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Structure of the F-Blowups of Simple Elliptic Singularities

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Dedicated to the centennial anniversary of Professor Kunihiko Kodaira

Abstract. As a natural continuation of the preceding paper [HSY], we study the F-blowups of simple elliptic singularities and determine their structure.

1. Introduction

In [Y1], Yasuda introduced the notion of the F-blowup, which is a canonical birational modification of varieties in characteristic $p > 0$. For a non-negative integer e, the e-th F-blowup of a variety X is defined as the universal birational flattening of the direct image $F^e_* \mathcal{O}_X$ of the structure sheaf by the e-iterated Frobenius morphism. The behavior of the F-blowups of mild singularities has been studied and is fairly well-understood: For $e \gg 0$, the e-th F-blowup of a tame quotient singularity coincides with the G-Hilbert scheme (Yasuda [Y1], Toda and Yasuda [TY]), and that of an F-regular surface singularity is the minimal resolution [Ha]. However, the behavior of the F-blowup in general is a mystery yet.

In [HSY], Sawada, Yasuda and the author studied the F-blowups of certain classes of surface singularities, that is, non-F-regular rational double points and simple elliptic singularities. The behavior of the F-blowups of these singularities turned out to be more pathological and unexpectedly complicated. As for simple elliptic singularities, an F-blowup may be non-normal, not dominated by the minimal resolution and the sequence of the F-blowups does not stabilize in general. To obtain these results we employed not only the classical theory of surface singularities but also computer-aided calculations using Macaulay2 [GS].

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Computation with Macaulay2 is very useful and gave a lot of suggestive examples, but we could not obtain an exhaustive list of the F-blowups of simple elliptic singularities in [HSY]. In this paper, we will determine the structure of the F-blowups of any simple elliptic singularity in characteristic $p > 0$ as a natural continuation of [HSY, Section 4]. Our main results are the following

Theorem 1.1. *Let (X, x) be a simple elliptic singularity in characteristic $p > 0$ with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$. Let $\mathrm{FB}_e(X)$ be the e-th F-blowup of (X, x). Then the following conditions are equivalent.*

(1) *The intersection number $-E^2$ is not a power of p.*

(2) *The F-blowup sequence $\{\mathrm{FB}_e(X) \mid e = 0, 1, 2, \ldots\}$ stabilizes.*

(3) $\mathrm{FB}_e(X) \cong \widetilde{X}$ *for all $e \geq 1$.*

We note that in the case of simple elliptic singularities of type $\widetilde{E}_8$, that is, the case where $E^2 = -1 = -p^0$, the F-blowup sequence does not stabilize in any characteristic $p > 0$.

Next we give a notation to state the result in the case when $-E^2$ is a power of p. Given an elliptic curve E with the zero element $P_0 \in E$ of the group law and an integer $q > 0$, let $E_{P_0}[q]$ denote the set of all q-torsion points on E. When q is a power of the characteristic p, it is known that $\sharp E_{P_0}[q] = q$ if E is ordinary; and that $E_{P_0}[q] = \{P_0\}$ if E is supersingular.

Theorem 1.2. *Let (X, x) be a simple elliptic singularity in characteristic $p > 0$ with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$. Suppose that $E^2 = -p^n$ for an integer $n \geq 0$ and choose a point $P_0 \in E$ such that $\mathcal{O}_{\tilde{X}}(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(p^n P_0)$ as the zero element of the group law of E.*

(1) *Suppose that (X, x) is F-pure, or equivalently, E is ordinary. If $E^2 = -1$ (resp. $E^2 = -p^n < -1$), then for all e with $p^e \geq \max\{3, p^n\}$, the e-th F-blow-up $\mathrm{FB}_e(X)$ coincides with the blowup $\mathrm{Bl}_{E_{P_0}[p^e] \setminus \{P_0\}}(\widetilde{X})$ (resp. $\mathrm{Bl}_{E_{P_0}[p^e]}(\widetilde{X})$) of $\widetilde{X}$ at the non-trivial p^e-torsion points (resp. all the p^e-torsion points) on E.*

(2) *Suppose that* (X, x) *is not F-pure, or equivalently,* E *is supersingular. If* $E^2 = -1$ *(resp.* $E^2 = -p^n < -1$*), then for all* e *with* $p^e \geq \max\{3, p^n\}$*, the* e*-th F-blowup* $\mathrm{FB}_e(X)$ *coincides with the blowup of* $\widetilde{X}$ *at an ideal defining a fat point at* P_0 *with local expression* (t, u^{p^e-1}) *(resp.* (t, u^{p^e})*), where* t, u *are local coordinates at* $P_0 \in \widetilde{X}$*.*

On the other hand, we have $\mathrm{FB}_e(X) \cong \widetilde{X}$ *for* $1 \leq e < n$*.*

Note that the sequence of F-blowups of any F-pure singularity is monotone [Y2], and this is the case for Theorem 1.2 (1). In Theorem 1.2 (2), the e-th F-blowup $\mathrm{FB}_e(X)$ has the exceptional set consisting of an elliptic curve $E_1 \cong E$ and a smooth rational curve $E_2 \cong \mathbb{P}^1$, and has an A_{p^e-2}-singularity (resp. A_{p^e-1}-singularity) on $E_2 \setminus E_1$. Thus the monotonicity of the F-blowup sequence breaks down in the non-F-pure case (2). We also remark that the F-blowups are normal except for the cases $p = 2$, $e = 1$ and $E^2 = -1, -2$, which are not covered by Theorem 1.2. Actually, we have examples of non-normal first F-blowups in these exceptional cases [HSY, Examples 4.6 and 4.10].

The above theorems are obtained by refining the arguments in [HSY, Section 4]. Since any simple elliptic singularity (X, x) is a cone singularity, we may assume that $X = \operatorname{Spec} R$ for a section ring $R = R(C, L)$ of an ample line bundle L on an elliptic curve $C \cong E$. Then for any $q = p^e$, the graded ring structure of R gives rise to a $\frac{1}{q}\mathbb{Z}$-grading of the R-module $R^{1/q} \cong F_*^e \mathcal{O}_X$ and its decomposition

$$R^{1/q} = [R^{1/q}]_{0 \bmod \mathbb{Z}} \oplus [R^{1/q}]_{1/q \bmod \mathbb{Z}} \oplus \cdots \oplus [R^{1/q}]_{(q-1)/q \bmod \mathbb{Z}}$$

into the R-summands $[R^{1/q}]_{i/q \bmod \mathbb{Z}} \cong \bigoplus_{m \geq 0} H^0(C, L^m \otimes F_*^e L^i)$, among which we especially focus on the 0-th summand $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ and a few others. We examine whether each of these summands are flattened on the minimal resolution $\widetilde{X}$ of X and whether $\widetilde{X}$ is the blowup (i.e., universal flattening) of it. It turns out that for $1 \leq i \leq q-1$, the i-th summand $[R^{1/q}]_{i/q \bmod \mathbb{Z}}$ is flattened on $\widetilde{X}$ if and only if $q \neq di$, where $d = \deg L = -E^2$. If $d \geq 2$, then the 0-th summand $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ is also flattened on $\widetilde{X}$, and the flattening $\widetilde{X}$ is universal unless $d = p = q = 2$. It follows that if $d = -E^2$ is not a power of p, then the e-th F-blowup $\mathrm{FB}_e(X)$ coincides with the minimal resolution $\widetilde{X}$ for all $e \geq 1$.

In the case when d is a power of p, $R^{1/q}$ may have a summand that is not flattened on $\widetilde{X}$, that is, the 0-th summand $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ if $d=1$; and the q/d-th summand $[R^{1/q}]_{1/d \bmod \mathbb{Z}}$ if $q \geq d \geq 2$, respectively. The structure of this summand depends on that of the vector bundle $F^e_*\mathcal{O}_C$ on C, which differs according to whether C is ordinary or supersingular. On the other hand, we see that $\widetilde{X}$ coincides with the blowup at another summand of $R^{1/q}$, that is, $[R^{1/q}]_{(q-1)/q \bmod \mathbb{Z}}$ if $d=1$; and $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ if $d \geq 2$, respectively. Hence the F-blowup $\mathrm{FB}_e(X)$ factors through $\widetilde{X}$, and we can determine the structure of $\mathrm{FB}_e(X)$ by a detailed study of the torsion-free pullback to $\widetilde{X}$ of the summand that is not flattened.

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2. Preliminaries

2.a. Blowups at modules and F-blowups

Let X be a Noetherian integral scheme and $\mathcal{M}$ a coherent sheaf on X. For a modification $f\colon Y \to X$, we denote the torsion-free pullback of $\mathcal{M}$ by $f^\star\mathcal{M} = f^*\mathcal{M}/\text{torsion}$.

Definition 2.1. A modification $f\colon Y \to X$ is called a *flattening* of $\mathcal{M}$ if $f^\star\mathcal{M}$ is flat, or equivalently locally free. A flattening f is said to be *universal* if every flattening $g\colon Z \to X$ of $\mathcal{M}$ factors as $g\colon Z \to Y \xrightarrow{f} X$. The universal flattening exists and is unique. It is also called the *blowup of X at $\mathcal{M}$* and denoted by $\mathrm{Bl}_{\mathcal{M}}(X)$.

