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Generalized Burniat Type surfaces and Bagnera-de Franchis Varieties

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Dedicated to the memory of Kunihiko Kodaira with great admiration

Abstract. In this article we construct three new families of surfaces of general type with $p_g = q = 0, K^2 = 6$, and seven new families of surfaces of general type with $p_g = q = 1, K^2 = 6$, realizing 10 new fundamental groups. We also show that these families correspond to pairwise distinct irreducible connected components of the Gieseker moduli space of surfaces of general type.

We achieve this using two different main ingredients. First we introduce a new class of surfaces, called generalized Burniat type surfaces, and we completely classify them (and the connected components of the moduli space containing them). Second, we introduce the notion of Bagnera-de Franchis varieties: these are the free quotients of an Abelian variety by a cyclic group (not consisting only of translations). For these we develop some basic results.

Introduction

The present paper continues, with new inputs, a research developed in a series of articles ([BC04], [BCG08], [BC10], [BC11a], [BC11b], [BC12], [BCGP12], [BC13a], [BC13b]) and dedicated to the discovery of new surfaces of general type with geometric genus $p_g = 0$, to their classification, and to the description of their moduli spaces (see the survey article [BCP11] for an account of what is known about surfaces wit $p_g = 0$, related conjectures and results).

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Indeed, in this article, we consider the more general case of surfaces of general type with $\chi = 1$, i.e., with $p_g = q$.

In the first part we focus again on the construction method originally due to Burniat (singular bidouble coverings), but in the reformulation done by Inoue (quotients by Abelian groups of exponent two), presenting it in a rather general fashion which shows how topological methods allow to describe explicitly connected components of moduli spaces. A first novelty here is a refined analysis of pencils of Del Pezzo surfaces admitting a certain group of symmetries, as we shall now explain.

In a more general approach (cf. [BC13b]) we consider quotients (cf. [BC12] for the case of a free action, treated there in an even greater generality), by some group G of the form $(\mathbb{Z}/m)^r$, of varieties $\hat{X}$ contained in a product of curves $\Pi_i C_i$, where each C_i is a maximal Abelian cover of the projective line with Galois group of exponent m and with fixed branch locus.

In the case $m = 2$ there is a connection with the Burniat surfaces: these are surfaces of general type with invariants $p_g = 0$ and $K^2 = 6, 5, 4, 3, 2$, whose birational models were constructed by Pol Burniat (cf. [Bur66]) in 1966 as singular bidouble covers of the projective plane. Later these surfaces were reconstructed by Inoue (cf. [Ino94]) as $G := (\mathbb{Z}/2\mathbb{Z})^3$-quotients of a ($G$-invariant) hypersurface $\hat{X}$ of multidegree $(2, 2, 2)$ in a product of three elliptic curves.

While Inoue writes the (affine) equation of $\hat{X}$ in terms of the uniformizing parameters of the respective elliptic curves using a variant of the Weierstrass' function (a Legendre function), we found it much more useful to write the elliptic curves as the complete intersection of two diagonal quadrics in three space.

This algebraic and systematic approach allows us, also with the aid of computer algebra, to find all the possible such constructions.

Our situation is as follows: we consider first the following diagram of quotient morphisms:

$$
\begin{array}{ll}
E_1 \times E_2 \times E_3 & E_1 := \{x_1^2 + x_2^2 + x_3^2 = 0,\ x_0^2 = a_1 x_1^2 + a_2 x_2^2\} \\
\Big\downarrow \mathcal{H}' := (\mathbb{Z}/2)^3 \ \pi' & E_2 := \{u_1^2 + u_2^2 + u_3^2 = 0,\ u_0^2 = b_1 u_1^2 + b_2 u_2^2\} \\
P_1 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & E_3 := \{z_1^2 + z_2^2 + z_3^2 = 0,\ z_0^2 = c_1 z_1^2 + c_2 z_2^2\} \\
\Big\downarrow \mathcal{H} := ((\mathbb{Z}/2)^2)^3 \ \pi & \\
P_2 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &
\end{array}
$$

where the map π' is given by "forgetting" the variables x_0, u_0, z_0, the map π is given by setting $x_j^2 = y_j$, $u_j^2 = v_j$, $z_j^2 = w_j$, $j = 1, 2, 3$, and where we view $P_2 \subset (\mathbb{P}^2)^3$ as the subvariety defined by the equations

$$y_1 + y_2 + y_3 = 0\,,\ v_1 + v_2 + v_3 = 0\,,\ w_1 + w_2 + w_3 = 0\,.$$

The Galois group for $\pi \circ \pi'$ is rather large, it is indeed $(\mathbb{Z}/2\mathbb{Z})^9 \cong \{\pm 1\}^9$.

We consider then P_1 with homogeneous coordinates $((s_1 : t_1), (s_2 : t_2), (s_3 : t_3))$ and for each $\lambda := (\lambda_1, \ldots, \lambda_8) \in \mathbb{C}^8 \setminus \{0\}$ we consider the hypersurface Y_λ of multidegree $(1, 1, 1)$ in P_1 given by the multihomogeneous equation

$$\begin{aligned}\lambda_1 s_1 s_2 s_3 + \lambda_2 s_1 s_2 t_3 + \lambda_3 s_1 t_2 s_3 + \lambda_4 s_1 t_2 t_3 + \\ \lambda_5 t_1 s_2 s_3 + \lambda_6 t_1 s_2 t_3 + \lambda_7 t_1 t_2 s_3 + \lambda_8 t_1 t_2 t_3 = 0.\end{aligned}$$

We then classify the subgroups H_1 (resp. H_0) of $\mathcal{H} \cong ((\mathbb{Z}/2\mathbb{Z})^2)^3$ which are isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ (resp. to $(\mathbb{Z}/2\mathbb{Z})^3$) and satisfy the property that there is an irreducible Del Pezzo surface Y_λ invariant under H_1 (resp. H_0).

We consider then $\hat{X}_\lambda := (\pi')^{-1}(Y_\lambda)$, which is then invariant under the subgroup $\mathcal{G}_1 \cong (\mathbb{Z}/2\mathbb{Z})^5 \subset (\mathbb{Z}/2\mathbb{Z})^9$ inverse image of H_1 (resp. $\mathcal{G}_0 \cong (\mathbb{Z}/2\mathbb{Z})^6$). We determine in this article all the subgroups $G \cong (\mathbb{Z}/2\mathbb{Z})^3 \subset \mathcal{G}_1$ (resp. $\mathcal{G}_0$), having the property that G acts freely on $\hat{X}_\lambda$.

This leads us to introduce a class of surfaces of general type, described by the following

Definition 0.1. Let $G \cong (\mathbb{Z}/2\mathbb{Z})^3 \leq \mathcal{G}_1$ (resp. $\mathcal{G}_0$) be such that G acts freely on $\hat{X}_\lambda$. Then the minimal resolution S of $X_\lambda := \hat{X}_\lambda/G$ is called a *generalized Burniat type surface*.

With the help of the computer algebra system MAGMA (cf. [BCP97]) we can classify all generalized Burniat type surfaces (=GBT surfaces for short) and can prove the following (see Proposition 3.4 and Theorem 3.6)

Main Theorem 1.

(1) *There are 16 irreducible families of GBT surfaces. These have* $K^2 = 6$ *and* $0 \leq p_g = q \leq 3$. *The families are listed in Tables 3 - 6, and the dimension of the irreducible family is 4 in cases* $\mathcal{S}_1$ *and* $\mathcal{S}_2$, *and 3 otherwise.*

(2) *Among the 16 families of generalized Burniat type surfaces four have* $p_g = q = 0$ *(Table 3), eight have* $p_g = q = 1$ *(Table 4), three have* $p_g = q = 2$ *(Table 5) and one has* $p_g = q = 3$ *(Table 6). Family* $\mathcal{S}_2$ *is the family of primary Burniat surfaces (the one due to Pol Burniat).*

(3) *The fundamental groups of these families are pairwise non isomorphic, except that* $\pi_1(S_{11}) \cong \pi_1(S_{12})$ *and* $\pi_1(S_{14}) \cong \pi_1(S_{15})$, *where* S_j *is in the family* $\mathcal{S}_j$.

(4) *The surfaces in the families* $\mathcal{S}_1$, $\mathcal{S}_3$ *and* $\mathcal{S}_4$ *realize new (i.e., up to now unknown) fundamental groups of surfaces with* $p_g = 0, K^2 = 6$, *while the surfaces in the families* $\mathcal{S}_5$-$\mathcal{S}_{11}$ *realize new fundamental groups for surfaces with* $p_g = q = 1, K^2 = 6$.

(5) *In cases* $\mathcal{S}_1$-$\mathcal{S}_{10}$, *each family of generalized Burniat type surfaces maps with a generically finite morphism onto an irreducible connected component of the Gieseker moduli space of surfaces of general type.*

We use indeed the techniques developed in [BC12] to determine the irreducible connected components of the moduli space containing the generalized Burniat type surfaces. We do not spell out all the details in the cases $\mathcal{S}_{13}$-$\mathcal{S}_{16}$, since the surfaces that we obtain in this way are not new and have already been classified by other authors.

In cases $\mathcal{S}_1$-$\mathcal{S}_{10}$ we can apply the general results of [BC12] concerning classical diagonal Inoue type varieties in order to describe the connected components of the moduli space containing the generalized Burniat type surfaces. As proved in [BF14], Bloch's conjecture holds for the surfaces in families $\mathcal{S}_1$-$\mathcal{S}_4$.

We then show that it is no coincidence that the fundamental groups of the families $\mathcal{S}_{11}$ and $\mathcal{S}_{12}$ in Table 4 are isomorphic. These families of surfaces are shown to be contained in a larger irreducible family, which corresponds to another realization as Inoue type varieties. This is done via the concept of a Bagnera-de Franchis variety, which we define simply as the quotient of an Abelian variety A by a nontrivial finite cyclic group G acting freely on A and not containing any translation.

We obtain in this way the following theorem

Main Theorem 2. *Define a Sicilian surface to be any minimal surface of general type S such that*

- *S has invariants $K_S^2 = 6$, $p_g(S) = q(S) = 1$,*
- *there exists an unramified double cover $\hat{S} \to S$ with $q(\hat{S}) = 3$,*
- *the Albanese morphism $\hat{\alpha}\colon \hat{S} \to A = \mathrm{Alb}(\hat{S})$ is birational onto its image Z, a divisor in A with $Z^3 = 12$.*

1) Then the canonical model of $\hat{S}$ is isomorphic to Z, and the canonical model of S is isomorphic to $Y = Z/(\mathbb{Z}/2\mathbb{Z})$. Y is a divisor in a Bagnera-de Franchis threefold $X := A/G$, where $A = (A_1 \times A_2)/T$, $G \cong T \cong \mathbb{Z}/2\mathbb{Z}$, and where the action is as in (6.1).
2) Sicilian surfaces exist, have an irreducible four dimensional moduli space, and their Albanese map $\alpha\colon S \to A_1 = A_1/A_1[2]$ has general fibre a non hyperelliptic curve of genus $g = 3$.
3) A GBT surface is a Sicilian surface if and only if it is in the family $\mathcal{S}_{11}$ or $\mathcal{S}_{12}$.
4) Any surface homotopically equivalent to a Sicilian surface is a Sicilian surface.

Indeed, one can replace the above assumption of homotopy equivalence by a weaker one, see Corollary 6.5.

In Section 5 we discuss the basic results of the theory of Bagnera-de Franchis varieties, and show how to describe concretely the effective divisors on them, thus solving in a special case one of the main technical difficulties in the general theory of Inoue type varieties, developed in [BC12].

1. Inoue's Description of Burniat Surfaces

We briefly recall the description of (primary) *Burniat surfaces* (those constructed by P. Burniat in [Bur66]) given by Inoue in [Ino94].

Inoue considers, for $j \in \{1,2,3\}$, a complex elliptic curve $E_j := \mathbb{C}/\langle 1, \tau_j\rangle$ with uniformizing parameter z_j, and then the following three commuting involutions on the Abelian variety $A^0 := E_1 \times E_2 \times E_3$:

$$\begin{aligned} g_1(z_1,z_2,z_3) &= (-z_1 + \tfrac{1}{2}, z_2 + \tfrac{1}{2}, z_3)\,, \\ g_2(z_1,z_2,z_3) &= (z_1, -z_2 + \tfrac{1}{2}, z_3 + \tfrac{1}{2})\,, \\ g_3(z_1,z_2,z_3) &= (z_1 + \tfrac{1}{2}, z_2, -z_3 + \tfrac{1}{2})\,. \end{aligned}$$

Note that $G := \langle g_1, g_2, g_3\rangle \cong (\mathbb{Z}/2\mathbb{Z})^3$.

Let $\mathcal{L}_j$, for $j = 1,2,3$, be a Legendre function for E_j: $\mathcal{L}_j \colon E_j \to \mathbb{P}^1$, a meromorphic function which makes E_j a double cover of $\mathbb{P}^1$ branched over the four distinct points: $\pm 1, \pm a_j \in \mathbb{P}^1 \setminus \{0,\infty\}$.

