d \in \mathbb{N} \) and a countable family \((\Gamma_i)_{i \in I}\) of curves on \( X \) such that

\[
0 < - (K_X + \omega) \cdot \Gamma_i \leq d
\]

and

\[
\overline{NE}(X) = \overline{NE}(X)_{(K_X + \omega) \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i].
\]

Then we have

\[
\overline{NA}(X) = \overline{NA}(X)_{(K_X + \omega) \geq 0} + \sum_{i \in I} \mathbb{R}^+ [\Gamma_i].
\]

Proof. Identical to the proof of [HP13, Prop.6.4]: simply replace \( K_X \) by \( K_X + \omega \) and note that the uniruledness of a surface \( S \subset X \) such that \((K_X + \omega)|_S\) is not pseudoeffective is proven in Lemma 3.4. \( \square \)

Finally, Theorem 3.6 follows from Theorem 3.13 in the same way as [HP13, Thm.1.2] is deduced from [HP13, Thm.6.3].
3.D. The adjoint contraction theorem

In this subsection we prove the contraction theorem:

3.15. Theorem. Under the Assumption 3.1, let $\mathbb{R}^+[\Gamma_i]$ be a $(K_X + \omega)$-negative extremal ray in $\overline{NA}(X)$. Then the contraction of $\mathbb{R}^+[\Gamma_i]$ exists in the Kähler category.

For the rest of this subsection we fix $R := \mathbb{R}^+[\Gamma_{i_0}]$ a $(K_X + \omega)$-negative extremal ray in $\overline{NA}(X)$.

3.16. Definition. We say that the $(K_X + \omega)$-negative extremal ray $R$ is small if every curve $C \subset X$ with $[C] \in R$ is very rigid in the sense of Definition 3.9. Otherwise we say that the extremal ray $R$ is divisorial.

3.17. Remark. Notice that, due to Assumption 3.1 and Lemma 3.3, through a general point $x \in X$ there is no curve $C$ belonging to $R$. Hence the curves belonging to $R$ cover at most a divisor.

If the extremal ray $R$ is small, standard arguments show that there are only finitely many curves $C \subset X$ such that $[C] \in R$ (cf. [HP13, Rem.7.2]).

If the extremal ray $R$ is divisorial, we can argue as in [HP13, Lemma 7.5] that there exists a unique surface $S \subset X$ such that

$$ S \cdot R < 0. $$

In particular any curve $C \subset X$ with $[C] \in R$ is contained in $S$.

The following proposition is a well-known consequence of the cone theorem 3.13, cf. [HP13, Prop.7.3] for details:

3.18. Proposition. There exists a nef class $\alpha \in N^1(X)$ such that

$$ R = \{ z \in \overline{NA}(X) \mid \alpha \cdot z = 0 \}, $$

and such that, using the notation of Theorem 3.13, the class $\alpha$ is strictly positive on

$$ \left( \overline{NA}(X)(K_X + \omega)_{\geq 0} + \sum_{i \in I, i \neq i_0} \mathbb{R}^+[\Gamma_i] \right) \setminus \{0\}. $$

We call $\alpha$ a nef supporting class for the extremal ray $R = \mathbb{R}^+[\Gamma_{i_0}]$. 

In what follows we will use at several places the following theorem, stated in [BCE+02, Thm.2.6] for projective manifolds:

\section{Theorem.} Let $X$ be a normal compact Kähler space, and let $\alpha$ be a nef cohomology class on $X$. Then there exists an almost holomorphic, dominant meromorphic map $f : X \dasharrow Y$ with connected fibers, such that

1) $\alpha$ is numerically trivial on all compact fibers $F$ of $f$ with $\dim F = \dim X - \dim Y$

2) for every general point $x \in X$ and every irreducible curve $C$ passing through $x$ with $\dim f(C) > 0$, we have $\alpha \cdot C > 0$.

In particular, if two general points of $X$ can be joined by a chain $C$ of curves such that $\alpha \cdot C = 0$, then $\alpha \equiv 0$.

For the convenience of the reader we sketch how to adapt the proof from [BCE+02] to this setting.