If $\mathcal{M}$ is an ideal sheaf, then the blowup at $\mathcal{M}$ defined above coincides with the usual blowup with respect to the ideal $\mathcal{M}$. We state a few basic properties of the blowup at a module; see [OZ], [Vi] and [HSY] for more detail.

Let r be the rank of $\mathcal{M}$, K the rational function field of X and fix an isomorphism $\bigwedge^r \mathcal{M} \otimes K \cong K$. Then we define a fractional ideal sheaf

$$\mathcal{I}_{\mathcal{M}} := \mathrm{Im}\Big(\bigwedge^r \mathcal{M} \to \bigwedge^r \mathcal{M} \otimes K \cong K\Big).$$

PROPOSITION 2.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be coherent sheaves on X.*

(1) $\mathrm{Bl}_{\mathcal{M}}(X) \cong \mathrm{Bl}_{\mathcal{I}_{\mathcal{M}}}(X) = \mathrm{Proj}\ \mathcal{O}_X[\mathcal{I}_{\mathcal{M}} t]$

(2) $\mathrm{Bl}_{\mathcal{M}\oplus\mathcal{N}}(X) \cong \mathrm{Bl}_{\varphi^\star\mathcal{M}}(\mathrm{Bl}_{\mathcal{N}}(X))$, *where* $\varphi\colon \mathrm{Bl}_{\mathcal{N}}(X) \to X$ *is the blowup at* $\mathcal{N}$*.*

PROOF. (1) See [OZ], [Vi]. (2) is easy. □

We recall the definition of the F-blowup in a modified form from the original one [Y1]. Let X be a Noetherian integral scheme of characteristic $p > 0$ and suppose that its (absolute) Frobenius morphism $F\colon X \to X$ is finite.

DEFINITION 2.3 (Yasuda [Y1]). For a non-negative integer e, we define the *e-th F-blowup* of X to be the blowup of X at $F^e_*\mathcal{O}_X$ and denote it by $\mathrm{FB}_e(X)$.

We now introduce the notation to be used throughout this paper. Since any simple elliptic singularity is a quasi-homogeneous by Hirokado [Hi, Theorem 4.2], we may and will work under the following setup; cf. [HSY, Section 4].

2.b. Setup

Let C be an elliptic curve defined over an algebraically closed field k of characteristic $p > 0$ and let L be a line bundle on C with $d = \deg L > 0$. Consider the graded k-algebra

$$R = R(C, L) = \bigoplus_{n\geq 0} H^0(C, L^n)t^n,$$

where $\deg t = 1$. Then $X = \mathrm{Spec}\, R$ has a simple elliptic singularity. The minimal resolution $f\colon \widetilde{X} \to X$ of X is described as follows: $\widetilde{X}$ has an $\mathbb{A}^1$-bundle structure $\pi\colon \widetilde{X} = \mathrm{Spec}_C(\bigoplus_{n\geq 0} L^n t^n) \to C$ over C, and its zero-section $E\,(\cong C)$ is the exceptional curve of f. Its self-intersection number is $E^2 = -\deg L$. Our situation is summarized in the following diagram:

$$\begin{array}{ccccc} E & \hookrightarrow & \widetilde{X} & \xrightarrow{f} & X \\ & \searrow & \downarrow \pi & & \\ & & C & & \end{array}$$

For $q = p^e$ we study the structure of the torsion-free pullback $f^\star R^{1/q}$ of $R^{1/q} \cong F^e_* \mathcal{O}_X$. We decompose $R^{1/q} = \bigoplus_{n \geq 0} H^0(C, F^e_* L^n) t^{n/q}$ as $R^{1/q} = \bigoplus_{i=0}^{q-1} [R^{1/q}]_{i/q \bmod \mathbb{Z}}$, where

$$[R^{1/q}]_{i/q \bmod \mathbb{Z}} = \bigoplus_{0 \leq n \equiv i \bmod q} H^0(C, F^e_* L^n) t^{n/q} \cong \bigoplus_{m \geq 0} H^0(C, L^m \otimes F^e_* L^i)$$

is an R-summand of $R^{1/q}$ for $i = 0, 1, \ldots, q-1$; cf. [SVdB, Example 3.1.7].

We are able to know whether any of the summands is flattened on the minimal resolution $\widetilde{X}$. We put $q = p^e$ and $d = \deg L = -E^2$ in what follows.

Lemma 2.4 (cf. [HSY]). *Let $q = p^e$ with $e \geq 1$ in the notation as above.*

(1) *$\widetilde{X}$ is a flattening of $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ if and only if $d \geq 2$.*

(2) *If $1 \leq i \leq q-1$, then $\widetilde{X}$ is a flattening of $[R^{1/q}]_{i/q \bmod \mathbb{Z}}$ if and only if $q \neq di$.*

(3) *Suppose $q = di$ with $d \geq 2$. Then for a point $Q \in E$ with $P = \pi(Q) \in C$, $f^\star [R^{1/q}]_{i/q \bmod \mathbb{Z}}$ is not flat at Q if and only if $L^i \cong \mathcal{O}_C(qP)$.*

Proof. (1) See [HSY, Lemma 4.4] if C is ordinary. The case where C is supersingular follows from subsections 4b1–4b2 of [HSY].

(2) The sufficiency is proved in [HSY, Lemma 4.1]. Suppose $q = di$ to prove the necessity (and also (3)). Then the line bundle L^i is divisible by its degree q, i.e., $L^i \cong \mathcal{O}_C(qP_0)$ for a point $P_0 \in C$, because the multiplication by q on the group structure of C identified with its Jacobian variety induces a finite (hence surjective) morphism $q_C \colon C \to C$. As in the proof of [HSY,

Lemma 4.1] we have

$$
\begin{aligned}
& f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}} \\
= \ & \mathrm{Im}([R^{1/q}]_{i/q \bmod \mathbb{Z}} \otimes_R \mathcal{O}_{\tilde{X}} \to F^e_* \mathcal{O}_{\tilde{X}}) \\
= \ & \mathrm{Im}\left(\bigoplus_{m \geq 0} H^0(C, L^m \otimes F^e_* L^i) \otimes_k \mathcal{O}_C \xrightarrow{\alpha} \bigoplus_{m \geq 0} L^m \otimes F^e_* L^i \right) \\
= \ & \mathrm{Im}(H^0(C, F^e_* L^i) \otimes \mathcal{O}_C \xrightarrow{\alpha_0} F^e_* L^i) \oplus \bigoplus_{m \geq 1} L^m \otimes F^e_* L^i,
\end{aligned}
$$

where α_m ($m \geq 0$) is the graded part of the map α of degree m. Now it is sufficient to show the following claim, which implies that $f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}}$ has a finite colength $q > 0$ in the locally free sheaf $\pi^* F^e_* L^i \cong \bigoplus_{m \geq 0} L^m \otimes F^e_* L^i$ on $\widetilde{X}$.

Claim. Coker(α_0) is supported on a finite closed subset of C and its length is q.

To prove the claim, note that the set $C_{P_0}[q]$ of q-torsion points with respect to the group law of C with the zero element P_0 is finite. If $P \in C$ is not q-torsion, then $H^1(C, L^i(-qP)) = 0$ and it follows as in the proof of [HSY, Lemma 4.1] that the map α_0 is surjective at P. Thus the map $\alpha_0 \colon H^0(C, F^e_* L^i) \otimes \mathcal{O}_C \to F^e_* L^i$ is a generically surjective map between locally free sheaves of the same rank q, so that it is injective and Coker(α_0) is a sheaf of finite length supported on q-torsion points. It follows from the exact sequence

$$0 \to \mathcal{O}_C^{\oplus q} \to F^e_* L^i \to \mathrm{Coker}(\alpha_0) \to 0$$

that the length of Coker(α_0) is equal to $\chi(F^e_* L^i) - \chi(\mathcal{O}_C^{\oplus q}) = q$.

(3) Continuing the argument above, we see that $f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}}$ is not flat at Q if and only if $P = \pi(Q) \in \mathrm{Supp}(\mathrm{Coker}(\alpha_0))$ and that $\mathrm{Supp}(\mathrm{Coker}(\alpha_0)) \subseteq C_{P_0}[q]$. In fact, all q-torsion points $P \in C_{P_0}[q]$ are in Supp(Coker(α_0)), since a translation $P_0 \mapsto P$ of q-torsion points preserves the sheaf $L^i \cong \mathcal{O}_C(qP_0)$ and so the map α_0. Thus $f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}}$ is not flat at Q if and only if $P \in C_{P_0}[q]$, or equivalently, if $L^i \cong \mathcal{O}_C(qP)$. $\square$

Remark 2.5. If one writes $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ as $[R^{1/q}]_{q/q \bmod \mathbb{Z}}$, then the assertion (1) of Lemma 2.4 can be included in (2).

The following is used to show the normality of F-blowups.

