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Some Remarks on the Minimal Model Program for Log Canonical Pairs

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In memory of Professor Kunihiko Kodaira

Abstract. We prove that the target space of an extremal Fano contraction from a log canonical pair has only log canonical singularities. We also treat some related topics, for example, the finite generation of canonical rings for compact Kähler manifolds, and so on. The main ingredient of this paper is the nefness of the moduli parts of lc-trivial fibrations. We also give some observations on the semi-ampleness of the moduli parts of lc-trivial fibrations. For the reader's convenience, we discuss some examples of non-Kähler manifolds, flopping contractions, and so on, in order to clarify our results.

Contents

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1. Introduction

Let $\pi : (X, \Delta) \to S$ be a projective morphism from a log canonical pair (X, Δ) to a variety S. Then the cone theorem

$$\overline{NE}(X/S) = \overline{NE}(X/S)_{K_X+\Delta\geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i]$$

holds for $\pi : (X, \Delta) \to S$. We take a $(K_X + \Delta)$-negative extremal ray $R = \mathbb{R}_{\geq 0}[C_i]$. Then there is a contraction morphism

$$\varphi_R : (X, \Delta) \to Y$$

over S associated to R. For the details of the cone and contraction theorem for log canonical pairs, see [A1], [F6], [F8], [F9], and [F10, Theorem 1.1] (see also [F13]).

From now on, let us consider a contraction morphism

$$f : (X, \Delta) \to Y$$

such that

(i) (X, Δ) is a $\mathbb{Q}$-factorial log canonical pair,

(ii) $-(K_X + \Delta)$ is f-ample, and

(iii) $\rho(X/Y) = 1$.

Then we have the following three cases.

Case 1 (Divisorial contraction). f is divisorial, that is, f is a birational contraction which contracts a divisor.

In this case, the exceptional locus $\mathrm{Exc}(f)$ of f is a prime divisor on X and (Y, Δ_Y) is a $\mathbb{Q}$-factorial log canonical pair with $\Delta_Y = f_*\Delta$.

Case 2 (Flipping contraction). f is flipping, that is, f is a birational contraction which is small.

In this case, we can take the flipping diagram:

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

where f^+ is a small projective birational morphism and (i′) (X^+, Δ^+) is a $\mathbb{Q}$-factorial log canonical pair with $\Delta^+ = \varphi_*\Delta$,

(ii′) $K_{X^+} + \Delta^+$ is f^+-ample, and

(iii′) $\rho(X^+/Y) = 1$.

For the existence of log canonical flips, see [B1, Corollary 1.2] and [HX, Corollary 1.8].

Case 3 (Fano contraction). f is a Fano contraction, that is, $\dim Y < \dim X$.

Then Y is $\mathbb{Q}$-factorial and has only log canonical singularities. Moreover, if every log canonical center of (X, Δ) is dominant onto Y, then Y has only log terminal singularities.

In Case 3, $f : (X, \Delta) \to Y$ is usually called a *Mori fiber space.*

The log canonicity of Y in Case 3 is missing in the literature. So we prove it in this paper. It is an easy consequence of the following theorem. For the other statements on singularities in the above three cases, see, for example, [KM, Propositions 3.36, 3.37, Corollaries 3.42, and 3.43] (see also [F13]).

Theorem 1.1 (cf. [F1, Theorem 1.2]). *Let (X, Δ) be a sub log canonical pair such that X is smooth and* $\mathrm{Supp}\Delta$ *is a simple normal crossing divisor on X. Let $f : (X, \Delta) \to Y$ be a proper surjective morphism such that*

$$f_*\mathcal{O}_X(\lceil -\Delta^{<1} \rceil) \simeq \mathcal{O}_Y$$

and that

$$K_X + \Delta \sim_{\mathbb{Q}, f} 0.$$

Assume that K_Y is $\mathbb{Q}$-Cartier. Then Y has only log canonical singularities. We further assume that every log canonical center of (X, Δ) is dominant onto Y. Then Y has only log terminal singularities.

Our proof of Theorem 1.1 depends on the nefness of the moduli parts of lc-trivial fibrations (cf. [Mr, Section 5, Part II], [Ka3], [A2], [F3], [Ko2], [FG3, Section 3], and so on). In this paper, we use Ambro's formulation in [A2] and its generalization in [FG3, Section 3] based on the semipositivity theorem in [F4]. For the details of the Hodge theoretic aspects of the semipositivity theorem, see also [FF] and [FFS]. It is conjectured that the moduli parts of lc-trivial fibrations are semi-ample (see Conjecture 3.9). We give some observations on the semi-ampleness of the moduli parts of lc-trivial fibrations in Section 3.

By the proof of [F1, Theorem 1.2] and [FG3, Section 3] (see Theorem 3.7), we have:

Theorem 1.2 (cf. [F1, Theorem 1.2] and [F3, Theorem 4.2.1]). *Let (X, Δ) be a sub log canonical pair such that X is smooth and* $\mathrm{Supp}\Delta$ *is a simple normal crossing divisor on X. Let $f : (X, \Delta) \to Y$ be a proper surjective morphism such that*

$$f_*\mathcal{O}_X(\lceil -\Delta^{<1}\rceil) \simeq \mathcal{O}_Y$$

and that

$$K_X + \Delta \sim_{\mathbb{Q}} f^*D$$

for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y. Assume that $\pi : Y \to S$ is a projective morphism onto a quasi-projective variety S. Let A be a π-ample Cartier divisor on Y and let ε be an arbitrary positive rational number. We further assume that every log canonical center of (X, Δ) is dominant onto Y. Then there is an effective $\mathbb{Q}$-divisor Δ_Y on Y such that

$$K_Y + \Delta_Y \sim_{\mathbb{Q},\pi} D + \varepsilon A$$

and that (Y, Δ_Y) is kawamata log terminal.

Let us recall some results in [R], [Ko1], and [F1] for the reader's convenience.

Remark 1.3 (Known results). Let $f : X \to Y$ be a contraction morphism associated to a K_X-negative extremal face such that X is a projective variety with only canonical singularities. Then it is well known that Y has only rational singularities by [Ko1, Corollary 7.4]. It was first proved by Reid when $\dim X \leq 3$ (see [R]).

Let $f : (X, \Delta) \to Y$ be a contraction morphism associated to a $(K_X+\Delta)$-negative extremal face such that (X, Δ) is a projective divisorial log terminal pair. Then there is an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is kawamata log terminal by [F1, Corollary 4.5]. In particular, Y has only rational singularities.

Note that the above results now easily follow from Theorem 1.2.

The following conjecture is related to Theorem 1.1 (cf. [Ka1, Conjecture 7.4]).

CONJECTURE 1.4. *Let (X, Δ) be a projective log canonical pair. Assume that the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra. We put

$$Y = \mathrm{Proj} R(X, \Delta).$$

Then there is an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is log canonical.

If (X, Δ) is kawamata log terminal in Conjecture 1.4, then we have:

THEOREM 1.5. *Let (X, Δ) be a projective kawamata log terminal pair such that Δ is a $\mathbb{Q}$-divisor. We put*

$$Y = \mathrm{Proj} R(X, \Delta).$$

Then there is an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is kawamata log terminal.

It is a generalization of Nakayama's result (see [N, Theorem]), which is a complete solution of [Ka1, Conjecture 7.4]. Theorem 1.6 is a partial answer to Conjecture 1.4.

THEOREM 1.6. *Let (X, Δ) be a projective log canonical pair such that the log canonical ring $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra. We assume that K_Y is $\mathbb{Q}$-Cartier where $Y = \mathrm{Proj} R(X, \Delta)$. Then Y has only log canonical singularities.*

Note that K_Y is not always $\mathbb{Q}$-Cartier in Conjecture 1.4. Therefore, Theorem 1.6 is far from a complete solution of Conjecture 1.4.

The following conjecture is open. It is closely related to Conjecture 1.4 and Theorem 1.1.

Conjecture 1.7. *Let (X, Δ) be a projective log canonical pair and let $f : X \to Y$ be a contraction morphism between normal projective varieties such that*

$$K_X + \Delta \sim_{\mathbb{R}, f} 0.$$

Then there is an effective $\mathbb{R}$-divisor Δ_Y on Y such that (Y, Δ_Y) is log canonical and

$$K_X + \Delta \sim_{\mathbb{R}} f^*(K_Y + \Delta_Y).$$

Of course, Conjecture 1.7 follows from the b-semi-ampleness conjecture of the moduli parts of lc-trivial fibrations (see Conjecture 3.9 and Remark 4.7).

From now on, the variety X is not always algebraic. We treat compact Kähler manifolds. The following theorem is also missing in the literature. Note that we can reduce the problem to the case when the variety is projective by taking the Iitaka fibration. When X is projective, Theorem 1.8 is well known (see [BCHM]).

Theorem 1.8 (cf. [BCHM] and [FM]). *Let X be a compact Kähler manifold and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is kawamata log terminal. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

As a special case of Theorem 1.8, we have:

Corollary 1.9. *Let X be a compact Kähler manifold. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \omega_X^{\otimes m})$$

is a finitely generated $\mathbb{C}$-algebra.

We note that there exists a compact complex non-Kähler manifold whose canonical ring is not a finitely generated $\mathbb{C}$-algebra (see Example 6.4).

The following conjecture is still open even when X is projective.

CONJECTURE 1.10. *Let X be a compact Kähler manifold and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is log canonical. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

We do not know if we can reduce Conjecture 1.10 to the case when the variety is projective or not (see Remark 5.9).

From Section 2 to Section 4, we assume that all the varieties are algebraic for simplicity, although some of the results can be generalized to analytic varieties. Section 2 collects some basic definitions. In Section 3, we discuss lc-trivial fibrations and give some new observations. Section 4 is devoted to the proofs of the main results. In Section 5, we discuss some analytic generalizations and related topics. We note that we just explain how to adapt the arguments to the analytic settings and discuss Theorem 1.8, Corollary 1.9, and so on. In Section 6, we discuss some examples of non-Kähler manifolds, which clarify the main difference between Kähler manifolds and non-Kähler manifolds. Note that Corollary 1.9 can not be generalized for non-Kähler manifolds (see Example 6.4). In Section 7: Appendix, we quickly discuss the minimal model program for log canonical pairs and describe some related examples by János Kollár for the reader's convenience.

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This paper is a supplement to the author's previous papers [F1], [F10], and so on. For some recent related topics, see, for example, [AB], [B3], [CHP], [HP1], and [HP2].

We will work over $\mathbb{C}$, the complex number field, throughout this paper. We will make use of the standard notation as in [KM] and [F10].

2. Preliminaries

Let us recall some basic definitions on singularities of pairs. For the details, see [KM] and [F10].