\section{Proof.} We define that two points $x, y \in X$ are equivalent if they can be joined by a connected curve $C$ such that $\alpha \cdot C = 0$. By [Cam04, Thm.1.1] there exists an almost holomorphic map $f : X \dasharrow Y$ with connected fibers to a normal compact Kähler space $Y$ such that two general points $x$ and $y$ are equivalent if and only if $f(x) = f(y)$. By construction a general $f$-fibre $F_0$ is a normal compact Kähler space such that two general points can be connected by a curve, thus $F_0$ is algebraic [Cam81, p.212, Cor.]. Hence we can apply [BCE+02, Thm.2.4] to see that $\alpha|_{F_0} = 0$. In particular for any Kähler form $\omega$ on $X$ we have $\alpha \cdot \omega^{d-1} \cdot F_0 = 0$ where $d := \dim X - \dim Y$. Since any compact $f$-fibre $F$ of dimension $d$ is homologous to some multiple of $F_0$ and $\alpha$ is nef we see that $\alpha|_F = 0$. \hfill $\square$

\section{Notation.} Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_{i_0}]$ is divisorial, and let $S$ be the surface such that $S \cdot R < 0$ (cf. Remark 3.17). Let $\nu : \tilde{S} \to S \subset X$ be the normalisation. By Lemma 3.11(a) there exists a curve $C \subset X$ such that $[C] \in R$ and $\dim_C \text{Chow}(X) > 0$. Since we have $S \cdot C < 0$, the deformations $(C_t)_{t \in T}$ of $C$ induce a dominating family $(\tilde{C}_t)_{t \in T'}$ of $\tilde{S}$ such that $\nu^*(\alpha) \cdot \tilde{C}_t = 0$. The class $\nu^*(\alpha)$ is a nef class on $\tilde{S}$ and we may consider the nef reduction

\[ \tilde{f} : \tilde{S} \to \tilde{B} \]
with respect to $\nu^*(\alpha)$, cf. Theorem 3.19. By definition of the nef reduction this implies
\[ n(\alpha) := \dim \tilde{B} \in \{0, 1\}. \]

3.21. **Lemma.**

a) Suppose that the extremal ray $R$ is divisorial and $n(\alpha) = 0$. Then the surface $S$ can be blown down to a point $p$: there exists a bimeromorphic morphism $\varphi : X \to Y$ to a normal compact threefold $Y$ with $\dim \varphi(S) = 0$ such that $\varphi|_{X \setminus S}$ is an isomorphism onto $Y \setminus \{p\}$.

b) Suppose that the extremal ray $R$ is divisorial and $n(\alpha) = 1$. Then there exists a fibration $f : S \to B$ onto a curve $B$ such that a curve $C \subset S$ is contracted if and only if $[C] \in R$. Moreover the surface $S$ can be contracted onto a curve: there exists a bimeromorphic morphism $\varphi : X \to Y$ to a normal compact threefold $Y$ such that $\varphi|_S = f$ and $\varphi|_{X \setminus S}$ is an isomorphism onto $Y \setminus B$.

**Proof.** The proof is identical to the proofs of [HP13, Cor.7.7, Lemma 7.8, Cor.7.9] which only use properties of the nef class $\alpha$ and $K_X \cdot R < 0$ which holds since $\omega \cdot R > 0$. □

3.22. **Notation.** Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_i]$ is small. Set
\[ C := \bigcup_{C_i \subset X, [C_i] \in R \Gamma_i}, \]
then $C$ is a finite union of curves by Remark 3.17. We say that $C$ is contractible if there exists a bimeromorphic morphism $\varphi : X \to Y$ onto a normal threefold $Y$ with $\dim \varphi(C) = 0$ such that $\varphi|_{X \setminus C}$ is an isomorphism onto $Y \setminus \varphi(C)$.

The following statement is a variant of [HP13, Prop.7.11].

3.23. **Proposition.** Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_i]$ is small. Let $S \subset X$ be an irreducible surface. Then we have $\alpha^2 \cdot S > 0$. 