Proposition 2.6. *Let n_0 be an integer with $n_0 \deg L \geq 3$ and I a fractional ideal of $R = R(C, L)$ of the form*

$$I = \bigoplus_{n \geq n_0} H^0(C, L_0 \otimes L^n) t^n$$

for a line bundle L_0 on C of degree zero. Then the Rees algebra $R[IT]$ is normal.

Proof. This is done similarly as in the proof of [HSY, Theorem 4.7], which we include for the sake of completeness. First note that the normalization of $R[IT]$ is

$$\widetilde{R[IT]} = \bigoplus_{m \geq 0} \overline{I^m} T^m,$$

where $\overline{I^m}$ is the integral closure of the fractional ideal I^m; see e.g., [L, 9.6.A]. Since $n_0 \deg L \geq 3$,

$$I\mathcal{O}_{\widetilde{X}} = f^{\star} I \cong \bigoplus_{n \geq n_0} (L_0 \otimes L^n) t^n \cong \mathcal{O}_{\widetilde{X}}(-n_0 E) \otimes \pi^* L_0$$

is an invertible sheaf on $\widetilde{X}$; see e.g., [Hart, IV, Corollary 3.2]. Hence

$$\overline{I^m} \cong H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(-mn_0 E) \otimes \pi^* L_0^m) \cong \bigoplus_{n \geq mn_0} H^0(C, L_0^m \otimes L^n) t^n$$

for all $m \geq 1$. Now, for any integers $n \geq mn_0$ and $n_1, \ldots, n_m \geq n_0$ with $n_1 + \cdots + n_m = n$, the map

$$H^0(C, L_0 \otimes L^{n_1}) \otimes \cdots \otimes H^0(C, L_0 \otimes L^{n_m}) \to H^0(C, L_0^m \otimes L^n)$$

is surjective by [HSY, Lemma 4.9]. This implies that the multiplication map $\overline{I}^{\otimes m} \to \overline{I^m}$ is surjective in all degrees n. Since $I = \overline{I}$ is integrally closed, we conclude that $I^m = \overline{I^m}$, from which the normality of the Rees algebra $R[IT]$ follows. □

The results on F-blowups of simple elliptic singularities obtained in [HSY] are summarized in the following

Theorem 2.7 ([HSY]). *Let (X, x) be a simple elliptic singularity in characteristic $p > 0$ with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$. Let $\widetilde{\mathrm{FB}}_e(X)$ be the normalization of the e-th F-blowup $\mathrm{FB}_e(X)$ of (X, x) for any $e \geq 1$.*

(1) *If (X, x) is F-pure with $E^2 = -1$, then $\widetilde{\mathrm{FB}}_e(X)$ coincides with the blowup of $\widetilde{X}$ at $p^e - 1$ non-trivial p^e-torsion points on E.*

(2) *If (X, x) is not F-pure with $E^2 = -1$, then $\widetilde{\mathrm{FB}}_e(X)$ coincides with the blowup of $\widetilde{X}$ at an ideal supported at a point $P_0 \in E$ with local expression (t, u^{p^e-1}), where t, u are local coordinates at $P_0 \in \widetilde{X}$.*

(3) *If $E^2 \leq -2$ and $-E^2$ is not a power of p, then $\widetilde{\mathrm{FB}}_e(X) \cong \widetilde{X}$ for all $e \geq 1$. Moreover, if (X, x) is F-pure and $E^2 \leq -3$, then $\mathrm{FB}_e(X) \cong \widetilde{X}$.*

In the above theorem, the behavior of the normalized F-blowups remains open in the case $E^2 \leq -2$ and $-E^2$ is a power of p. Also, we could not determine the normality of the F-blowups except for a special case in (3), where $X = \operatorname{Spec} R$ is assumed to be F-pure.

We note that $R = R(C, L)$ is F-pure if and only if C is ordinary. This is related to the structure of the vector bundle $F^e_* \mathcal{O}_C$ as follows.

Lemma 2.8 ([HSY, Lemma 4.12], cf. Atiyah [At], Tango [T]). *Let C be an elliptic curve in characteristic p and let $q = p^e$ for $e \geq 0$.*

(1) *If C is ordinary, then $F^e_* \mathcal{O}_C$ splits into a direct sum of q distinct q-torsion line bundles.*

(2) *If C is supersingular, then $F^e_* \mathcal{O}_C$ is isomorphic to Atiyah's indecomposable bundle $\mathcal{F}_q$ of rank q; see Section 3 for the definition.*

3. Some Surjectivity Results

In this section we prove some surjectivity results, which are used to prove that the F-blowups of certain simple elliptic singularities coincide with the minimal resolution. Among them we have the following

Theorem 3.1. *Let L be a line bundle on an elliptic curve C of* $\deg L \geq 2$. *Then the natural map*

$$\bigwedge^{p^e} H^0(C, L \otimes F^e_* \mathcal{O}_C) \to H^0(C, \det(L \otimes F^e_* \mathcal{O}_C))$$

is surjective for all $e \geq 0$.

Note that the Frobenius push-forward $F_*^e \mathcal{O}_C$ is a vector bundle of rank p^e and of degree zero. Its structure differs according to whether C is ordinary or supersingular as we have seen in Lemma 2.8. In both cases, however, $F_*^e \mathcal{O}_C$ is obtained by starting from $\mathcal{O}_C$ and taking extension by a line bundle of degree zero repeatedly. (The difference is whether the extensions are trivial or not.)

In order to handle the supersingular case, we recall the construction of Atiyah's vector bundle $\mathcal{F}_r$ studied in [At]. For any elliptic curve C and an integer $r > 0$, there exists an indecomposable vector bundle $\mathcal{F}_r$ on C of rank r and degree zero with $h^0(\mathcal{F}_r) = h^1(\mathcal{F}_r) = 1$, determined inductively by $\mathcal{F}_1 = \mathcal{O}_C$ and the unique non-trivial extension

$$0 \to \mathcal{O}_C \to \mathcal{F}_r \to \mathcal{F}_{r-1} \to 0. \tag{1}$$

It is easy to see that $\det \mathcal{F}_r = \mathcal{O}_C$. We state an easy lemma for later use.

Lemma 3.2. *Let L be a line bundle on an elliptic curve C.*

(1) *If* $\deg L \geq 1$, *then* $H^1(C, \mathcal{F}_r \otimes L) = 0$.

(2) *If* $\deg L \geq 2$, *then* $\mathcal{F}_r \otimes L$ *is generated by its global sections.*

Proof. (1) If $\deg L \geq 1$, then $H^1(X, L) = 0$, so that the assertion follows by induction on r with exact sequence (1).

(2) For any point $P \in C$ with residue field $\kappa(P)$ we have an exact sequence $0 \to \mathcal{F}_r \otimes L(-P) \to \mathcal{F}_r \otimes L \to \mathcal{F}_r \otimes L \otimes \kappa(P) \to 0$. If $\deg L \geq 2$, then $H^1(C, \mathcal{F}_r \otimes L(-P)) = 0$ by (1), so that the map $H^0(X, \mathcal{F}_r \otimes L) \to H^0(X, \mathcal{F}_r \otimes L \otimes \kappa(P))$ is surjective, that is, $\mathcal{F}_r \otimes L$ is globally generated at P. $\square$

The following lemma slightly improves [HSY, Lemma 4.9].

Lemma 3.3. *Let $L_1, \ldots, L_n$ be line bundles on an elliptic curve C of $\deg L_i \geq 2$ for $i = 1, \ldots, n$. Assume in addition that if $\deg L_1 = \cdots = \deg L_n = 2$, then $L_i \not\cong L_j$ for some i, j. Then the natural map*

$$H^0(C, L_1) \otimes \cdots \otimes H^0(C, L_n) \to H^0(C, L_1 \otimes \cdots \otimes L_n)$$

is surjective.

PROOF. The case that $\deg L_i \geq 3$ for $i = 1, \ldots, n$ is done in [HSY, Lemma 4.9]. Thus we may assume that $2 = \deg L_1 < \deg L_2$ or $\deg L_1 = \deg L_2 = 2$ and $L_1 \not\cong L_2$. Since L_1 is generated by global sections and $h^0(L_1) = 2$, we have a short exact sequence

$$0 \to L_1^{-1} \to H^0(C, L_1) \otimes \mathcal{O}_C \to L_1 \to 0.$$

The cohomology long exact sequence of this sequence tensorized by L_2 is

$$H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_1 \otimes L_2) \to H^1(C, L_1^{-1} \otimes L_2).$$

Since $H^1(C, L_1^{-1} \otimes L_2) = 0$ by our assumption, we obtain the surjectivity of the map $H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_1 \otimes L_2)$. On the other hand, since $\deg(L_1 \otimes L_2) \geq 3$, the map $H^0(C, L_1 \otimes L_2) \otimes H^0(C, L_3) \to H^0(C, L_1 \otimes L_2 \otimes L_3)$ is also surjective. Thus the map $H^0(C, L_1) \otimes H^0(C, L_2) \otimes H^0(C, L_3) \to H^0(C, L_1 \otimes L_2 \otimes L_3)$ is surjective. Now the required surjectivity follows inductively. $\square$

LEMMA 3.4. *Let $L_1, \ldots, L_n$ be line bundles on an elliptic curve C of $\deg L_i \geq 3$ for $i = 1, \ldots, n$. Then the natural map*

$$H^0(C, \mathcal{F}_r \otimes L_1) \otimes \cdots \otimes H^0(C, \mathcal{F}_r \otimes L_n) \to H^0(C, \mathcal{F}_r^{\otimes n} \otimes L_1 \otimes \cdots \otimes L_n)$$

is surjective.