It is well known that the following statements hold (see [Ino94, Lemma 3-2] and [BC11a, Section 1] for an algebraic treatment):

- $\mathcal{L}_j(0) = 1$, $\mathcal{L}_j(\frac{1}{2}) = -1$, $\mathcal{L}_j(\frac{\tau_j}{2}) = a_j$, $\mathcal{L}_j(\frac{\tau_j+1}{2}) = -a_j$;
- set $b_j := \mathcal{L}_j(\frac{\tau_j}{4})$: then $b_j^2 = a_j$;
- $\frac{\mathrm{d}\mathcal{L}_j}{\mathrm{d}z_j}(z_j) = 0$ if and only if $z_j \in \{0, \frac{1}{2}, \frac{\tau_j}{2}, \frac{\tau_j+1}{2}\}$ since these are the ramification points of $\mathcal{L}_j$.

Moreover,

$$\mathcal{L}_j(z_j) = \mathcal{L}_j(z_j+1) = \mathcal{L}_j(z_j+\tau_j) = \mathcal{L}_j(-z_j) = -\mathcal{L}_j\left(z_j + \frac{1}{2}\right),$$

$$\mathcal{L}_j\left(z_j + \frac{\tau_j}{2}\right) = \frac{a_j}{\mathcal{L}_j(z_j)}\,.$$

For $c \in \mathbb{C} \setminus \{0\}$, Inoue considers the surface

$$\hat{X}_c := \{[z_1,z_2,z_3] \in A^0 \mid \mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = c\}$$

inside the Abelian variety A^0. Then he shows:

- $\hat{X}_c$ is a hypersurface in A^0 of multidegree $(2,2,2)$ and is invariant under the action of G, $\forall c$.
- For a general choice of c, $\hat{X}_c$ is smooth, and G acts freely on $\hat{X}_c$, whence $X_c := \hat{X}_c/G$ is a smooth minimal surface of general type with $p_g = 0$ and $K^2 = 6$.
- For special values of c, the hypersurface $\hat{X}_c$ has $4, 8, 12, 16$ nodes, which are isolated fixed points of G; in these cases the minimal resolution of singularities of $X_c := \hat{X}_c/G$ is a minimal surface of general type with $p_g = 0$ and $K^2 = 5, 4, 3, 2$.

REMARK 1.1. The minimal resolution of singularities S_c of X_c is called a *Burniat surface*. If X_c is already smooth, or equivalently if $K^2_{S_c} = 6$, then S_c is called a *primary* Burniat surface. For an extensive treatment of Burniat surfaces and their moduli spaces we refer to [BC11a], [BC10], [BC13a].

2. Intersection of Diagonal Quadrics and $(\mathbb{Z}/2\mathbb{Z})^n$-actions

As already in [BC13b, Section 3], we exhibit A^0 as a Galois covering of $(\mathbb{P}^1)^3$ with Galois group $\cong (\mathbb{Z}/2)^9$. This is done via the following diagram.

The main purpose of this section is to find irreducible Del Pezzo surfaces in P_1 which are left invariant under large subgroups of the group $\mathcal{H} \cong (\mathbb{Z}/2)^6$.

(2.1)
$$\begin{array}{lll} E_1 \times E_2 \times E_3 & E_1 := \{x_1^2 + x_2^2 + x_3^2 = 0, \quad x_0^2 = a_1 x_1^2 + a_2 x_2^2\} \\ \Big\downarrow {\scriptstyle \mathcal{H}' := (\mathbb{Z}/2\mathbb{Z})^3} \; {\scriptstyle \pi'} & E_2 := \{u_1^2 + u_2^2 + u_3^2 = 0, \quad u_0^2 = b_1 u_1^2 + b_2 u_2^2\} \\ P_1 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & E_3 := \{z_1^2 + z_2^2 + z_3^2 = 0, \quad z_0^2 = c_1 z_1^2 + c_2 z_2^2\} \\ \Big\downarrow {\scriptstyle \mathcal{H} := ((\mathbb{Z}/2\mathbb{Z})^2)^3} \; {\scriptstyle \pi} & \\ P_2 := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & \end{array}$$

The map π' is given by "forgetting" the variables x_0, u_0, z_0, whereas the map π is given by setting $x_j^2 = y_j$, $u_j^2 = v_j$, $z_j^2 = w_j$, $j = 1, 2, 3$, and viewing $P_2 \subset (\mathbb{P}^2)^3$ as the subvariety defined by the equations

$$y_1 + y_2 + y_3 = 0\,,\ v_1 + v_2 + v_3 = 0\,,\ w_1 + w_2 + w_3 = 0\,.$$

The Galois group for $\pi \circ \pi'$, is $(\mathbb{Z}/2\mathbb{Z})^9 \cong \{\pm 1\}^9$.

Restricting diagram (2.1) to one (w.l.o.g. the first) factor we get:

$$\begin{array}{ccc} E_1 = E & & \\ \downarrow{\scriptstyle \mathbb{Z}/2\mathbb{Z}} & & \\ \mathbb{P}^1 & = & \{x_1^2 + x_2^2 + x_3^2 = 0\} =: C \subset \mathbb{P}^2 \\ \downarrow{\scriptstyle (\mathbb{Z}/2\mathbb{Z})^2} & & \\ \mathbb{P}^1 & = & \{y_1 + y_2 + y_3 = 0\} \subset \mathbb{P}^2 \end{array} \tag{2.2}$$

Since

$$x_1^2 + x_2^2 + x_3^2 = 0 \Longleftrightarrow \det \begin{pmatrix} x_1 + ix_2 & -x_3 \\ x_3 & x_1 - ix_2 \end{pmatrix} = 0\,,$$

we get an isomorphism of C with $\mathbb{P}^1$:

$$(s : t) = (x_1 + ix_2 : x_3) = (-x_3 : x_1 - ix_2)$$

and a parametrization of C

$$(x_1 : x_2 : x_3) = (i(s^2 - t^2) : (s^2 + t^2) : 2ist).$$

With this parametrization, we can rewrite the action of $(\mathbb{Z}/2\mathbb{Z})^2$ on $\mathbb{P}^1$ in the following way (on the left hand side we use the convenient notation by which all variables not mentioned in a transformation are left unchanged by the transformation):

a) $x_1 \mapsto -x_1$ corresponds to $A_1 \colon (s : t) \mapsto (t : s)$;

b) $x_2 \mapsto -x_2$ corresponds to $A_{-1} \colon (s : t) \mapsto (-t : s)$;

c) $x_3 \mapsto -x_3$ corresponds to $B \colon (s : t) \mapsto (s : -t)$.

The fixed points of these three involutions are respectively:

a) $s = \pm t$, equivalently, $x_1 = x_3 \pm ix_2 = 0$;

b) $s = \pm it$, equivalently, $x_2 = x_3 \pm ix_1 = 0$;

c) $st = 0$, equivalently, $x_3 = x_1 \pm ix_2 = 0$.

For each $\lambda := (\lambda_1, \dots, \lambda_8) \in \mathbb{C}^8 \setminus \{0\}$ we consider the hypersurface Y_λ of multidegree $(1,1,1)$ in $P_1 = \mathbb{P}^1_{(s_1:t_1)} \times \mathbb{P}^1_{(s_2:t_2)} \times \mathbb{P}^1_{(s_3:t_3)}$ given by the multihomogeneous equation

$$\begin{aligned}(2.3)\qquad &\lambda_1 s_1 s_2 s_3 + \lambda_2 s_1 s_2 t_3 + \lambda_3 s_1 t_2 s_3 + \lambda_4 s_1 t_2 t_3 + \\ &\lambda_5 t_1 s_2 s_3 + \lambda_6 t_1 s_2 t_3 + \lambda_7 t_1 t_2 s_3 + \lambda_8 t_1 t_2 t_3 = 0.\end{aligned}$$

Clearly, Y_λ is a Del Pezzo surface of degree 6. Since we shall be looking for Del Pezzo surfaces Y_λ which are left invariant by certain subgroups of $\mathcal{H}$ (the Galois group of π), we first need to establish conditions ensuring that the hypersurface Y_λ is left invariant by an element $h = (h_1, h_2, h_3) \in \mathcal{H}$.

This is done in the next lemma, which is easy to verify and which takes care of the normal form of a transformation $(h_1, h_2, h_3) \in \mathcal{H}$, taken up to a permutation of the three factors (here Id is the identity map of $\mathbb{P}^1$, while A_1, A_{-1} and B are the maps defined above).

Table 1.

h	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	λ_8	c^2
$\mathrm{Id}, \mathrm{Id}, A_{\alpha_3}$		$c\lambda_1$		$c\lambda_3$		$c\lambda_5$		$c\lambda_7$	α_3
$\mathrm{Id}, \mathrm{Id}, B$	0	0	0	0	0	0	0	0	
$\mathrm{Id}, A_{\alpha_2}, A_{\alpha_3}$			$c\alpha_3\lambda_2$	$c\lambda_1$			$c\alpha_3\lambda_6$	$c\lambda_5$	$\alpha_2\alpha_3$
$\mathrm{Id}, A_{\alpha_2}, B$			$c\lambda_1$	$-c\lambda_2$			$c\lambda_5$	$-c\lambda_6$	α_2
Id, B, B	0	0	0	0	0	0	0	0	
$A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}$					$c\alpha_2\alpha_3\lambda_4$	$c\alpha_2\lambda_3$	$c\alpha_3\lambda_2$	$c\lambda_1$	$\alpha_1\alpha_2\alpha_3$
$A_{\alpha_1}, A_{\alpha_2}, B$					$c\alpha_2\lambda_3$	$-c\alpha_2\lambda_4$	$c\lambda_1$	$-c\lambda_2$	$\alpha_1\alpha_2$
A_{α_1}, B, B					$c\lambda_1$	$-c\lambda_2$	$-c\lambda_3$	$c\lambda_4$	α_1
B, B, B	0	0	0	0	0	0	0	0	

Lemma 2.1. *Let $h = (h_1, h_2, h_3) \in \mathcal{H} \setminus \{\mathrm{Id}\}$ be one of the transformations listed in the first column of Table 1.*

Then Y_λ is h-invariant if and only if the coefficients λ_j satisfy the linear conditions listed in Table 1.

Note that in Table 1, the numbers $\alpha_i \in \{\pm 1\}$, since they are labelling A_1 and A_{-1}. If for a given case there appear two rows, this means that there are two alternatives, one for each row.

Remark 2.2. Consider the following matrices:

$$\Gamma_1 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \Gamma_{-1} := \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}, \tag{2.4}$$

and denote by f_1, respectively f_{-1}, the induced projectivities in $\mathrm{Aut}(\mathbb{P}^1)$ (observe that $f_1 = f_1^{-1}$).

It is straightforward to verify the following conjugacies

- $B = f_1^{-1} \circ A_1 \circ f_1 = f_{-1}^{-1} \circ A_{-1} \circ f_{-1}$,
- $A_1 = f_1^{-1} \circ B \circ f_1 = f_{-1}^{-1} \circ B \circ f_{-1}$,
- $A_{-1} = f_1^{-1} \circ A_{-1} \circ f_1 = f_{-1}^{-1} \circ A_1 \circ f_{-1}$.

Remark 2.3. If Y_λ is invariant under $h = (\mathrm{Id}, \mathrm{Id}, A_\alpha)$ ($\alpha = \pm 1$), or under $h = (\mathrm{Id}, \mathrm{Id}, B)$ then the equation of Y_λ is reducible. Since these projectivities are conjugate, it suffices to consider the case $h = (\mathrm{Id}, \mathrm{Id}, B)$, when the equation of Y_λ is

$$\begin{aligned} &s_3(\lambda_1 s_1 s_2 + \lambda_3 s_1 t_2 + \lambda_5 t_1 s_2 + \lambda_7 t_1 t_2) = 0 \quad \text{or} \\ &t_3(\lambda_2 s_1 s_2 + \lambda_4 s_1 t_2 + \lambda_6 t_1 s_2 + \lambda_8 t_1 t_2) = 0 \end{aligned}$$

The above enable us to prove the following:

Proposition 2.4. *Let $\lambda \in \mathbb{C}^8 \setminus \{0\}$ be such that Y_λ is irreducible. Assume moreover that there is a subgroup $H_1 \cong (\mathbb{Z}/2\mathbb{Z})^2$ of $\mathcal{H}$, such that Y_λ is H_1-invariant. Then, up to the action of $\mathbb{P}\,\mathrm{GL}(2, \mathbb{C})^3$ and up to a permutation of the factors of $(\mathbb{P}^1)^3$, there are exactly two possibilities:*

i) $H_1 = \langle (A_1, A_1, A_1), (\mathrm{Id}, B, B) \rangle$, *or*

ii) $H_1 = \langle (\mathrm{Id}, B, B), (B, B, \mathrm{Id}) \rangle$.