Proof. By hypothesis, the cohomology class \( \alpha - (K_X + \omega) \) is positive on the extremal ray \( R \), moreover we know by Proposition 3.18 that \( \alpha \) is positive on
\[
\left( \text{NA}(X)_{(K_X + \omega) \geq 0} + \sum_{i \in I, i \neq i_0} \mathbb{R}^+[\Gamma_i] \right) \setminus \{0\}.
\]
Thus, up to replacing \( \alpha \) by some positive multiple, we may suppose that \( \alpha - (K_X + \omega) \) is positive on \( \text{NA}(X) \setminus \{0\} \). Since \( X \) is a Kähler space, this implies by [HP13, Cor.3.16] that
\[
\eta := \alpha - (K_X + \omega)
\]
is a Kähler class. Arguing by contradiction we suppose that \( \alpha^2 \cdot S = 0 \).

We first claim that \( (K_X + \omega)|_S \) is not pseudoeffective. If \( \alpha|_S = 0 \) this is obvious, so suppose \( \alpha|_S \neq 0 \). Then we have
\[
0 = \alpha^2 \cdot S = (K_X + \omega) \cdot \alpha \cdot S + \eta \cdot \alpha \cdot S
\]
and
\[
\eta \cdot \alpha \cdot S = \eta|_S \cdot \alpha|_S > 0
\]
by the Hodge index theorem (note that if \( \pi : S' \to S \) is a desingularisation, then \( \pi^*(\eta|_S) \) is nef and big and \( \pi^*(\alpha|_S) \) is nef, so the “smooth” Hodge index theorem applies). Thus we have
\[
(K_X + \omega) \cdot \alpha \cdot S = (K_X + \omega)|_S \cdot \alpha|_S < 0.
\]
(7)
In particular \( (K_X + \omega)|_S \) is not pseudoeffective, the class \( \alpha|_S \) being nef.

Since \( (K_X + \omega)|_S \) is not pseudoeffective, we know by Lemma 3.4 that \( S \) is uniruled and one of the surfaces in the Zariski decomposition (5). In particular we cannot have \( \alpha|_S = 0 \) since \( S \) contains infinitely many curves (recall that the ray \( R \) is small, hence \( \alpha \cdot C = 0 \) can occur only for finitely many curves \( C \)). Using the decomposition (5) and (7) we obtain \( \alpha \cdot S^2 < 0 \), hence
\[
(K_X + \omega + S) \cdot \alpha \cdot S < 0.
\]
(8)
Let \( \pi : \hat{S} \to S \) be the composition of the normalisation and the minimal resolution (cf. Subsection 3.A), then (3) and (8) imply that
\[
(K_{\hat{S}} + \pi^*\omega|_S) \cdot \pi^*\alpha|_S < 0.
\]
(9)
Since the surface $\hat{S}$ is projective, the nef class $\pi^*\alpha|_S$ is represented by an $\mathbb{R}$-divisor. The extremal ray $R$ contains only the classes of finitely many curves, so $\pi^*\alpha$ is strictly positive on every movable curve in $\hat{S}$.

Fix an ample divisor $A$ on $\hat{S}$. By [Ara10, Thm.1.3] for every $\varepsilon > 0$ we have a decomposition

$$\pi^*\alpha|_S = C_\varepsilon + \sum \lambda_{i,\varepsilon} M_{i,\varepsilon}$$

where $\lambda_{i,\varepsilon} \geq 0$, the $M_{i,\varepsilon}$ are movable curves and $(K_{\hat{S}} + \varepsilon A) \cdot C_\varepsilon \geq 0$. The class $\pi^*\alpha|_S$ is strictly positive on every movable curve in $\hat{S}$, so we have $\pi^*\alpha|_S \cdot M_{i,\varepsilon} > 0$. Since $(\pi^*\alpha|_S)^2 = 0$ and $\pi^*\alpha|_S \cdot M_{i,\varepsilon} > 0$ we must have $\pi^*\alpha|_S = C_\varepsilon$ for all $\varepsilon > 0$. Passing to the limit we obtain $K_{\hat{S}} \cdot \pi^*\alpha|_S \geq 0$, a contradiction to (9). □

3.24. Theorem. Suppose that the extremal ray $R$ is small. Then $C$ is contractible.

Proof. Let $\alpha \in N^1(X)$ be the nef class supporting $R$ as in Proposition 3.18. We claim that the class $\alpha$ is big, i.e., if $\pi : X' \to X$ is a desingularisation then the pull-back $\pi^*\alpha$ is a big cohomology class. Once we have shown this property, the proof of [HP13, Thm.7.12] applies.

