J. Math. Sci. Univ. Tokyo
22 (2015), 219–246.

Mori Fibre Spaces for Kähler Threefolds

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In memory of Professor Kunihiko Kodaira

Abstract. Let X be a compact Kähler threefold such that the base of the MRC-fibration has dimension two. We prove that X is bimeromorphic to a Mori fibre space. Together with our earlier result [HP13] this completes the MMP for compact Kähler threefolds: let X be a non-projective compact Kähler threefold. Then X has a minimal model or X is bimeromorphic to a Mori fibre space over a non-projective Kähler surface.

1. Introduction

This paper continues our study of the minimal model program (MMP) for compact Kähler threefolds. In [HP13] we established the existence of minimal models for compact Kähler threefolds such that K_X is pseudoeffective. More precisely, minimal models are obtained, as in the projective setting, by a sequence of contractions of extremal rays (in a suitable cone) and flips. By a theorem of Brunella [Bru06] a smooth compact Kähler threefold has pseudoeffective K_X if and only if X is not uniruled. In the present work we deal with the remaining case where X is uniruled. The general fibre of the MRC-fibration $X \dashrightarrow Z$ is rationally connected, so carries no holomorphic forms [Deb01, Cor.4.18]. Thus if the base Z has dimension at most one, then we obtain $H^2(X, \mathcal{O}_X) = H^0(X, \Omega^2_X) = 0$. In particular the Kähler manifold X is projective by Kodaira's criterion. Since our main interest is the study of non-projective Kähler threefolds, we focus on the case where Z has dimension two:

1.1. Theorem. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dashrightarrow Z$ has dimension two.*

2010 *Mathematics Subject Classification.* 32J27, 14E30, 14J30, 32J17, 32J25.
Key words: MMP, rational curves, Zariski decomposition, Kähler manifolds.

Then X is bimeromorphic to a Mori fibre space, i.e. there exists a MMP

$$X \dashrightarrow X',$$

consisting of contractions of extremal rays and flips, such that X' admits a fibration $\varphi : X' \to S$ onto a normal compact $\mathbb{Q}$−factorial Kähler surface with at most klt singularities such that $-K_{X'}$ is φ-ample and $\rho(X'/S) = 1$.

It will be important to work with a special type of Kähler classes:

1.2. Definition. Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dashrightarrow Z$ has dimension two, and let $F \simeq \mathbb{P}^1$ be a general fibre. A Kähler class ω on X is normalised if $\omega \cdot F = 2$.

Since the canonical class K_X has degree -2 on F, the adjoint class $K_X + \omega$ is trivial on F. Using a recent result of Păun [Pău12] we first prove that $K_X + \omega$ is pseudoeffective. The proof of Theorem 1.1 then proceeds in two steps, the first being the existence of a MMP for the adjoint class $K_X + \omega$:

1.3. Theorem. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dashrightarrow Z$ has dimension two. Then there exists a MMP*

$$X \dashrightarrow X'$$

such that for every normalised Kähler class ω' on X' the adjoint class $K_{X'} + \omega'$ is nef.

Once we have a normalised Kähler class ω such that $K_X + \omega$ is nef, the adjoint class $K_X + \omega$ is a natural candidate for the "nef supporting class" that defines a Mori fibre space structure.

The second step is to prove an analogue of the base-point free theorem for the adjoint class $K_X + \omega$.

1.4. Theorem. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dashrightarrow Z$ has dimension two. Let ω be a normalised Kähler class on X such that $K_X + \omega$ is nef.*

Then there exists a holomorphic *fibration* $\varphi : X \to S$ *onto a normal compact Kähler surface* S *such that* $K_X + \omega$ *is* φ*-trivial.*

By construction, the anticanonical class $-K_X$ is ample with respect to the fibration $X \to S$, so we can use the cone and contraction theorem for projective morphisms ([Nak87], [KM98]) to run a relative MMP. This MMP terminates with the Mori fibre space we are looking for.

In the situation of Theorem 1.4 one can prove that S is $\mathbb{Q}$-factorial with at most rational singularities, but it is not quite clear whether S is klt. However we can prove this property for an elementary contraction of fibre type, cf. Lemma 4.1.

Acknowledgements. We thank the Forschergruppe 790 "Classification of algebraic surfaces and compact complex manifolds" of the Deutsche Forschungsgemeinschaft for financial support. A. Höring was partially also supported by the A.N.R. project CLASS[1].

2. Notation

We use the same notation as in [HP13]. For the convenience of the reader we recall the most important definitions and basic results.

2.1. Definition. An irreducible and reduced complex space X is Kähler if there exists a Kähler form ω, i.e. a positive closed real $(1,1)$-form $\omega \in \mathcal{A}^{1,1}_{\mathbb{R}}(X)$, such that the following holds: for every point $x \in X_{\text{sing}}$ there exists an open neighbourhood $x \in U \subset X$ and a closed embedding $i_U : U \subset V$ into an open set $V \subset \mathbb{C}^N$, and a strictly plurisubharmonic C^∞-function $f : V \to \mathbb{C}$ with $\omega|_{U \cap X_{\text{nons}}} = (i\partial\overline{\partial} f)|_{U \cap X_{\text{nons}}}$.

In the same manner one can define $(p,q)-$forms on an irreducible reduced complex space [Dem85], by duality we obtain the usual notions of currents.

We will next define the appropriate analogue of the Néron-Severi space $N^1(X)$ for a normal compact Kähler space, as well as the cones $\overline{\text{NE}}(X)$ and $\overline{\text{NA}}(X)$ contained in its dual $N_1(X)$. For any details we refer to [HP13, Sect.3].

[1]ANR-10-JCJC-0111.

2.2. Definition. [BPEG13, Defn. 4.6.2] [HP13, Defn.3.6] Let X be an irreducible reduced complex space. Let $\mathcal{H}_X$ be the sheaf of real parts of holomorphic functions multiplied with i. A $(1,1)$-form with local potentials on X is a global section of the quotient sheaf $\mathcal{A}^0_X/\mathcal{H}_X$. We define the Bott-Chern cohomology

$$N^1(X) := H^1(X, \mathcal{H}_X).$$

2.3. Remark. Using the exact sequence

$$0 \to \mathcal{H}_X \to \mathcal{A}^0_X \to \mathcal{A}^0_X/\mathcal{H}_X \to 0,$$

and the fact that $\mathcal{A}^0_X$ is acyclic, we obtain a surjective map

$$H^0(X, \mathcal{A}^0_X/\mathcal{H}_X) \to H^1(X, \mathcal{H}_X).$$

Thus we can see an element of the Bott-Chern cohomology group as a closed $(1,1)$-form with local potentials modulo all the forms that are globally of the form $dd^c u$.

Let $\mathcal{D}_X$ be the sheaf of distributions. Using the exact sequence

$$0 \to \mathcal{H}_X \to \mathcal{D}_X \to \mathcal{D}_X/\mathcal{H}_X \to 0,$$

we see that one obtains the same Bott-Chern group, considering $(1,1)-$currents T with local potentials, which is to say that locally $T = dd^c u$ with u a distribution.