Proof. Let $H_1 = \langle h, h' \rangle$. Then, by Remarks 2.2 and 2.3, after possibly changing the coordinates of $(\mathbb{P}^1)^3$, we may assume that $h = (B, B, B)$ or $= (\mathrm{Id}, B, B)$.
1) $h = (B, B, B)$: in this case $h' \in \{(\mathrm{Id}, B, B), (A_{\alpha_1}, B, B)\}$ implies that $(B, \mathrm{Id}, \mathrm{Id}) \in H_1$ or $(A_{\alpha_1}, \mathrm{Id}, \mathrm{Id}) \in H_1$, contradicting the irreducibility of Y_λ (cf. Remark 2.3).

If we assume that $h' \in \{(\mathrm{Id}, A_{\alpha_2}, B), (A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3})\}$, $\alpha_i \in \{\pm 1\}$, then we see (cf. Table 1) that the invariance of Y_λ under h and h' implies that $\lambda = 0$: this is a contradiction.

Assuming instead that $h' = (\mathrm{Id}, A_{\alpha_2}, A_{\alpha_3})$, then conjugating h' by $(f_1, f_{\alpha_2}, f_{\alpha_3})$, we see that in the new coordinates we have:

$$h = (f_1^{-1}\, B\, f_1, f_{\alpha_2}^{-1}\, B\, f_{\alpha_2}, f_{\alpha_3}^{-1}\, B\, f_{\alpha_3}) = (A_1, A_1, A_1)$$

and

$$h' = (f_1^{-1}\ \mathrm{Id}\ f_1, f_{\alpha_2}^{-1}\, A_{\alpha_2}\, f_{\alpha_2}, f_{\alpha_3}^{-1}\, A_{\alpha_3}\, f_{\alpha_3}) = (\mathrm{Id}, B, B)\,,$$

i.e., we are in case i).

Assume finally that $h' = (A_{\pm 1}, A_{\pm 1}, B)$. Then $h \cdot h' = (A_{\mp 1}, A_{\mp 1}, \mathrm{Id})$ and we reduce to the previous case showing that we are in case i).
2) $h = (\mathrm{Id}, B, B)$: in this case if $h' = (\mathrm{Id}, A_{\alpha_2}, A_{\alpha_3})$, the equation of Y_λ is (cf. Table 1):

$$(\lambda_1 s_1 + \lambda_5 t_1)(s_2 s_3 + c t_2 t_3) = 0,$$

contradicting the irreducibility of Y_λ.

If $h' \in \{(B, B, B), (\mathrm{Id}, A_{\alpha_2}, B), (\mathrm{Id}, B, A_{\alpha_3}), (A_{\alpha_1}, B, B)\}$, we obtain that Y_λ is not irreducible by Remark 2.3.

Assume that $h' \in \{(A_{\alpha_1}, \mathrm{Id}, A_{\alpha_3}), (A_{\alpha_1}, A_{\alpha_2}, \mathrm{Id}), (B, A_{\alpha_2}, \mathrm{Id}), (B, \mathrm{Id}, A_{\alpha_3}), (A_{\alpha_1}, A_{\alpha_2}, B), (A_{\alpha_1}, B, A_{\alpha_3}), (B, A_{\alpha_2}, B), (B, B, A_{\alpha_3})\}$. Then one checks easily, consulting Table 1, that $\lambda = 0$, hence also these cases can be excluded.

If $h' \in \{(A_\alpha, \mathrm{Id}, B), (A_\alpha, B, \mathrm{Id})\}$, $\alpha \in \{\pm 1\}$, after changing the coordinates by $(f_\alpha, \mathrm{Id}, \mathrm{Id})$ we get $H_1 = \langle (\mathrm{Id}, B, B), (B, \mathrm{Id}, B) \rangle$, hence we are in case ii).

Assume now that $h' = (A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3})$. Changing coordinates by conjugating with $(\gamma_1, \gamma_2, \gamma_3)$, where $\gamma_j := \mathrm{Id}$ if $\alpha_j = 1$ and $\gamma_j := (f_{-1} \circ f_1)$ if $\alpha_j = -1$ and using the fact that

$$(f_{-1} \circ f_1)^{-1} \circ B \circ (f_{-1} \circ f_1) = B, \quad (f_{-1} \circ f_1)^{-1} \circ A_{-1} \circ (f_{-1} \circ f_1) = A_1,$$

we see that (in the new coordinates) we are in case i).

If $h' = (B, A_{\alpha_2}, A_{\alpha_3})$, then changing the coordinates by conjugating with $(f_1, \gamma_2, \gamma_3)$, where γ_j is defined as above, we are in case i).

Finally, if $h' \in \{(B, \mathrm{Id}, B), (B, B, \mathrm{Id})\}$, then we are in case ii). □

Remark 2.5. It is seen immediately that in case i) each Del Pezzo surface $Y_\lambda = \{\lambda_1 s_1 s_2 s_3 + \lambda_8 t_1 t_2 t_3 = 0\}$ is invariant under H_1, whereas in case ii) each surface $Y_\lambda = \{\lambda_1(s_1 s_2 s_3 + t_1 t_2 t_3) + \lambda_4(s_1 t_2 t_3 + t_1 s_2 s_3) = 0\}$ is invariant under H_1. In particular, in both respective cases i) and ii), we obtain a linear action of H_1 on the vector space $V := H^0((\mathbb{P}^1)^3, \mathcal{O}_{(\mathbb{P}^1)^3}(1,1,1))$, which is independent of the chosen invariant surface in the pencil (see Proposition 5.12).

Proposition 2.6. *With the same notation as in Proposition 2.4, the respective decompositions of V in character spaces with respect to the above action of $H_1 \cong (\mathbb{Z}/2\mathbb{Z})^2$ are as follows:*

i) $H_1 = \langle (A_1, A_1, A_1), (\mathrm{Id}, B, B) \rangle$:

- $V^{++} = \{\lambda_1(s_1 s_2 s_3 + t_1 t_2 t_3) + \lambda_4(s_1 t_2 t_3 + t_1 s_2 s_3) \mid \lambda_1, \lambda_4 \in \mathbb{C}\} \cong \mathbb{C}^2$;
- $V^{+-} = \{\lambda_2(s_1 s_2 t_3 + t_1 t_2 s_3) + \lambda_3(s_1 t_2 s_3 + t_1 s_2 t_3) \mid \lambda_2, \lambda_3 \in \mathbb{C}\}$;
- $V^{-+} = \{\lambda_1(s_1 s_2 s_3 - t_1 t_2 t_3) + \lambda_4(s_1 t_2 t_3 - t_1 s_2 s_3) \mid \lambda_1, \lambda_4 \in \mathbb{C}\}$;
- $V^{--} = \{\lambda_2(s_1 s_2 t_3 - t_1 t_2 s_3) + \lambda_3(s_1 t_2 s_3 - t_1 s_2 t_3) \mid \lambda_2, \lambda_3 \in \mathbb{C}\}$.

ii) $H_1 = \langle (\mathrm{Id}, B, B), (B, B, \mathrm{Id}) \rangle$:

- $V^{++} = \{\lambda_1 s_1 s_2 s_3 + \lambda_8 t_1 t_2 t_3 \mid \lambda_1, \lambda_8 \in \mathbb{C}\} \cong \mathbb{C}^2$;
- $V^{+-} = \{\lambda_4 s_1 t_2 t_3 + \lambda_5 t_1 s_2 s_3 \mid \lambda_4, \lambda_5 \in \mathbb{C}\}$;
- $V^{-+} = \{\lambda_2 s_1 s_2 t_3 + \lambda_7 t_1 t_2 s_3 \mid \lambda_2, \lambda_7 \in \mathbb{C}\}$;
- $V^{--} = \{\lambda_3 s_1 t_2 s_3 + \lambda_6 t_1 s_2 t_3 \mid \lambda_3, \lambda_6 \in \mathbb{C}\}$.

PROOF. This is a simple calculation using Table 1. □

The same arguments as in the proof of Proposition 2.4 yield the following statement:

PROPOSITION 2.7. *Let $\lambda \in \mathbb{C}^8 \setminus \{0\}$ be such that Y_λ is irreducible. Assume moreover that there is a subgroup $H_0 \cong (\mathbb{Z}/2\mathbb{Z})^3$ of $\mathcal{H}$, such that Y_λ is H_0-invariant. Then, up to the action of $\mathbb{P}\,\mathrm{GL}(2,\mathbb{C})^3$ and up to a permutation of the factors of $(\mathbb{P}^1)^3$, we have:*

$$H_0 = \langle (\mathrm{Id}, B, B), (A_1, A_1, A_1), (B, B, \mathrm{Id}) \rangle .$$

REMARK 2.8. Again we see immediately that the Del Pezzo surface $Y_\lambda = \{s_1s_2s_3 + t_1t_2t_3\}$ is invariant under H_0, hence we get again a linear action of H_0 on the vector space $V := H^0((\mathbb{P}^1)^3, \mathcal{O}_{(\mathbb{P}^1)^3}(1,1,1))$.

PROPOSITION 2.9. *Use the same notation as in Proposition 2.7; then V decomposes in 8 one-dimensional character spaces for the action of $H_0 \cong (\mathbb{Z}/2\mathbb{Z})^3$, as follows:*

- $V^{+++} = \{\lambda(s_1s_2s_3 + t_1t_2t_3) \mid \lambda \in \mathbb{C}\}$;
- $V^{+-+} = \{\lambda(s_1s_2s_3 - t_1t_2t_3) \mid \lambda \in \mathbb{C}\}$;
- $V^{++-} = \{\lambda(s_1t_2t_3 + t_1s_2s_3) \mid \lambda \in \mathbb{C}\}$;
- $V^{+--} = \{\lambda(s_1t_2t_3 - t_1s_2s_3) \mid \lambda \in \mathbb{C}\}$;
- $V^{-++} = \{\lambda(s_1s_2t_3 + t_1t_2s_3) \mid \lambda \in \mathbb{C}\}$;
- $V^{--+} = \{\lambda(s_1s_2t_3 - t_1t_2s_3) \mid \lambda \in \mathbb{C}\}$;
- $V^{-+-} = \{\lambda(t_1s_2t_3 + s_1t_2s_3) \mid \lambda \in \mathbb{C}\}$;
- $V^{---} = \{\lambda(t_1s_2t_3 - s_1t_2s_3) \mid \lambda \in \mathbb{C}\}$.

REMARK 2.10. Case 1):

$$H_1 := \langle (\mathrm{Id}, B, B), (A_1, A_1, A_1) \rangle \cong (\mathbb{Z}/2\mathbb{Z})^2 \ \triangleleft\, \mathcal{H}.$$

Then there are four pencils of Del Pezzo surfaces, which are left invariant by H_1 (cf. Proposition 2.6); their inverse images under π' (see (2.1)) $\pi'^{-1}(Y_\nu)$ (resp. $\pi'^{-1}(Y'_\nu)$, $\pi'^{-1}(Y''_\nu)$, $\pi'^{-1}(Y'''_\nu)$) are pencils of hypersurfaces of multidegree $(2,2,2)$ in $A^0 = E_1 \times E_2 \times E_3$ invariant under $\mathcal{G}'_1 \cong (\mathbb{Z}/2\mathbb{Z})^5 \subset (\mathbb{Z}/2\mathbb{Z})^9$.

We list now the four pencils ($\nu = (\nu_1 : \nu_2) \in \mathbb{P}^1$):

$$Y_\nu := \{\nu_1(s_1s_2s_3 + t_1t_2t_3) + \nu_2(s_1t_2t_3 + t_1s_2s_3) = 0\}, \tag{2.5}$$

$$Y'_\nu := \{\nu_1(s_1s_2t_3 + t_1t_2s_3) + \nu_2(s_1t_2s_3 + t_1s_2t_3) = 0\}, \tag{2.6}$$

$$Y''_\nu := \{\nu_1(s_1s_2s_3 - t_1t_2t_3) + \nu_2(s_1t_2t_3 - t_1s_2s_3) = 0\}, \tag{2.7}$$

$$Y'''_\nu := \{\nu_1(s_1s_2t_3 - t_1t_2s_3) + \nu_2(s_1t_2s_3 - t_1s_2t_3) = 0\}. \tag{2.8}$$

It is immediate to see that the 4 pencils are transformed to each other by the elements of the group $\mathcal{H} = ((\mathbb{Z}/2)^2)^3$ (for instance we pass from the first to the second via $s_3 \leftrightarrow t_3$, from the first to the third via $t_1 \leftrightarrow -t_1$, and so on). Therefore, in the future we shall only consider the first pencil: (2.5).

Case 2):

$$H_1 := \langle(\mathrm{Id}, B, B), (B, B, \mathrm{Id})\rangle \cong (\mathbb{Z}/2\mathbb{Z})^2 \ \triangleleft\, \mathcal{H}.$$

Then there are four pencils of Del Pezzo surfaces, which are left invariant by H_1; their respective inverse images under π' yield four pencils, invariant under $\mathcal{G}_1 \cong (\mathbb{Z}/2\mathbb{Z})^5 \subset (\mathbb{Z}/2\mathbb{Z})^9$.