Proof of the Claim. By definition of the class $\alpha$, the class $-(K_X + \omega) + \alpha$ is positive on the extremal ray $R$. Since $\alpha$ is strictly positive on

$$\left( \overline{NA(X)(K_X + \omega)} \geq 0 + \sum_{i \in I, \delta \neq i_0} \mathbb{R}^+ [\Gamma_i] \right) \setminus \{0\},$$

we may suppose, up to replacing $\alpha$ by some positive multiple, that $-(K_X + \omega) + \alpha$ is strictly positive on this cone. In total, $-(K_X + \omega) + \alpha$ is strictly positive on $\overline{NA(X)} \setminus \{0\}$. Thus $-(K_X + \omega) + \alpha$ is a Kähler class by [HP13, Cor.3.16], i.e., we may write

$$\alpha = (K_X + \omega) + \eta,$$

where $\eta$ is a Kähler class. We know by Lemma 3.3 that $K_X + \omega$ is pseudoeffective. Thus $\pi^*\alpha$ is the sum of the pseudoeffective class $\pi^*(K_X + \omega)$ and the nef and big class $\pi^*\eta$, hence it is big. □
Proof of Theorem 3.15. The existence of a morphism $\varphi : X \to Y$ contracting exactly the curves in the extremal ray is established in Lemma 3.21 and in Theorem 3.24. Since $\omega$ is nef, the extremal ray $\mathbb{R}^+ [\Gamma_i]$ is $K_X$-negative. Therefore, applying [HP13, Cor.8.2], it follows that $Y$ is a Kähler space. □

3.E. Proof of Theorem 1.3

Proof. Step 1 Running the MMP. If $K_X + \omega$ is nef for every normalised Kähler class $\omega$, we are finished. Suppose therefore that $K_X + \omega$ is not nef. Then there exists by Theorem 3.6 a $(K_X + \omega)$-negative extremal ray $R$ in $\text{NA}(X)$. By Theorem 3.15 the contraction $\varphi : X \to Y$ of $R$ exists in the Kähler category. Note that since $\omega$ is nef, the canonical class $K_X$ is negative on the extremal ray $R$.

If $R$ is divisorial we can continue the MMP with $Y$ by [HP13, Prop.8.1.c)]. If $R$ is small, we know by Mori’s flip theorem [Mor88, Thm.0.4.1] that the flip $\varphi^+ : X^+ \to Y$ exists, and by [HP13, Prop.8.1.d)] we can continue the MMP with $X^+$ (which is again Kähler).

Step 2 Termination of the MMP. Recall that for a normal compact threefold $X$ with at most terminal singularities, the difficulty $d(X)$ [Sho85] is defined by

$$d(X) := \#\{i \mid a_i < 1\},$$

where $K_Y = \mu^* K_X + \sum a_i E_i$ and $\mu : Y \to X$ is any resolution of singularities. Recall that any contraction in our MMP is a $K_X$-negative contraction, so by [KMM87, Lemma 5.1.16]4, [Sho85] we have $d(X) > d(X^+)$, if $X^+$ is the flip of a small contraction. Since the Picard number and the difficulty are non-negative integers, any MMP terminates after finitely many steps. □

4. The Base-point Free Theorem

We first prove Theorem 1.4, which is the analogue of the base point free theorem in the non-algebraic case.

4The proof is local in a neighbourhood of the flipping locus, so it holds without change in the analytic setting.
4.A. Proof of Theorem 1.4

Proof. We will use the nef reduction of $X$ with respect to the cohomology class $K_X + \omega$, cf. Theorem 3.19. We denote by $n(K_X + \omega)$ the dimension of the base of the nef reduction of $K_X + \omega$ and claim that

$$n(K_X + \omega) = 2.$$ 

Notice first that the general fibres of the MRC-fibration provide a dominating family of curves which is $K_X + \omega$-trivial, so $n(K_X + \omega) \leq 2$.