2.1 (Pairs). A pair (X, Δ) consists of a normal variety X and an $\mathbb{R}$-divisor Δ on X such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A pair (X, Δ) is called *sub kawamata log terminal* (resp. *sub log canonical*) if for any proper birational morphism $g : Z \to X$ from a normal variety Z, every coefficient of Δ_Z is < 1 (resp. ≤ 1) where

$$K_Z + \Delta_Z := g^*(K_X + \Delta).$$

A pair (X, Δ) is called *kawamata log terminal* (resp. *log canonical*) if (X, Δ) is sub kawamata log terminal (resp. sub log canonical) and Δ is effective. If $(X, 0)$ is kawamata log terminal, then we simply say that X has only *log terminal singularities*.

Let (X, Δ) be a sub log canonical pair and let W be a closed subset of X. Then W is called a *log canonical center* of (X, Δ) if there are a proper birational morphism $g : Z \to X$ from a normal variety Z and a prime divisor E on Z such that $\mathrm{mult}_E \Delta_Z = 1$ and $g(E) = W$.

We note that $-\mathrm{mult}_E \Delta_Z$ is denoted by $a(E, X, \Delta)$ and is called the *discrepancy coefficient* of E with respect to (X, Δ).

Let $D = \sum_i d_i D_i$ be an $\mathbb{R}$-divisor on X such that D_i is a prime divisor for every i and that $D_i \neq D_j$ for $i \neq j$. Then $\lceil D \rceil$ (resp. $\lfloor D \rfloor$) denotes the *round-up* (resp. *round-down*) of D. We put

$$D^{<1} = \sum_{d_i < 1} d_i D_i.$$

In this paper, we use the notion of *b-divisors* introduced by Shokurov. For the details, see, for example, [F11, Section 3].

2.2 (Canonical b-divisors and discrepancy b-divisors). Let X be a normal variety and let ω be a top rational differential form of X. Then (ω) defines a b-divisor $\mathbf{K}$. We call $\mathbf{K}$ the *canonical b-divisor* of X. The *discrepancy b-divisor* $\mathbf{A} = \mathbf{A}(X, \Delta)$ of a pair (X, Δ) is the $\mathbb{R}$-b-divisor of X with the trace $\mathbf{A}_Y$ defined by the formula

$$K_Y = f^*(K_X + \Delta) + \mathbf{A}_Y,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties. Similarly, we define $\mathbf{A}^* = \mathbf{A}^*(X, \Delta)$ by

$$\mathbf{A}^*_Y = \sum_{a_i > -1} a_i A_i$$

for

$$K_Y = f^*(K_X + \Delta) + \sum a_i A_i,$$

where $f : Y \to X$ is a proper birational morphism of normal varieties.

2.3 (b-nef and b-semi-ample $\mathbb{Q}$-b-divisors). Let X be a normal variety and let $X \to S$ be a proper surjective morphism onto a variety S. A $\mathbb{Q}$-b-divisor $\mathbf{D}$ of X is *b-nef over* S (resp. *b-semi-ample over* S) if there exists a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ and $\mathbf{D}_{X'}$ is nef (resp. semi-ample) relative to the induced morphism $X' \to S$. A $\mathbb{Q}$-b-divisor $\mathbf{D}$ of X is $\mathbb{Q}$-*b-Cartier* if there is a proper birational morphism $X' \to X$ from a normal variety X' such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$.

3. Lc-Trivial Fibrations

Let us recall the definition of *lc-trivial fibrations.*

Definition 3.1 (Lc-trivial fibrations). An *lc-trivial fibration* $f : (X, \Delta) \to Y$ consists of a proper surjective morphism between normal varieties with connected fibers and a pair (X, Δ) satisfying the following properties:

(i) (X, Δ) is sub log canonical over the generic point of Y,

(ii) $\mathrm{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, \Delta) \rceil) = 1$, and

(iii) there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that

$$K_X + \Delta \sim_{\mathbb{Q}} f^*D.$$

REMARK 3.2. Let $f : X \to Y$ be a proper surjective morphism between normal varieties with $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$. Assume that (X, Δ) is log canonical over the generic point of Y. Then we have

$$\mathrm{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, \Delta)\rceil) = 1.$$

We give a standard example of lc-trivial fibrations.

Example 3.3. Let (X, Δ) be a sub log canonical pair such that X is smooth and that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X. Let $f : X \to Y$ be a proper surjective morphism onto a normal variety Y such that

$$K_X + \Delta \sim_{\mathbb{Q}, f} 0$$

and that

$$f_*\mathcal{O}_X(\lceil -\Delta^{<1}\rceil) \simeq \mathcal{O}_Y.$$

Then $f : (X, \Delta) \to Y$ is an lc-trivial fibration.

We give a remark on the definition of lc-trivial fibrations for the reader's convenience.

REMARK 3.4 (Lc-trivial fibrations and klt-trivial fibrations). In [A2, Definition 2.1], (X, Δ) is assumed to be sub kawamata log terminal over the generic point of Y. Therefore, Definition 3.1 is wider than Ambro's original definition of lc-trivial fibrations. When (X, Δ) is sub kawamata log terminal over the generic point of Y in Definition 3.1, we call $f : (X, \Delta) \to Y$ a *klt-trivial fibration* (see [FG3, Definition 3.1]).

We need the notion of *induced lc-trivial fibrations*, *discriminant* $\mathbb{Q}$-*divisors*, *moduli* $\mathbb{Q}$-*divisors*, and so on in order to discuss lc-trivial fibrations.

3.5 (Induced lc-trivial fibrations by base changes). Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration and let $\sigma : Y' \to Y$ be a generically finite morphism.

Then we have an induced lc-trivial fibration $f' : (X', \Delta_{X'}) \to Y'$, where $\Delta_{X'}$ is defined by $\mu^*(K_X + \Delta) = K_{X'} + \Delta_{X'}$:

$$\begin{tikzcd}
(X', \Delta_{X'}) \arrow[r, "\mu"] \arrow[d, "f'"'] & (X, \Delta) \arrow[d, "f"] \\
Y' \arrow[r, "\sigma"'] & Y,
\end{tikzcd}$$

Note that X' is the normalization of the main component of $X \times_Y Y'$. We sometimes replace X' with X'' where X'' is a normal variety such that there is a proper birational morphism $\varphi : X'' \to X'$. In this case, we set $K_{X''} + \Delta_{X''} = \varphi^*(K_{X'} + \Delta_{X'})$.

3.6 (Discriminant $\mathbb{Q}$-b-divisors and moduli $\mathbb{Q}$-b-divisors). Let us consider an lc-trivial fibration $f : (X, \Delta) \to Y$ as in Definition 3.1. We take a prime divisor P on Y. By shrinking Y around the generic point of P, we assume that P is Cartier. We set

$$b_P = \max \left\{ t \in \mathbb{Q} \;\middle|\; \begin{array}{l} (X, \Delta + tf^*P) \text{ is sub log canonical} \\ \text{over the generic point of } P \end{array} \right\}$$

and set

$$B_Y = \sum_P (1 - b_P)P,$$

where P runs over prime divisors on Y. Then it is easy to see that B_Y is a well-defined $\mathbb{Q}$-divisor on Y and is called the *discriminant* $\mathbb{Q}$*-divisor* of $f : (X, \Delta) \to Y$. We set

$$M_Y = D - K_Y - B_Y$$

and call M_Y the *moduli* $\mathbb{Q}$*-divisor* of $f : (X, \Delta) \to Y$. Let $\sigma : Y' \to Y$ be a proper birational morphism from a normal variety Y' and let $f' : (X', \Delta_{X'}) \to Y'$ be the induced lc-trivial fibration by $\sigma : Y' \to Y$ (see 3.5). We can define $B_{Y'}$, $K_{Y'}$ and $M_{Y'}$ such that

$$\sigma^* D = K_{Y'} + B_{Y'} + M_{Y'},$$

$\sigma_* B_{Y'} = B_Y$, $\sigma_* K_{Y'} = K_Y$, and $\sigma_* M_{Y'} = M_Y$. Hence there exist a unique $\mathbb{Q}$-b-divisor $\mathbf{B}$ such that $\mathbf{B}_{Y'} = B_{Y'}$ for every $\sigma : Y' \to Y$ and a unique

$\mathbb{Q}$-b-divisor $\mathbf{M}$ such that $\mathbf{M}_{Y'} = M_{Y'}$ for every $\sigma : Y' \to Y$. Note that $\mathbf{B}$ is called the *discriminant* $\mathbb{Q}$*-b-divisor* and that $\mathbf{M}$ is called the *moduli* $\mathbb{Q}$*-b-divisor* associated to $f : (X, \Delta) \to Y$. We sometimes simply say that $\mathbf{M}$ is the *moduli part* of $f : (X, \Delta) \to Y$.

Theorem 3.7 is the most fundamental result on lc-trivial fibrations. It is the main ingredient of this paper. Ambro [A2] obtained Theorem 3.7 for klt-trivial fibrations. Theorem 3.7 is a direct generalization of [A2, Theorem 0.2].

Theorem 3.7 ([FG3, Theorem 3.6]). *Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration and let $\pi : Y \to S$ be a proper morphism. Let $\mathbf{B}$ and $\mathbf{M}$ be the induced discriminant and moduli $\mathbb{Q}$-b-divisors of f. Then,*

(1) *$\mathbf{K} + \mathbf{B}$ is $\mathbb{Q}$-b-Cartier,*

(2) *$\mathbf{M}$ is b-nef over S.*

Remark 3.8. Theorem 3.7 says that there is a proper birational morphism $Y' \to Y$ from a normal variety Y' such that $\mathbf{K} + \mathbf{B} = \overline{K_{Y'} + B_{Y'}}$, $\mathbf{M} = \overline{M_{Y'}}$, and $M_{Y'}$ is nef over S. We note that the arguments in [A2, Section 5] show how to construct Y'. For the details, see [A2, p. 245, set-up] and [A2, Proof of Theorem 2.7].

The following conjecture is one of the most important open problems on lc-trivial fibrations. It was conjectured by Fujita, Mori, Shokurov and others (see [N, Problem], [PS, Conjecture 7.13] and so on).

Conjecture 3.9 (b-semi-ampleness conjecture). *Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration and let $\pi : Y \to S$ be a proper morphism. Then the moduli part $\mathbf{M}$ is b-semi-ample over S.*

Conjecture 3.9 was only solved for some special cases (see [Ka2], [F2], and [PS, Section 8]). The arguments in [Ka2] (see also [PS]) and [F2] use the theory of moduli spaces of curves, $K3$ surfaces, and Abelian varieties.

We give some observations on Conjecture 3.9.