Dually we define

2.4. Definition. Let X be a normal compact complex space. Then $N_1(X)$ is the vector space of real closed currents of bidimension $(1,1)$ modulo the following equivalence relation: $T_1 \sim T_2$ if and only if

$$T_1(\eta) = T_2(\eta)$$

for all real closed $(1,1)$-forms η.

In [HP13, Prop.3.9] we established a canonical isomorphism

$$\Phi : N^1(X) \to N_1(X)^* \tag{1}$$

for any normal compact complex space X in the Fujiki class $\mathcal{C}$, i.e., for those X which are bimeromorphic to a Kähler space.

2.5. Definition. Let X be a normal compact complex space in class $\mathcal{C}$. We define $\overline{NA}(X) \subset N_1(X)$ as the cone generated by the positive closed currents of bidimension $(1,1)$.

Given an irreducible curve $C \subset X$, we associate to C the current of integration T_C. In the case of isolated singularities, which is the only case relevant in our setting, we define

$$T_C(\omega) = \int_C \omega = \int_{\hat{C}} \pi^*(\omega),$$

where $\pi : \hat{X} \to X$ is a resolution of singularities, the curve $\hat{C}$ is the strict transform of C, and ω a d-closed $(1,1)$-form on X. We define the Mori cone $\overline{NE}(X) \subset N_1(X)$ as the closure of the cone generated by the currents T_C and clearly have an inclusion

$$\overline{NE}(X) \subset \overline{NA}(X).$$

2.6. Definition. Let X be an irreducible reduced compact complex space in class $\mathcal{C}$. We denote by $\mathrm{Nef}(X) \subset N^1(X)$ the cone generated by cohomology classes which are nef in the sense of [Pău98, Defn.3]: let $u \in N^1(X)$ be a class represented by a form α with local potentials. Then u is nef if for some positive $(1,1)-$form ω on X and for every $\epsilon > 0$ there exists $f_\epsilon \in \mathcal{A}^0(X)$ such that

$$\alpha + i\partial\overline{\partial} f_\epsilon \geq -\epsilon\omega.$$

The class u is pseudo-effective, if it can be represented by a current T which is locally of the form $T = \partial\overline{\partial}\varphi$ with φ a plurisubharmonic function.

If X is a normal compact Kähler space, we can also consider the open cone $\mathcal{K}$ generated by the classes of Kähler forms. In this case we know by [Dem92, Prop.6.1][2] that

$$\mathrm{Nef}(X) = \overline{\mathcal{K}}.$$

[2]The statement in [Dem92, Prop.6.1.iii)] is for compact manifolds, but the proof works in the singular setting, cf. also [HP13, Rem.3.5] for regularisation arguments in the singular setting.

As in the projective setting, we have a duality statement:

2.7. Proposition. [HP13, Prop.3.15] *Let X be a normal compact threefold in class $\mathcal{C}$. Then the cones* $\mathrm{Nef}(X)$ *and* $\overline{NA}(X)$ *are dual via the canonical isomorphism* $\Phi : N^1(X) \to N_1(X)^*$ *given by (1).*

Finally we define the notion of the contraction of an extremal ray R. It is very important to consider extremal rays in the dual Kähler cone $\overline{\mathrm{NA}}(X)$ rather than in the Mori cone $\overline{\mathrm{NE}}(X)$.

2.8. Definition. Let X be a normal $\mathbb{Q}$-factorial compact Kähler space with at most terminal singularities, and let ω be a Kähler class on X. Let R be a $(K_X + \omega)$-negative extremal ray in $\overline{\mathrm{NA}}(X)$. A contraction of the extremal ray R is a morphism $\varphi : X \to Y$ onto a normal compact Kähler space Y, such that $-(K_X + \omega)$ is a Kähler class on every fibre and a curve $C \subset X$ is contracted if and only if $[C] \in R$.

3. MMP for the Adjoint Class

In order to simplify the statements we will work under the following

3.1. Assumption. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Suppose that the base of the MRC-fibration $X \dashrightarrow Z$ has dimension two, and let ω be a normalised Kähler class on X.*

3.2. Remark. Observe that the surface Z is not uniruled: since this is a bimeromorphic statement we can suppose that X and Z are smooth and the MRC-fibration is a morphism $\varphi : X \to Z$. If $(C_t)_{t \in T} \subset Z$ is a dominant family of rational curves, the surface $\varphi^{-1}(C_t)$ is uniruled by the fibres of $\varphi^{-1}(C_t) \to C_t$. Thus it carries no holomorphic 2-form, in particular $\varphi^{-1}(C_t)$ is projective by Kodaira' s criterion. Thus Tsen's theorem applies and we obtain that $\varphi^{-1}(C_t)$ is rationally connected. Now we conclude as in the algebraic case that φ is not the MRC-fibration. The same line of arguments also shows that the theorem of Graber-Harris-Starr [GHS03] is also true in Kähler category.

3.A. Remarks on adjunction

Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Let $S \subset X$ be a prime divisor, i.e. an irreducible and reduced compact surface. Let $m \in \mathbb{N}$ be the smallest positive integer such that both mK_X and mS are Cartier divisors on X. Then the canonical class $K_S \in \mathrm{Pic}(S) \otimes \mathbb{Q}$ is defined by

$$K_S := \frac{1}{m}(mK_X + mS)|_S.$$

Since X is smooth in codimension two, there exist at most finitely many points $\{p_1, \dots p_q\}$ where K_X and S are not Cartier. Thus by the adjunction formula K_S is isomorphic to the dualising sheaf ω_S on $S \setminus \{p_1, \dots p_q\}$.

Let now $\nu : \tilde{S} \to S$ be the normalisation. Then we have

$$K_{\tilde{S}} \sim_{\mathbb{Q}} \nu^* K_S - N, \tag{2}$$

where N is an effective Weil divisor defined by the conductor ideal. Indeed this formula holds by [Rei94] for the dualising sheaves. Since $\mathcal{O}_{\tilde{S}}(\nu^* K_S)$ is isomorphic to $\nu^* \omega_S$ on the complement of $\nu^{-1}(p_1, \dots p_q)$, the formula holds for the canonical classes.