The four pencils are given by the following equations ($\mu \in \mathbb{C}, \mu \neq 0$):

$$Y_\mu := \{s_1s_2s_3 + \mu\, t_1t_2t_3 = 0\}, \tag{2.9}$$

$$Y'_\mu := \{s_1t_2s_3 + \mu\, t_1s_2t_3 = 0\}, \tag{2.10}$$

$$Y''_\mu := \{s_1t_2t_3 + \mu\, t_1s_2s_3 = 0\}, \tag{2.11}$$

$$Y'''_\mu := \{s_1s_2t_3 + \mu\, t_1t_2s_3 = 0\}. \tag{2.12}$$

Also in this case the 4 pencils are transformed to each other by the elements of the group $\mathcal{H} = ((\mathbb{Z}/2)^2)^3$, hence in the future we shall only consider the first pencil: (2.9).

Case 3):

$$H_0 := \langle(\mathrm{Id}, B, B), (A_1, A_1, A_1), (B, B, \mathrm{Id})\rangle \cong ((\mathbb{Z}/2\mathbb{Z}))^3 \triangleleft\, \mathcal{H}.$$

Then there are eight Del Pezzo surfaces which are are left invariant by H_0; their respective inverse images under π' are invariant under $\mathcal{G}_0 \cong (\mathbb{Z}/2\mathbb{Z})^6 \subset (\mathbb{Z}/2\mathbb{Z})^9$.

Their respective equations are the following ones:

$$Y_1 := \{s_1s_2s_3 + t_1t_2t_3 = 0\}, \quad Y_{-1} := \{s_1s_2s_3 - t_1t_2t_3 = 0\}, \tag{2.13}$$

$$Y_1' := \{s_1t_2t_3 + t_1s_2s_3 = 0\}, \quad Y_{-1}' := \{s_1t_2t_3 - t_1s_2s_3 = 0\}, \tag{2.14}$$

$$Y_1'' := \{s_1s_2t_3 + t_1t_2s_3 = 0\}, \quad Y_{-1}'' := \{s_1s_2t_3 - t_1t_2s_3 = 0\}, \tag{2.15}$$

$$Y_1''' := \{t_1s_2t_3 + s_1t_2s_3 = 0\}, \quad Y_{-1}''' := \{t_1s_2t_3 - s_1t_2s_3 = 0\}. \tag{2.16}$$

Also here the 8 hypersurfaces are transformed to each other by the elements of the group $\mathcal{H} = ((\mathbb{Z}/2)^2)^3$, hence in the future we shall only consider the first one: (2.13).

Definition 2.11. Let $\hat{X}$ be an irreducible hypersurface, in the product of three smooth elliptic curves $A^0 := E_1 \times E_2 \times E_3$, which is the inverse image under π' of a Del Pezzo surface Y of degree 6, invariant under a subgroup $H \cong (\mathbb{Z}/2\mathbb{Z})^2 \triangleleft \mathcal{H}$.

Then we call $\hat{X}$ a *Burniat hypersurface in* A^0.

Lemma 2.12. *Let* $\mathcal{G}_0 \cong (\mathbb{Z}/2\mathbb{Z})^6 \triangleleft (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3 \times (\mathbb{Z}/2\mathbb{Z})^3$ *be the group:*

$$\mathcal{G}_0 := \{(\epsilon_0, \eta_1, \epsilon_1, \eta_0, \epsilon_2, \zeta_0, \epsilon_3) \subset \{\pm 1\}^7 \cong (\mathbb{Z}/2\mathbb{Z})^7 \mid \epsilon_1\epsilon_2\epsilon_3 = 1\},$$

which acts on $E_1 \times E_2 \times E_3$ *by:*

$$\begin{array}{lll} x_0 \mapsto \epsilon_0 x_0\,, & u_0 \mapsto \eta_0 u_0\,, & z_0 \mapsto \zeta_0 z_0\,, \\ x_3 \mapsto \epsilon_1 x_3\,, & u_3 \mapsto \epsilon_2 u_3\,, & z_3 \mapsto \epsilon_3 z_3 \end{array} \qquad \textit{and} \qquad \begin{pmatrix} x_1 \\ u_1 \\ z_1 \end{pmatrix} \mapsto \eta_1 \begin{pmatrix} x_1 \\ u_1 \\ z_1 \end{pmatrix}.$$

With the same notation as in Remark 2.10:

(1) $\pi'^{-1}(Y_\nu)$ *is invariant under the group*

$$\mathcal{G}_1' := \{(\epsilon_0, \eta_1, 1, \eta_0, \epsilon_2, \zeta_0, \epsilon_3) \mid \epsilon_2\epsilon_3 = 1\} \cong (\mathbb{Z}/2\mathbb{Z})^5 \triangleleft \mathcal{G}_0\,.$$

(2) $\pi'^{-1}(Y_\mu)$ *is invariant under the group*

$$\mathcal{G}_1 := \{(\epsilon_0, 1, \epsilon_1, \eta_0, \epsilon_2, \zeta_0, \epsilon_3) \mid \epsilon_1\epsilon_2\epsilon_3 = 1\} \cong (\mathbb{Z}/2\mathbb{Z})^5 \triangleleft \mathcal{G}_0\,.$$

(3) *If $\mu = \pm 1$, then $\pi'^{-1}(Y_\mu)$ is invariant under $\mathcal{G}_0$.*

Proof. Just note that multiplication of (x_1, u_1, z_1) by -1 corresponds to $(s_j : t_j) \mapsto (t_j : s_j)$ for each $j = 1, 2, 3$. □

2.1. Fixed points

In order to systematically search for all the subgroups $G \cong (\mathbb{Z}/2\mathbb{Z})^3 \triangleleft \mathcal{G}_0$ acting freely on a Burniat hypersurface in $A^0 := E_1 \times E_2 \times E_3$, we need to determine which elements of $\mathcal{G}_0$ have fixed points on A^0 .

Remark 2.13. Fix $a_1, a_2 \in \mathbb{C}$ pairwise distinct such that the curve

$$E := \{x_1^2 + x_2^2 + x_3^3 = 0,\ x_0^2 = a_1 x_1^2 + a_2 x_2^2\} \subset \mathbb{P}^2$$

is smooth. Then

$$g(x_0 : x_1 : x_2 : x_3) := (\alpha_0 x_0 : \alpha_1 x_1 : x_2 : \alpha_3 x_3)\,, \quad \alpha_j \in \{\pm 1\}$$

has fixed points on E if and only if

- either $\alpha_0 = \alpha_1 = \alpha_3 = -1$, or
- exactly one $\alpha_i = -1$ and the others are equal to 1.

From now on, we change to an additive notation in which $\mathbb{Z}/2\mathbb{Z}$ is the additive group $\{0, 1\}$.

Let $g \in \mathcal{G}_0$ be an element fixing points on A^0. By [BC13b, Proposition 3.3], g is an element in Table 2.

Remark 2.14. 1) Let $\hat{X} := \pi'^{-1}(Y_{\pm 1})$. In Table 2, the elements 1-3 fix pointwise a surface $S \subset A^0$. Each element 4-9 fixes pointwise a curve $C \subset A^0$ and its fixed locus has non trivial intersection with $\hat{X}$ since $\hat{X} \subset A^0$ is an ample divisor. Finally, the elements 10-17 have isolated fixed points on A^0; arguing as in [BC13b, Proposition 3.3] one proves that the elements 11-17 have fixed points on $\hat{X}$, while the fixed locus of element 10 intersects $\hat{X}$ only for special choices of the three elliptic curves.

2) The same holds for $\hat{X} := \pi'^{-1}(Y_\nu)$ (resp. $\pi'^{-1}(Y_\mu)$), considering only the elements 1-7,10,11,14,15 (resp. 1-13), i.e. the ones belonging to $\mathcal{G}_1'$ (resp. $\mathcal{G}_1$). In particular, the fixed locus of element 10 intersects $\hat{X}$ only for special choices of the three elliptic curves and of the parameter ν (resp. μ).

Table 2. The elements of $\mathcal{G}_0$ having fixed points on A^0, written additively!

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
ϵ_0	0	0	1	0	1	1	0	0	0	1	1	0	0	0	0	1	1
η_1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1
ϵ_1	0	0	0	0	0	0	0	1	1	0	0	1	1	0	0	1	1
η_0	0	1	0	1	0	1	0	0	0	1	0	1	0	0	1	0	1
ϵ_2	0	0	0	0	0	0	1	0	1	0	1	0	1	0	1	0	1
ζ_0	1	0	0	1	1	0	0	0	0	1	0	0	1	0	1	1	0
ϵ_3	0	0	0	0	0	0	1	1	0	0	1	1	0	0	1	1	0

2.2. Description in terms of Legendre families

We now describe the families of Burniat hypersurfaces in A^0 in terms of Legendre functions $\mathcal{L}$ (see Section 1).

To this purpose, we consider the following 1-parameter family of intersections of two quadrics:

$$E(b) := \{x_1^2 + x_2^2 + x_3^2 = 0, \quad x_0^2 = (b^2+1)^2 x_1^2 + (b^2-1)^2 x_2^2\},$$

where $b \in \mathbb{C} \setminus \{0, 1, -1, i, -i\}$.

We set

$$\xi := \frac{bs}{t},$$

and in this way the family of genus one curves $E(b)$ is the Legendre family of elliptic curves in Legendre normal affine form:

$$y^2 = (\xi^2 - 1)(\xi^2 - a^2), \quad a := b^2.$$

In fact,

$$\begin{aligned}
x_0^2 &= (b^2+1)^2 x_1^2 + (b^2-1)^2 x_2^2 = \\
&= -(a+1)^2(s^2-t^2)^2 + (a-1)^2(s^2+t^2)^2 = \\
&= 4[(a^2+1)s^2t^2 - a(t^4+s^4)] = \\
&= 4t^4\left[(a^2+1)\left(\frac{\xi}{b}\right)^2 - a\left(1+\left(\frac{\xi}{b}\right)^4\right)\right] = \\
&= -4t^4\frac{1}{b^2}[-(a^2+1)\xi^2 + (a^2+\xi^4) = \frac{-4t^4}{b^2}[(\xi^2-1)(\xi^2-a^2)]
\end{aligned}$$

and it suffices to set

$$y := \frac{ibx_0}{2t^2}\,.$$

The group $(\mathbb{Z}/2)^3$ acts fibrewise on the family $E(b)$ via the commuting involutions:

$$x_0 \longleftrightarrow -x_0, \quad x_3 \longleftrightarrow -x_3, \quad x_1 \longleftrightarrow -x_1,$$

which on the birational model given by the Legendre family act as

$$y \longleftrightarrow -y, \quad \xi \longleftrightarrow -\xi, \quad \xi \longleftrightarrow \frac{a}{\xi}\,.$$

Consider the subgroup

$$\Gamma_{2,4} := \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \mathbb{P}SL(2,\mathbb{Z}) \;\middle|\; \begin{array}{llll} \alpha \equiv 1 & \mod 4, & \beta \equiv 0 & \mod 4, \\ \gamma \equiv 0 & \mod 2, & \delta \equiv 1 & \mod 2 \end{array} \right\}$$

a subgroup of index 2 of the congruence subgroup

$$\Gamma_2 := \left\{ \left(\begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \in \mathbb{P}SL(2,\mathbb{Z}) \;\middle|\; \begin{array}{llll} \alpha \equiv 1 & \mod 2, & \beta \equiv 0 & \mod 2, \\ \gamma \equiv 0 & \mod 2, & \delta \equiv 1 & \mod 2 \end{array} \right\}.$$

To the chain of inclusions

$$\Gamma_{2,4} < \Gamma_2 < \mathbb{P}SL(2,\mathbb{Z})$$

corresponds a chain of fields of invariants

$$\mathbb{C}(j) \subset \mathbb{C}(\lambda) = \mathbb{C}(\tau)^{\Gamma_2} \subset \mathbb{C}(\tau)^{\Gamma_{2,4}}\,,$$

where the respective degrees of the extensions are 6, 2.

Here, λ is the cross-ratio of the four points $\mathfrak{p}(0), \mathfrak{p}(\frac{1}{2}), \mathfrak{p}(\frac{\tau}{2}), \mathfrak{p}(\frac{1+\tau}{2})$, where $\mathfrak{p}$ is the Weierstrass function, and $j(\lambda) = \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2}$ is the j-invariant.

If $\lambda(a)$ is the cross ratio of the four points $1, -1, a, -a$, $\lambda(a) = \frac{(a-1)^2}{(a+1)^2}$, thus $\mathbb{C}(a) = \mathbb{C}(\sqrt{\lambda})$ is a quadratic extension and there are two values of a for which we get a Legendre function for the elliptic curve. Setting $b := \mathcal{L}(\frac{\tau}{4})$, we have that $a = b^2$, hence $\mathbb{C}(b)$ is a quadratic extension of $\mathbb{C}(\tau)^{\Gamma_{2,4}}$.

In other words, the parameter $b \in \mathbb{C} \setminus \{0, 1, -1, i, -i\}$ yields an unramified covering of degree 4 of $\lambda \in \mathbb{C}\setminus\{0,1\}$, hence the field $\mathbb{C}(b)$ is the invariant field for a subgroup $\Gamma_{2,8}$ of index 2 in $\Gamma_{2,4}$.