PROOF. More generally, we prove the surjectivity of the map

$$H^0(C, \mathcal{F}_{r_1} \otimes L_1) \otimes \cdots \otimes H^0(C, \mathcal{F}_{r_n} \otimes L_n) \to H^0(C, \mathcal{F}_{r_1} \otimes \cdots \otimes \mathcal{F}_{r_n} \otimes L_1 \otimes \cdots \otimes L_n)$$

by n-fold induction on $r_1, \ldots, r_n$. First, the case $r_1 = \cdots = r_n = 1$ is nothing but Lemma 3.3. We now suppose $r_i > 1$ and prove that the above map is surjective assuming the surjectivity of the maps up to $r_1, \ldots, r_i - 1, \ldots, r_n$. For this purpose we may assume without loss of generality that $i = 1$. Let $V_{i,r} = H^0(C, \mathcal{F}_r \otimes L_i)$ and $H = L_1 \otimes \cdots \otimes L_n$. Since $H^1(C, L_1) = 0$, we can derive from exact sequence (1) the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & V_{1,1} \otimes V_{2,r_2} \otimes \cdots \otimes V_{n,r_n} & \to & V_{1,r_1} \otimes \cdots \otimes V_{n,r_n} & \to & V_{1,r_1-1} \otimes V_{2,r_2} \otimes \cdots \otimes V_{n,r_n} \to 0 \\
& \downarrow & & \downarrow & & \downarrow \\
0 \to & H^0(\mathcal{F}_{r_2} \otimes \cdots \otimes \mathcal{F}_{r_n} \otimes H) & \to & H^0(\mathcal{F}_{r_1} \otimes \cdots \otimes \mathcal{F}_{r_n} \otimes H) & \to & H^0(\mathcal{F}_{r_1-1} \otimes \mathcal{F}_{r_2} \otimes \cdots \otimes \mathcal{F}_{r_n} \otimes H)
\end{array}$$

Since the vertical maps on the left- and right-hand sides are surjective by induction hypothesis, the required surjectivity in the middle follows from the five-lemma. □

LEMMA 3.5. *Let H be a line bundle on an elliptic curve C of* $\deg H \geq 2r+1$ *for an integer $r \geq 1$. Then the natural map*

$$H^0(C, \mathcal{F}_r^{\otimes r} \otimes H) \to H^0(C, \det(\mathcal{F}_r) \otimes H) \cong H^0(C, H)$$

is surjective.

PROOF. Let L be any line bundle with $\deg L = 2$. Then $\mathcal{F}_r \otimes L$ is generated by its global sections by Lemma 3.2 (2). It then follows as in Step 2 of the proof of [Ha, Lemma 1.8] that

$$I_r = \operatorname{Ker}(\mathcal{F}_r^{\otimes r} \otimes L^r \to \det(\mathcal{F}_r \otimes L) \cong L^r)$$

is also generated by its global sections. Then $H^1(C, I_r \otimes H \otimes L^{-r}) = 0$ since $\deg H \otimes L^{-r} > 0$. Hence the required surjectivity follows from the exact sequence

$$H^0(C, \mathcal{F}_r^{\otimes r} \otimes H) \to H^0(C, \det(\mathcal{F}_r) \otimes H) \to H^1(C, I_r \otimes H \otimes L^{-r}))$$

associated to

$$0 \to I_r \otimes H \otimes L^{-r} \to \mathcal{F}_r^{\otimes r} \otimes H \to \det(\mathcal{F}_r) \otimes H \to 0. \ \square$$

LEMMA 3.6. *Let L be a line bundle on an elliptic curve C with* $\deg L \geq 2$*. Then the natural map*

$$\varphi_r \colon \bigwedge^r H^0(C, \mathcal{F}_r \otimes L) \to H^0(C, \det(\mathcal{F}_r \otimes L))$$

is surjective for all $r \geq 1$.

PROOF. When $\deg L \geq 3$, the assertion immediately follows from Lemmas 3.4 and 3.5.[1]

[1]Actually, we do not use the non-triviality of the extension (1) in this case. But we do use the non-triviality of the extension (1) in considering the case $\deg L = 2$ with direct computation.

Let $\deg L = 2$ and let $s, t \in H^0(C, L)$ be a basis of the 2-dimensional k-vector space $H^0(C, L)$. Let $U = C \backslash (s)_0$ (resp. $V = C \backslash (t)_0$), the complement of the divisor of zeros of s (resp. t). Then C is covered by the affine open subsets U, V.

We first consider the case $r = 2$. Let $\mathcal{E} = \mathcal{F}_2 \otimes L$. Then $\mathcal{E}$ is given by a non-trivial extension

$$0 \to L \to \mathcal{E} \to L \to 0, \tag{2}$$

which gives rise to an exact sequence

$$0 \to H^0(C, L) \xrightarrow{\imath} H^0(C, \mathcal{E}) \xrightarrow{\rho} H^0(C, L) \to 0.$$

We choose a basis s_1, s_2, t_1, t_2 of the 4-dimensional vector space $H^0(C, \mathcal{E})$ so that s and t map to s_1 and t_1 under $\imath$ and s_2 and t_2 map to s and t under ρ, respectively. Then s_1, s_2 (resp. t_1, t_2) give a local basis of $\mathcal{E}$ on U (resp. V), and there exist regular functions f, g on U such that $t_1 = f s_1$ and $t_2 = g s_1 + f s_2$. Then the image of the map $\bigwedge^2 H^0(C, \mathcal{E}) \to H^0(C, \det \mathcal{E})$ contains $s_1 \wedge s_2$, $t_1 \wedge s_2 = f(s_1 \wedge s_2)$, $t_1 \wedge t_2 = f^2(s_1 \wedge s_2)$, $t_2 \wedge s_2 = g(s_1 \wedge s_2)$. Since $H^0(C, \det \mathcal{E}) \cong H^0(C, L^2)$ is 4-dimensional, it is sufficient to show the following

Claim. $1, f, f^2, g$ are linearly independent over k.

To prove the claim, note that $f \notin k$, since s_1, t_1 are linearly independent over k. Hence f is transcendental over the algebraically closed field k, so that $1, f, f^2$ are linearly independent over k. Thus, if the claim fails, then there exist $a, b, c \in k$ such that $g = a + bf + cf^2$. Then we define an $\mathcal{O}_U$-module homomorphism $\phi_U \colon L|_U = \mathcal{O}_U s \to \mathcal{E}|_U$ and an $\mathcal{O}_V$-module homomorphism $\phi_V \colon L|_V = \mathcal{O}_V t \to \mathcal{E}|_V$ by

$$\phi_U(s) = s_2 + ct_1 \text{ and } \phi_V(t) = t_2 - as_1 - bt_1.$$

Then ϕ_U and ϕ_V give splittings of the surjection $\mathcal{E} \to L$ on U and V, respectively, and

$$\phi_U(t) = \phi_U(fs) = f(s_2 + ct_1) = t_2 - gs_1 + cf^2 s_1 = t_2 - (a + bf)s_1 = \phi_V(t)$$

on $U \cap V$. Thus ϕ_U and ϕ_V glue together to give a global splitting $\phi \colon L \to \mathcal{E}$ of the surjection $\mathcal{E} \to L$. This contradicts to the non-triviality of the extension (2) and the claim follows.

Next let $r \geq 3$. In view of the exact sequence

$$0 \to H^0(C,L) \xrightarrow{\imath} H^0(C,\mathcal{F}_r \otimes L) \xrightarrow{\rho} H^0(C,\mathcal{F}_{r-1} \otimes L) \to 0,$$

we have the following diagram, whose commutativity is verified with an explicit computation.

$$\begin{array}{ccc}
H^0(L) \otimes \bigwedge^{r-1} H^0(\mathcal{F}_r \otimes L) & \xrightarrow{\mathrm{id}\otimes\bigwedge^{r-1}\rho} & H^0(L) \otimes \bigwedge^{r-1} H^0(\mathcal{F}_{r-1} \otimes L) \\
{\scriptstyle \imath\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\varphi_{r-1}} \\
H^0(\mathcal{F}_r \otimes L) \otimes \bigwedge^{r-1} H^0(\mathcal{F}_r \otimes L) & & H^0(L) \otimes H^0(\det(\mathcal{F}_{r-1} \otimes L)) \\
\downarrow & & \downarrow{\scriptstyle\cong} \\
\bigwedge^{r} H^0(\mathcal{F}_r \otimes L) & & H^0(L) \otimes H^0(L^{r-1}) \\
{\scriptstyle \varphi_r}\downarrow & & \downarrow \\
H^0(\det(\mathcal{F}_r \otimes L)) & \xrightarrow{\cong} & H^0(L^r)
\end{array}$$

Here the upper horizontal map $\mathrm{id} \otimes \bigwedge^{r-1}\rho$ is surjective since ρ is, and the map $\mathrm{id} \otimes \varphi_{r-1}$ is surjective since so is φ_{r-1} by induction. Since $r \geq 3$, it follows from Lemma 3.3 that the multiplication map $H^0(C,L) \otimes H^0(C,L^{r-1}) \to H^0(C,L^r)$ is also surjective. Thus the map φ_r is surjective as required. □

Proof of Theorem 3.1. Suppose that C is an ordinary elliptic curve. Then $F^e_*\mathcal{O}_C$ is a direct sum of $q = p^e$ non-isomorphic q-torsion line bundles by Lemma 2.8(1). Hence $L \otimes F^e_*\mathcal{O}_C$ is a direct sum of q non-isomorphic line bundles $L = L_1, L_2, \ldots, L_q$ of degree equal to $\deg L \geq 2$. Then the vector space $\bigwedge^q H^0(C, L \otimes F^e_*\mathcal{O}_C)$ contains a subspace isomorphic to $H^0(C,L_1) \otimes \cdots \otimes H^0(C,L_q)$, which surjects onto $H^0(C, L_1 \otimes \cdots \otimes L_q) \cong H^0(C, \det(L \otimes F^e_*\mathcal{O}_C))$ by Lemma 3.3.