Let $\mu : \hat{S} \to \tilde{S}$ be the minimal resolution of the normal surface $\tilde{S}$, then we have

$$K_{\hat{S}} \sim_{\mathbb{Q}} \mu^* K_{\tilde{S}} - N',$$

where N' is an effective μ-exceptional $\mathbb{Q}$-divisor [Sak84, 4.1]. Thus if $\pi : \hat{S} \to S$ is the composition $\nu \circ \mu$, there exists an effective, canonically defined $\mathbb{Q}$-divisor $E \subset \hat{S}$ such that

$$K_{\hat{S}} \sim_{\mathbb{Q}} \pi^* K_S - E. \tag{3}$$

Let $C \subset S$ be a curve such that $C \not\subset S_{\mathrm{sing}}$. Then the morphism π is an isomorphism at the general point of C, and we can define the strict transform $\hat{C} \subset \hat{S}$ as the closure of $C \setminus S_{\mathrm{sing}}$. Since $\hat{C}$ is an (irreducible) curve that is not contained in the divisor N defined by the conductor, we have $\hat{C} \not\subset E$. By the projection formula and (3) we obtain

$$K_{\hat{S}} \cdot \hat{C} \leq K_S \cdot C. \tag{4}$$

3.B. Divisorial Zariski decomposition for $K_X + \omega$

The starting point of our investigation is the following observation:

3.3. Lemma. *Under the Assumption 3.1 the adjoint class $K_X + \omega$ is pseudoeffective.*

Proof. Being pseudoeffective is a closed property in $N^1(X)$, so it is sufficient to prove that for every $\varepsilon > 0$, the class $K_X + (1+\varepsilon)\omega$ is pseudoeffective. Let $\mu : X' \to X$ be a bimeromorphic morphism from a smooth Kähler threefold X' such that the MRC-fibration is a morphism $\varphi' : X' \to Z'$ onto a smooth surface Z'. The projection formula yields

$$\mu_* \left(K_{X'} + (1+\varepsilon)\mu^*\omega\right) = K_X + (1+\varepsilon)\omega,$$

so it is sufficient to prove that $K_{X'} + (1+\varepsilon)\mu^*\omega$ is pseudoeffective. However by a recent result of Păun [Pău12, Thm.1.1], the class $K_{X'/Z'} + (1+\varepsilon)\mu^*\omega$ is pseudoeffective. Since the surface Z' is not uniruled (cf. Remark 3.2) and Kähler by [Var86, Thm.3], the canonical class $K_{Z'}$ is pseudoeffective. Thus $K_{X'} + (1+\varepsilon)\mu^*\omega$ is pseudoeffective. □

Since $K_X + \omega$ is pseudoeffective, we may apply [Bou04, Thm.3.12] to obtain a divisorial Zariski decomposition[3]

$$K_X + \omega = \sum_{j=1}^{r} \lambda_j S_j + P_\omega, \tag{5}$$

where the S_j are integral surfaces in X, the coefficients $\lambda_j \in \mathbb{R}^+$ and P_ω is a pseudoeffective class which is nef in codimension one [Bou04, Prop.2.4], that is for every surface $S \subset X$ the restriction $P_\omega|_S$ is pseudoeffective.

3.4. Lemma. *Under the Assumption 3.1, let S be a surface such that $(K_X + \omega)|_S$ is not pseudoeffective. Then S is one of the surfaces S_j in*

[3] The statements in [Bou04] are for complex compact manifolds, but generalise immediately to our situation: take $\mu : X' \to X$ a desingularisation, and let $m \in \mathbb{N}$ be the Cartier index of K_X. Then $\mu^*(m(K_X + \omega))$ is a pseudoeffective class with divisorial Zariski decomposition $\mu^*(m(K_X + \omega)) = \sum \eta_j S'_j + P'_\omega$. The decomposition of $K_X + \omega$ is defined by the push-forwards $\mu_*(\frac{1}{m} \sum \eta_j S'_j)$ and $\mu_*(\frac{1}{m} P'_\omega)$. Since a prime divisor $D \subset X$ is not contained in the singular locus of X, the decomposition has the stated properties.

the divisorial Zariski decomposition (5) *of* $K_X + \omega$. *Moreover* $S = S_j$ *is Moishezon and any desingularisation* $\hat{S}_j$ *is a uniruled projective surface.*

PROOF. The proof that $S = S_j$ for some j is analogous to the proof in [HP13, Lemma 4.1], thus (up to renumbering) we may suppose that $S = S_1$. We have

$$S = S_1 = \frac{1}{\lambda_1}(K_X + \omega) - \frac{1}{\lambda_1}(\sum_{j=2}^{r} \lambda_j S_j + P_\omega),$$

so by adjunction

$$K_S = (K_X + S)|_S = (\frac{\lambda_1 + 1}{\lambda_1} K_X|_S + \frac{1}{\lambda_1}\omega|_S) - \frac{1}{\lambda_1}(\sum_{j=2}^{r} \lambda_j (S_j \cap S) + P_\omega|_S).$$

Note now that $\frac{\lambda_1+1}{\lambda_1} K_X|_S + \frac{1}{\lambda_1}\omega|_S$ is not pseudoeffective: otherwise

$$\left(\frac{\lambda_1 + 1}{\lambda_1} K_X|_S + \frac{1}{\lambda_1}\omega|_S\right) + \omega|_S = \frac{\lambda_1 + 1}{\lambda_1}(K_X + \omega)|_S$$

would be pseudoeffective, in contradiction to our assumption. Since

$$\frac{1}{\lambda_1}(\sum_{j=2}^{r} \lambda_j (S_j \cap S) + P_\omega|_S)$$

is pseudoeffective, the class K_S cannot be pseudoeffective.

Let now $\pi : \hat{S} \to S$ be the composition of the normalisation and the minimal resolution of the surface S, then by (3) there exists an effective divisor E such that

$$K_{\hat{S}} \sim_{\mathbb{Q}} \pi^* K_S - E.$$

Thus $K_{\hat{S}}$ is not pseudoeffective, in particular $\kappa(\hat{S}) = -\infty$. It follows from the Castelnuovo-Kodaira classification that $\hat{S}$ is covered by rational curves, in particular $\hat{S}$ is a projective surface [BHPVdV04]. Thus S is Moishezon. □

3.5. COROLLARY. *Under the Assumption 3.1, the adjoint class* $K_X + \omega$ *is nef if and only if*

$$(K_X + \omega) \cdot C \geq 0$$

for every curve $C \subset X$.

PROOF. We prove the non-trivial implication by contradiction, so suppose that $K_X + \omega$ is not nef, but $(K_X + \omega) \cdot C \geq 0$ for all curves $C \subset X$. Since $K_X + \omega$ is pseudoeffective by Lemma 3.3 and the restriction to every curve is nef, there exists by [Pău98], [Bou04, Prop.3.4] an irreducible surface $S \subset X$ such that $(K_X + \omega)|_S$ is not pseudoeffective. Fix a desingularisation $\pi : \hat{S} \to S$ of the surface S. By Lemma 3.4 the surface $\hat{S}$ is projective and uniruled. The class $\pi^*(K_X + \omega)|_S$ is not pseudoeffective and, since $H^2(\hat{S}, \mathcal{O}_{\hat{S}}) = 0$, the class is represented by an $\mathbb{R}$-divisor. Thus there exists a covering family of curves $C_t \subset S$ such that

$$(K_X + \omega) \cdot C_t = \pi^*(K_X + \omega)|_S \cdot \hat{C}_t < 0,$$

where $\hat{C}_t$ denotes the strict transform of C_t in $\hat{S}$. This contradicts our assumption that $(K_X + \omega) \cdot C \geq 0$ for all curves $C \subset X$. □

3.C. The adjoint cone theorem

The goal of this subsection is to prove a cone theorem for the adjoint class $K_X + \omega$:

3.6. THEOREM. *Under the Assumption 3.1 there exists a countable family* $(\Gamma_i)_{i \in I}$ *of rational curves on* X *such that*

$$0 < -(K_X + \omega) \cdot \Gamma_i \leq 4$$

and

$$\overline{NA}(X) = \overline{NA}(X)_{(K_X+\omega)\geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i]$$

The proof of Theorem 3.6 is quite similar to the proof of [HP13, Thm.1.2.]; for sakes of completeness we explain the main steps:

3.7. LEMMA. *Under the Assumption 3.1, let* $C \subset X$ *be a curve such that* $(K_X + \omega) \cdot C < 0$ *and* $\dim_C \mathrm{Chow}(X) > 0$.