By [Bia16, §182], b is invariant under the subgroup of Γ_2 given by the transformation such that $\alpha^2 + \alpha\beta \equiv 1 \mod 8$. Since $\alpha \equiv 1 \mod 2$, this equation is equivalent to require that $\beta \equiv 0 \mod 8$, i.e.:

$$\Gamma_{2,8} := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbb{P}SL(2,\mathbb{Z}) \;\middle|\; \begin{array}{llll} \alpha \equiv 1 & \mod 4, & \beta \equiv 0 & \mod 8, \\ \gamma \equiv 0 & \mod 2, & \delta \equiv 1 & \mod 2 \end{array} \right\}.$$

Consider now the following family

$$\mathcal{A}^0 = E(b_1) \times E(b_2) \times E(b_3).$$

It is the family of products of three elliptic curves with a $\Gamma_{2,8}$-level structure: $\mathcal{A}^0$ is the quotient of $(\mathbb{C} \times \mathbb{H})^3$, with coordinates $((z_1,\tau_1),(z_2,\tau_2),(z_3,\tau_3))$, by the action of the group (a semidirect product) generated by $(\mathbb{Z}^2)^3$ which acts by

$$\begin{aligned} &((m_1,n_1),(m_2,n_2),(m_3,n_3)) \circ ((z_1,\tau_1),(z_2,\tau_2),(z_3,\tau_3)) = \\ &= ((z_1+m_1+n_1\tau_1,\tau_1),(z_2+m_2+n_2\tau_2,\tau_2),(z_3+m_3+n_3\tau_3,\tau_3)) \end{aligned}$$

and by ${\Gamma_{2,8}}^3 \subset \mathbb{P}SL(2,\mathbb{Z})^3$.

The fibre of $f : \mathcal{A}^0 \to \mathcal{E} := (\mathbb{H}/\Gamma_{2,8})^3$ is the product of the three elliptic curves, for $k = 1,2,3$, $E_k := \mathbb{C}/\langle 1, \tau_k \rangle$.

Let $\mathcal{L}_k \colon E_k \to \mathbb{P}^1$ be a Legendre function for E_k. We have seen that the relation between $\mathcal{L}_k(z_k)$ and the coordinates $(s_k : t_k)$ of $\mathbb{P}^1$ is

$$\frac{\mathcal{L}_k(z_k)}{b_k} = \frac{s_k}{t_k}$$

where $b_k := \mathcal{L}_k(\frac{\tau_k}{4})$. A basis for the $(\mathbb{Z}/2\mathbb{Z})^3$-action on E_k, $k = 1,2,3$, is given by:

$$\begin{array}{lcccl} x_0 \mapsto -x_0 & \hat{=} & (z_k \mapsto -z_k) & \hat{=} & (1,0,0) \\ x_1 \mapsto -x_1 & \hat{=} & (z_k \mapsto -z_k + \frac{\tau_k}{2}) & \hat{=} & (0,1,0) \\ x_3 \mapsto -x_3 & \hat{=} & (z_k \mapsto -z_k + \frac{1}{2}) & \hat{=} & (0,0,1) \end{array} \tag{2.17}$$

The above formulae define an action of $((\mathbb{Z}/2)^3)^3$ on the fibration $f : \mathcal{A}^0 \to \mathcal{E}$, which acts trivially on the basis.

It follows from (2.5, 2.9, 2.13) that it suffices to consider only the families of Burniat hypersurfaces defined by:

$$\begin{aligned}&\hat{\mathcal{X}}_\nu = \{([(z_1,\tau_1),(z_2,\tau_2),(z_3,\tau_3)],(\nu_1:\nu_2)) \in \mathcal{A}^0 \times \mathbb{P}^1 \mid \\ &\nu_1(\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) + b_1b_2b_3) \\ &+\nu_2(\mathcal{L}_1(z_1)b_2b_3 + b_1\mathcal{L}_2(z_2)\mathcal{L}_3(z_3)) = 0\}\,,\end{aligned} \tag{2.18}$$

$$\begin{aligned}&\hat{\mathcal{X}}_\mu = \{([(z_1,\tau_1),(z_2,\tau_2),(z_3,\tau_3)],\mu) \in \mathcal{A}^0 \times \mathbb{C}^* \mid \\ &\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = \mu\}\,,\end{aligned} \tag{2.19}$$

$$\begin{aligned}&\hat{\mathcal{X}}_b = \{[(z_1,\tau_1),(z_2,\tau_2),(z_3,\tau_3)] \in \mathcal{A}^0 \mid \\ &\mathcal{L}_1(z_1)\mathcal{L}_2(z_2)\mathcal{L}_3(z_3) = b_1b_2b_3\}\,,\end{aligned} \tag{2.20}$$

where the meaning of the subscript is to refer to the variables: $\nu = (\nu_1 : \nu_2) \in \mathbb{P}^1$, $\mu \in \mathbb{C}^*$, $b := b_1b_2b_3$.

Remark 2.15. There is also an obvious action of the symmetric group $\mathfrak{S}_3$ on the family $f : \mathcal{A}^0 \to \mathcal{E}$.

Let $\hat{X}$ be a Burniat hypersurface in A^0 (see Definition 2.11). An explicit calculation using the above equations shows that $\hat{X}$ has at most finitely many nodes as singularities.

Let $\epsilon\colon X' \to \hat{X}$ be the minimal resolution of its singularities. Since $\hat{X}$ has at most canonical singularities, $K_{X'} = \epsilon^* K_{\hat{X}}$ and X' is a minimal surface of general type with $K^2_{X'} = 48$ and $\chi(X') = 8$ (cf. [Ino94]).

3. Generalized Burniat Type Surfaces

Using the notation introduced in the previous sections, we give the following definition.

Definition 3.1. Let $\hat{X}$ be a Burniat hypersurface in $A^0 := E_1 \times E_2 \times E_3$, let $G \cong (\mathbb{Z}/2\mathbb{Z})^3$ be a subgroup of $\mathcal{G}_0$ acting freely on $\hat{X}$.

The minimal resolution S of the quotient surface $X := \hat{X}/G$ is called a *generalized Burniat type (GBT) surface*. We call X the *quotient model of* S (indeed, we easily see that X is the canonical model of S).

REMARK 3.2. 1) Since G acts freely and $\hat{X}$ has at most nodes as singularities (we assume Y, hence also $\hat{X}$, to be irreducible!), a generalized Burniat type surface S is a smooth minimal surface of general type with $K_S^2 = 6$ and $\chi(S) = 1$.

2) If $G \triangleleft \mathcal{G}_1$ or $G \triangleleft \mathcal{G}_1'$, then there is a pencil of Burniat hypersurfaces which are left invariant by the G-action, and the family of quotients of the hypersurfaces on which the action is free is then a one parameter family of GBT G-quotient surfaces (if we vary also E_1, E_2, E_3 we obtain a four dimensional family).

Let Δ be the subgroup of $\mathrm{Aut}(((\mathbb{Z}/2\mathbb{Z})^3)^3)$ generated by:

$$\begin{array}{rcl} l_1(g_1,g_2,g_3) & = & (g_2,g_1,g_3) \\ l_2(g_1,g_2,g_3) & = & (g_3,g_2,g_1) \end{array} \qquad \begin{array}{rcl} h_1(g_1,g_2,g_3) & = & (f(g_1),f(g_2),g_3) \\ h_2(g_1,g_2,g_3) & = & (f(g_1),g_2,f(g_3)) \\ h_3(g_1,g_2,g_3) & = & (g_1,f(g_2),f(g_3)) \end{array}$$

where $g_j \in (\mathbb{Z}/2\mathbb{Z})^3\,(j \in \{1,2,3\})$ and where f is defined by:

$$\begin{array}{rcl} f\colon (\mathbb{Z}/2\mathbb{Z})^3 & \longrightarrow & (\mathbb{Z}/2\mathbb{Z})^3 \\ f\colon (a,b,c) & \longmapsto & (a+b,b,b+c) \end{array}$$

REMARK 3.3. 1) It is easy to see that $\Delta(\mathcal{G}_0) = (\mathcal{G}_0)$.

2) We claim now that, as it can be verified, for each $\delta \in \Delta$, $\delta(g)$ is conjugate to g via an element ϕ of the group of automorphisms of $\mathcal{A}^0$.

For example, let $E := \mathbb{C}/\langle 1, \tau\rangle$ be a complex elliptic curve and let $\tau' := \tau + 1$. Then $E = \mathbb{C}/\langle 1, \tau'\rangle$ and the $(\mathbb{Z}/2\mathbb{Z})^3$-action, defined in (2.17), is:

$$\begin{array}{rcl} (z \mapsto -z) & = & (1,0,0) \\ (z \mapsto -z + \frac{\tau'}{2}) = (z \mapsto -z + \frac{\tau+1}{2}) & = & (1,1,1) \\ (z \mapsto -z + \frac{1}{2}) & = & (0,0,1) \end{array} \tag{3.1}$$

This shows that the groups G and $G' := h_j(G) \subset \mathcal{G}_0\,, j = 1,2,3$ are conjugate via an automorphism of $\mathcal{A}^0$.

It follows then that $g \in \mathcal{G}_0$ acts freely on one of the families $\hat{\mathcal{X}}$ if and only if $\delta(g)$ acts freely on the transformed family $\phi(\hat{\mathcal{X}})$.

It follows also that two groups in the same Δ-orbit yield isomorphic families of GBT surfaces, hence we can restrict our attention to a single representative for each Δ-orbit.

Proposition 3.4.

(1) *There are exactly 16 irreducible families of generalized Burniat type surfaces, listed in Tables 3-6.*

(2) *The family of generalized Burniat type surfaces has dimension 4 in cases $\mathcal{S}_1$ and $\mathcal{S}_2$, and dimension 3 otherwise.*

Proof. 1) The MAGMA script below searches for subgroups of $G \leq \mathcal{G}_0$, which satisfy the following

- $G \cong (\mathbb{Z}/2\mathbb{Z})^3$;
- G does not contain the elements 1-9, 11-17 of Table 2.

The 161 groups of the output therefore act freely on $\hat{X}_b \subset E(b_1) \times E(b_2) \times E(b_3)$, except for a finite number of values of $b_1, b_2, b_3 \in \mathbb{C}$ (cf. Remark 2.14).

The following script moreover proves that the 161 groups G belong to exactly 16 Δ-orbits.

```
F:=FiniteField(2); V6:=VectorSpace(F,6);
V3:=VectorSpace(F,3); H3:=Hom(V6,V3);
U:={ V6![0,0,0,0,0,1], V6![0,0,0,1,0,0], V6![1,0,0,0,0,0],
V6![1,0,0,0,1,0], V6![0,1,0,0,0,0], V6![0,1,0,1,1,1],
V6![0,0,1,1,0,0], V6![0,0,1,0,1,1], V6![1,1,1,0,0,1],
V6![1,1,1,1,1,0], V6![0,0,0,0,1,0], V6![0,0,0,1,0,1],
V6![0,0,1,0,0,0],  V6![1,0,0,0,0,1], V6![0,0,1,0,1,0],
V6![1,0,0,1,0,0]};
S3:={Kernel(f): f in H3 | Dimension(Kernel(f)) eq 3};
M:=[**];
for k in S3 do K:=Set(k);
    if #(K meet U) eq 0 then Append(~M, k ); end if;
end for;
#M;
161
P:={1..#M}; Q:={}; // Delta-action
```

```
g1:=hom<V6->V6| V6![0,0,0,1,0,0],V6![0,1,0,0,0,0],V6![0,0,0,0,1,0],
 V6![1,0,0,0,0,0],V6![0,0,1,0,0,0], V6![0,0,0,0,0,1]>;
g2:=hom<V6->V6| V6![0,0,0,0,0,1],V6![0,1,0,0,0,0], V6![0,0,1,0,0,0],
 V6![0,0,0,1,0,0],V6![0,0,1,0,1,0], V6![1,0,0,0,0,0]>;
f1:=hom<V6->V6| V6![1,0,0,0,0,0],V6![1,1,1,1,1,0], V6![0,0,1,0,0,0],
 V6![0,0,0,1,0,0],V6![0,0,0,0,1,0], V6![0,0,0,0,0,1]>;
f2:=hom<V6->V6| V6![1,0,0,0,0,0],V6![1,1,1,0,0,1], V6![0,0,1,0,0,0],
 V6![0,0,0,1,0,0],V6![0,0,0,0,1,0], V6![0,0,0,0,0,1]>;
f3:=hom<V6->V6| V6![1,0,0,0,0,0],V6![0,1,0,1,1,1], V6![0,0,1,0,0,0],
 V6![0,0,0,1,0,0],V6![0,0,0,0,1,0], V6![0,0,0,0,0,1]>;
L1:=Transpose(Matrix([g1(x): x in Basis(V6)]));
L2:=Transpose(Matrix([g2(x): x in Basis(V6)]));
H1:=Transpose(Matrix([f1(x): x in Basis(V6)]));
H2:=Transpose(Matrix([f2(x): x in Basis(V6)]));
H3:=Transpose(Matrix([f3(x): x in Basis(V6)]));
GL6:=GeneralLinearGroup(6,F); PG:=sub<GL6|L1,L2,H1,H2,H3>;
while not IsEmpty(P) do
i:=Rep( P ); Exclude(~P,i);  Include(~Q,i);
for m in PG do
  f:=map<V6->V6| x:->[(m[1],x),(m[2],x),(m[3],x),
                          (m[4],x),(m[5],x),(m[6],x)]>;
  test:=sub<V6|f(Set(M[i]))>;
  if  exists(x){x: x in P | M[x] eq test } then
        Exclude(~P, x); end if;
end for; end while;
#Q;
16
```

This proves the first assertion.