If C is supersingular, then $F^e_*\mathcal{O}_C \cong \mathcal{F}_{p^e}$ by Lemma 2.8(2), and the result follows from Lemma 3.6. □

Corollary 3.7. *Let L be a line bundle on an elliptic curve C with $\deg L \geq 2$. Then for any $n \geq r \geq 2$, the natural map*

$$\bigwedge^r H^0(C,\mathcal{F}_r \otimes L) \otimes H^0(C,L)^{\otimes n-r} \to H^0(C,\det(\mathcal{F}_r) \otimes L^n) \cong H^0(C,L^n)$$

is surjective.

Proof. This follows from Lemmas 3.3 and 3.6. □

4. Comparing "Partial" F-Blowups with the Minimal Resolution

By "partial" F-blowup, we mean the blowup of $X = \operatorname{Spec} R$ at a direct summand of $R^{1/q}$ of the form $M = [R^{1/q}]_{i/q \bmod \mathbb{Z}}$. If a partial F-blowup $\mathrm{Bl}_M(X)$ coincides with the minimal resolution $\widetilde{X}$, then we can study the F-blowup as a further blowup of $\widetilde{X}$. For this purpose we study partial F-blowups in two cases: We treat the case $E^2 \le -2$ with the blowup at $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ in subsection 4.a and the case $E^2 = -1$ with $[R^{1/q}]_{\frac{q-1}{q} \bmod \mathbb{Z}}$ in subsection 4.b.

4.a.

We work under the notation in the setup (2.b). In this subsection we will study the structure of the blowup of $X = \operatorname{Spec} R$ at the R-summand

$$[R^{1/q}]_{0 \bmod \mathbb{Z}} \cong \bigoplus_{m \ge 0} H^0(C, L^m \otimes F^e_* \mathcal{O}_C)$$

of $R^{1/q}$, under the assumption that $E^2 \le -2$. It follows from Lemma 2.8 that if C is supersingular, then $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ is isomorphic to $\Gamma_*(\mathcal{F}_q) = \bigoplus_{n \ge 0} H^0(C, \mathcal{F}_q \otimes L^n) t^n$ as a graded R-module. In order to study the blowup at this module, we put for the moment

$$M_r = \Gamma_*(\mathcal{F}_r) = \bigoplus_{n \ge 0} H^0(C, \mathcal{F}_r \otimes L^n) t^n$$

for each $r \ge 1$ and regard its torsion-free pullback $\widetilde{M_r} = f^\star M_r$ to the minimal resolution $\widetilde{X}$ of $X = \operatorname{Spec} R$ as a subsheaf of

$$\mathcal{M}_r = \bigoplus_{n \ge 0} (\mathcal{F}_r \otimes L^n) t^n.$$

Note that $M_r = H^0(\widetilde{X}, \widetilde{M_r})$ since M_r is a reflexive R-module, and $\widetilde{M_r}$ is locally free by our assumption that $\deg L = -E^2 \ge 2$; see Lemma 2.4 (1).

Lemma 4.1. *Suppose that the self-intersection number of $E \subset \widetilde{X}$ is $E^2 \le -2$ and let $M_r = \Gamma_*(\mathcal{F}_r)$ as above. Then* $\det \widetilde{M_r} = \mathcal{O}_{\widetilde{X}}((1-r)E)$ *and the natural map*

$$\varphi_r \colon \bigwedge^r M_r \to H^0(\widetilde{X}, \det \widetilde{M_r})$$

is surjective.

Proof. Let $V \subset C$ be an open subset on which L and $\mathcal{F}_r$ trivialize and let s be a local basis of L on V. We choose a local basis $e_0, \dots, e_{r-1}$ of $\mathcal{F}_r|_V$ with respect to the exact sequence restricted from (1),

$$0 \to \mathcal{O}_V \to \mathcal{F}_r|_V \to \mathcal{F}_{r-1}|_V \to 0$$

as follows: e_0 is a local basis of $\mathcal{O}_C \subset \mathcal{F}_r$ corresponding to its global section 1, which we also denote by $e_0 = 1 \in H^0(C, \mathcal{O}_C)$, and $e_1, \dots, e_{r-1}$ give a local basis $\bar{e}_1, \dots, \bar{e}_{r-1}$ of $\mathcal{F}_{r-1}|_V$. Let $U = \pi^{-1}V \subset \widetilde{X}$. Then, with the local trivialization $L|_V = \mathcal{O}_V s$ and $\mathcal{F}_r|_V = \bigoplus_{i=0}^{r-1} \mathcal{O}_V e_i \cong \mathcal{O}_V^{\oplus r}$ as above, we have

$$\mathcal{M}_r|_U = \bigoplus_{i=0}^{r-1} \mathcal{O}_U e_i \cong \mathcal{O}_U^{\oplus r},$$

where $\mathcal{O}_U = \bigoplus_{n \geq 0} (L|_V)^n t^n = \bigoplus_{n \geq 0} \mathcal{O}_V (st)^n = \mathcal{O}_V[st]$. On the other hand, the degree zero piece of M_r is $H^0(C, \mathcal{F}_r) = H^0(C, \mathcal{O}_C)$ and the positively graded parts of $\widetilde{M_r}$ and $\mathcal{M}_r$ coincide since $\mathcal{F}_r \otimes L^n$ is generated by global sections for $n \geq 1$ by Lemma 3.2 (2). Hence $\widetilde{M_r} = \mathcal{O}_E e_0 \oplus \bigoplus_{n \geq 1} (\mathcal{F}_r \otimes L^n) t^n$ and

$$\widetilde{M_r}|_U = \mathcal{O}_U \langle e_0, ste_1, \dots, ste_{r-1} \rangle.$$

Thus $\det(\widetilde{M_r})|_U = \mathcal{O}_U (st)^{r-1} e_0 \wedge \cdots \wedge e_{r-1}$, from which we obtain

$$\det \widetilde{M_r} \cong \mathcal{O}_{\tilde{X}}((1-r)E) \otimes \pi^* \det \mathcal{F}_r \cong \mathcal{O}_{\tilde{X}}((1-r)E) = \bigoplus_{n \geq r-1} L^n t^n.$$

Now, to prove the surjectivity of the map φ_r, note that the target $H^0(X, \det \widetilde{M_r})$ of φ_r sits in degree $n \geq r-1$ and its n-th graded piece is $H^0(C, L^n) t^n \cong H^0(C, L^n)$. Then it immediately follows from Corollary 3.7 that φ_r is surjective in degree $n \geq r$.

To show the surjectivity in degree $n = r-1$, note that $\bigwedge^r M_r$ contains the vector subspace

$$H^0(C, \mathcal{F}_r) \otimes \bigwedge^{r-1} H^0(C, \mathcal{F}_r \otimes L).$$

On the other hand, it follows from Lemma 3.6 that the natural map

$$\bigwedge^{r-1} H^0(C, \mathcal{F}_{r-1} \otimes L) \to H^0(C, \det(\mathcal{F}_{r-1} \otimes L)) \cong H^0(C, L^{r-1})$$

is surjective. Now the surjectivity of φ_r in degree $r-1$ follows from the surjectivity of $H^0(C, \mathcal{F}_r \otimes L) \to H^0(C, \mathcal{F}_{r-1} \otimes L)$ and the identification $H^0(C, \mathcal{O}_C) = H^0(C, \mathcal{F}_r)$. □

COROLLARY 4.2. *Suppose that the self-intersection number of $E \subset \widetilde{X}$ is $E^2 \le -2$ and let $M_q = [R^{1/q}]_{0 \bmod \mathbb{Z}}$ for any $q = p^e$. Then the natural map*

$$\varphi_q \colon \bigwedge^q M_q \to H^0(\widetilde{X}, \det \widetilde{M_q})$$

is surjective.