3.10 (Observation I). Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration. For simplicity, we assume that X is smooth and $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X. We write

$$\Delta = \Delta^+ - \Delta^-$$

where Δ^+ and Δ^- are effective $\mathbb{Q}$-divisors on X such that Δ^+ and Δ^- have no common irreducible components. In this situation, we have

$$\mathcal{O}_X(\lceil \mathbf{A}^*(X, \Delta)\rceil) \simeq \mathcal{O}_X(\lceil \Delta^-\rceil)$$

over the generic point of Y (see [F11, Lemma 3.22]). Therefore, the condition

$$\mathrm{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, \Delta)\rceil) = 1$$

is equivalent to

$$\mathrm{rank} f_*\mathcal{O}_X(\lceil \Delta^-\rceil) = 1.$$

For Conjecture 3.9, it seems to be reasonable to assume that

$$\mathrm{rank} f_*\mathcal{O}_X(\lceil m\Delta^-\rceil) = 1$$

holds for every nonnegative integer m. This condition is equivalent to

$$\kappa(X_\eta, K_{X_\eta} + \Delta^+|_{X_\eta}) = 0$$

where X_η is the generic fiber of $f : X \to Y$. The condition

$$\mathrm{rank} f_*\mathcal{O}_X(\lceil \Delta^-\rceil) = 1$$

seems to be insufficient for Conjecture 3.9.

If there are an lc-trivial fibration $f^\dagger : (X^\dagger, \Delta^\dagger) \to Y$ such that $(X^\dagger, \Delta^\dagger)$ is log canonical and a proper birational morphism $\mu : X \to X^\dagger$ such that $K_X+\Delta = \mu^*(K_{X^\dagger}+\Delta^\dagger)$ and $f = f^\dagger\circ\mu$, then Δ^- is μ-exceptional. Therefore we have

$$\mu_*\mathcal{O}_X(\lceil m\Delta^-\rceil) \simeq \mathcal{O}_{X^\dagger}$$

for every nonnegative integer m. This implies

$$f_*\mathcal{O}_X(\lceil m\Delta^-\rceil) \simeq \mathcal{O}_Y$$

for every nonnegative integer m. Consequently, this extra assumption that

$$\mathrm{rank} f_* \mathcal{O}_X(\lceil m\Delta^- \rceil) = 1$$

for every nonnegative integer m is harmless for many applications.

3.11 (Observation II). Assume that the minimal model program and the abundance conjecture hold.

Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration such that X is smooth and $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X. Assume that

$$\kappa(X_\eta, K_{X_\eta} + \Delta^+|_{X_\eta}) = 0$$

as in 3.10. By [AK], we can construct the following commutative diagram:

$$\begin{tikzcd}
X \arrow[d, "f"'] & X' \arrow[l, "\mu"'] \arrow[d, "f'"] & U_{X'} \arrow[l, hook'] \arrow[d] \\
Y & Y' \arrow[l, "\sigma"] & U_{Y'}, \arrow[l, hook']
\end{tikzcd}$$

satisfying:

(1) μ and σ are projective birational morphisms.

(2) $f' : (U_{X'} \subset X') \to (U_{Y'} \subset Y')$ is a projective equidimensional toroidal morphism.

(3) $K_{X'} + \Delta_{X'} = \mu^*(K_X + \Delta)$, $\mathrm{Supp}\Delta_{X'} \subset \Sigma_{X'} = X' \setminus U_{X'}$, and X' is $\mathbb{Q}$-factorial.

(4) Y' is a smooth quasi-projective variety and $\Sigma_{Y'} = Y' \setminus U_{Y'}$ is a simple normal crossing divisor on Y'.

(5) f' is smooth over $U_{Y'}$ and $\Sigma_{X'}$ is a relatively normal crossing divisor over $U_{Y'}$.

We can write

$$K_{X'} + \Delta_{X'} \sim_{\mathbb{Q}} f'^*(K_{Y'} + B_{Y'} + M_{Y'})$$

as in 3.6. Let $\Sigma_{Y'} = \sum_i P_i$ be the irreducible decomposition. Then we can write

$$B_{Y'} = \sum_i (1 - b_{P_i}) P_i$$

as in 3.6. We put

$$\Delta_{X'} + \sum_i b_{P_i} f'^* P_i = \Theta - E$$

where Θ and E are effective $\mathbb{Q}$-divisors on X' such that Θ and E have no common irreducible components. Then

$$K_{X'} + \Theta \sim_{\mathbb{Q}} f'^*(K_{Y'} + \Sigma_{Y'} + M_{Y'}) + E \sim_{\mathbb{Q}, f'} E \geq 0.$$

We run the minimal model program with respect to $K_{X'} + \Theta$ over Y' (cf. [FG3, Proof of Theorem 1.1]). Note that (X', Θ) is a $\mathbb{Q}$-factorial log canonical pair. Then we obtain a minimal model $\widetilde{f} : (\widetilde{X}, \widetilde{\Theta}) \to Y'$ such that

$$K_{\widetilde{X}} + \widetilde{\Theta} \sim_{\mathbb{Q}, \widetilde{f}} 0.$$

It is easy to see that

$$K_{\widetilde{X}} + \widetilde{\Theta} \sim_{\mathbb{Q}} \widetilde{f}^*(K_{Y'} + \Sigma_{Y'} + M_{Y'}),$$

that is, $\Sigma_{Y'}$ is the discriminant $\mathbb{Q}$-divisor of $\widetilde{f} : (\widetilde{X}, \widetilde{\Theta}) \to Y'$ and $M_{Y'}$ is the moduli part of $\widetilde{f} : (\widetilde{X}, \widetilde{\Theta}) \to Y'$. Therefore, if we assume that the minimal model program and the abundance conjecture hold, then we can replace (X, Δ) with a log canonical pair $(\widetilde{X}, \widetilde{\Theta})$ when we prove the b-semi-ampleness of $\mathbf{M}$ under the assumption that $\kappa(X_\eta, K_{X_\eta} + \Delta^+|_{X_\eta}) = 0$. We note that the b-semi-ampleness conjecture of $\mathbf{M}$ for $\widetilde{f} : (\widetilde{X}, \widetilde{\Theta}) \to Y'$ can be reduced to the case when $g : (V, \Delta_V) \to W$ is an lc-trivial fibration such that (V, Δ_V) is kawamata log terminal over the generic point of W and Δ_V is effective. For the details, see [FG3, Proof of Theorem 1.1].

We also note that the existence of a good minimal model of $(X'_\eta, \Theta|_{X'_\eta})$, where X'_η is the generic fiber of $f' : X' \to Y'$, is sufficient to construct a relative good minimal model $\widetilde{f} : (\widetilde{X}, \widetilde{\Theta}) \to Y'$. Let us go into details. By replacing (X', Θ) with its dlt blow-up, we may assume that (X', Θ) is a $\mathbb{Q}$-factorial divisorial log terminal pair. We run the minimal model program on $K_{X'} + \Theta$ with ample scaling over Y'. After finitely many steps, all the horizontal components of E are contracted if $(X'_\eta, \Theta|_{X'_\eta})$ has a good minimal model. Thus we assume that E has no horizontal components. Then it is easy to see that E is very exceptional over Y'. For the definition of *very exceptional divisors*, see, for example, [B1, Definition 3.1]. Therefore, by [B1, Theorem 3.4], we obtain a relative minimal model $\widetilde{f} : (\widetilde{X}, \widetilde{\Theta}) \to Y'$

with $K_{\widetilde{X}} + \widetilde{\Theta} \sim_{\mathbb{Q}, \tilde{f}} 0$. Note that the existence of a good minimal model of $(X'_\eta, \Theta|_{X'_\eta})$ is equivalent to the existence of a good minimal model of $(X_\eta, \Delta^+|_{X_\eta})$.

3.12 (Observation III). Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration such that X and Y are quasi-projective and that (X, Δ) is log canonical. By taking a dlt blow-up, we may assume that (X, Δ) is a $\mathbb{Q}$-factorial divisorial log terminal pair. Let $\overline{Y}$ be a normal projective variety which is a compactification of Y. By using the minimal model program, we can construct a projective $\mathbb{Q}$-factorial divisorial log terminal pair $(\overline{X}, \overline{\Delta})$ which is a compactification of (X, Δ) such that $\overline{X} \setminus X$ contains no log canonical centers of $(\overline{X}, \overline{\Delta})$ and that $f : X \to Y$ is extended to $\overline{f} : \overline{X} \to \overline{Y}$.

$$\begin{tikzcd}
(\overline{X}, \overline{\Delta}) \arrow[d, "\overline{f}"'] & (X, \Delta) \arrow[l, hook'] \arrow[d, "f"] \\
\overline{Y} & Y \arrow[l, hook']
\end{tikzcd}$$

By [B1, Theorem 1.4], we have a good minimal model $(\overline{X}', \overline{\Delta}')$ over $\overline{Y}$. See also [HX, Theorem 1.1 and Corollary 1.2]. Let $\overline{f}' : \overline{X}' \to \overline{Y}'$ be the contraction morphism over $\overline{Y}$ associated to $K_{\overline{X}'} + \overline{\Delta}'$. Then $\overline{f}' : (\overline{X}', \overline{\Delta}') \to \overline{Y}'$ is an lc-trivial fibration which is a compactification of $f : (X, \Delta) \to Y$.

Therefore, the b-semi-ampleness of $\mathbf{M}$ of $\overline{f}' : (\overline{X}', \overline{\Delta}') \to \overline{Y}'$ implies that the moduli part of $f : (X, \Delta) \to Y$ is b-semi-ample over S, where $Y \to S$ is a proper morphism as in Conjecture 3.9.

By combining the above observations with the results in [A3, Theorem 3.3] and [PS, Theorem 8.1], we have:

THEOREM 3.13. *Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration such that X is smooth and* SuppΔ *is a simple normal crossing divisor on X and let $Y \to S$ be a proper morphism. We write $\Delta = \Delta^+ - \Delta^-$ where Δ^+ and Δ^- are effective $\mathbb{Q}$-divisors and have no common irreducible components. Assume that $\kappa(X_\eta, K_{X_\eta} + \Delta^+|_{X_\eta}) = 0$ where X_η is the generic fiber of f and that $(X_\eta, \Delta^+|_{X_\eta})$ has a good minimal model. Then the moduli part* $\mathbf{M}$ *of $f : (X, \Delta) \to Y$ is b-nef and abundant over S. This means that there is a proper birational morphism $Y' \to Y$ from a normal variety Y' such that*

$\mathbf{M} = \overline{\mathbf{M}_{Y'}}$ *and* $\mathbf{M}_{Y'}$ *is nef and abundant relative to the induced morphism* $Y' \to S$.

We further assume that $\dim X = \dim Y + 1$. *Then the moduli part* $\mathbf{M}$ *of* $f : (X, \Delta) \to Y$ *is b-semi-ample over* S.

We note that we do not use Theorem 3.13 in the subsequent sections.

Sketch of Proof of Theorem 3.13. By the arguments in 3.11, we may assume that X and Y are quasi-projective and that (X, Δ) is log canonical. By the arguments in 3.12, we may further assume that X and Y are projective. Then, by [FG3, Theorem 1.1], we obtain that $\mathbf{M}$ is b-nef and abundant over S. When $\dim X = \dim Y + 1$, we see that $\mathbf{M}$ is b-semi-ample over S by [PS, Theorem 8.1]. □

For the details of lc-trivial fibrations, see also [A2] and [FG3, Section 3].