2) In Tables 3-6 we list one representative G for each of the 16 Δ-orbits.

Observe that the dimension of the family is 3 (the number of moduli of the three elliptic curves $E(b_1) \times E(b_2) \times E(b_3)$) if and only if the group G stabilizes only a finite number of Burniat hypersurfaces, equivalently iff G is neither contained in $\mathcal{G}_1$ nor in $\mathcal{G}_1'$. Otherwise G is contained in $\mathcal{G}_1$ or in $\mathcal{G}_1'$ and, by Lemma 2.12, fixes a pencil of Burniat hypersurfaces. Therefore in this case the dimension of the family of generalized Burniat type surfaces is 4.

It is now easy to verify that in case $\mathcal{S}_1$ (of Table 3) $G \subset \mathcal{G}_1'$, in case $\mathcal{S}_2$ $G \subset \mathcal{G}_1$, whereas in cases $\mathcal{S}_3$-$\mathcal{S}_{16}$ of Tables 3-6 G is not contained in any of the two groups $\mathcal{G}_1, \mathcal{G}_1'$. $\square$

3.1. The fundamental groups

To determine the fundamental group of a GBT surface $S \to X = \hat{X}/G$, we preliminarily observe that, by van Kampen's theorem and since X has only nodes as singularities, $\pi_1(X) = \pi_1(S)$. Then we argue as follows.

Let $E_j = \mathbb{C}/\langle e_j, \tau_j e_j \rangle$, $j = 1, 2, 3$ and denote by Λ the fundamental group of $A^0 := E_1 \times E_2 \times E_3$. In particular, $\Lambda = \Lambda_1 \oplus \Lambda_2 \oplus \Lambda_3$, where $\Lambda_j = \langle e_j, \tau_j e_j \rangle$.

Lemma 3.5. *Consider the affine group*

$$\Gamma := \langle \gamma_1, \gamma_2, \gamma_3, e_1, \tau_1 e_1, e_2, \tau_2 e_2, e_3, \tau_3 e_3 \rangle \leq \mathbb{A}(3, \mathbb{C}),$$

generated by Λ and by lifts γ_j of the generators g_j of G as affine transformations.

Then $\Gamma = \pi_1(X) = \pi_1(S)$.

Proof. Observe that by the Lefschetz' hyperplane section theorem (see [Mil63, Theorem 7.4]) follows that $\pi_1(\hat{X}) \cong \pi_1(A^0) = \Lambda$.

The universal covering $\tilde{X}$ of $\hat{X} \subset A^0$ has then a natural inclusion $\tilde{X} \subset \mathbb{C}^3$ and the affine group Γ acts on $\mathbb{C}^3$ leaving $\tilde{X}$ invariant. Since the action of Γ on $\tilde{X}$ is free, and $X = \hat{X}/G = \tilde{X}/\Gamma$ we conclude that $\Gamma = \pi_1(X) = \pi_1(S)$. □

The following MAGMA script, which continues the previous one, computes the fundamental group of each GBT surface. Observe that the fundamental group does only depend on G: since it does not change within the same connected family, and since each group G determines an irreducible family.

```
V9:=VectorSpace(F,9); T:=[* *];
h:=hom<V6->V9| V9![1,0,0,0,0,0,0,0,0], V9![0,1,0,0,1,0,0,1,0],
 V9![0,0,1,0,0,0,0,0,1], V9![0,0,0,1,0,0,0,0,0],
 V9![0,0,0,0,0,1,0,0,1], V9![0,0,0,0,0,0,1,0,0]>;
G1:=DirectProduct([CyclicGroup(2),CyclicGroup(2),CyclicGroup(2)]);
G2:=DirectProduct([CyclicGroup(2),CyclicGroup(2),CyclicGroup(2)]);
G3:=DirectProduct([CyclicGroup(2),CyclicGroup(2),CyclicGroup(2)]);
H:=DirectProduct([G1,G2,G3]);
PolyGroup:=func<seq|Group<a1,a2,a3,a4|
           a1^seq[1], a2^seq[2],a3^seq[3],a4^seq[4], a1*a2*a3*a4>>;
```

```
P1:=PolyGroup([2,2,2,2]);
P2:=PolyGroup([2,2,2,2]);
P3:=PolyGroup([2,2,2,2]);
P:=DirectProduct([P1,P2,P3]);
f:=hom<P->H | H!(1,2),H!(3,4),H!(5,6),H!(1,2)(3,4)(5,6),
H!(7,8),H!(9,10),H!(11,12),H!(7,8)(9,10)(11,12),
H!(13,14),H!(15,16),H!(17,18),H!(13,14)(15,16)(17,18)>;
for i in Q do  G:=h(M[i]);
  s:=[]; for j in [1..3] do s[j]:=Id(H); end for;
  for i  in {1..3} do
    for j in [1..9]  do
      if (G.i)[j] eq 1 then s[i]:=s[i]* H!(2*j-1,2*j);
  end if; end for; end for;
  GG1:=sub<H|s>;
  Pi1:=Simplify(Rewrite(P,GG1@@f));
  Append(~T, [* G, Pi1, AbelianQuotient(Pi1) *]);
end for;
```

Since the fundamental groups are infinite and the presentations given as output are quite long, we only list the respective first homology groups for the 16 families of surfaces in Tables 3, 4, 5 and 6.

It is not obvious, from the presentation given as output of the MAGMA script, whether two of these fundamental groups are isomorphic or not. To check whether two different families have different fundamental groups, we

Table 3. $q = 0$.

	ϵ_0	η_1	ϵ_1	η_0	η_1	ϵ_2	ζ_0	η_1	ϵ_3	H_1
	1	0	0	1	0	0	1	0	0	
$\mathcal{S}_1$	0	1	0	1	1	0	1	1	0	$(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^2$
	0	0	0	0	0	1	1	0	1	
	1	0	0	0	0	1	1	0	1	
$\mathcal{S}_2$	0	0	1	0	0	0	1	0	1	$(\mathbb{Z}/2\mathbb{Z})^6$
	0	0	0	1	0	1	0	0	1	
	1	0	0	0	0	1	1	0	1	
$\mathcal{S}_3$	0	1	0	0	1	0	1	1	0	$(\mathbb{Z}/4\mathbb{Z})^3$
	0	0	1	1	0	1	1	0	0	
	1	0	1	0	0	1	1	0	0	
$\mathcal{S}_4$	0	1	0	0	1	0	1	1	0	$(\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})^2$
	0	0	0	1	0	1	1	0	1	

can compare the number of normal subgroups of the fundamental group of index $k \leq m$ (in our case $m = 6$). This can be done easily using the MAGMA function: `LowIndexNormalSubgroups(H, m)` which returns a sequence containing the normal subgroups of the finitely presented group H of index $k \leq m$. This allows us to see that the fundamental groups of the families we constructed are pairwise non-isomorphic, except for two pairs: $(\mathcal{S}_{11}, \mathcal{S}_{12})$ and $(\mathcal{S}_{14}, \mathcal{S}_{15})$.

Indeed in these cases, the fundamental groups are isomorphic. We verified this using the MAGMA function `SearchForIsomorphism(H, K, n)` which attempts to find an isomorphism of the finitely presented group H with the finitely presented group K. The search is restricted to those homomorphisms for which the sum of the word-lengths of the images of the

Table 4. $q = 1$.

	ϵ_0	η_1	ϵ_1	η_0	η_1	ϵ_2	ζ_0	η_1	ϵ_3	H_1	π_1
	1	0	1	0	0	0	1	0	1		
$\mathcal{S}_5$	0	1	0	0	1	0	1	1	0	$(\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}^2$	
	0	0	0	0	0	1	1	0	1		
	0	1	0	1	1	0	1	1	0		
$\mathcal{S}_6$	0	0	1	1	0	0	1	0	1	$(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}^2$	
	0	0	0	0	0	1	1	0	1		
	1	0	0	0	0	1	1	0	1		
$\mathcal{S}_7$	0	1	0	0	1	0	1	1	0	$(\mathbb{Z}/4\mathbb{Z}) \times \mathbb{Z}^2$	
	0	0	1	1	0	1	0	0	0		
	1	0	0	0	0	1	1	0	1		
$\mathcal{S}_8$	0	1	0	1	1	0	1	1	0	$(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}^2$	
	0	0	1	1	0	0	1	0	1		
	1	0	0	1	0	1	1	0	1		
$\mathcal{S}_9$	0	1	0	0	1	0	1	1	0	$(\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}) \times \mathbb{Z}^2$	
	0	0	1	1	0	1	0	0	0		
	1	0	1	1	0	0	1	0	1		
$\mathcal{S}_{10}$	0	1	0	1	1	0	1	1	0	$(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}^2$	
	0	0	0	0	0	1	1	0	1		
	1	0	0	1	0	1	0	0	1		
$\mathcal{S}_{11}$	0	1	0	1	1	0	0	1	0	$(\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}^2$	
	0	0	1	1	0	1	1	0	0		
	1	0	1	0	0	0	1	0	1		
$\mathcal{S}_{12}$	0	1	0	0	1	0	1	1	0	$(\mathbb{Z}/2\mathbb{Z})^3 \times \mathbb{Z}^2$	$\cong \pi_1(S_{11})$
	0	0	0	1	0	1	1	0	1		

generators of H in K is at most n (in our case $n = 8$). The answer is given as follows: if an isomorphism ϕ is found, then the output is the triple (true, ϕ, ϕ^{-1}); otherwise, the output is 'false'.

That these isomorphisms exist is no coincidence: we shall in fact show later that in both cases we have two families of surfaces which are contained in a larger irreducible family (see Sections 4 and 6).

Since for a smooth projective surface S it holds $q(S) = \frac{1}{2} \operatorname{rk} H_1(S, \mathbb{Z})$, we have proved the following:

THEOREM 3.6. *Among the 16 families of generalized Burniat type surfaces four have $p_g = q = 0$ (Table 3), eight have $p_g = q = 1$ (Table 4), three have $p_g = q = 2$ (Table 5) and one has $p_g = q = 3$ (Table 6).*

Moreover, the fundamental groups of these families are pairwise non isomorphic, except for $\pi_1(S_{11}) \cong \pi_1(S_{12})$ and $\pi_1(S_{14}) \cong \pi_1(S_{15})$, where S_j is in the family $\mathcal{S}_j$.

Table 5. $q = 2$.

	ϵ_0	η_1	ϵ_1	η_0	η_1	ϵ_2	ζ_0	η_1	ϵ_3	H_1	π_1
	1	0	0	1	0	1	1	0	1		
$\mathcal{S}_{13}$	0	1	0	1	1	0	1	1	0	$\mathbb{Z}^4$	
	0	0	1	1	0	1	0	0	0		
	1	0	1	0	0	0	0	0	1		
$\mathcal{S}_{14}$	0	1	1	0	1	1	0	1	0	$(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^4$	
	0	0	0	1	0	1	0	0	1		
	1	0	1	0	0	0	1	0	1		
$\mathcal{S}_{15}$	0	1	1	0	1	1	0	1	0	$(\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}^4$	$\cong \pi_1(S_{14})$
	0	0	0	1	0	1	1	0	1		

Table 6. $q = 3$.

	ϵ_0	η_1	ϵ_1	η_0	η_1	ϵ_2	ζ_0	η_1	ϵ_3	π_1
	1	0	1	0	0	0	1	0	1	
$\mathcal{S}_{16}$	0	1	1	0	1	1	1	1	0	$\mathbb{Z}^6$
	0	0	0	1	0	1	1	0	1	

Remark 3.7. We observe that the family $\mathcal{S}_2$ in Table 3 corresponds to the family of *primary Burniat surfaces* (cf. Section 1).

4. The Moduli Space of Generalized Burniat Type Surfaces

The aim of this section is to describe the connected components of the Gieseker moduli space of surfaces of general type containing the isomorphism classes of the generalized Burniat type surfaces.

First we shall prove the following result:

Theorem 4.1. *Let X be the canonical model of a generalized Burniat type surface S. Then the base of the Kuranishi family of X is smooth.*

Proof. Recall that X is the quotient model of a generalized Burniat type surface S, and let $\hat{X} \to X$ be the canonical $G \cong (\mathbb{Z}/2\mathbb{Z})^3$-cover. Then $\hat{X} \subset A^0 = E_1 \times E_2 \times E_3$ is a hypersurface of multidegree $(2,2,2)$ having at most nodes as singularities.

It suffices to show (cf. [Cat13, Proposition 4.5]) that the base of the Kuranishi family of $\hat{X}$ is smooth (since the base of Kuranishi family of X is given by the G-invariant part of the base of the Kuranishi family of $\hat{X}$).