Then there exists a unique surface S_j *from the divisorial Zariski decomposition* (5) *such that* C *and its deformations are contained in the surface* S_j. *Moreover we have*

$$K_{S_j} \cdot C < K_X \cdot C. \tag{6}$$

PROOF. Identical to the proof of [HP13, Lemma 5.4], simply replace K_X by $K_X + \omega$. □

3.8. LEMMA. *Under the Assumption 3.1, let $S_1, \dots, S_r$ be the surfaces appearing in the divisorial Zariski decomposition* (5). *Set*

$$b := \max\{1, -(K_X+\omega)\cdot Z \mid Z \textit{ a curve s.t. } Z \subset S_{j,\text{sing}} \textit{ or } Z \subset S_j \cap S_{j'}, j \neq j'\}.$$

If $C \subset X$ is a curve such that

$$-(K_X + \omega) \cdot C > b,$$

then we have $\dim_C \operatorname{Chow}(X) > 0$.

In the proof we will use the following deformation property:

3.9. DEFINITION. [HP13, Defn.4.3] Let X be a normal $\mathbb{Q}$-factorial Kähler threefold with at most terminal singularities. We say that a curve C is very rigid if

$$\dim_{mC} \operatorname{Chow}(X) = 0$$

for all $m > 0$.

PROOF OF LEMMA 3.8. Since ω is nef, we have $-K_X \cdot C > b$. The condition $b \geq 1$ implies that the curve C is not very rigid (cf. [HP13, Thm.4.5]). We can now argue exactly as in [HP13, Lemma 5.6] to deduce

$$P_\omega \cdot C \geq 0.$$

Since $(K_X + \omega) \cdot C < 0$, the divisorial Zariski decomposition implies that there exists a number $j \in \{1, \dots, r\}$ such that $S_j \cdot C < 0$. In particular we have $C \subset S_j$. The class ω being nef, we thus obtain

$$K_{S_j} \cdot C < K_X \cdot C < -b.$$

By definition of b, the curve C is not contained in the singular locus of S_j. Let $\pi_j : \hat{S}_j \to S_j$ be the composition of normalisation and minimal resolution (cf. Subsection 3.A). Then the strict transform $\hat{C}$ of C is well-defined and from (4) we deduce

$$K_{\hat{S}_j} \cdot \hat{C} \leq K_{S_j} \cdot C < -b.$$

Since $b \geq 1$, [Kol96, Thm.1.15] yields

$$\dim_{\hat{C}} \mathrm{Chow}(\hat{S}) > 0,$$

so $\hat{C}$ deforms. Thus its push-forward $\pi_* \hat{C} = C$ deforms. □

3.10. Corollary. *Under the Assumption 3.1, let b be the constant from Lemma 3.8 and set*

$$d := \max\{3, b\}.$$

If $C \subset X$ is a curve such that $-(K_X + \omega) \cdot C > d$, we have

$$[C] = [C_1] + [C_2]$$

with C_1 and C_2 effective 1-cycles (with integer coefficients) on X.

Proof. Since ω is nef, we have $-K_X \cdot C > d$. Using the Lemmas 3.7 and 3.8, the proof of [HP13, Cor.5.7] applies without changes. □

3.11. Lemma. *Under the Assumption 3.1, let $\mathbb{R}^+[\Gamma_i]$ be a $(K_X + \omega)$-negative extremal ray in $\overline{NE}(X)$, where Γ_i is a curve that is not very rigid (cf. Definition 3.9). Then the following holds:*

a) There exists a curve $C \subset X$ such that $[C] \in \mathbb{R}^+[\Gamma_i]$ and $\dim_C \mathrm{Chow}(X) > 0$.

b) There exists a rational curve $C \subset X$ such that $[C] \in \mathbb{R}^+[\Gamma_i]$.

Proof. This is completely analogous to [HP13, Lemma 5.8] since the existence of the rational curve $C \subset X$ such that $[C] \in \mathbb{R}^+[\Gamma_i]$ is a consequence of [HP13, Lemma 5.5 a)] which contains no assumption on K_X. □

Following the strategy of [HP13, Thm.6.2] we first establish the cone theorem for the Mori cone.

3.12. Theorem. *Under the Assumption 3.1, there exists a number $d \in \mathbb{N}$ and a countable family $(\Gamma_i)_{i \in I}$ of curves on X such that*

$$0 < -(K_X + \omega) \cdot \Gamma_i \leq d$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\omega)\geq 0} + \sum_{i\in I} \mathbb{R}^+[\Gamma_i].$$

If the ray $\mathbb{R}^+[\Gamma_i]$ *is extremal in* $\overline{NE}(X)$*, there exists a rational curve* C_i *on* X *such that* $[C_i] \in \mathbb{R}^+[\Gamma_i]$*.*

Proof. Let $d \in \mathbb{N}$ be the bound from Corollary 3.10. There are only countably many curve classes $[C] \in \overline{\mathrm{NE}}(X)$, such that

$$0 < -(K_X + \omega) \cdot C \leq d.$$

We choose a representative Γ_i for each such class $[C]$ and set

$$V := \overline{\mathrm{NE}}(X)_{(K_X+\omega)\geq 0} + \sum_{0<-(K_X+\omega)\cdot\Gamma_i\leq d} \mathbb{R}^+[\Gamma_i].$$

Fix a Kähler class η on X such that $\eta \cdot C \geq 1$ for every curve $C \subset X$

Step 1 We have $\overline{\mathrm{NE}}(X) = V$. By [HP13, Lemma 6.1] it is sufficient to prove that $\overline{\mathrm{NE}}(X) = \overline{V}$, i.e. the class $[C]$ of every irreducible curve $C \subset X$ is contained in V. We will prove the statement by induction on the degree $l := \eta \cdot C$. The start of the induction for $l = 0$ is trivial. Suppose now that we have shown the statement for all curves of degree at most $l-1$ and let C be a curve such that $l-1 < \eta \cdot C \leq l$. If $-(K_X+\omega)\cdot C \leq d$ we have $[C] \in V$ by definition. Otherwise there exists by Corollary 3.10 a decomposition

$$[C] = [C_1] + [C_2]$$

with C_1 and C_2 effective 1-cycles (with integer coefficients) on X. Since $\eta \cdot C_i \geq 1$ for $i = 1, 2$ we have $\eta \cdot C_i \leq l-1$ for $i = 1, 2$. By induction both classes are in V, so $[C]$ is in V.