4. Proof of the Main Results

First, let us prove the log canonicity of Y in Case 3 in the introduction by using Theorem 1.1.

Proof of the Log Canonicity of Y in Case 3. It is easy to see that Y is $\mathbb{Q}$-factorial (see, for example, [KM, Proposition 3.36]). By perturbing Δ, we may assume that Δ is a $\mathbb{Q}$-divisor. By shrinking Y, we may assume that Y is affine. We can take an effective $\mathbb{Q}$-divisor Δ' on X such that $(X, \Delta + \Delta')$ is log canonical and that

$$K_X + \Delta + \Delta' \sim_{\mathbb{Q}, f} 0.$$

Let $g : Z \to X$ be a resolution such that

$$K_Z + \Delta_Z = g^*(K_X + \Delta + \Delta')$$

and that $\mathrm{Supp}\Delta_Z$ is a simple normal crossing divisor on Z. Then

$$g_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_X.$$

Therefore,

$$h_*\mathcal{O}_Z(\lceil -\Delta_Z^{<1} \rceil) \simeq \mathcal{O}_Y$$

and

$$K_Z + \Delta_Z \sim_{\mathbb{Q},h} 0$$

where $h = f \circ g$. By Theorem 1.1, Y has only log canonical singularities.

If every log canonical center of (X, Δ) is dominant onto Y, then we can take Δ' such that every log canonical center of (Z, Δ_Z) is dominant onto Y. Thus Y is log terminal by Theorem 1.1 when every log canonical center of (X, Δ) is dominant onto Y. $\square$

Let us prove Theorem 1.1. We use the framework of lc-trivial fibrations.

PROOF OF THEOREM 1.1. Without loss of generality, we may assume that Y is affine. We can write

$$K_X + \Delta \sim_{\mathbb{Q}} f^*(K_Y + B_Y + M_Y)$$

where B_Y is the discriminant and M_Y is the moduli part of the lc-trivial fibration $f : (X, \Delta) \to Y$ (see 3.6). Note that B_Y is effective (see, for example, the proof of [F1, Theorem 1.2]) and the coefficients of B_Y are ≤ 1. Let E be an arbitrary prime divisor over Y. We take a resolution $\sigma : Y' \to Y$ with

$$K_{Y'} + B_{Y'} + M_{Y'} = \sigma^*(K_Y + B_Y + M_Y)$$

such that E is a prime divisor on Y' and that $E \cup \mathrm{Supp} B_{Y'} \cup \mathrm{Exc}(\sigma)$ is a simple normal crossing divisor on Y'. Note that $f' : (X', \Delta') \to Y'$ is an induced lc-trivial fibration by $\sigma : Y' \to Y$ (see 3.5) and that $B_{Y'}$ is the discriminant and $M_{Y'}$ is the moduli part of $f' : (X', \Delta') \to Y'$.

$$\begin{array}{ccc} (X', \Delta_{X'}) & \longrightarrow & (X, \Delta) \\ {\scriptstyle f'} \downarrow & & \downarrow {\scriptstyle f} \\ Y' & \xrightarrow[\sigma]{} & Y \end{array}$$

By taking $\sigma : Y' \to Y$ suitably, we may assume that $M_{Y'}$ is σ-nef (see Theorem 3.7) and there is an effective exceptional $\mathbb{Q}$-divisor F on Y' which is anti-σ-ample. Without loss of generality, we may assume that the coefficients of F are ≤ 1. Let ε be an arbitrary positive rational number.

Then

$$\begin{aligned} K_{Y'} + B_{Y'} + M_{Y'} &= K_{Y'} + B_{Y'} + \varepsilon F + M_{Y'} - \varepsilon F \\ &\sim_{\mathbb{Q}} K_{Y'} + B_{Y'} + \varepsilon F + G \end{aligned}$$

where G is a general effective $\mathbb{Q}$-divisor on Y' such that $\lfloor G \rfloor = 0$, $G \sim_{\mathbb{Q}} M_{Y'} - \varepsilon F$, and $\mathrm{Supp} B_{Y'} \cup \mathrm{Supp} F \cup \mathrm{Supp} G$ is a simple normal crossing divisor. Note that $M_{Y'} - \varepsilon F$ is σ-ample and that Y is affine. We put

$$\Theta_{E,\varepsilon} = \sigma_*(B_{Y'} + \varepsilon F + G).$$

Then $\Theta_{E,\varepsilon}$ is an effective $\mathbb{Q}$-divisor on Y whose coefficients are ≤ 1 such that $K_Y + \Theta_{E,\varepsilon}$ is $\mathbb{Q}$-Cartier and

$$\begin{aligned} a(E, Y, \Theta_{E,\varepsilon}) &= -\mathrm{mult}_E B_{Y'} - \varepsilon \mathrm{mult}_E F \\ &\geq -1 - \varepsilon. \end{aligned}$$

Therefore,

$$a(E, Y, 0) \geq a(E, Y, \Theta_{E,\varepsilon}) \geq -1 - \varepsilon.$$

This means that $a(E, Y, 0) \geq -1$. Thus Y has only log canonical singularities.

When every log canonical center of (X, Δ) is dominant onto Y, $\mathrm{mult}_E B_{Y'} < 1$ always holds by the construction of $B_{Y'}$. Therefore, we obtain $a(E, Y, 0) > -1$. Thus Y has only log terminal singularities. □

Remark 4.1. In the proof of Theorem 1.1, if $M_{Y'}$ is σ-semi-ample and Y is quasi-projective, then we can take a general effective $\mathbb{Q}$-divisor G on Y' such that

$$K_{Y'} + B_{Y'} + M_{Y'} \sim_{\mathbb{Q},\sigma} K_{Y'} + B_{Y'} + G.$$

Thus (Y, Δ_Y) is log canonical where $\Delta_Y = \sigma_*(B_{Y'} + G)$. Therefore, the b-semi-ampleness of $\mathbf{M}$ is desirable (see Conjecture 3.9). Of course, if $M_{Y'}$ is semi-ample, then we can choose G such that

$$K_{Y'} + B_{Y'} + M_{Y'} \sim_{\mathbb{Q}} K_{Y'} + B_{Y'} + G.$$

Note that Y in Theorem 1.1 has a quasi-log structure in the sense of Ambro (see [A1]).

Remark 4.2 (Quasi-log structure). We use the same notation as in Theorem 1.1. We can write

$$K_X + \Delta \sim_{\mathbb{Q}} f^*\omega$$

for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor ω on Y. Then the pair $[Y, \omega]$ has a quasi-log structure with only qlc singularities (see [A1], [F6], [F8], and [F13]). Therefore, the cone and contraction theorem holds for Y with respect to ω. It is a complete generalization of [F1, Theorem 4.1].

Proof of Theorem 1.2. By using Theorem 3.7, the proof of Theorem 1.2 in [F1] works. We leave the details as an exercise for the reader. □

Let us start the proof of Theorem 1.6.

Proof of Theorem 1.6. By taking a suitable resolution, we may assume that $f : X \to Y$ is a morphism such that

$$m_0(K_X + \Delta) = f^*A + E$$

where m_0 is a sufficiently large and divisible positive integer, A is a very ample Cartier divisor on Y, and E is an effective divisor on X satisfying

$$|mm_0(K_X + \Delta)| = |mf^*A| + mE$$

for every positive integer m (see, for example, [B2, Lemma 3.2]). Without loss of generality, we may further assume that $\mathrm{Supp}\Delta \cup \mathrm{Supp}E$ is a simple normal crossing divisor on X. We put

$$\Delta_X = \Delta - \frac{1}{m_0}E.$$

Then we have

$$K_X + \Delta_X \sim_{\mathbb{Q},f} 0.$$

It is easy to see that $f_*\mathcal{O}_X(\lceil -\Delta_X^{<1} \rceil) \simeq \mathcal{O}_Y$ (see, for example, the proof of [B1, Lemma 3.2]). Note that

$$0 \leq \lceil -\Delta_X^{<1} \rceil = \lceil \frac{1}{m_0}E \rceil \leq E.$$

Therefore, by Theorem 1.1, we have that Y has only log canonical singularities. $\square$

Remark 4.3. In the proof of Theorem 1.6, we have $\kappa(X_\eta, K_{X_\eta} + \Delta|_{X_\eta}) = 0$ where X_η is the generic fiber of $f : X \to Y$. Therefore, if Conjecture 3.9 holds under the extra assumption that

$$\kappa(X_\eta, K_{X_\eta} + \Delta_X^+|_{X_\eta}) = \kappa(X_\eta, K_{X_\eta} + \Delta|_{X_\eta}) = 0$$

as in 3.10, then Conjecture 1.4 also holds (see Proof of Theorem 1.1 and Remark 4.1).

Remark 4.4. If (X, Δ) has a good minimal model in Conjecture 1.4, then we may assume that there is a morphism $f : X \to Y$ such that $f_*\mathcal{O}_X \simeq \mathcal{O}_Y$ and $K_X + \Delta \sim_{\mathbb{Q},f} 0$ by replacing (X, Δ) with its good minimal model. In this case, Conjecture 1.4 follows from Conjecture 1.7.

Proof of Theorem 1.5. By combining the proof of Theorem 1.6 with Theorem 1.2, we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that (Y, Δ_Y) is kawamata log terminal. We leave the details as an exercise for the reader. $\square$

We give a remark on the finite generation of $R(X, \Delta)$.

Remark 4.5 (Finite generation of $R(X, \Delta)$). Let (X, Δ) be a projective log canonical pair such that Δ is a $\mathbb{Q}$-divisor. It is conjectured that the log canonical ring $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra. It is one of the most important conjectures for higher-dimensional algebraic varieties. For the details and various related conjectures, see [FG4]. It is known that $R(X, \Delta)$ is finitely generated for $\dim X \leq 4$ (see [F7, Theorem 1.2]). Note that $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra when (X, Δ) is kawamata log terminal and Δ is a $\mathbb{Q}$-divisor. It was established by Birkar–Cascini–Hacon–M^{c}Kernan ([BCHM]) and is now well known.

We close this section with remarks on Conjecture 1.7.

Remark 4.6. If (X, Δ) is kawamata log terminal and Δ is a $\mathbb{Q}$-divisor in Conjecture 1.7, then we can take a $\mathbb{Q}$-divisor Δ_Y such that (Y, Δ_Y) is kawamata log terminal and $K_X+\Delta \sim_{\mathbb{Q}} f^*(K_Y+\Delta_Y)$ by [A2, Theorem 0.2], which is a complete solution of [F1, Problem 1.1]. Theorem 3.1 in [FG1] generalizes [A2, Theorem 0.2] for $\mathbb{R}$-divisors.