Now, since $\hat{X}$ moves in a family with smooth base of dimension $13 = 6 + 7$, it it is enough to show that

$$\dim \mathrm{Ext}^1_{\mathcal{O}_{\hat{X}}}(\Omega^1_{\hat{X}}, \mathcal{O}_{\hat{X}}) = 13.$$

Moreover, the Kodaira-Spencer map of the above family is a bijection, but we omit the verification here.

Indeed $\hat{X} \subset A^0$ is an ample divisor, and it suffices to apply the following lemma. □

Lemma 4.2. *Let A be an Abelian variety of dimension n and let $D \subset A$ be an ample divisor. Then:*

$$\dim \mathrm{Ext}^1_{\mathcal{O}_D}(\Omega^1_D, \mathcal{O}_D) = \frac{1}{2}n(n+1) + \dim |D|.$$

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_D(-D) \to \Omega^1_A \otimes \mathcal{O}_D \cong \mathcal{O}_D^{\oplus n} \to \Omega^1_D \to 0.$$

Applying the functor $\mathrm{Hom}_{\mathcal{O}_D}(\_, \mathcal{O}_D)$, we obtain the long exact sequence:

$$\begin{aligned} 0 \ &\to \mathrm{Hom}(\Omega^1_D, \mathcal{O}_D) = 0 \to \mathrm{Hom}(\mathcal{O}_D^{\oplus n}, \mathcal{O}_D) \to \\ &\to \mathrm{Hom}(\mathcal{O}_D(-D), \mathcal{O}_D) \to \mathrm{Ext}^1(\Omega^1_D, \mathcal{O}_D) \to \\ &\to \mathrm{Ext}^1(\mathcal{O}_D^{\oplus n}, \mathcal{O}_D) \to \mathrm{Ext}^1(\mathcal{O}_D(-D), \mathcal{O}_D) \to \\ &\to \mathrm{Ext}^2(\Omega^1_D, \mathcal{O}_D) \to \mathrm{Ext}^2(\mathcal{O}_D^{\oplus n}, \mathcal{O}_D) \to \\ &\to \mathrm{Ext}^2(\mathcal{O}_D(-D), \mathcal{O}_D) \to \dots . \end{aligned} \tag{4.1}$$

We have that

i) $\mathrm{Ext}^i(\mathcal{O}_D^{\oplus n}, \mathcal{O}_D) = H^i(D, \mathcal{O}_D)^{\oplus n}$;

ii) $\omega_D = \omega_A \otimes \mathcal{O}_D(D) = \mathcal{O}_D(D)$;

iii) $\mathrm{Ext}^i(\mathcal{O}_D(-D), \mathcal{O}_D) \cong \mathrm{Ext}^i(\mathcal{O}_D, \mathcal{O}_D(D)) = \mathrm{Ext}^i(\mathcal{O}_D, \omega_D) \cong H^{n-1-i}(D, \mathcal{O}_D)^*$, where the last equality holds by Serre duality.

Next, we consider the short exact sequence:

$$0 \to \mathcal{O}_A(-D) \to \mathcal{O}_A \to \mathcal{O}_D \to 0$$

and the associated long cohomology sequence

$$\begin{aligned} 0 \to H^0(\mathcal{O}_A) \to H^0(\mathcal{O}_D) \to H^1(\mathcal{O}_A(-D)) \to H^1(\mathcal{O}_A) \to \\ \to H^1(\mathcal{O}_D) \to H^2(\mathcal{O}_A(-D)) \to H^2(\mathcal{O}_A) \to \dots \to H^{n-1}(\mathcal{O}_D) \to \\ \to H^n(\mathcal{O}_A(-D)) \to H^n(\mathcal{O}_A) \to 0. \end{aligned} \tag{4.2}$$

Note that by Serre duality $H^i(\mathcal{O}_A(-D)) \cong H^{n-i}(\mathcal{O}_A(D))^*$, and since $D \subset A$ is an ample divisor, we get that these cohomology groups are trivial for $i \leq n-1$ by the Kodaira vanishing theorem.

This implies that

- $\dim H^i(\mathcal{O}_D) = \dim H^i(\mathcal{O}_A) = \binom{n}{i}$, for $0 \leq i \leq n-2$,
- $\dim H^{n-1}(\mathcal{O}_D) = \dim |D| + n$.

Inserting this information in the long exact sequence (4.1), we see that

$$\dim \mathrm{Ext}^1_{\mathcal{O}_D}(\Omega^1_D, \mathcal{O}_D) = \frac{1}{2} n(n+1) + \dim |D| \,,$$

once we show that

$$\varphi\colon \operatorname{Ext}^1(\mathcal{O}_D^{\oplus n}, \mathcal{O}_D) \to \operatorname{Ext}^1(\mathcal{O}_D(-D), \mathcal{O}_D)$$

is surjective.

But

$$\operatorname{Ext}^1(\mathcal{O}_D^{\oplus n}, \mathcal{O}_D) \cong H^1(\mathcal{O}_D)^{\oplus n} \cong H^1(\mathcal{O}_A^{\oplus n}) \cong H^1(\Theta_A)$$

and

$$\operatorname{Ext}^1(\mathcal{O}_D(-D), \mathcal{O}_D) \cong H^{n-1-1}(\mathcal{O}_D)^* \cong H^{n-2}(\mathcal{O}_A)^* \cong H^2(\mathcal{O}_A),$$

where the first and third equality follow from Serre duality.

Composing with these isomorphisms, φ becomes

$$H^1(\Theta_A) \to H^2(\mathcal{O}_A),$$

the contraction with the first Chern class of D, an element of $H^1(A, \Omega^1_A)$, which is represented by a non degenerate alternating form. Hence the surjectivity of φ follows. □

4.1. Surfaces in $\mathcal{S}_j$ with $j \leq 10$, i.e., with $p_g = q \leq 1$

Recall the following definition.

Definition 4.3 ([BC12, Definition 0.2-0.3]). A complex projective manifold X is said to be a *diagonal classical Inoue-type manifold* if

(1) $\dim(X) \geq 2$;

(2) there is a finite group G and a Galois étale G-covering $\hat{X} \to X\,(= \hat{X}/G)$ such that:

(3) $\hat{X}$ is an ample divisor inside a $K(\Gamma, 1)$-projective manifold Z (hence by Lefschetz $\pi_1(\hat{X}) \cong \pi_1(Z) \cong \Gamma$) and moreover

(4) the action of G on $\hat{X}$ yields a faithful action on $\pi_1(\hat{X}) \cong \Gamma$: in other words the exact sequence

$$1 \to \Gamma \cong \pi_1(\hat{X}) \to \pi_1(X) \to G \to 1$$

gives an injection $G \to \operatorname{Out}(\Gamma)$, defined by conjugation;

(5) $Z = (A_1 \times \ldots \times A_r) \times (C_1 \times \ldots \times C_s)$ where each A_j is an Abelian variety and each C_k is a curve of genus $g(C_k) \geq 2$;

(6) the action of G on $\hat{X}$ is induced by a diagonal action on Z;

(7) the faithful action on $\pi_1(\hat{X}) \cong \Gamma$, induced by conjugation by lifts of elements of G, has the Schur property:

$$\text{(SP)} \qquad \mathrm{Hom}(V_j, V_k)^G = 0\,, \quad \forall k \neq j\,,$$

where $V_j := \Lambda_j \otimes \mathbb{Q}$, being $\Lambda_j := \pi_1(A_j)$ (it suffices to verify that, for each Λ_j, there is a subgroup H_j of G for which $\mathrm{Hom}(V_j, V_k)^{H_j} = 0\,, \forall k \neq j$).

We say instead that X is a *diagonal classical Inoue-type variety* if we replace the assumption of smoothness of X by the assumption that X has canonical singularities.

To fix the notation, let us call a surface S a *generalized Burniat type (GBT) surface of type j* if S belongs to the family $\mathcal{S}_j$ in Tables 3-6.

Lemma 4.4. *Let X_j be the canonical model of a GBT surface S_j of type j. Then the embedding $\hat{X}_j \subset A^0 = E_1 \times E_2 \times E_3$ realizes X_j as a diagonal classical Inoue-type variety if and only if $1 \leq j \leq 10$.*

Proof. It is trivial to see that the canonical model of a generalized Burniat type surface $X_j = \hat{X}/G_j$ satisfies conditions (1-6) in Definition 4.3. Hence there remains only to determine which surfaces fulfill the Schur Property (SP).

To verify the Schur Property one has to find, for each pair $j \neq k \in \{1, 2, 3\}$ an element $g \in G$ such that $\mathrm{d}g$ being the derivative of g, $\mathrm{d}g_{|E_j} \cdot \mathrm{d}g_{|E_k} = -1$. Let $j = 1$ and $g = (0, 1, 0, 1, 1, 0, 1, 1, 0) \in G_1$: then $\mathrm{d}g_{|E_1} = -1$, $\mathrm{d}g_{|E_2} = \mathrm{d}g_{|E_3} = 1$, while for $g' = (0, 0, 0, 0, 0, 1, 1, 0, 1) \in G_1$ one has $\mathrm{d}g'_{|E_2} = -1$ and $\mathrm{d}g'_{|E_3} = 1$. Hence X_1 satisfies (SP). Considering a suitable pair of generators of G_j, one can prove in the same way that X_j satisfies (SP) for $j = 2, \ldots, 10$.

Consider now the case $j = 11$ and let g be one of the three generators of G_{11} in Table 4. Then $\mathrm{d}g_{|E_1} = \mathrm{d}g_{|E_3} = -1$ and $\mathrm{d}g_{|E_2} = 1$; this means that $\mathrm{Hom}(V_1, V_3)^G \neq 0$, hence X_{11} does not fulfill the Schur property.

In the same way one can show that S_j does not fulfill (SP) for $j = 12, \dots, 16$. □

We are now in the position to prove the following result.

Theorem 4.5. *Let S be a smooth projective surface homotopically equivalent to a GBT surface S_j of type j with $1 \leq j \leq 10$. Then S is a GBT surface of type j, i.e., contained in the same irreducible family as S_j.*

Proof. Assume that S is homotopically equivalent to S_j ($1 \leq j \leq 10$), hence in particular S has the same fundamental group as S_j. Consider the étale $G_j \cong (\mathbb{Z}/2\mathbb{Z})^3$-cover $\hat{S} \to S$. Then by [BC12, Theorem 0.5] we have a splitting of the Albanese variety and an Albanese map $f\colon \hat{S} \to E_1 \times E_2 \times E_3$ which is generically finite onto its image W. By loc. cit. Lemma 1.2, G_j acts on $E_1 \times E_2 \times E_3$ with the same action as for a GBT surface of type j. It is now easy to verify that there is no effective G_j-invariant divisor of numerical type $(1,1,1)$ on $E_1 \times E_2 \times E_3$, hence W has class $2F_1 + 2F_2 + 2F_3$, where F_i is the class of a fibre of the projection of $E_1 \times E_2 \times E_3$ on the j-th factor. Therefore f is birational onto its image and one verifies as in loc. cit. that W has at most rational double points as singularities and is therefore the canonical model $\hat{X}$ of $\hat{S}$.

Claim. W is the pull-back of a Del Pezzo surface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ for a suitable degree $(\mathbb{Z}/2\mathbb{Z})^3$-covering π'.

Proof of the Claim. The pull back of a divisor of multidegree $(1,1,1)$ on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ under any

$$\pi' : E_1 \times E_2 \times E_3 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$

is a divisor which has the same class as W: hence the two divisors are linearly equivalent to a translate of each other. Since the corresponding linear systems have the same dimension we infer that W is the translate of such an effective divisor. Changing the origin of the Abelian variety A^0 we obtain another action of $(\mathbb{Z}/2\mathbb{Z})^3$ such that W is invariant; the claim is thus proven.

We have therefore seen that S is a GBT surface and has the same fundamental group as S_j. Thus by our classification S is in the same irreducible family as S_j, whence S is a GBT surface of type j. □

REMARK 4.6. The same conclusion holds under the weaker assumptions:

1) $\pi_1(S) \cong \pi_1(S_j)$

2) the corresponding covering $\hat{S}$, whose Albanese is a product of 3 elliptic curves because of the Schur property, satisfies that the image of the Albanese map has class $(2, 2, 2)$.

We can now summarize our results in the following theorem

THEOREM 4.7. *The connected component $\mathfrak{N}_j$ of the Gieseker moduli space $\mathfrak{M}^{can}_{1,6}$ corresponding to generalized Burniat type surfaces of type j ($1 \leq j \leq 10$) is irreducible, normal and unirational, of dimension 4 if $j = 1$ or 2, else of dimension 3.*

PROOF. We have shown that the Kuranishi family is smooth, hence the moduli space is normal. By the previous theorem each family of GBT surfaces with $j \leq 10$ surjects onto a connected component of the Gieseker moduli space: since the family has a rational base (a projective bundle over a rational variety), follows the assertion about the unirationality. □

4.2. Surfaces in $\mathcal{S}_j$ with $j = 11, 12$, having $p_g = q = 1$

Since these surfaces do not fulfill the Schur property, the family constructed as $(\mathbb{Z}/2\mathbb{Z})^3$-quotient of a Burniat hypersurface in a product of three elliptic curves is not complete. We will study these surfaces in greater generality in Section 5 and Section 6. In fact, it turns out that the families $\mathcal{S}_{11}$, $\mathcal{S}_{12}$ yield two irreducible subsets each of codimension one in an irreducible connected component of dimension 4 of the moduli space of surfaces of general type with $p_g = q = 1$, $K^2 = 6$.