Step 2 Every extremal ray contains the class of a rational curve. If the ray $\mathbb{R}^+[\Gamma_i]$ is extremal in $\overline{\mathrm{NE}}(X)$, we know by [HP13, Thm.4.5] and Lemma 3.11 that there exists a rational curve C_i such that $[C_i]$ is in the extremal ray. □

We next pass from $\overline{\mathrm{NE}}(X)$ to $\overline{\mathrm{NA}}(X)$:

3.13. Theorem. *Under the Assumption 3.1 there exists a number* $d \in \mathbb{N}$ *and a countable family* $(\Gamma_i)_{i \in I}$ *of curves on* X *such that*

$$0 < -(K_X + \omega) \cdot \Gamma_i \leq d$$

and

$$\overline{NA}(X) = \overline{NA}(X)_{(K_X+\omega)\geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].$$

If the ray $\mathbb{R}^+[\Gamma_i]$ *is extremal in* $\overline{NA}(X)$, *there exists a rational curve* C_i *on* X *such that* $[C_i] \in \mathbb{R}^+[\Gamma_i]$.

Theorem 3.13 is a consequence of Theorem 3.12 and the following proposition.

3.14. Proposition. *Under the Assumption 3.1, suppose that there exists a* $d \in \mathbb{N}$ *and a countable family* $(\Gamma_i)_{i \in I}$ *of curves on* X *such that*

$$0 < -(K_X + \omega) \cdot \Gamma_i \leq d$$

and

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\omega)\geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].$$

Then we have

$$\overline{NA}(X) = \overline{NA}(X)_{(K_X+\omega)\geq 0} + \sum_{i \in I} \mathbb{R}^+[\Gamma_i].$$

Proof. Identical to the proof of [HP13, Prop.6.4]: simply replace K_X by $K_X + \omega$ and note that the uniruledness of a surface $S \subset X$ such that $(K_X + \omega)|_S$ is not pseudoeffective is proven in Lemma 3.4. □

Finally, Theorem 3.6 follows from Theorem 3.13 in the same way as [HP13, Thm.1.2] is deduced from [HP13, Thm.6.3].

3.D. The adjoint contraction theorem

In this subsection we prove the contraction theorem:

3.15. THEOREM. *Under the Assumption 3.1, let* $\mathbb{R}^+[\Gamma_i]$ *be a* $(K_X+\omega)$*-negative extremal ray in* $\overline{NA}(X)$*. Then the contraction of* $\mathbb{R}^+[\Gamma_i]$ *exists in the Kähler category.*

For the rest of this subsection we fix $R := \mathbb{R}^+[\Gamma_{i_0}]$ a $(K_X+\omega)$-negative extremal ray in $\overline{\mathrm{NA}}(X)$.

3.16. DEFINITION. We say that the $(K_X+\omega)$-negative extremal ray R is small if every curve $C \subset X$ with $[C] \in R$ is very rigid in the sense of Definition 3.9. Otherwise we say that the extremal ray R is divisorial.

3.17. REMARK. Notice that, due to Assumption 3.1 and Lemma 3.3, through a general point $x \in X$ there is no curve C belonging to R. Hence the curves belonging to R cover at most a divisor.

If the extremal ray R is small, standard arguments show that there are only finitely many curves $C \subset X$ such that $[C] \in R$ (cf. [HP13, Rem.7.2]).

If the extremal ray R is divisorial, we can argue as in [HP13, Lemma 7.5] that there exists a unique surface $S \subset X$ such that

$$S \cdot R < 0.$$

In particular any curve $C \subset X$ with $[C] \in R$ is contained in S.

The following proposition is a well-known consequence of the cone theorem 3.13, cf. [HP13, Prop.7.3] for details:

3.18. PROPOSITION. *There exists a nef class* $\alpha \in N^1(X)$ *such that*

$$R = \{z \in \overline{NA}(X) \mid \alpha \cdot z = 0\},$$

and such that, using the notation of Theorem 3.13, the class α *is strictly positive on*

$$\left(\overline{NA}(X)_{(K_X+\omega)\geq 0} + \sum_{i\in I, i\neq i_0} \mathbb{R}^+[\Gamma_i]\right) \setminus \{0\}.$$

We call α *a nef supporting class for the extremal ray* $R = \mathbb{R}^+[\Gamma_{i_0}]$.

In what follows we will use at several places the following theorem, stated in [BCE$^+$02, Thm.2.6] for projective manifolds:

3.19. THEOREM. *Let X be a normal compact Kähler space, and let α be a nef cohomology class on X. Then there exists an almost holomorphic, dominant meromorphic map $f : X \dashrightarrow Y$ with connected fibers, such that*

a) α is numerically trivial on all compact fibers F of f with $\dim F = \dim X - \dim Y$

b) for every general point $x \in X$ and every irreducible curve C passing through x with $\dim f(C) > 0$, we have $\alpha \cdot C > 0$.

In particular, if two general points of X can be joined by a chain C of curves such that $\alpha \cdot C = 0$, then $\alpha \equiv 0$.

For the convenience of the reader we sketch how to adapt the proof from [BCE$^+$02] to this setting.

PROOF. We define that two points $x, y \in X$ are equivalent if they can be joined by a connected curve C such that $\alpha \cdot C = 0$. By [Cam04, Thm.1.1] there exists an almost holomorphic map $f : X \dashrightarrow Y$ with connected fibers to a normal compact Kähler space Y such that two general points x and y are equivalent if and only if $f(x) = f(y)$. By construction a general f-fibre F_0 is a normal compact Kähler space such that two general points can be connected by a curve, thus F_0 is algebraic [Cam81, p.212, Cor.]. Hence we can apply [BCE$^+$02, Thm.2.4] to see that $\alpha|_{F_0} = 0$. In particular for any Kähler form ω on X we have $\alpha \cdot \omega^{d-1} \cdot F_0 = 0$ where $d := \dim X - \dim Y$. Since any compact f-fibre F of dimension d is homologous to some multiple of F_0 and α is nef we see that $\alpha|_F = 0$. □

3.20. *Notation.* Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_{i_0}]$ is divisorial, and let S be the surface such that $S \cdot R < 0$ (cf. Remark 3.17). Let $\nu : \tilde{S} \to S \subset X$ be the normalisation. By Lemma 3.11(a) there exists a curve $C \subset X$ such that $[C] \in R$ and $\dim_C \mathrm{Chow}(X) > 0$. Since we have $S \cdot C < 0$, the deformations $(C_t)_{t \in T}$ of C induce a dominating family $(\tilde{C}_t)_{t \in T'}$ of $\tilde{S}$ such that $\nu^*(\alpha) \cdot \tilde{C}_t = 0$. The class $\nu^*(\alpha)$ is a nef class on $\tilde{S}$ and we may consider the nef reduction

$$\tilde{f} : \tilde{S} \to \tilde{B}$$

with respect to $\nu^*(\alpha)$, cf. Theorem 3.19. By definition of the nef reduction this implies

$$n(\alpha) := \dim \tilde{B} \in \{0, 1\}.$$

3.21. LEMMA.

a) *Suppose that the extremal ray R is divisorial and $n(\alpha) = 0$. Then the surface S can be blown down to a point p: there exists a bimeromorphic morphism $\varphi : X \to Y$ to a normal compact threefold Y with* $\dim \varphi(S) = 0$ *such that $\varphi|_{X \setminus S}$ is an isomorphism onto $Y \setminus \{p\}$.*

b) *Suppose that the extremal ray R is divisorial and $n(\alpha) = 1$. Then there exists a fibration $f : S \to B$ onto a curve B such that a curve $C \subset S$ is contracted if and only if $[C] \in R$. Moreover the surface S can be contracted onto a curve: there exists a bimeromorphic morphism $\varphi : X \to Y$ to a normal compact threefold Y such that $\varphi|_S = f$ and $\varphi|_{X \setminus S}$ is an isomorphism onto $Y \setminus B$.*