Remark 4.7 (cf. the proof of Theorem 3.1 in [FG1]). In Conjecture 1.7, we can write

$$K_X + \Delta = \sum_{i=1}^{k} r_i(K_X + \Delta_i)$$

such that

(a) Δ_i is an effective $\mathbb{Q}$-divisor for every i,

(b) (X, Δ_i) is log canonical and $K_X + \Delta_i$ is f-nef for every i, and

(c) $0 < r_i < 1$, $r_i \in \mathbb{R}$ for every i, and $\sum_{i=1}^{k} r_i = 1$.

Since $K_X + \Delta$ is numerically f-trivial, so is $K_X + \Delta_i$ for every i. By [FG2, Theorem 4.9], we obtain that $K_X+\Delta_i \sim_{\mathbb{Q},f} 0$ for every i. Therefore, we can reduce Conjecture 1.7 to the case when Δ is a $\mathbb{Q}$-divisor with $K_X+\Delta \sim_{\mathbb{Q},f} 0$. Then we can use the framework of lc-trivial fibrations. We can easily check that Conjecture 1.7 follows from Conjecture 3.9 such that S is a point (see also Remark 4.1).

5. Some Analytic Generalizations

In this section, we give some remarks on complex analytic varieties in Fujiki's class $\mathcal{C}$ and compact Kähler manifolds. The following theorem easily follows from [BCHM] and [FM]. Note that Theorem 5.1 is equivalent to Theorem 1.8 by taking a resolution.

Theorem 5.1 (cf. [BCHM] and [FM]). *Let X be a normal complex analytic variety in Fujiki's class $\mathcal{C}$ and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is kawamata log terminal. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta)\rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

As a special case of Theorem 5.1, we have:

COROLLARY 5.2. *Let X be a compact Kähler manifold. Then the canonical ring*

$$R(X) = \bigoplus_{m \geq 0} H^0(X, \omega_X^{\otimes m})$$

is a finitely generated $\mathbb{C}$-algebra.

Theorem 5.1 and Corollary 5.2 do not hold for varieties which are not in Fujiki's class $\mathcal{C}$ (see Example 6.4 below).

Note that the proof of Theorem 5.1 is not related to the minimal model theory for compact Kähler manifolds. We do not discuss the minimal models for compact Kähler manifolds here (see [CHP], [HP1], and [HP2]).

5.3 (Ideas). Let m be a large and divisible positive integer and let

$$\Phi_{|m(K_X+\Delta)|} : X \dashrightarrow Y$$

be the Iitaka fibration. Then Y is *projective* even when X is only a complex analytic variety. By taking suitable resolutions, it is sufficient to treat the case where $\Phi : X \to Y$ is a proper surjective morphism from a compact Kähler manifold X to a normal projective variety Y with connected fibers. In this situation, the arguments in [FM], [A2], [FG3, Section 3], and so on work with some minor modifications. This is because we can use the theory of variations of (mixed) $\mathbb{R}$-Hodge structure for $\Phi : X \to Y$. In general, the general fibers of Φ are not projective. They are only Kähler. Therefore, the natural polarization of the variation of (mixed) Hodge structure is defined only on $\mathbb{R}$.

Anyway, by the arguments in [FM, Sections 4 and 5], we can find an effective $\mathbb{Q}$-divisor Δ_Y on Y such that the finite generation of $R(X, \Delta)$ is equivalent to the finite generation of $R(Y, \Delta_Y)$ where (Y, Δ_Y) is kawamata log terminal and $K_Y + \Delta_Y$ is big. Therefore, by [BCHM], $R(X, \Delta)$ is finitely generated.

REMARK 5.4. Let $f : X \to Y$ be a surjective morphism from a compact Kähler manifold (or, more generally, a complex analytic variety in Fujiki's class $\mathcal{C}$) X to a projective variety Y. In this setting, we can prove various fundamental results, for example, Kollár type vanishing theorem, torsion-free theorem, weak positivity theorem, and so on. For the details, see [F12].

By the arguments in [A2, Sections 4 and 5] and the semipositivity theorem in [F4] (see also [FF], [FFS], and [F12, Theorem 1.5]), we can prove an analytic generalization of Theorem 3.7 without any difficulties (see Example 6.1 and Remark 6.3 for the case when the varieties are not in Fujiki's class $\mathcal{C}$).

Theorem 5.5 (cf. [FG3, Theorem 3.6]). *Let X be a normal complex analytic variety in Fujiki's class $\mathcal{C}$ and let Δ be a $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Let $f : X \to Y$ be a surjective morphism onto a normal projective variety Y. Assume that $f : (X, \Delta) \to Y$ is an lc-trivial fibration, that is,*

(i) *$(F, \Delta|_F)$ is sub log canonical for a general fiber F of $f : X \to Y$,*

(ii) $\mathrm{rank} f_*\mathcal{O}_X(\lceil \mathbf{A}^*(X, \Delta)\rceil) = 1$, *and*

(iii) *there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that*

$$K_X + \Delta \sim_{\mathbb{Q}} f^*D.$$

Let $\mathbf{B}$ and $\mathbf{M}$ be the induced discriminant and moduli $\mathbb{Q}$-b-divisor of $f : (X, \Delta) \to Y$. Then

(1) *$\mathbf{K}+\mathbf{B}$ is $\mathbb{Q}$-b-Cartier, that is, there exists a proper birational morphism $Y' \to Y$ from a normal variety Y' such that $\mathbf{K} + \mathbf{B} = \overline{K_{Y'} + \mathbf{B}_{Y'}}$,*

(2) *$\mathbf{M}$ is b-nef.*

In the setting of Theorem 5.5, we have:

Conjecture 5.6 (cf. Conjecture 3.9). *In Theorem 5.5, $\mathbf{M}$ is b-semi-ample.*

Conjecture 5.6 may be harder than Conjecture 3.9 because there are no good moduli theory for compact Kähler manifolds.

Proof of Theorem 5.1. Let $f : X \dashrightarrow Y$ be the Iitaka fibration with respect to $K_X + \Delta$. By replacing X and Y bimeromorphically, we may assume that Y is a smooth projective variety, X is a compact Kähler manifold, (X, Δ) is kawamata log terminal such that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X, and f is a morphism. By using the theory of log-canonical bundle formulas discussed in [FM, Section 4] with the aid of Theorem 5.5, we can apply [FM, Theorem 5.2]. Then there are a smooth projective variety Y', which is birationally equivalent to Y, and an effective $\mathbb{Q}$-divisor Δ' on Y' such that (Y', Δ') is a kawamata log terminal pair, $K_{Y'} + \Delta'$ is big, and

$$R(X, \Delta)^{(e)} \simeq R(Y', \Delta')^{(e')}$$

for some positive integers e and e', where

$$R(Y', \Delta') = \bigoplus_{m \geq 0} H^0(Y', \mathcal{O}_{Y'}(\lfloor m(K_{Y'} + \Delta') \rfloor)).$$

Note that $R^{(e)} = \bigoplus_{m \geq 0} R_{em}$ for a graded ring $R = \bigoplus_{m \geq 0} R_m$. By [BCHM], $R(Y', \Delta')$ is a finitely generated $\mathbb{C}$-algebra. This implies that $R(X, \Delta)$ is a finitely generated $\mathbb{C}$-algebra. □

Remark 5.7. By using Theorem 5.5, we can prove some analytic generalizations of Theorem 1.1, Theorem 1.2, and so on. We leave the details for the interested reader.

Conjecture 5.8 is obviously equivalent to Conjecture 1.10 by taking a resolution.

Conjecture 5.8. *Let X be a normal complex analytic variety in Fujiki's class $\mathcal{C}$ and let Δ be an effective $\mathbb{Q}$-divisor on X such that (X, Δ) is log canonical. Then the log canonical ring*

$$R(X, \Delta) = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(\lfloor m(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

We give a remark on Conjecture 5.8.

REMARK 5.9. Conjecture 5.8 is still open even when X is projective. If Conjecture 5.6 holds true, then we can reduce Conjecture 5.8 to the case when X is projective by the same way as in [FM, Sections 4 and 5] (see also 5.3, Proof of Theorem 5.1, and Remark 4.1). Note that Theorem 5.5 is not sufficient for this reduction argument.

We close this section with an observation on Conjecture 5.6.

5.10 (Observation IV). Let $f : (X, \Delta) \to Y$ be an lc-trivial fibration as in Theorem 5.5. By taking a resolution, we assume that X is a compact Kähler manifold and that $\mathrm{Supp}\Delta$ is a simple normal crossing divisor on X. For Conjecture 5.6, it seems to be reasonable to assume that

$$\mathrm{rank} f_*\mathcal{O}_X(\lceil m\Delta^-\rceil) = 1$$

for every nonnegative integer m as in 3.10, equivalently,

$$\kappa(F, K_F + \Delta^+|_F) = 0$$

where F is a sufficiently general fiber of f. The extra assumption $\kappa(F, K_F + \Delta^+|_F) = 0$ is harmless for [FM, Sections 4 and 5]. Therefore, Remark 5.9 works even if we add the extra assumption $\kappa(F, K_F + \Delta^+|_F) = 0$ to Conjecture 5.6. Unfortunately, the reduction arguments in 3.11 based on the minimal model program have not been established for compact Kähler manifolds.

Anyway, Conjecture 5.6 looks harder than Conjecture 3.9.

6. Examples of Non-Kähler Manifolds

In this section, we discuss some examples of compact complex non-Kähler manifolds constructed by Atiyah ([A]) and Wilson ([W]) for the reader's convenience. These examples clarify the reason why we have to assume that the varieties are in Fujiki's class $\mathcal{C}$ in Section 5. For some related examples, see [U, Remark 15.3] and [Mg].

The following example is due to Atiyah (see [A, §10, Specific examples]). This example shows that the Fujita–Kawamata semipositivity theorem does not hold for non-Kähler manifolds. For the details of the theory of fiber spaces of complex tori, see [A].

Example 6.1 (cf. [A, §10]). Let us construct an analytic family of tori $f : X \to Y = \mathbb{P}^1$ such that $f_*\omega_{X/Y}$ is not semipositive.