If $n(K_X + \omega) = 1$ the nef reduction is a holomorphic fibration $X \to C$ (cf. [BCE+02, 2.4.3]) and $K_X + \omega$ is numerically trivial on the general fibre by Theorem 3.19. In particular the general fiber is a smooth Fano surface, hence rationally connected, a contradiction to our assumption on the base of the MRC-fibration.

If $n(K_X + \omega) = 0$, then $K_X + \omega \equiv 0$, hence $X$ is Fano and rationally connected, again a contradiction.

Let $Z$ be a resolution of singularities of the unique irreducible component of $\text{Chow}(X)$ such that the general point corresponds to the general fibre of the MRC-fibration. Let $\Gamma$ be the normalisation of the pull-back of the universal family and denote by $p : \Gamma \to X$ and $q : \Gamma \to Z$ the natural morphisms. Since $\Gamma$ is in Fujiki’s class $\mathcal{C}$, the surface $Z$ is in the class $\mathcal{C}$ by [Var86, Thm. 3]. A smooth surface in the class $\mathcal{C}$ is Kähler, so $Z$ is Kähler.

We claim that there exists a big and nef class $\alpha$ on $Z$ such that $p^*(K_X + \omega) = q^*\alpha$.

Step 1 Construction of the class $\alpha$. Set $\Gamma_z = q^{-1}(z)$ for $z \in Z$. Note first that we have $R^1q_*(\mathcal{O}_\Gamma) = 0$ (the morphism $q$ is projective, so we can apply [Kol96, II, 2.8.6.2]). Using the exponential sequence this implies $R^1q_*(\mathcal{Z}) = 0$ and hence $R^1q_*(\mathbb{R}) = 0$ by the universal coefficient theorem. Now we apply the Leray spectral sequence for $q$ and the sheaf $\mathbb{R}$. By what precedes we have

$$E_2^{0,1} = H^0(Z, R^1q_*(\mathbb{R})) = 0$$

and

$$E_2^{1,1} = H^1(Z, R^1q_*(\mathbb{R})) = 0.$$
Therefore $E_2^{2,0} = H^2(Z, \mathbb{R})$ embeds into $H^2(\Gamma, \mathbb{R})$, and it suffices to show that the section $s \in E_2^{0,2} = H^0(Z, R^2q_* (\mathbb{R}))$ which is given by the class 

$$s(z) = [p^*(K_X + \omega)|\Gamma_z] \in H^2(\Gamma_z, \mathbb{R}),$$

vanishes for every $z \in Z$. By definition of a normalised Kähler class we have $s(z) = 0$ for $z \in Z$ general. Since $p^*(K_X + \omega)$ is nef, this implies that the class $p^*(K_X + \omega)$ is zero on all the irreducible components of any fibre $\Gamma_z$. Thus we have $s(z) = 0$ for all $z \in Z$ proving the existence of $\alpha$. Note that since $q^*\alpha$ is nef, the class $\alpha$ is nef [Pău98, Thm.1].

Step 2 Intersection numbers. Let $D \subset \Gamma$ be an irreducible component (possibly of dimension 1) of the $p$-exceptional locus. Since $p$ is finite on the fibres of $q$, there exists a curve $C \subset D$ that is contracted by $p$ and such that $q(C)$ is not a point. In particular we have

$$\alpha \cdot q(C) = q^*\alpha \cdot C = p^*(K_X + \omega) \cdot C = 0.$$

Since the meromorphic map $X \rightarrow Z$ is almost holomorphic, $D$ does not surject onto $Z$. Thus we have $q(D) = q(C)$, and by what precedes we obtain

$$(q^*\alpha)|_D = 0.$$

Note now that, $\Gamma$ being a modification of a threefold which has a finite singular locus, the singular locus of $\Gamma$ is a union of curves which are contained in the $p$-exceptional locus and finitely many points. Let $\mu : \hat{X} \rightarrow \Gamma$ be a desingularisation such the exceptional set of $\hat{p} := p \circ \mu$ has pure codimension one. Set moreover $\hat{q} := q \circ \mu$. By what precedes,

$$\hat{q}^*\alpha \cdot \hat{D} = 0 \text{ in } N_1(\hat{X})$$

for every irreducible component $\hat{D}$ of the $\hat{p}$-exceptional locus.