PROOF. The proof is identical to the proofs of [HP13, Cor.7.7, Lemma 7.8, Cor.7.9] which only use properties of the nef class α and $K_X \cdot R < 0$ which holds since $\omega \cdot R > 0$. □

3.22. *Notation.* Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_{i_0}]$ is small. Set

$$C := \cup_{C_l \subset X, [C_l] \in R} C_l,$$

then C is a finite union of curves by Remark 3.17. We say that C is contractible if there exists a bimeromorphic morphism $\varphi : X \to Y$ onto a normal threefold Y with $\dim \varphi(C) = 0$ such that $\varphi|_{X \setminus C}$ is an isomorphism onto $Y \setminus \varphi(C)$.

The following statement is a variant of [HP13, Prop.7.11].

3.23. PROPOSITION. *Suppose that the extremal ray $R = \mathbb{R}^+[\Gamma_i]$ is small. Let $S \subset X$ be an irreducible surface. Then we have $\alpha^2 \cdot S > 0$.*

PROOF. By hypothesis, the cohomology class $\alpha - (K_X + \omega)$ is positive on the extremal ray R, moreover we know by Proposition 3.18 that α is positive on

$$\left(\overline{\mathrm{NA}}(X)_{(K_X+\omega)\geq 0} + \sum_{i \in I, i \neq i_0} \mathbb{R}^+[\Gamma_i] \right) \setminus \{0\}.$$

Thus, up to replacing α by some positive multiple, we may suppose that $\alpha - (K_X + \omega)$ is positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$. Since X is a Kähler space, this implies by [HP13, Cor.3.16] that

$$\eta := \alpha - (K_X + \omega)$$

is a Kähler class. Arguing by contradiction we suppose that $\alpha^2 \cdot S = 0$.

We first claim that $(K_X + \omega)|_S$ is not pseudoeffective. If $\alpha|_S = 0$ this is obvious, so suppose $\alpha|_S \neq 0$. Then we have

$$0 = \alpha^2 \cdot S = (K_X + \omega) \cdot \alpha \cdot S + \eta \cdot \alpha \cdot S$$

and

$$\eta \cdot \alpha \cdot S = \eta|_S \cdot \alpha|_S > 0$$

by the Hodge index theorem (note hat if $\pi : S' \to S$ is a desingularisation, then $\pi^*(\eta|_S)$ is nef and big and $\pi^*(\alpha|_S)$ is nef, so the "smooth" Hodge index theorem applies). Thus we have

$$(K_X + \omega) \cdot \alpha \cdot S = (K_X + \omega)|_S \cdot \alpha|_S < 0. \tag{7}$$

In particular $(K_X + \omega)|_S$ is not pseudoeffective, the class $\alpha|_S$ being nef.

Since $(K_X + \omega)|_S$ is not pseudoeffective, we know by Lemma 3.4 that S is uniruled and one of the surfaces in the Zariski decomposition (5). In particular we cannot have $\alpha|_S = 0$ since S contains infinitely many curves (recall that the ray R is small, hence $\alpha \cdot C = 0$ can occur only for finitely many curves C). Using the decomposition (5) and (7) we obtain $\alpha \cdot S^2 < 0$, hence

$$(K_X + \omega + S) \cdot \alpha \cdot S < 0. \tag{8}$$

Let $\pi : \hat{S} \to S$ be the composition of the normalisation and the minimal resolution (cf. Subsection 3.A), then (3) and (8) imply that

$$(K_{\hat{S}} + \pi^*\omega|_S) \cdot \pi^*\alpha|_S < 0. \tag{9}$$

Since the surface $\hat{S}$ is projective, the nef class $\pi^*\alpha|_S$ is represented by an $\mathbb{R}$-divisor. The extremal ray R contains only the classes of finitely many curves, so $\pi^*\alpha$ is strictly positive on every movable curve in $\hat{S}$.

Fix an ample divisor A on $\hat{S}$. By [Ara10, Thm.1.3] for every $\varepsilon > 0$ we have a decomposition

$$\pi^*\alpha|_S = C_\varepsilon + \sum \lambda_{i,\varepsilon} M_{i,\varepsilon}$$

where $\lambda_{i,\varepsilon} \geq 0$, the $M_{i,\varepsilon}$ are movable curves and $(K_{\hat{S}} + \varepsilon A) \cdot C_\varepsilon \geq 0$. The class $\pi^*\alpha|_S$ is strictly positive on every movable curve in $\hat{S}$, so we have $\pi^*\alpha|_S \cdot M_{i,\varepsilon} > 0$. Since $(\pi^*\alpha|_S)^2 = 0$ and $\pi^*\alpha|_S \cdot M_{i,\varepsilon} > 0$ we must have $\pi^*\alpha|_S = C_\varepsilon$ for all $\varepsilon > 0$. Passing to the limit we obtain $K_{\hat{S}} \cdot \pi^*\alpha|_S \geq 0$, a contradiction to (9). □

3.24. Theorem. *Suppose that the extremal ray R is small. Then C is contractible.*

Proof. Let $\alpha \in N^1(X)$ be the nef class supporting R as in Proposition 3.18. *We claim that the class α is big*, i.e., if $\pi : X' \to X$ is a desingularisation then the pull-back $\pi^*\alpha$ is a big cohomology class. Once we have shown this property, the proof of [HP13, Thm.7.12] applies.

Proof of the Claim. By definition of the class α, the class $-(K_X + \omega) + \alpha$ is positive on the extremal ray R. Since α is strictly positive on

$$\left(\overline{\mathrm{NA}}(X)_{(K_X+\omega)\geq 0} + \sum_{i \in I, i \neq i_0} \mathbb{R}^+[\Gamma_i] \right) \setminus \{0\},$$

we may suppose, up to replacing α by some positive multiple, that $-(K_X + \omega) + \alpha$ is strictly positive on this cone. In total, $-(K_X + \omega) + \alpha$ is strictly positive on $\overline{\mathrm{NA}}(X) \setminus \{0\}$. Thus $-(K_X + \omega) + \alpha$ is a Kähler class by [HP13, Cor.3.16], i.e., we may write

$$\alpha = (K_X + \omega) + \eta,$$

where η is a Kähler class. We know by Lemma 3.3 that $K_X + \omega$ is pseudoeffective. Thus $\pi^*\alpha$ is the sum of the pseudoeffective class $\pi^*(K_X + \omega)$ and the nef and big class $\pi^*\eta$, hence it is big. □

Proof of Theorem 3.15. The existence of a morphism $\varphi : X \to Y$ contracting exactly the curves in the extremal ray is established in Lemma 3.21 and in Theorem 3.24. Since ω is nef, the extremal ray $\mathbb{R}^+[\Gamma_i]$ is K_X-negative. Therefore, applying [HP13, Cor.8.2], it follows that Y is a Kähler space. □

3.E. Proof of Theorem 1.3

Proof. *Step* 1 Running the MMP. If $K_X + \omega$ is nef for every normalised Kähler class ω, we are finished. Suppose therefore that $K_X + \omega$ is not nef. Then there exists by Theorem 3.6 a $(K_X + \omega)$-negative extremal ray R in $\overline{\mathrm{NA}}(X)$. By Theorem 3.15 the contraction $\varphi : X \to Y$ of R exists in the Kähler category. Note that since ω is nef, the canonical class K_X is negative on the extremal ray R.