PROOF. It remains to consider the F-pure case. For this purpose we decompose $F^e_* \mathcal{O}_C$ into the direct sum of q non-isomorphic q-torsion line bundles $\mathcal{O}_C = L_1, L_2, \ldots, L_q$. Then $M_q = \bigoplus_{i=1}^q J_i$, where $J_1 = R$, and for $2 \le i \le q$,

$$J_i = \bigoplus_{n \ge 1} H^0(C, L_i \otimes L^n) t^n$$

with the torsion-free pullback $\widetilde{J_i} = \bigoplus_{n \ge 1} (L_i \otimes L^n) t^n$ to $\widetilde{X}$. We then have that $\det \widetilde{M_q} = \bigoplus_{n \ge q-1} (L_1 \otimes \cdots \otimes L_q \otimes L^n) t^n$, and the required surjectivity is an easy consequence of Lemma 3.3. □

We are now able to improve Theorem 2.7 (3).

THEOREM 4.3. *Let (X, x) be a simple elliptic singularity with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$ with $E^2 \le -2$.*

(1) *Suppose either that $E^2 \le -2$ and $p^e \ge 3$, or that $E^2 \le -3$ and $p^e \ge 2$. Then the blowup of X at $[R^{1/p^e}]_{0 \bmod \mathbb{Z}}$ coincides with the minimal resolution $\widetilde{X}$.*

(2) *If $-E^2$ is not a power of the characteristic p, then $\mathrm{FB}_e(X) \cong \widetilde{X}$ for all $e \ge 1$.*

PROOF. (1) Let Y be the blowup of X at the R-module $[R^{1/p^e}]_{0 \bmod \mathbb{Z}}$. We know that $\widetilde{X}$ dominates Y by Lemma 2.4 (1) and that $Y \to X$ has an exceptional curve, which must be the image of E, since $[R^{1/p^e}]_{0 \bmod \mathbb{Z}}$ is not flat; see the proof of [HSY, Corollary 4.3]. This implies that $\widetilde{X}$ is the normalization of Y. On the other hand, it follows from Lemma 4.1 and Corollary 4.2 that Y is the blowup of $X = \operatorname{Spec} R$ at a fractional ideal I of the form

$$I = \bigoplus_{n \geq q-1} H^0(C, L_0 \otimes L^n)t^n,$$

where L_0 is a line bundle on C of degree zero. Thus, to prove (1) it is sufficient to show that the Rees algebra $R[IT]$ is normal. This follows from Proposition 2.6 since $(q-1)\deg L \geq 3$ by our assumption.

(2) follows from (1) and Lemma 2.4 (2). $\square$

4.b.

In this subsection, we assume that $\deg L = -E^2 = 1$ in the notation of (2.b), that is, $X = \operatorname{Spec} R$ has a simple elliptic singularity of type $\widetilde{E}_8$. We will study the structure of the blowup of X at the R-summand

$$[R^{1/q}]_{\frac{q-1}{q} \bmod \mathbb{Z}} \cong \bigoplus_{m \geq 0} H^0(C, L^m \otimes F_*^e L^{q-1})t^m$$

of $R^{1/q}$. We denote this module by $M = M_q$ throughout this subsection and embed its torsion-free pullback $\widetilde{M}$ to the minimal resolution $\widetilde{X}$ into the graded $\mathcal{O}_{\widetilde{X}}$-module

$$\mathcal{M} = \bigoplus_{m \geq 0} (L^m \otimes F_*^e L^{q-1})t^m$$

as in 4.a.

Lemma 4.4. *For a power $q = p^e \geq 2$ of p, let $\widetilde{M}$ denote the torsion-free pullback of the R-module $M = M_q := [R^{1/q}]_{\frac{q-1}{q} \bmod \mathbb{Z}}$ to the minimal resolution $\widetilde{X}$. Then $\det \widetilde{M} \cong \mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^* \det F_*^e(L^{q-1})$ and the natural map*

$$\varphi_q \colon \bigwedge^q M \to H^0(\widetilde{X}, \det \widetilde{M})$$

is surjective.

Proof. First note that the cokernel of the map $H^0(C, F_*^e L^{q-1}) \otimes \mathcal{O}_C \to F_*^e(L^{q-1})$ is locally free of rank one as claimed in the proof of [HSY, Lemma 4.1]. Hence we have an exact sequence

$$0 \to \mathcal{O}_C^{\oplus q-1} \xrightarrow{\alpha} F_*^e(L^{q-1}) \to L' \to 0$$

with L' a line bundle on C, and it is easy to see that $\deg L' = q-1$ by Riemann-Roch. As before, let $V \subset C$ be an open subset on which L and L' trivialize and let s be a local basis of L on V. We choose a local basis $e_1, \dots, e_q$ of $F_*^e(L^{q-1})|_V$ so that $e_1, \dots, e_{q-1}$ is the standard global basis of $\mathrm{Im}(\alpha) \cong \mathcal{O}_C^{\oplus q-1}$ corresponding to a k-basis of $H^0(C, F_*^e L^{q-1}) = H^0(C, L^{q-1})$ and e_q maps to a local basis of L'. Let $U = \pi^{-1}V$ and embed $\widetilde{M}|_U$ into $\mathcal{M}|_U = \bigoplus_{i=1}^q \mathcal{O}_U e_i$, where $\mathcal{O}_U = \mathcal{O}_V[st]$. We now express homogeneous generators of $\widetilde{M}|_U$ in each degree with the local basis $e_1, \dots, e_q$ of $\mathcal{M}|_U$. It is clear that $\widetilde{M}|_U$ has generators $e_1, \dots, e_{q-1}$ in degree zero. In order to find a new generator in degree one, we look at the exact sequence

$$0 \to H^0(C, L^{\oplus q-1}) \to H^0(C, L \otimes F_*^e L^{q-1}) \to H^0(C, L \otimes L') \to 0.$$

Here elements of degree one coming from $H^0(C, L^{\oplus q-1})$ are generated by $e_1, \dots, e_{q-1}$. Thus a new generator in degree one lifts from a global section in $H^0(C, L \otimes L')$ that generates $(L \otimes L')|_U$; it has a local expression $st(e_q + \sum_{i=1}^{q-1} a_i e_i)$ with $a_i \in \mathcal{O}_V$. Finally, $\widetilde{M}|_U$ and $\mathcal{M}|_U$ coincide and are generated by $(st)^m e_1, \dots, (st)^m e_q$ in degree $m \geq 2$. Hence there are no new generators in degree $m \geq 2$. Consequently we have

$$\widetilde{M}|_U = \mathcal{O}_U \langle e_1, \dots, e_{q-1}, st e_q \rangle.$$

It follows that $\det(\widetilde{M})|_U = \mathcal{O}_U(st) e_1 \wedge \cdots \wedge e_q$, and

$$\det \widetilde{M} \cong \mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^* \det(F_*^e L^{q-1}) \cong \mathcal{O}_{\widetilde{X}}(-E) \otimes \pi^* L' \cong \bigoplus_{n \geq 1} (L' \otimes L^n) t^n.$$

Now the surjectivity of the map

$$\varphi_q \colon \bigwedge^q M \to H^0(\widetilde{X}, \det \widetilde{M}) \cong \bigoplus_{n \geq 1} H^0(C, L' \otimes L^n) t^n$$

follows easily: For any $\lambda \in H^0(C, L' \otimes L^n)$ with $n \geq 1$, let $\widetilde{\lambda}$ be its lifting to $H^0(C, L^n \otimes F_*^e L^{q-1})$. Then $e_1 \wedge \ldots \wedge e_{q-1} \wedge \widetilde{\lambda} t^n \in \bigwedge^q M$ maps to λt^n via φ_q. $\square$

PROPOSITION 4.5. *Let (X, x) be a simple elliptic singularity of type $\widetilde{E}_8$ and suppose $q = p^e \geq 3$. Then the minimal resolution $\widetilde{X}$ coincides with the the blowup of $X = \operatorname{Spec} R$ at the R-module $M_q = [R^{1/q}]_{\frac{q-1}{q} \bmod \mathbb{Z}}$.*

Proof. By the previous lemma, the fractional ideal $I = I_M$ attached to $M = M_q$ is $I = H^0(\widetilde{X}, \det \widetilde{M}) \cong \bigoplus_{n\geq 1} H^0(C, L' \otimes L^n)t^n$. Since $\deg(L' \otimes L) \geq q \geq 3$ by our assumption, the result follows from Proposition 2.6 as in Theorem 4.3. $\square$

5. The Case $-E^2$ is a Power of p

In this section we consider the exceptional case $d \mid q = p^e$ to prove Theorem 1.2. In this case, it turns out that the F-blowup sequence does not stabilize.

5.a.