Step 3 The class $\alpha$ is big, i.e. we have $\alpha^2 > 0$. Since $\omega$ is a Kähler class, we know that, up to replacing $\hat{X}$ by some further blowup, there exists an effective $\mathbb{Q}$-divisor $F$ with support in the $\hat{p}$-exceptional locus such that

$$\hat{p}^*\omega - F$$
is a Kähler class. Being a Kähler class is an open property, so there exists a Kähler class \( \eta_Z \) on \( Z \) such that

\[ \hat{p}^* \omega - F - \hat{q}^* \eta_Z \]

is a Kähler class. Using Păun’s theorem [Pău12, Thm.1.1] as in the proof of Lemma 3.3, we conclude that

\[ K_{\hat{X}/Z} + \hat{p}^* \omega - F - \hat{q}^* \eta_Z \]

is pseudoeffective. Since \( X \) has terminal singularities,

\[ K_{\hat{X}} = \hat{p}^* K_X + E \]

with \( E \) an effective \( \mathbb{Q} \)-divisor supported on the \( \hat{p} \)-exceptional locus. Consider now the decomposition

\[ \hat{p}^*(K_X + \omega) = [K_{\hat{X}/Z} + \hat{p}^* \omega - F - \hat{q}^* \eta_Z] - E + F + \hat{q}^* K_Z + \hat{q}^* \eta_Z. \]

We are going to intersect this equation with \( \hat{q}^*(\alpha) \) in order to compute

\[ \hat{q}^* \alpha^2 = \hat{q}^* \alpha \cdot \hat{p}^*(K_X + \omega). \]

Since \( \alpha \) is nef, the intersection product

\[ \hat{q}^* \alpha \cdot [K_{\hat{X}/Z} + \hat{p}^* \omega - F - \hat{q}^* \eta_Z] \]

is an element of \( \overline{\text{NA}}(\hat{X}) \). By (10) we have \( \hat{q}^* \alpha \cdot (-E + F) = 0 \). The surface \( Z \) is not uniruled since it is the base of the MRC-fibration (cf. Remark 3.2). Thus \( K_Z \) is pseudoeffective, in particular the intersection product \( \hat{q}^* \alpha \cdot \hat{q}^* K_Z \) is an element of \( \overline{\text{NA}}(\hat{X}) \). Recall now that \( \alpha \neq 0 \) since \( K_X + \omega \neq 0 \). Since \( \eta_Z \) is a Kähler class and \( \alpha \) is a non-zero nef class, the Hodge index theorem yields \( \eta_Z \cdot \alpha > 0 \). Thus

\[ q^* \alpha \cdot q^* \eta_Z \]

is a non-zero element of \( \overline{\text{NA}}(\hat{X}) \). In total we obtain that

\[ q^* \alpha^2 = q^* \alpha \cdot \hat{p}^*(K_X + \omega) \]

is a non-zero element of \( \overline{\text{NA}}(\hat{X}) \). Thus we have \( \alpha^2 \neq 0 \).
Step 4 Construction of the fibration $\varphi$. Let

$$E := \cup E_j \subset Z$$

be the union of curves $E_j \subset Z$ such that $\alpha \cdot E_j = 0$. Since $\alpha$ is nef and big, the Hodge index theorem implies that the intersection form on $E$ is negative definite. In particular $E$ is a finite set. By Grauert’s criterion there exists a bimeromorphic morphism $\nu : Z \to S$ such that $E$ equals the $\nu$-exceptional locus. Since $Z$ is a Kähler surface and $\nu$ contracts only subvarieties onto points, the surface $S$ is Kähler. In fact, take any Kähler form $\omega$ on $Z$. Then the class of the Kähler current $\nu_* (\omega)$ contains a Kähler form by [DP04, Prop.3.3(iii)].