If R is divisorial we can continue the MMP with Y by [HP13, Prop.8.1.c)]. If R is small, we know by Mori's flip theorem [Mor88, Thm.0.4.1] that the flip $\varphi^+ : X^+ \to Y$ exists, and by [HP13, Prop.8.1.d)] we can continue the MMP with X^+ (which is again Kähler).

Step 2 Termination of the MMP. Recall that for a normal compact threefold X with at most terminal singularities, the difficulty $d(X)$ [Sho85] is defined by

$$d(X) := \#\{i \mid a_i < 1\},$$

where $K_Y = \mu^* K_X + \sum a_i E_i$ and $\mu : Y \to X$ is any resolution of singularities. Recall that any contraction in our MMP is a K_X-negative contraction, so by [KMM87, Lemma 5.1.16][4], [Sho85] we have $d(X) > d(X^+)$, if X^+ is the flip of a small contraction. Since the Picard number and the difficulty are non-negative integers, any MMP terminates after finitely many steps. □

4. The Base-point Free Theorem

We first prove Theorem 1.4, which is the analogue of the base point free theorem in the non-algebraic case.

[4]The proof is local in a neighbourhood of the flipping locus, so it holds without change in the analytic setting.

4.A. Proof of Theorem 1.4

PROOF. We will use the nef reduction of X with respect to the cohomology class $K_X + \omega$, cf. Theorem 3.19. We denote by $n(K_X + \omega)$ the dimension of the base of the nef reduction of $K_X + \omega$ and claim that

$$n(K_X + \omega) = 2.$$

Notice first that the general fibres of the MRC-fibration provide a dominating family of curves which is $K_X + \omega$-trivial, so $n(K_X + \omega) \leq 2$.

If $n(K_X + \omega) = 1$ the nef reduction is a holomorphic fibration $X \to C$ (cf. [BCE$^+$02, 2.4.3]) and $K_X + \omega$ is numerically trivial on the general fibre by Theorem 3.19. In particular the general fiber is a smooth Fano surface, hence rationally connected, a contradiction to our assumption on the base of the MRC-fibration.

If $n(K_X + \omega) = 0$, then $K_X + \omega \equiv 0$, hence X is Fano and rationally connected, again a contradiction.

Let Z be a resolution of singularities of the unique irreducible component of Chow(X) such that the general point corresponds to the general fibre of the MRC-fibration. Let Γ be the normalisation of the pull-back of the universal family and denote by $p : \Gamma \to X$ and $q : \Gamma \to Z$ the natural morphisms. Since Γ is in Fujiki's class $\mathcal{C}$, the surface Z is in the class $\mathcal{C}$ by [Var86, Thm. 3]. A smooth surface in the class $\mathcal{C}$ is Kähler, so Z is Kähler.

We claim that there exists a big and nef class α on Z such that

$$p^*(K_X + \omega) = q^*\alpha.$$

Step 1 Construction of the class α. Set $\Gamma_z = q^{-1}(z)$ for $z \in Z$. Note first that we have $R^1q_*(\mathcal{O}_\Gamma) = 0$ (the morphism q is projective, so we can apply [Kol96, II, 2.8.6.2]). Using the exponential sequence this implies $R^1q_*(\mathbb{Z}) = 0$ and hence $R^1q_*(\mathbb{R}) = 0$ by the universal coefficient theorem. Now we apply the Leray spectral sequence for q and the sheaf $\mathbb{R}$. By what precedes we have

$$E_2^{0,1} = H^0(Z, R^1q_*(\mathbb{R})) = 0$$

and

$$E_2^{1,1} = H^1(Z, R^1q_*(\mathbb{R})) = 0.$$

Therefore $E_2^{2,0} = H^2(Z, \mathbb{R})$ embeds into $H^2(\Gamma, \mathbb{R})$, and it suffices to show that the section $s \in E_2^{0,2} = H^0(Z, R^2 q_*(\mathbb{R}))$ which is given by the class

$$s(z) = [p^*(K_X + \omega)|\Gamma_z] \in H^2(\Gamma_z, \mathbb{R}),$$

vanishes for every $z \in Z$. By definition of a normalised Kähler class we have $s(z) = 0$ for $z \in Z$ general. Since $p^*(K_X + \omega)$ is nef, this implies that the class $p^*(K_X + \omega)$ is zero on all the irreducible components of any fibre Γ_z. Thus we have $s(z) = 0$ for all $z \in Z$ proving the existence of α. Note that since $q^*\alpha$ is nef, the class α is nef [Pău98, Thm.1].

Step 2 Intersection numbers. Let $D \subset \Gamma$ be an irreducible component (possibly of dimension 1) of the p-exceptional locus. Since p is finite on the fibres of q, there exists a curve $C \subset D$ that is contracted by p and such that $q(C)$ is not a point. In particular we have

$$\alpha \cdot q(C) = q^*\alpha \cdot C = p^*(K_X + \omega) \cdot C = 0.$$

Since the meromorphic map $X \dashrightarrow Z$ is almost holomorphic, D does not surject onto Z. Thus we have $q(D) = q(C)$, and by what precedes we obtain

$$(q^*\alpha)|_D = 0.$$

Note now that, Γ being a modification of a threefold which has a finite singular locus, the singular locus of Γ is a union of curves which are contained in the p-exceptional locus and finitely many points. Let $\mu : \hat{X} \to \Gamma$ be a desingularisation such the exceptional set of $\hat{p} := p \circ \mu$ has pure codimension one. Set moreover $\hat{q} := q \circ \mu$. By what precedes,

$$\hat{q}^*\alpha \cdot \hat{D} = 0 \text{ in } N_1(\hat{X}) \tag{10}$$

for every irreducible component $\hat{D}$ of the $\hat{p}$-exceptional locus.