Let $q = p^e$ with $e \geq 1$ and suppose that q is divisible by $d = -E^2$. Then by Lemma 2.4, the R-module $R^{1/q}$ has a unique summand that is not flattened by torsion-free pullback to $\widetilde{X}$, that is, $[R^{1/q}]_{0 \bmod \mathbb{Z}}$ if $d = 1$; and $[R^{1/q}]_{1/d \bmod \mathbb{Z}}$ if $d \geq 2$. If we further assume that $q = p^e \geq 3$, then

(a) $\mathrm{FB}_e(X)$ is the blowup of $\widetilde{X}$ at the $\mathcal{O}_{\widetilde{X}}$-module $f^\star[R^{1/q}]_{0 \bmod \mathbb{Z}}$ if $d = 1$; and

(b) $\mathrm{FB}_e(X)$ is the blowup of $\widetilde{X}$ at the $\mathcal{O}_{\widetilde{X}}$-module $f^\star[R^{1/q}]_{1/d \bmod \mathbb{Z}}$ if $d \geq 2$,

by Proposition 2.2 (2), Theorem 4.3 (1) and Proposition 4.5. Case (a) is already treated in the proof of [HSY, Theorems 4.5 and 4.13]: The case $E^2 = -1$ in Theorem 1.2 follows from Proposition 4.5 and the descriptions of the torsion-free pullback $f^\star[R^{1/q}]_{0 \bmod \mathbb{Z}}$ in [HSY, Lemma 4.4 (1) and subsection 4b1]. Thus we assume in addition that $E^2 \leq -2$ and let $i = q/d$. Then $1 \leq i \leq q-1$, and Lemma 2.4 (3) tells us that

$$f^\star[R^{1/q}]_{1/d \bmod \mathbb{Z}} = f^\star[R^{1/q}]_{i/q \bmod \mathbb{Z}}$$

is flat exactly off the points $P \in E \subset \widetilde{X}$ such that $L^i \cong \mathcal{O}_C(qP)$, where we identify points on C and E via $C \cong E$. Fix one such point $P_0 \in E$ as the zero element of the group law of $E \cong C$. Then $L^i \cong \mathcal{O}_C(qP)$ if and only if P is a q-torsion point. It is well-known that there are exactly q distinct q-torsion points if C is ordinary and that there is no non-trivial q-torsion point if C is supersingular. By this reason the structure of F-blowups differs according to whether $R = R(C, L)$ is F-pure or not.

Now to study the structure of the torsion-free pullback $f^\star[R^{1/q}]_{1/d \bmod \mathbb{Z}}$, note that $F^e_*L^{q/d} \cong \mathcal{O}_C(P_0) \otimes F^e_*\mathcal{O}_C$ since $L^{q/d} \cong \mathcal{O}_C(qP_0)$ by our assumption, so that

$$[R^{1/q}]_{1/d \bmod \mathbb{Z}} \;\cong\; \bigoplus_{m \geq 0} H^0(C, L^m(P_0) \otimes F^e_*\mathcal{O}_C)t^m. \tag{3}$$

We first consider the F-pure case.

Theorem 5.1. *Let (X, x) be an F-pure simple elliptic singularity with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$ such that $-E^2 = p^n$ for an integer $n \geq 1$. Fix a point $P_0 \in E$ such that $\mathcal{O}_{\widetilde{X}}(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(p^nP_0)$, and for a power $q = p^e$ of p, let $Z_e = \{P_0, P_1, \dots, P_{q-1}\} \subset \widetilde{X}$ be the set of q-torsion points on $E \subset \widetilde{X}$ with respect to the group structure (E, P_0). Then for all $e \geq n$, the normalization $\widetilde{\mathrm{FB}}_e(X)$ of the e-th F-blowup $\mathrm{FB}_e(X)$ coincides with the blowup $\mathrm{Bl}_{Z_e}(\widetilde{X})$ of $\widetilde{X}$ at the q-torsion points. Moreover, we have*

$$\mathrm{FB}_e(X) \cong \mathrm{Bl}_{Z_e}(\widetilde{X})$$

except for the case $p = 2$, $e = 1$.

Proof. Let $d = \deg L = p^n \geq 2$. We will look at the torsion-free pullback to $\widetilde{X}$ of the R-module $[R^{1/q}]_{1/d \bmod \mathbb{Z}}$ as in (3). Since $C \cong E$ is ordinary by the F-purity, $F^e_*\mathcal{O}_C$ splits into line bundles as $F^e_*\mathcal{O}_C \cong \bigoplus_{i=0}^{q-1} \mathcal{O}_C(P_i - P_0)$; see Lemma 2.8 and also [HSY]. Accordingly we have a splitting $[R^{1/q}]_{1/d \bmod \mathbb{Z}} = \bigoplus_{i=0}^{q-1} J_i$ into q non-isomorphic reflexive R-modules $J_0, J_1, \dots, J_{q-1}$ of rank one, where

$$J_i = \bigoplus_{m \geq 0} H^0(C, \mathcal{O}_C(P_i) \otimes L^m).$$

As in the proof of [HSY, Lemma 4.4] we see that the torsion-free pullback of J_i is

$$\begin{aligned} f^\star J_i &= \mathrm{Im}\left(\bigoplus_{m \geq 0} H^0(C, \mathcal{O}_C(P_i) \otimes L^m) \otimes \mathcal{O}_C \to \bigoplus_{m \geq 0} \mathcal{O}_C(P_i) \otimes L^m\right) \\ &= \mathcal{O}_C \oplus \bigoplus_{m \geq 1} \mathcal{O}_C(P_i) \otimes L^m \subset \bigoplus_{m \geq 0} \mathcal{O}_C(P_i) \otimes L^m \cong \pi^*\mathcal{O}_C(P_i), \end{aligned}$$

where $\mathcal{O}_C \subset \mathcal{O}_C(P_i)$ is the graded part of degree $m = 0$. Thus we have the following exact sequence of $\mathcal{O}_{\widetilde{X}}$-modules:

$$0 \to f^\star J_i \to \pi^* \mathcal{O}_C(P_i) \to \kappa(P_i) \to 0.$$

It follows that $f^\star J_i \cong \mathcal{I}_{P_i} \otimes \pi^* \mathcal{O}_C(P_i)$, where $\mathcal{I}_{P_i} \subset \mathcal{O}_{\widetilde{X}}$ is the ideal sheaf defining P_i viewed as a point on $E \subset \widetilde{X}$. Hence $f^\star J_i$ is flattened by blowing up at $P_i \in \widetilde{X}$. It follows that the normalized F-blowup $\widetilde{\mathrm{FB}}_e(X)$ is obtained by blowing up the points $P_0, \ldots, P_{q-1}$; see the proof of [HSY, Corollary 4.3]. If we assume further that $q = p^e \geq 3$, then $\mathrm{FB}_e(X) \cong \mathrm{Bl}_{Z_e}(\widetilde{X})$ by Proposition 2.2(2) and Theorem 4.3(1). □

Remark 5.2. In the exceptional case $p = 2$, $e = 1$ of the theorem, the normality of the F-blowup may break down. Indeed, we have an example of $\widetilde{E}_7$-singularity in characteristic 2 whose first F-blowup is not normal [HSY, Example 4.10]. In this example, the exceptional set of the first F-blowup consists of three $\mathbb{P}^1$'s, one of which is the image of the elliptic exceptional curve on $\widetilde{\mathrm{FB}}_1(X)$.

5.b. Non-F-pure case

We now consider the non-F-pure case. We assume that the exceptional elliptic curve $E \cong C$ is supersingular with $-E^2 = p^n \geq 2$ throughout the remainder of this subsection. Then we have a unique point $P_0 \in C$ such that $L \cong \mathcal{O}_{\widetilde{X}}(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_C(p^n P_0)$, since the multiplication by p^n on the group structure of $C = \mathrm{Jac}\, C$ induces a purely inseparable endomorphism on C.

For each $r > 0$ let $\mathcal{F}_r$ be Atiyah's indecomposable bundle of rank r on C. To proceed along the same line as in [HSY, 4b1], we note that $\mathcal{F}_r$ is self-dual, and consider the dual sequence

$$0 \to \mathcal{F}_{r-1} \to \mathcal{F}_r \to \mathcal{O}_C \to 0 \tag{4}$$

of the non-split exact sequence (1) in Section 2. We consider the graded R-module

$$M_r = \bigoplus_{m \geq 0} H^0(C, \mathcal{F}_r(P_0) \otimes L^m) t^m$$

and embed its torsion-free pullback $\widetilde{M_r} = f^\star M_r$ into $\mathcal{M}_r = \bigoplus_{m \geq 0} (\mathcal{F}_r(P_0) \otimes L^m) t^m$. If $q = p^e \geq d = p^n$ then by Lemma 2.8, M_q is isomorphic to the R-module $[R^{1/q}]_{1/d \bmod \mathbb{Z}}$ under consideration.

We fix any point $P \in C$ and let $V \subset C$ be a sufficiently small open neighborhood of P on which L, $\mathcal{O}_C(P_0)$ and $\mathcal{F}_r$ trivialize. We choose a local basis $e_1, \ldots, e_r$ of $\mathcal{F}_r$ on V inductively as follows. For $r = 1$, let e_1 be a (local) basis of $\mathcal{F}_1 = \mathcal{O}_C$ corresponding to its global section $1 \in H^0(C, \mathcal{O}_C)$. For $r \geq 2$, we think of $\mathcal{F}_{r-1}$ as a subbundle of $\mathcal{F}_r$ via the exact sequence (4), and extend the local basis $e_1, \ldots, e_{r-1}$ of $\mathcal{F}_{r-1}$ on V to a local basis $e_1, \ldots, e_r$ of $\mathcal{F}_r$.