We claim that the fibration $\nu \circ q : \Gamma \to S$ factors through the bimeromorphic map $p$, i.e., there exists a holomorphic fibration $\varphi : X \to S$ such that $\nu \circ q = \varphi \circ p$. By the rigidity lemma [BS95, Lemma 4.1.13] it is sufficient to prove that every $p$-fibre is contracted by $\nu \circ q$. Since $p$ is a Moishezon morphism, it moreover suffices to show that every curve $C \subset \Gamma$ such that $p(C)$ is a point is contracted by $\nu \circ q$. Yet for such a curve $C$ we have

$$q^* \alpha \cdot C = p^*(K_X + \omega) \cdot C = 0.$$ 

It follows that $q(C) \subset E$, hence $q(C)$ is a point. This shows the existence of the fibration $\varphi$; by construction the class $K_X + \omega$ is $\varphi$-trivial. □

4.B. MMP for uniruled Kähler threefolds

Recall that in our context a normal Kähler space $X$ is $\mathbb{Q}$-factorial if every Weil divisor $D \subset X$ is $\mathbb{Q}$-Cartier and some reflexive power $\omega^m_X$ of the dualising sheaf $\omega_X$ is locally free.

4.1. Lemma. Let $X$ be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Let $\varphi : X \to S$ be an elementary Mori contraction onto a normal compact surface, i.e., $\rho(X/S) = 1$ and $-K_X$ is $\varphi$-ample.

Then $S$ is $\mathbb{Q}$-factorial and has at most klt singularities.

4.2. Remark. In the situation above, the fibration $\varphi$ is equidimensional since an elementary contraction of fibre type does not contract a divisor. For a point $s \in S$ denote by $X_s$ the fibre over $s$, and let $A \subset S$
be the set of all $s$ such that the fiber $X_s$ is singular at some point $x_0$ and such that $X$ is not smooth at $x_0$. Then $A$ is finite, set $S_0 = S \setminus A$ and $X_0 = X \setminus \varphi^{-1}(A)$. The fiber space $f_0 : X_0 \to S_0$ is a conic bundle. The sheaf $f_*(\omega_{X/S})$ is reflexive, but might have singularities on $A$, so that $f$ might globally not be a conic bundle. However, $H^1(X_s, O_{X_s}) = 0$, in particular, every irreducible component of any fiber $X_s$ is a smooth rational curve.

**Proof of Lemma 4.1.** Arguing as in [KMM87, 5-1-5], every Weil divisor $D \subset S$ is $\mathbb{Q}$-Cartier.

In order to see that $S$ has at most klt singularities we proceed as in the algebraic case. The claim is local on the base $S$, so given a point $0 \in S_{\text{sing}}$ we fix a small analytic neighbourhood $0 \in U \subset S$. Since $X$ is smooth in codimension two and, by Remark 4.2 the projective morphism $\varphi$ is a conic bundle in the complement of the fibre $X_0$, there exists a smooth analytic subvariety $H \subset \varphi^{-1}(U)$ such that $H \to U$ is finite and étale in codimension one. By [KM98, Prop.5.20] the surface $U$ has at most klt singularities. In particular some reflexive power $\omega_X^{[m]}$ of the dualising sheaf $\omega_X$ is locally free. □

**Proof of Theorem 1.1.** By Theorem 1.3, there exists a MMP $X \dasharrow X'$ such that $K_{X'} + \omega'$ is nef for all normalised Kähler classes $\omega'$ on $X'$. Fix such a Kähler class $\omega'$. Then apply the base point free theorem 1.4 to the variety $X'$ to obtain a fibration $\varphi : X' \to S'$ onto a surface $S'$ such that $-K_{X'}$ is $\varphi$-ample. In particular $\varphi$ is a projective morphism. Thus we can run the MMP of $X'$ over $S'$ using the relative version of the cone and contraction theorem as in [Nak87, Sect.4], [KM98, Sect.3.6]. As in the proof of Theorem 1.3 we can use the Picard number $\rho(X')$ and the difficulty $d(X')$ to show that the MMP terminates. Since $K_{X'}$ is not pseudoeffective over $S'$, the outcome of the MMP

$$X' \dasharrow X''$$

is a Mori fibre space $X'' \to S''$ over $S'$, with $S''$ a normal compact complex surface that dominates $S'$. Since $S'$ is Kähler, and the bimeromorphic morphism $S'' \to S'$ is projective (we can always find an anti-effective exceptional divisor that is relatively ample), the surface $S''$ is Kähler. The properties of $S''$ are proven in Lemma 4.1. □
References


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