Step 3 The class α is big, i.e. we have $\alpha^2 > 0$. Since ω is a Kähler class, we know that, up to replacing $\hat{X}$ by some further blowup, there exists an effective $\mathbb{Q}$-divisor F with support in the $\hat{p}$-exceptional locus such that

$$\hat{p}^*\omega - F$$

is a Kähler class. Being a Kähler class is an open property, so there exists a Kähler class η_Z on Z such that

$$\hat{p}^*\omega - F - \hat{q}^*\eta_Z$$

is a Kähler class. Using Păun's theorem [Pău12, Thm.1.1] as in the proof of Lemma 3.3, we conclude that

$$K_{\hat{X}/Z} + \hat{p}^*\omega - F - \hat{q}^*\eta_Z$$

is pseudoeffective. Since X has terminal singularities,

$$K_{\hat{X}} = \hat{p}^*K_X + E$$

with E an effective $\mathbb{Q}$-divisor supported on the $\hat{p}$-exceptional locus. Consider now the decomposition

$$\hat{p}^*(K_X + \omega) = [K_{\hat{X}/Z} + \hat{p}^*\omega - F - \hat{q}^*\eta_Z] - E + F + \hat{q}^*K_Z + \hat{q}^*\eta_Z. \tag{11}$$

We are going to intersect this equation with $\hat{q}^*(\alpha)$ in order to compute

$$\hat{q}^*\alpha^2 = \hat{q}^*\alpha \cdot \hat{p}^*(K_X + \omega).$$

Since α is nef, the intersection product

$$\hat{q}^*\alpha \cdot [K_{\hat{X}/Z} + \hat{p}^*\omega - F - \hat{q}^*\eta_Z]$$

is an element of $\overline{\mathrm{NA}}(\hat{X})$. By (10) we have $\hat{q}^*\alpha \cdot (-E + F) = 0$. The surface Z is not uniruled since it is the base of the MRC-fibration (cf. Remark 3.2). Thus K_Z is pseudoeffective, in particular the intersection product $\hat{q}^*\alpha \cdot \hat{q}^*K_Z$ is an element of $\overline{\mathrm{NA}}(\hat{X})$. Recall now that $\alpha \neq 0$ since $K_X + \omega \neq 0$. Since η_Z is a Kähler class and α is a non-zero nef class, the Hodge index theorem yields $\eta_Z \cdot \alpha > 0$. Thus

$$q^*\alpha \cdot q^*\eta_Z$$

is a non-zero element of $\overline{\mathrm{NA}}(\hat{X})$. In total we obtain that

$$\hat{q}^*\alpha^2 = \hat{q}^*\alpha \cdot \hat{p}^*(K_X + \omega)$$

is a non-zero element of $\overline{\mathrm{NA}}(\hat{X})$. Thus we have $\alpha^2 \neq 0$.

Step 4 Construction of the fibration φ. Let

$$E := \cup E_j \subset Z$$

be the union of curves $E_j \subset Z$ such that $\alpha \cdot E_j = 0$. Since α is nef and big, the Hodge index theorem implies that the intersection form on E is negative definite. In particular E is a finite set. By Grauert's criterion there exists a bimeromorphic morphism $\nu : Z \to S$ such that E equals the ν-exceptional locus. Since Z is a Kähler surface and ν contracts only subvarieties onto points, the surface S is Kähler. In fact, take any Kähler form ω on Z. Then the class of the Kähler current $\nu_*(\omega)$ contains a Kähler form by [DP04, Prop.3.3(iii)].

We claim that the fibration $\nu \circ q : \Gamma \to S$ factors through the bimeromorphic map p, i.e., there exists a holomorphic fibration $\varphi : X \to S$ such that $\nu \circ q = \varphi \circ p$. By the rigidity lemma [BS95, Lemma 4.1.13] it is sufficient to prove that every p-fibre is contracted by $\nu \circ q$. Since p is a Moishezon morphism, it moreover suffices to show that every curve $C \subset \Gamma$ such that $p(C)$ is a point is contracted by $\nu \circ q$. Yet for such a curve C we have

$$q^*\alpha \cdot C = p^*(K_X + \omega) \cdot C = 0.$$

It follows that $q(C) \subset E$, hence $q(C)$ is a point. This shows the existence of the fibration φ; by construction the class $K_X + \omega$ is φ-trivial. □

4.B. MMP for uniruled Kähler threefolds

Recall that in our context a normal Kähler space X is $\mathbb{Q}$-factorial if every Weil divisor $D \subset X$ is $\mathbb{Q}$-Cartier and some reflexive power $\omega_X^{[m]}$ of the dualising sheaf ω_X is locally free.

4.1. LEMMA. *Let X be a normal $\mathbb{Q}$-factorial compact Kähler threefold with at most terminal singularities. Let $\varphi : X \to S$ be an elementary Mori contraction onto a normal compact surface, i.e., $\rho(X/S) = 1$ and $-K_X$ is φ-ample.*

Then S is $\mathbb{Q}$-factorial and has at most klt singularities.

4.2. REMARK. In the situation above, the fibration φ is equidimensional since an elementary contraction of fibre type does not contract a divisor. For a point $s \in S$ denote by X_s the fibre over s, and let $A \subset S$

be the set of all s such that the fiber X_s is singular at some point x_0 and such that X is not smooth at x_0. Then A is finite, set $S_0 = S \setminus A$ and $X_0 = X \setminus \varphi^{-1}(A)$. The fiber space $f_0 : X_0 \to S_0$ is a conic bundle. The sheaf $f_*(\omega_{X/S})$ is reflexive, but might have singularities on A, so that f might globally not be a conic bundle. However, $H^1(X_s, \mathcal{O}_{X_s}) = 0$, in particular, every irreducible component of any fiber X_s is a smooth rational curve.

Proof of Lemma 4.1. Arguing as in [KMM87, 5-1-5], every Weil divisor $D \subset S$ is $\mathbb{Q}$-Cartier.

In order to see that S has at most klt singularities we proceed as in the algebraic case. The claim is local on the base S, so given a point $0 \in S_{\text{sing}}$ we fix a small analytic neighbourhood $0 \in U \subset S$. Since X is smooth in codimension two and, by Remark 4.2 the projective morphism φ is a conic bundle in the complement of the fibre X_0, there exists a smooth analytic subvariety $H \subset \varphi^{-1}(U)$ such that $H \to U$ is finite and étale in codimension one. By [KM98, Prop.5.20] the surface U has at most klt singularities. In particular some reflexive power $\omega_X^{[m]}$ of the dualising sheaf ω_X is locally free. □

Proof of Theorem 1.1. By Theorem 1.3, there exists a MMP $X \dashrightarrow X'$ such that $K_{X'} + \omega'$ is nef for all normalised Kähler classes ω' on X'. Fix such a Kähler class ω'. Then apply the base point free theorem 1.4 to the variety X' to obtain a fibration $\varphi : X' \to S'$ onto a surface S' such that $-K_{X'}$ is φ-ample. In particular φ is a projective morphism. Thus we can run the MMP of X' over S' using the relative version of the cone and contraction theorem as in [Nak87, Sect.4], [KM98, Sect.3.6]. As in the proof of Theorem 1.3 we can use the Picard number $\rho(X')$ and the difficulty $d(X')$ to show that the MMP terminates. Since $K_{X'}$ is not pseudoeffective over S', the outcome of the MMP

$$X' \dashrightarrow X''$$

is a Mori fibre space $X'' \to S''$ over S', with S'' a normal compact complex surface that dominates S'. Since S' is Kähler, and the bimeromorphic morphism $S'' \to S'$ is projective (we can always find an anti-effective exceptional divisor that is relatively ample), the surface S'' is Kähler. The properties of S'' are proven in Lemma 4.1. □

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(Received July 9, 2014)
(Revised October 6; December 17, 2014)

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