Let $U = \pi^{-1}V \subset \widetilde{X}$. Then, with the local trivialization $L|_V \cong \mathcal{O}_C(P_0)|_V \cong \mathcal{O}_V$ and $\mathcal{F}_r|_V \cong \bigoplus_{i=1}^r \mathcal{O}_V e_i \cong \mathcal{O}_V^{\oplus r}$ as above, we have

$$\mathcal{M}_r|_U \cong \bigoplus_{i=1}^{r} \mathcal{O}_U e_i \cong \mathcal{O}_U^{\oplus r},$$

where $\mathcal{O}_U = \bigoplus_{n \geq 0} (L|_V)^n t^n \cong \bigoplus_{n \geq 0} \mathcal{O}_V t^n = \mathcal{O}_V[t]$. Note that the fiber coordinate t and a regular parameter u at $P \in C \cong E$ form a system of coordinates of U. With this notation we shall express generators of the $\mathcal{O}_U$-module $\widetilde{M_r}|_U \subseteq \mathcal{M}_r|_U$, which come from homogeneous elements of the graded R-module M_r.

First note that the graded parts of $\widetilde{M_r}|_U$ and $\mathcal{M}_r|_U$ coincide in degree ≥ 1 and are generated by $te_1, \ldots, te_r$, since $\mathcal{F}_r(P_0) \otimes L^n$ is generated by global sections for $n \geq 1$. It remains to consider the contribution of the degree zero piece $[M_r]_0 = H^0(C, \mathcal{F}_r(P_0))$ to the generation of $\widetilde{M_r}|_U$. To this end, note that we have an exact sequence

$$0 \to H^0(C, \mathcal{F}_i(P_0)) \to H^0(C, \mathcal{F}_{i+1}(P_0)) \to H^0(C, \mathcal{O}(P_0)) \to 0$$

for $1 \leq i \leq r-1$, via which we regard $H^0(C, \mathcal{F}_i(P_0))$ as a subspace of $H^0(C, \mathcal{F}_r(P_0))$. Then, since $h^0(\mathcal{F}_i(P_0)) = i$ by Riemann-Roch, we can choose a basis $s_1, \ldots, s_r$ of $H^0(C, \mathcal{F}_r(P_0))$ so that $s_1, \ldots, s_i$ form a basis of $H^0(C, \mathcal{F}_i(P_0))$ for $1 \leq i \leq r$. It also follows from exact sequence (4)$\otimes \mathcal{O}_C(P_0)$ that the global sections $s_1, \ldots, s_i$ generate $\mathcal{F}_i(P_0)$ on $C \setminus \{P_0\}$, so that they give a basis of $\mathcal{F}_i(P_0) \otimes K$ as a vector space over the function field K of C. On the other hand, $e_1, \ldots, e_i$ can also be viewed as a basis of $\mathcal{F}_i(P_0) \otimes K \cong K^{\oplus i}$ under the local trivialization $\mathcal{F}_i(P_0)|_V \cong \bigoplus_{j=1}^i \mathcal{O}_V e_j \cong \mathcal{O}_V^{\oplus i}$ induced from $\mathcal{F}_i|_V \cong \mathcal{O}_V^{\oplus i}$ and $\mathcal{O}_E(P_0)|_V \cong \mathcal{O}_V$. We will compare the basis consisting of $s_i \otimes 1$ and the standard basis $e_1, \ldots, e_r$ of $\mathcal{F}_r(P_0) \otimes K \cong K^{\oplus r}$ using the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & H^0(\mathcal{F}_{i-1}(P_0)) \otimes \mathcal{O}_V & \to & H^0(\mathcal{F}_i(P_0)) \otimes \mathcal{O}_V & \to & H^0(\mathcal{O}_C(P_0)) \otimes \mathcal{O}_V & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 \to & \mathcal{F}_{i-1}(P_0)|_V & \to & \mathcal{F}_i(P_0)|_V & \to & \mathcal{O}_C(P_0)|_V & \to 0 \\
& \downarrow\cong & & \downarrow\cong & & \downarrow\cong & \\
0 \to & \mathcal{O}_V^{\oplus i-1} & \to & \mathcal{O}_V^{\oplus i} & \to & \mathcal{O}_V & \to 0
\end{array}$$

Suppose now that $P = P_0$. Since $\mathrm{Bs}|\mathcal{O}_C(P_0)| = \{P_0\}$, we may choose a regular parameter u at $P_0 \in C$ so that $s_1 \otimes 1 = u$. It then follows from the above diagram that

$$s_i \otimes 1 = ue_i + \sum_{j=1}^{i-1} a_{i,j} e_j,$$

where a_{ij}'s are local regular functions on V. Arguing with the non-triviality of the extension (4) as in [HSY, 4b1 (4)], we see that we can replace $s_1, \ldots, s_r$ so that they satisfy the condition:

(5) $\quad u | a_{i,j}$ for $1 \le j \le i-2$ but $a_{i,i-1}$ is not divisible by u.

Therefore, local generators of $\widetilde{M}_r$ on a neighborhood U_0 of P_0 are described as

$$\begin{aligned}
\widetilde{M}_r|_{U_0} &= \mathcal{O}_{U_0}\langle ue_1, ue_i + a_{i,i-1}e_{i-1}, te_i \,|\, 2 \le i \le r\rangle \\
&= \mathcal{O}_{U_0}\langle ue_1, ue_i + a_{i,i-1}e_{i-1}, te_r \,|\, 2 \le i \le r\rangle,
\end{aligned}$$

where $a_{i,i-1}(P_0) \ne 0$. Accordingly the ideal $\mathcal{I}_{\widetilde{M}_r} \subset \mathcal{O}_{\widetilde{X}}$ defined in Section 2 has the following local expression:

$$\mathcal{I}_{\widetilde{M}_r}|_{U_0} \cong (t, u^r).$$

If $P_0 \ne P \in U$ then $\widetilde{M}_r|_U = \mathcal{O}_U\langle e_1, \ldots, e_r\rangle \cong \mathcal{O}_U^{\oplus r}$ by a similar argument. As in Theorem 5.1 we are led to the following

Theorem 5.3. *Let (X, x) be a non-F-pure simple elliptic singularity with the elliptic exceptional curve E on the minimal resolution $\widetilde{X}$ such that $-E^2 = p^n$ for an integer $n \ge 1$. Let $P_0 \in E$ be the point such that $\mathcal{O}_{\widetilde{X}}(-E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(p^n P_0)$, and let $\mathcal{I}_e \subseteq \mathcal{O}_{\widetilde{X}}$ be the ideal sheaf defining a fat point supported at $P_0 \in \widetilde{X}$ whose local expression at P_0 is*

$$(\mathcal{I}_e)_{P_0} = (t, u^{p^e})$$

as above. Then for all $e \geq n$, the normalization $\widetilde{\mathrm{FB}}_e(X)$ *of the e-th F-blowup* $\mathrm{FB}_e(X)$ *coincides with the blowup* $\mathrm{Bl}_{\mathcal{I}_e}(\widetilde{X})$ *of* $\widetilde{X}$ *at* $\mathcal{I}_e$*. Moreover, we have*

$$\mathrm{FB}_e(X) \cong \mathrm{Bl}_{\mathcal{I}_e}(\widetilde{X})$$

except for the case $p = 2$, $e = 1$.

Remark 5.4 (cf. [HSY, Remark 4.14]). As is mentioned in the introduction, the e-th normalized F-blowup $\mathrm{FB}_e(X)$ in Theorem 5.3 has the exceptional set consisting of an elliptic curve $E_1 \cong E$ and a smooth rational curve $E_2 \cong \mathbb{P}^1$, and has an A_{p^e-1}-singularity on $E_2 \setminus E_1$. Thus the monotonicity and stabilization of the F-blowup sequence break down in Theorem 5.3, whereas the F-blowup sequence in Theorem 5.1 is monotone and does not stabilize.

5.c. Proofs of the main theorems

Proof of Theorem 1.1. The implication (1) $\Rightarrow$ (3) is Theorem 4.3 (2), and (3) $\Rightarrow$ (2) is trivial. The implication (2) $\Rightarrow$ (1) follows as soon as we prove Theorem 1.2, in which the F-blowup sequence does not stabilize. □

Proof of Theorem 1.2. Suppose first that $1 \leq e < n$. Then $E^2 \leq -4$, and $\widetilde{X}$ is a flattening of R^{1/p^e} by Lemma 2.4. It follows from Theorem 4.3 (1) that $\mathrm{FB}_e(X) \cong \widetilde{X}$.

Suppose now that $e \geq n$. If $E^2 \leq -2$, then assertions (1) and (2) of the theorem follow from Theorems 5.1 and 5.3, respectively. If $E^2 = -1$, then the assertions follow by combining Propositions 2.2 and 4.5 with [HSY, Theorems 4.5 and 4.13] (cf. Theorem 2.7). □

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