We put

$$I = \begin{pmatrix} 0, & 1 \\ -1, & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0, & \sqrt{-1} \\ \sqrt{-1}, & 0 \end{pmatrix}, \quad K = \begin{pmatrix} \sqrt{-1}, & 0 \\ 0, & -\sqrt{-1} \end{pmatrix},$$

and

$$E = \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}.$$

We take $s_1, s_2 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)) \setminus \{0\}$ such that s_1 and s_2 have no common zeros. We consider the analytic family of tori $f : X \to Y := \mathbb{P}^1$ where

$$X = \mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))/\Lambda.$$

Note that $\mathbb{V}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)) = \mathrm{Spec}_{\mathbb{P}^1}\mathrm{Sym}((\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))^*)$ is the total space of $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ and

$$\Lambda = \left\langle E\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \ I\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \ J\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \ K\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right\rangle.$$

In other words, the fiber $X_p = f^{-1}(p)$ for $p \in Y$ is $\mathbb{C}^2/\Lambda(p)$, where $\Lambda(p)$ is the lattice

$$\left\langle E\begin{pmatrix} s_1(p) \\ s_2(p) \end{pmatrix}, \ I\begin{pmatrix} s_1(p) \\ s_2(p) \end{pmatrix}, \ J\begin{pmatrix} s_1(p) \\ s_2(p) \end{pmatrix}, \ K\begin{pmatrix} s_1(p) \\ s_2(p) \end{pmatrix} \right\rangle_{\mathbb{Z}}$$

in $\mathbb{C}^2$. For the details of the construction, see [A]. Then $\omega_{X/Y} \simeq f^*\mathcal{O}_{\mathbb{P}^1}(-2)$ by [A, Proposition 10]. Therefore, we have

$$f_*\omega_{X/Y} \simeq \mathcal{O}_{\mathbb{P}^1}(-2).$$

This means that $f_*\omega_{X/Y}$ is not always semipositive when X is not Kähler. Note that f is smooth in this example.

Remark 6.2. Let $f : V \to W$ be a surjective morphism from a compact complex manifold V in Fujiki's class $\mathcal{C}$ to a smooth projective curve W. Then we can easily check that $f_*\omega_{V/W}$ is semipositive by [Ft]. This means that X in Example 6.1 is not in Fujiki's class $\mathcal{C}$.

REMARK 6.3. Example 6.1 shows that Theorem 5.5 does not hold without assuming that X is in Fujiki's class $\mathcal{C}$. Therefore, the proof of Theorem 5.1 does not work for varieties which are not in Fujiki's class $\mathcal{C}$.

The following example is essentially the same as Wilson's example (see [W, Example 4.3]). It is a compact complex non-Kähler 4-fold whose canonical ring is not a finitely generated $\mathbb{C}$-algebra. Wilson's example is very important. Unfortunately, [W, Example 4.3] omitted some technical details. Moreover, we can not find it in the standard literature for the minimal model program. So we explain a slightly simplified example in details for the reader's convenience.

Example 6.4 (cf. [W, Example 4.3]). Let us construct a 4-dimensional compact complex non-Kähler manifold X whose canonical ring $R(X)$ is not a finitely generated $\mathbb{C}$-algebra.

Let $C \subset \mathbb{P}^2$ be a smooth elliptic curve and let H be a line on $\mathbb{P}^2$. We blow up 12 general points $P_1, \cdots, P_{12}$ on C and one point $P \notin C$. Let $\pi : Z \to \mathbb{P}^2$ denote this birational modification and let E be the exceptional curve $\pi^{-1}(P)$. Let C' be the strict transform of C. We put $H' = \pi^*H - E$. Then the linear system $|H'|$ is free and $(H')^2 = 0$. Note that $K_Z \sim -C' + E$.

CLAIM 1. The linear system $|n\pi^*H + (n-1)C'|$ is free for every $n \geq 1$ and the base locus $\mathrm{Bs}|n\pi^*H + nC'| = C'$ for every $n \geq 1$. Therefore, we have

$$|n\pi^*H + nC'| = |n\pi^*H + (n-1)C'| + C'$$

for every $n \geq 1$.

PROOF OF CLAIM 1. It is obvious that $|n\pi^*H|$ is free. We consider the following short exact sequence:

$$(\spadesuit) \qquad \begin{aligned} 0 &\to \mathcal{O}_Z(n\pi^*H + (r-1)C') \to \mathcal{O}_Z(n\pi^*H + rC') \\ &\to \mathcal{O}_{C'}(n\pi^*H + rC') \to 0 \end{aligned}$$

for $1 \leq r \leq n$. Note that $\deg \mathcal{O}_{C'}(n\pi^*H + rC') \geq 3$ for $1 \leq r \leq n-1$. Therefore, $|\mathcal{O}_{C'}(n\pi^*H + rC')|$ is very ample for $1 \leq r \leq n-1$ since C' is an elliptic curve. On the other hand,

$$\begin{aligned} n\pi^*H + (r-1)C' - K_Z &\sim n\pi^*H + rC' - E \\ &= (n-1)\pi^*H + rC' + H' \end{aligned}$$

is nef and big for $1 \le r \le n-1$. By the Kawamata–Viehweg vanishing theorem, we obtain

$$H^1(Z, \mathcal{O}_Z(n\pi^*H + (r-1)C')) = 0$$

for $1 \le r \le n-1$. By using the long exact sequence associated to (♠), we have that $|n\pi^*H + rC'|$ is free for $1 \le r \le n-1$ by induction on r. Note that

$$H^0(C', \mathcal{O}_{C'}(n\pi^*H + nC')) = 0$$

for every $n \neq 0$ since $P_1, \cdots, P_{12}$ are general points on C. Precisely speaking, we take $P_1, \cdots, P_{12}$ such that $\mathcal{O}_{C'}(\pi^*H + C')$ is not a torsion element in $\mathrm{Pic}^0(C')$. This means that the natural inclusion

$$0 \to H^0(Z, \mathcal{O}_Z(n\pi^*H + (n-1)C')) \to H^0(Z, \mathcal{O}_Z(n\pi^*H + nC'))$$

is an isomorphism for every $n \ge 1$. Thus we have

$$|n\pi^*H + nC'| = |n\pi^*H + (n-1)C'| + C'$$

for every $n \ge 1$. □

Similarly, we can check the following statement.

Claim 2. The linear system $|4\pi^*H + 4H' + 6C' - 2E|$ is free.

Proof of Claim 2. We note that

$$4\pi^*H + 4H' + 6C' - 2E = 6H' + 2\pi^*H + 6C'.$$

We also note that $|H'|$ and $|\pi^*H|$ are free. We consider the linear system $|6H' + 2\pi^*H + rC'|$ for $0 \le r \le 6$. If $r = 0$, then the linear system $|6H' + 2\pi^*H|$ is free. We consider the following short exact sequence:

$$(\clubsuit) \qquad \begin{aligned} 0 \to \mathcal{O}_Z(6H' + 2\pi^*H + (r-1)C') &\to \mathcal{O}_Z(6H' + 2\pi^*H + rC') \\ \to \mathcal{O}_{C'}(6H' + 2\pi^*H + rC') &\to 0. \end{aligned}$$

Note that

$$\begin{aligned} 6H' + 2\pi^*H + (r-1)C' - K_Z &\sim 6H' + 2\pi^*H - E + rC' \\ &= 7H' + \pi^*H + rC' \end{aligned}$$

is nef and big for $1 \le r \le 6$. Therefore, by the Kawamata–Viehweg vanishing theorem, we obtain

$$H^1(Z, \mathcal{O}_Z(6H' + 2\pi^*H + (r-1)C')) = 0$$

for $1 \le r \le 6$. On the other hand,

$$\deg \mathcal{O}_{C'}(6H' + 2\pi^*H + rC') \ge 6$$

for $1 \le r \le 6$. Thus $|\mathcal{O}_{C'}(6H' + 2\pi^*H + rC')|$ is very ample for $1 \le r \le 6$. Note that C' is an elliptic curve. By considering the long exact sequence associated to (♣) and by induction on r, we obtain that $|6H' + 2\pi^*H + rC'|$ is free for $0 \le r \le 6$. In particular, $|4\pi^*H + 4H' + 6C' - 2E|$ is free. □

We take a general member C_0 of the free linear system $|4\pi^*H + 4H' + 6C' - 2E|$ and take the double cover $g : Y \to Z$ ramified along C_0. Then we have

$$\begin{aligned} K_Y &= g^*(K_Z + 2\pi^*H + 2H' + 3C' - E) \\ &\sim g^*(2\pi^*H + 2H' + 2C'). \end{aligned}$$

Note that $|g^*H'|$ is free on a smooth projective surface Y such that $\kappa(Y, g^*H') = 1$. Then we can take $s_1, s_2 \in H^0(Y, \mathcal{O}_Y(g^*H')) \setminus \{0\}$ such that s_1 and s_2 have no common zeros. By using s_1 and s_2, we can construct the analytic family of tori $f : X \to Y$ as in Example 6.1, that is,

$$X = \mathbb{V}(\mathcal{O}_Y(g^*H') \oplus \mathcal{O}_Y(g^*H'))/\Lambda$$

and

$$\Lambda = \left\langle E\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \ I\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \ J\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}, \ K\begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right\rangle.$$

Then X is a compact complex 4-fold. By [A, Proposition 10], we can check that

$$\begin{aligned} \omega_X &= f^*\mathcal{O}_Y(K_Y - 2g^*H') \\ &= f^*\mathcal{O}_Y(g^*(2\pi^*H + 2H' + 2C') - 2g^*H') \\ &\simeq f^*\mathcal{O}_Y(g^*(2\pi^*H + 2C')). \end{aligned}$$

Therefore, if the canonical ring $R(X) = \bigoplus_{m\geq 0} H^0(X, \omega_X^{\otimes m})$ is a finitely generated $\mathbb{C}$-algebra, then so is

$$R(Z, 2\pi^*H + 2C') = \bigoplus_{m\geq 0} H^0(Z, \mathcal{O}_Z(2m(\pi^*H + C'))).$$

Claim 3. $R(Z, 2\pi^*H + 2C') = \bigoplus_{m\geq 0} H^0(Z, \mathcal{O}_Z(2m(\pi^*H + C')))$ is not a finitely generated $\mathbb{C}$-algebra.

Proof of Claim 3. By Claim 1,

$$\bigoplus_{a=1}^{m-1} H^0(Z, \mathcal{O}_Z(2a(\pi^*H + C'))) \otimes H^0(Z, \mathcal{O}_Z(2(m-a)(\pi^*H + C'))) \\ \to H^0(Z, \mathcal{O}_Z(2m(\pi^*H + C')))$$

is not surjective for any $m \geq 1$. This implies that $R(Z, 2\pi^*H + 2C')$ is not a finitely generated $\mathbb{C}$-algebra. $\square$

Alternative Proof of Claim 3. Since $2\pi^*H + 2C'$ is nef and big, we know that $R(Z, 2\pi^*H + 2C')$ is a finitely generated $\mathbb{C}$-algebra if and only if $2\pi^*H + 2C'$ is semi-ample (see, for example, [L, Theorem 2.3.15]). On the other hand, $\mathcal{O}_{C'}(\pi^*H + C')$ is not a torsion element in $\mathrm{Pic}^0(C')$. This implies that $\pi^*H + C'$ is not semi-ample. Therefore, $R(Z, 2\pi^*H + 2C')$ is not a finitely generated $\mathbb{C}$-algebra. $\square$

Therefore, the canonical ring $R(X)$ of X is not finitely generated as a $\mathbb{C}$-algebra. Since $f_*\omega_{X/Y} \simeq \mathcal{O}_Y(-2g^*H')$ is not nef, X is non-Kähler. Note that X is a compact complex manifold which is not in Fujiki's class $\mathcal{C}$ by Theorem 5.1.

Example 6.4 shows that the finite generation of canonical rings does not always hold for compact complex manifolds which are not in Fujiki's class $\mathcal{C}$.

Remark 6.5. Wilson's original example (see [W, Example 4.3]) uses the fact that $nH' + (n-1)C'$ is very ample for all $n \geq 1$ (see [W, Claim in Example 4.3]). Since H' is not a big divisor, the statement in [W, Claim in

Example 4.3] has to be changed suitably. So, we modified his construction slightly. Note that $f : X \to Y$ constructed in Example 6.4 does not coincide with Wilson's original example $V \to \widetilde{S}$ in [W, Example 4.3].

Example 6.4 also shows that there are no generalizations of the abundance conjecture for compact complex non-Kähler manifolds.

Remark 6.6. In Example 6.4, we can check that $\pi^*H + C'$ is nef and big. Therefore, ω_X is a pull-back of a nef and big line bundle on a smooth projective variety Y. So X should be recognized to be a minimal model. However, ω_X is not semi-ample. This means that the abundance conjecture can not be generalized for compact complex manifolds which are not in Fujiki's class $\mathcal{C}$.

We close this section with a comment on Moriwaki's result (see [Mw]).

Remark 6.7. Let X be a three-dimensional compact complex manifold. Moriwaki proved that the canonical ring $R(X)$ of X is always a finitely generated $\mathbb{C}$-algebra even when X is not Kähler (see [Mw, (3.5) Theorem]).

7. Appendix

In this appendix, we quickly discuss the minimal model program for log canonical pairs and describe some related examples by János Kollár for the reader's convenience. We assume that all the varieties are algebraic throughout this section.

7.1. Minimal model program for log canonical pairs

Let $\pi : (X, \Delta) \to S$ be a projective morphism such that (X, Δ) is a $\mathbb{Q}$-factorial log canonical pair. Then we can run the minimal model program on (X, Δ) over S since we have the cone and contraction theorem (see, for example, [F10, Theorem 1.1]) and the flip theorem for log canonical pairs (see [B1, Corollary 1.2] and [HX, Corollary 1.8]). We can also run the minimal model program on (X, Δ) over S with scaling by [F10, Theorem 1.1]. Unfortunately, we do not know if the minimal model program terminates or not.

CONJECTURE 7.1 (Flip conjecture II). *A sequence of flips*

$$(X_0, \Delta_0) \dashrightarrow (X_1, \Delta_1) \dashrightarrow (X_2, \Delta_2) \dashrightarrow \cdots$$

terminates after finitely many steps. Namely, there exists no infinite sequence of flips.

Note that each flip in Conjecture 7.1 is a flip described in Case 2 in the introduction.

If Conjecture 7.1 is true, then we can freely use the minimal model program in full generality. In order to prove Conjecture 7.1 in dimension n, it is sufficient to solve Conjecture 7.1 for kawamata log terminal pairs in dimension $\leq n$. This reduction is an easy consequence of the existence of dlt blow-ups and the special termination theorem by induction on the dimension. For the details, see [F5] and [F13].

More generally, by the cone and contraction theorem (see [F10, Theorem 1.1]), [B1, Theorem 1.1] and [HX, Theorem 1.6], we can run the minimal model program for non-$\mathbb{Q}$-factorial log canonical pairs (see [F6, Subsection 3.1.2] and [F13]). Note that the termination of flips in this more general setting also follows from Conjecture 7.1 for kawamata log terminal pairs by using the existence of dlt blow-ups and the special termination theorem as explained above (see [F5] and [F13]).

Anyway, Conjecture 7.1 for $\mathbb{Q}$-factorial kawamata log terminal pairs is one of the most important open problems for the minimal model program.

7.2. On log canonical flops

In this subsection, we discuss some examples, which show the differences between kawamata log terminal pairs and log canonical pairs. The following result is well known to the experts (see, for example, [F6, Theorem 3.24]).

THEOREM 7.2. *Let (X, Δ) be a kawamata log terminal pair and let D be a $\mathbb{Q}$-divisor on X. Then $\bigoplus_{m \geq 0} \mathcal{O}_X(\lfloor mD \rfloor)$ is a finitely generated $\mathcal{O}_X$-algebra.*

PROOF. If D is $\mathbb{Q}$-Cartier, then this theorem is obvious. So we assume that D is not $\mathbb{Q}$-Cartier. Since the statement is local, we may assume that X is affine. By replacing D with D' such that $D' \sim D$ and $D' \geq 0$, we may further assume that D is effective. By [BCHM], we can take a small projective birational morphism $f : Y \to X$ such that Y is $\mathbb{Q}$-factorial, $K_Y + \Delta_Y = f^*(K_X + \Delta)$, and (Y, Δ_Y) is kawamata log terminal. Let D_Y be the strict transform of D on Y. Note that D_Y is $\mathbb{Q}$-Cartier because Y is $\mathbb{Q}$-factorial. Let ε be a small positive number. By running the minimal model program on $(Y, \Delta_Y + \varepsilon D_Y)$ over X with scaling, we may assume that D_Y is f-nef. Then, by the basepoint-free theorem, D_Y is f-semi-ample. Therefore,

$$\bigoplus_{m \geq 0} f_*\mathcal{O}_Y(\lfloor mD_Y \rfloor)$$

is a finitely generated $\mathcal{O}_X$-algebra. Since we have an $\mathcal{O}_X$-algebra isomorphism

$$\bigoplus_{m \geq 0} f_*\mathcal{O}_Y(\lfloor mD_Y \rfloor) \simeq \bigoplus_{m \geq 0} \mathcal{O}_X(\lfloor mD \rfloor),$$

$\bigoplus_{m \geq 0} \mathcal{O}_X(\lfloor mD \rfloor)$ is a finitely generated $\mathcal{O}_X$-algebra. □

The next example shows that Theorem 7.2 does not always hold for log canonical pairs. In other words, if (X, Δ) is log canonical, then $\bigoplus_{m \geq 0} \mathcal{O}_X(\lfloor mD \rfloor)$ is not necessarily finitely generated as an $\mathcal{O}_X$-algebra.

Example 7.3 ([Ko3, Exercise 95]). Let $E \subset \mathbb{P}^2$ be a smooth cubic curve. Let S be a surface obtained by blowing up nine sufficiently general points on E and let $E_S \subset S$ be the strict transform of E. Let H be a very ample divisor on S giving a projectively normal embedding $S \subset \mathbb{P}^N$. Let $X \subset \mathbb{A}^{N+1}$ be the cone over S and let $D \subset X$ be the cone over E_S. Then (X, D) is log canonical by Lemma 7.4 below since $K_S + E_S \sim 0$. Let $P \in D \subset X$ be the vertex of the cones D and X. Since X is normal, we have

$$\begin{aligned} H^0(X, \mathcal{O}_X(mD)) &= H^0(X \setminus P, \mathcal{O}_X(mD)) \\ &\simeq \bigoplus_{r \in \mathbb{Z}} H^0(S, \mathcal{O}_S(mE_S + rH)). \end{aligned}$$

By construction, $\mathcal{O}_S(mE_S)$ has only the obvious section which vanishes along mE_S for every $m > 0$. It can be checked by induction on m using the following exact sequence

$$\begin{aligned} 0 &\to H^0(S, \mathcal{O}_S((m-1)E_S)) \to H^0(S, \mathcal{O}_S(mE_S)) \\ &\to H^0(E_S, \mathcal{O}_{E_S}(mE_S)) \to \cdots \end{aligned}$$

since $\mathcal{O}_{E_S}(E_S)$ is not a torsion element in $\mathrm{Pic}^0(E_S)$. Therefore,

$$H^0(S, \mathcal{O}_S(mE_S + rH)) = 0$$

for every $r < 0$. So, we have

$$\bigoplus_{m\geq 0} \mathcal{O}_X(mD) \simeq \bigoplus_{m\geq 0}\bigoplus_{r\geq 0} H^0(S, \mathcal{O}_S(mE_S + rH)).$$

Since E_S is nef, $\mathcal{O}_S(mE_S+4H) \simeq \mathcal{O}_S(K_S+E_S+mE_S+4H)$ is very ample for every $m \geq 0$. Therefore, by replacing H with $4H$, we may assume that $\mathcal{O}_S(mE_S + rH)$ is very ample for every $m \geq 0$ and every $r > 0$. In this setting, the multiplication maps

$$\begin{aligned}\bigoplus_{a=0}^{m-1} H^0(S, \mathcal{O}_S(aE_S + H)) \otimes H^0(S, \mathcal{O}_S((m-a)E_S))\\ \to H^0(S, \mathcal{O}_S(mE_S + H))\end{aligned}$$

are never surjective. This implies that $\bigoplus_{m\geq 0} \mathcal{O}_X(mD)$ is not finitely generated as an $\mathcal{O}_X$-algebra.

Let us recall an easy lemma for the reader's convenience.

LEMMA 7.4. *Let (V, Δ) be a log canonical pair such that V is smooth,* $\mathrm{Supp}\Delta$ *is a simple normal crossing divisor on V, and $K_V + \Delta \sim_{\mathbb{Q}} 0$. Let $V \subset \mathbb{P}^N$ be a projectively normal embedding. Let $W \subset \mathbb{A}^{N+1}$ be the cone over V and let Δ_W be the cone over Δ. Then (W, Δ_W) is log canonical.*

PROOF. Let $g : W' \to W$ be the blow-up at $0 \in \mathbb{A}^{N+1}$ and let E be the exceptional divisor of g. Note that W' is smooth and $E \simeq V$. Then we can check that

$$K_{W'} + \Delta_{W'} + E = g^*(K_W + \Delta_W)$$

where $\Delta_{W'}$ is the strict transform of Δ_W. Note that $\mathrm{Supp}(\Delta_{W'} + E)$ is a simple normal crossing divisor on W'. Thus, (W, Δ_W) is log canonical. □

Let us recall the definition of log canonical flops.

DEFINITION 7.5 (Log canonical flop). Let (X, Δ) be a log canonical pair. Let D be a Cartier divisor on X. Let $f : X \to Y$ be a small contraction such that $K_X+\Delta$ is numerically f-trivial and $-D$ is f-ample. The opposite of f with respect to D is called a *flop with respect to* D *for* (X, Δ) or simply a D*-flop*. We sometimes call it *flop* or a *log canonical flop* if there is no risk of confusion.

Remark 7.6. Without loss of generality, we may assume that Δ is a $\mathbb{Q}$-divisor and $K_X+\Delta \sim_{\mathbb{Q},f} 0$ in Definition 7.5 (see, for example, Remark 4.7 and [FG2, Theorem 4.9 and Subsection 4.1]). Furthermore, if $(X, \Delta+\varepsilon D)$ is log canonical for some positive number ε, then a D-flop always exists by [B1, Theorem 1.1 and Corollary 1.2] and [HX, Theorem 1.6 and Corollary 1.8].

The following example shows that log canonical flops do not always exist. Of course, flops always exist for kawamata log terminal pairs by [BCHM].

Example 7.7 ([Ko3, Exercise 96]). Let E be an elliptic curve and let L be a degree zero line bundle on E. We put

$$S=\mathbb{P}_E(\mathcal{O}_E\oplus L).$$

Let C_1 and C_2 be the sections of the $\mathbb{P}^1$-bundle $p: S\to E$. We note that $K_S+C_1+C_2\sim 0$. As in Example 7.3, we take a sufficiently ample divisor $H=aF+bC_1$ on S giving a projectively normal embedding $S\subset \mathbb{P}^N$, where F is a fiber of the $\mathbb{P}^1$-bundle $p: S\to E$, $a>0$, and $b>0$. We may assume that $\mathcal{O}_S(mC_i+rH)$ is very ample for $i=1,2$, every $m\geq 0$, and every $r>0$. Moreover, we may assume that $\mathcal{O}_S(M+rH)$ is very ample for any nef Cartier divisor M and every $r>0$. Let $X\subset \mathbb{A}^{N+1}$ be the cone over S and let $D_i\subset X$ be the cones over C_i for $i=1$ and 2. Since $K_S+C_1+C_2\sim 0$, (X, D_1+D_2) is log canonical by Lemma 7.4. We can check $K_X+D_1+D_2\sim 0$ by construction. By the same arguments as in Example 7.3, we can prove the following statement.

Claim 1. If L is a non-torsion element in $\mathrm{Pic}^0(E)$, then

$$\bigoplus_{m\geq 0}\mathcal{O}_X(mD_i)$$

is not a finitely generated $\mathcal{O}_X$-algebra for $i=1$ and 2.

We note that $\mathcal{O}_S(mC_i)$ has only the obvious section which vanishes along mC_i for every $m > 0$.

Let $B \subset X$ be the cone over F. Then we have the following result.

CLAIM 2. The graded $\mathcal{O}_X$-algebra $\bigoplus_{m\geq 0}\mathcal{O}_X(mB)$ is a finitely generated $\mathcal{O}_X$-algebra.

PROOF OF CLAIM 2. By the same arguments as in Example 7.3, we have

$$\bigoplus_{m\geq 0}\mathcal{O}_X(mB) \simeq \bigoplus_{m\geq 0}\bigoplus_{r\geq 0} H^0(S, \mathcal{O}_S(mF + rH)).$$

We consider $V = \mathbb{P}_S(\mathcal{O}_S(F) \oplus \mathcal{O}_S(H))$. Then $\mathcal{O}_V(1)$ is semi-ample. Therefore,

$$\bigoplus_{n\geq 0} H^0(V, \mathcal{O}_V(n)) \simeq \bigoplus_{m\geq 0}\bigoplus_{r\geq 0} H^0(S, \mathcal{O}_S(mF + rH))$$

is finitely generated. $\square$

Let $P \in X$ be the vertex of the cone X and let $f : Y \to X$ be the blow-up at P. Let $A \simeq S$ be the exceptional divisor of f. We consider the $\mathbb{P}^1$-bundle $\pi : \mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{O}_S(H)) \to S$. Then

$$Y \simeq \mathbb{P}_S(\mathcal{O}_S \oplus \mathcal{O}_S(H)) \setminus G,$$

where G is the section of π corresponding to

$$\mathcal{O}_S \oplus \mathcal{O}_S(H) \to \mathcal{O}_S(H) \to 0.$$

We consider π^*F on Y. Then $\mathcal{O}_Y(\pi^*F)$ is obviously f-semi-ample. So, we obtain a contraction morphism $g : Y \to Z$ over X. We can check that

$$Z \simeq \mathrm{Proj}_X \bigoplus_{m\geq 0}\mathcal{O}_X(mB)$$

over X and that $h : Z \to X$ is a small projective contraction. On Y, we have

$$-A \sim \pi^*H = a\pi^*F + b\pi^*C_1.$$

Therefore, we obtain $aB + bD_1 \sim 0$ on X. Let B' be the strict transform of B on Z and let D'_i be the strict transform of D_i on Z for $i = 1$ and 2. Note that

B' is h-ample, $aB'+bD'_1 \sim 0$, and $K_Z+D'_1+D'_2 = h^*(K_X+D_1+D_2) \sim 0$. If L is not a torsion element, then the flop of $h: Z \to X$ with respect to D'_1 for $(Z, D'_1+D'_2)$ does not exist since $\bigoplus_{m\geq 0}\mathcal{O}_X(mD_1)$ is not finitely generated as an $\mathcal{O}_X$-algebra.

Let C be any Cartier divisor on Z such that $-C$ is h-ample. Then the flop of $h: Z \to X$ with respect to C exists if and only if

$$\bigoplus_{m\geq 0} h_*\mathcal{O}_Z(mC)$$

is a finitely generated $\mathcal{O}_X$-algebra. We can take positive integers m_0 and m_1 such that m_1C is numerically equivalent to $m_0D'_1$ over X. Note that $\mathrm{Exc}(h) \simeq E$. Therefore, we can find a degree zero Cartier divisor N on E such that

$$m_1C - m_0D'_1 \sim_h g_*(\pi|_Y)^*(p^*N).$$

Thus,

$$\bigoplus_{m\geq 0} h_*\mathcal{O}_Z(mm_1C)$$

is a finitely generated $\mathcal{O}_X$-algebra if and only if

$$R = \bigoplus_{m\geq 0} h_*\mathcal{O}_Z(m(m_0D'_1 + g_*(\pi|_Y)^*(p^*N)))$$

is so. Since h is small, R is isomorphic to

$$\bigoplus_{m\geq 0} \mathcal{O}_X(m(m_0D_1+\widetilde{N})),$$

where $\widetilde{N} \subset X$ is the cone over p^*N. Anyway,

$$\bigoplus_{m\geq 0} h_*\mathcal{O}_Z(mC)$$

is a finitely generated $\mathcal{O}_X$-algebra if and only if

$$\bigoplus_{m\geq 0} \mathcal{O}_X(m(m_0D_1+\widetilde{N}))$$

is so, where $\widetilde{N}$ is the cone over p^*N. We note the following commutative diagram:

$$\begin{tikzcd}
 & Z \arrow[dl, "h"'] & \\
X \arrow[drr, dashed] & & Y \arrow[ul, "g"'] \arrow[ll, "f"'] \arrow[d, "\pi|_Y"] \\
 & & S \arrow[d, "p"] \\
 & & E
\end{tikzcd}$$

where $X \dashrightarrow S$ is the natural projection from the vertex P of X.

Claim 3. If L is not a torsion element in $\mathrm{Pic}^0(E)$, then

$$\bigoplus_{m\geq 0} \mathcal{O}_X(m(m_0D_1+\widetilde{N}))$$

is not finitely generated as an $\mathcal{O}_X$-algebra. In particular, the flop of $h: Z \to X$ with respect to C does not exist.

Proof of Claim 3. By the same arguments as in Example 7.3, we have

$$\begin{aligned}
&\bigoplus_{m\geq 0} \mathcal{O}_X(m(m_0D_1+\widetilde{N})) \\
&\simeq \bigoplus_{m\geq 0}\bigoplus_{r\in\mathbb{Z}} H^0(S, \mathcal{O}_S(m(m_0C_1+p^*N)+rH)).
\end{aligned}$$

By considering

$$\begin{aligned}
0 &\to H^0(S,\mathcal{O}_S((l-1)C_1+mp^*N)) \to H^0(S,\mathcal{O}_S(lC_1+mp^*N)) \\
&\to H^0(C_1,\mathcal{O}_{C_1}(lC_1+mp^*N)) \to \cdots
\end{aligned}$$

for $1\leq l\leq mm_0$, we obtain that

$$\dim H^0(S,\mathcal{O}_S(m(m_0C_1+p^*N)))\leq 1$$

for every $m\geq 0$. Therefore, we can check that the above $\mathcal{O}_X$-algebra is not finitely generated by the same arguments as in Example 7.3. We note that

$\mathcal{O}_S(m(m_0C_1 + p^*N) + rH)$ is very ample for every $m \geq 0$ and every $r > 0$ because $m_0C_1 + p^*N$ is nef. $\square$

Anyway, if L is not a torsion element in $\mathrm{Pic}^0(E)$, then the flop of $h : Z \to X$ does not exist with respect to any divisor.

From now on, in the above setting, we assume that L is a torsion element in $\mathrm{Pic}^0(E)$. Then $\mathcal{O}_Y(\pi^*C_1)$ is f-semi-ample. So, we obtain a contraction morphism $g' : Y \to Z^+$ over X. It is easy to see that

$$\bigoplus_{m\geq 0} \mathcal{O}_X(mD_i)$$

is finitely generated as an $\mathcal{O}_X$-algebra for $i = 1, 2$ (cf. Claim 2),

$$Z^+ \simeq \mathrm{Proj}_X \bigoplus_{m\geq 0} \mathcal{O}_X(mD_1),$$

over X and that $Z^+ \to X$ is the flop of $Z \to X$ with respect to D'_1.

Let C be any Cartier divisor on Z such that $-C$ is h-ample. If $-C \sim_{\mathbb{Q},h} cB'$ for some positive rational number c, then it is obvious that the above $Z^+ \to X$ is the flop of $h : Z \to X$ with respect to C. If $-C \not\sim_{\mathbb{Q},h} cB'$ for any positive rational number c, then the flop of $h : Z \to X$ with respect to C does not exist. As above, we take positive integers m_0 and m_1 such that m_1C is numerically equivalent to $m_0D'_1$ over X. Then we can find a degree zero Cartier divisor N on E such that

$$m_1C - m_0D'_1 \sim_h g_*(\pi|_Y)^*(p^*N).$$

Since $-C \not\sim_{\mathbb{Q},h} cB'$ for any positive rational number c, N is a non-torsion element in $\mathrm{Pic}^0(E)$. Thus,

$$\bigoplus_{m\geq 0} h_*\mathcal{O}_Z(mC)$$

is finitely generated if and only if

$$\bigoplus_{m\geq 0} \mathcal{O}_X(m(m_0D_1 + \widetilde{N}))$$

is so, where $\widetilde{N} \subset X$ is the cone over $p^*N \subset S$. By the same arguments as in the proof of Claim 3, we can check that

$$\bigoplus_{m \geq 0} \mathcal{O}_X(m(m_0 D_1 + \widetilde{N}))$$

is not finitely generated as an $\mathcal{O}_X$-algebra. We note that

$$\dim H^0(S, \mathcal{O}_S(m(m_0 C_1 + p^*N))) = 0$$

for every $m > 0$ since N is a non-torsion element in $\mathrm{Pic}^0(E)$ and L is a torsion element in $\mathrm{Pic}^0(E)$ (see the proof of Claim 3).

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