4.3. Surfaces in $\mathcal{S}_j$ with $j = 13, 14, 15$, i.e., those with $p_g = q = 2$

We have three families (each of dimension 3, the number of moduli of the triple of elliptic curves) of GBT surfaces with $p_g = q = 2$. We have already observed that the embedding $\hat{X}_j \subset A^0 = E_1 \times E_2 \times E_3$ does not fulfill the Schur property. In fact, it is not difficult to show that each of

the three families is a subfamily of a four dimensional irreducibile family, where the product of the two elliptic curves on which the projection of G_j acts freely deforms to an Abelian surface A_2. In this case the embedding $\hat{X}_j \subset E_1 \times A_2$ fulfills the Schur property and we can show that we obtain in this way exactly two irreducible connected components of the moduli space of surfaces of general type.

We do not give more details here since these surfaces have already been classified in [PP13].

Observe in fact the following:

Proposition 4.8. *Let S be a GBT surface with $p_g(S) = q(S) = 2$. Then S is of Albanese general type and the Albanese map is generically of degree 2.*

Proof. Assume S to be of type 13 (the proof in the other two cases is exactly the same) and consider the following diagram:

$$\begin{array}{ccccc} & & \hat{X} \subset E_1 \times E_2 \times E_3 & \xrightarrow{p_{23}} & E_2 \times E_3 \\ & & \Big\downarrow{\scriptstyle G_{13}} & & \Big\downarrow{\scriptstyle p_{23}(G_{13})} \\ S & \longrightarrow & X & \xrightarrow{a} & (E_2 \times E_3)/p_{23}(G_{13}) \end{array}$$

Since $p_{23}\colon \hat{X} \to E_2 \times E_3$ is generically finite of degree 2 (as $\hat{X}$ is a divisor of multidegree $(2,2,2)$), and since $G_{13} \cong p_{23}(G_{13})$, one sees immediately that a is generically finite of degree 2 and that $\mathrm{Alb}(S) = (E_2 \times E_3)/p_{23}(G_{13})$. □

We recall the following result due to Penegini and Polizzi:

Theorem 4.9 ([PP13, Theorem 31]). *Let $\mathcal{M}$ be the moduli space of minimal surfaces S of general type with $p_g = q = 2$, $K_S^2 = 6$ and Albanese map of degree 2. Then the following holds:*

(i) *$\mathcal{M}$ is the union of three irreducible connected components, namely $\mathcal{M}_{Ia}$, $\mathcal{M}_{Ib}$ and $\mathcal{M}_{II}$.*

(ii) *$\mathcal{M}_{Ia}$, $\mathcal{M}_{Ib}$ and $\mathcal{M}_{II}$ are generically smooth of respective dimensions 4, 4, 3.*

(iii) *The general surface in $\mathcal{M}_{Ia}$ and $\mathcal{M}_{Ib}$ has ample canonical class; all surfaces in $\mathcal{M}_{II}$ have ample canonical class.*

It is immediately clear that the subset of the moduli space corresponding to GBT surfaces with $p_g = q = 2$ cannot be contained in $\mathcal{M}_{II}$, since it has dimension 3, while the families $13, 14, 15$ yield irreducible families of dimension at least four. We have the following

LEMMA 4.10. *Let S_j be a GBT surface of type j with $p_g = q = 2$, i.e., $j \in \{13, 14, 15\}$. Consider the pencil $f_j \colon S_j \to \mathbb{P}^1 \cong E_k/p_k(G_j)$, where $k = 1$ for S_{13} and $k = 3$ for $j = 14, 15$. Then the general fibre of f_j is a smooth curve of genus 5 if $j = 13$ and of genus 3 if $j = 14, 15$.*

PROOF. Consider the diagram

$$\begin{array}{ccccc} & & \hat{X}_j \subset E_1 \times E_2 \times E_3 & \xrightarrow{p_k} & E_k \\ & & \Big\downarrow{\scriptstyle G_j \cong (\mathbb{Z}/2)^3} & & \Big\downarrow{\scriptstyle p_k(G_j)} \\ S_j & \longrightarrow & X_j & \xrightarrow{f_j} & E_k/p_k(G_j) \end{array}$$

Note that the general fibre of p_k is a divisor of bidegree $(2, 2)$ in the product of two elliptic curves, whence has genus 5. Since $p_1(G_{13}) \cong (\mathbb{Z}/2\mathbb{Z})^3$, the genus of a general fibre of f_{13} is 5, whereas $p_3(G_{14}), p_3(G_{15}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, whence the genus of a general fibre of f_{14} and f_{15} is 3. □

This allows us to conclude the following:

PROPOSITION 4.11. *Let S_j be a GBT surface of type j with $p_g = q = 2$. Then the point of the Gieseker moduli space corresponding to S_{13} lies in $\mathcal{M}_{Ia}$, whereas the point corresponding to S_{14}, resp. S_{15}, lies in $\mathcal{M}_{Ib}$.*

In particular, GBT surfaces with $p_g = q = 2$ of type 13 (resp. 14, 15) form a three dimensional subset of the four dimensional connected component $\mathcal{M}_{Ia}$ (resp. $\mathcal{M}_{Ib}$).

PROOF. This follows from Lemma 4.10 and [PP13, Proposition 22]. □

REMARK 4.12. Consider for $j = 13$ (the same holds also for $j = 14, 15$) the irrational pencil $f \colon S_{13} \to E_2/p_2(G_{13})$. Observe that $E_2/p_2(G_{13})$ is an

elliptic curve and that the genus of the fibres of f is 3. This implies that f is not isotrivial (otherwise it would be contained in the table of [Pen11]). This contradicts Theorem A of [Zuc03].

4.4. Surfaces in $\mathcal{S}_{16}$, i.e., those with $p_g = q = 3$

Minimal surfaces of general type with $p_g = q = 3$ are completely classified by the work of several authors (cf. [CCML98, Pir02, HP02]).

Theorem 4.13. *A minimal surface of general type with $p_g = q = 3$ has $K^2 = 6$ or $K^2 = 8$ and, more precisely:*

- *if $K^2 = 6$, S is the minimal resolution of the symmetric square of a curve of genus 3;*
- *otherwise $S = (C_2 \times C_3)/\sigma$, where C_g denotes a curve of genus g and σ is an involution of product type acting on C_2 as an elliptic involution (i.e., with elliptic quotient), and on C_3 as a fixed point free involution.*

In particular, the moduli space of minimal surfaces of general type with $p_g = q = 3$ is the disjoint union of $\mathcal{M}_{6,3,3}$ and $\mathcal{M}_{8,3,3}$, which are both irreducible of respective dimension 6 and 5.

We get:

Proposition 4.14. *Generalized Burniat type surfaces with $p_g = q = 3$ (i.e. of type 16) form a three dimensional subset of the six dimensional connected component $\mathcal{M}_{6,3,3}$.*

5. Bagnera-de Franchis Varieties

Definition 5.1. A *Generalized Hyperelliptic Variety (GHV) X* is defined to be a quotient $X = A/G$ of an Abelian Variety A by a nontrivial finite group G acting freely, and with the property that G contains no translations.

Remark that, if G is any group acting freely on A, which is not a subgroup of the group of translations, then the quotient $X = A/G$ is a GHV. Because the subgroup G_T of translations in G is a normal subgroup of G,

and, if we denote $G' = G/G_T$, then $A/G = A'/G'$, where A' is the Abelian variety $A' := A/G_T$.

Definition 5.2. 1) A *Bagnera-de Franchis variety* (for short: BdF variety) is a GHV $X = A/G$ such that $G \cong \mathbb{Z}/m\mathbb{Z}$ is a cyclic group.

2) A *Bagnera-de Franchis variety of product type* is a Bagnera-de Franchis variety $X = A/G$, where $A = (A_1 \times A_2)$, A_1, A_2 are Abelian Varieties, and $G \cong \mathbb{Z}/m\mathbb{Z}$ is generated by an automorphism of the form

$$g(a_1, a_2) = (a_1 + \beta_1, \alpha_2(a_2)),$$

where $\beta_1 \in A_1[m]$ is an element of order exactly m, and similarly $\alpha_2 : A_2 \to A_2$ is a linear automorphism of order exactly m without 1 as eigenvalue (these conditions guarantee that the action is free).

3) If moreover all eigenvalues of α_2 are primitive m-th roots of 1, we shall say that $X = A/G$ is a *primary Bagnera-de Franchis variety.*

Remark 5.3. 1) One can give a similar definition of Bagnera-de Franchis manifolds, requiring only that A, A_1, A_2 be complex tori.

2) If A has dimension $n = 2$, the Bagnera-de Franchis manifolds coincide with the Generalized Hyperelliptic varieties, due to the classification result of Bagnera-de Franchis in [BdF08].

We have the following proposition, giving a characterization of Bagnera-de Franchis varieties.

Proposition 5.4. *Every Bagnera-de Franchis variety $X = A/G$ is the quotient of a Bagnera-de Franchis variety of product type, $(A_1 \times A_2)/G$ by any finite subgroup T of $(A_1 \times A_2)$ which satisfies the following properties:*

(1) *T is the graph of an isomorphism between two respective subgroups $T_1 \subset A_1,\ T_2 \subset A_2$,*

(2) *$(\alpha_2 - \mathrm{Id})T_2 = 0$,*

(3) *if $g(a_1, a_2) = (a_1 + \beta_1, \alpha_2(a_2))$, then the subgroup of order m generated by β_1 intersects T_1 only in $\{0\}$.*

In particular, we may write X as the quotient $X = (A_1 \times A_2)/(G \times T)$ of $A_1 \times A_2$ by the Abelian group $G \times T$.

Proof. We refer to [Cat14]. □

5.1. Actions of a finite group on an Abelian variety

Assume that we have the action of a finite group G on a complex torus $A = V/\Lambda$. Since every holomorphic map between complex tori lifts to a complex affine map of the respective universal covers, we can attach to the group G the group of affine transformations Γ, which consists of all affine maps of V which lift transformations of G. Then Γ fits into an exact sequence:

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G \longrightarrow 1\,.$$

The following is a slight improvement of [BC12, Lemma 1.2]:

Proposition 5.5. *The group Γ determines the real affine type of the action of Γ on V (respectively: the rational affine type of the action of Γ on $\Lambda \otimes \mathbb{Q}$), in particular the above exact sequence determines the action of G up to real affine isomorphism of A (resp.: rational affine isomorphism of $(\Lambda \otimes \mathbb{Q})/\Lambda$).*

Proof. It is clear that $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ as a real vector space, and we denote by $V_{\mathbb{Q}} := \Lambda \otimes \mathbb{Q}$. Let

$$\Lambda' := \ker(\alpha_L \colon \Gamma \to \mathrm{GL}(V_{\mathbb{Q}}) \subset \mathrm{GL}(V)),$$
$$\overline{G}_1 := \mathrm{im}(\alpha_L \colon \Gamma \to \mathrm{GL}(V_{\mathbb{Q}}))\,.$$

The group Λ' is obviously Abelian, contains Λ, and maps isomorphically onto a lattice $\Lambda' \subset V$.

In turn $V = \Lambda' \otimes_{\mathbb{Z}} \mathbb{R}$, and, if $G' := \Gamma/\Lambda'$, then $G' \cong \overline{G}_1$ and the exact sequence

$$1 \longrightarrow \Lambda' \longrightarrow \Gamma \longrightarrow G' \longrightarrow 1\,,$$

since we have an embedding $G' \subset \mathrm{GL}(\Lambda')$, shows that the affine group $\Gamma \subset \mathrm{Aff}(\Lambda') \subset \mathrm{Aff}(V)$ is uniquely determined (Γ is the inverse image of G' under $\mathrm{Aff}(\Lambda') \to \mathrm{GL}(\Lambda')$).

There remains only to show that Λ' is determined by Γ as an abstract group, independently of the exact sequence we started with. In fact, one property of Λ' is that it is a maximal Abelian subgroup, normal and of finite index.

Assume that Λ'' has the same property: then their intersection $\Lambda^0 := \Lambda' \cap \Lambda''$ is a normal subgroup of finite index, in particular $\Lambda^0 \otimes_{\mathbb{Z}} \mathbb{R} = \Lambda' \otimes_{\mathbb{Z}}$

$\mathbb{R} = V$; hence $\Lambda'' \subset \ker(\alpha_L \colon \Gamma \to \mathrm{GL}(V)) = \Lambda'$, where α_L is induced by conjugation on Λ^0.

By maximality $\Lambda' = \Lambda''$. □

Observe that, in order to obtain the structure of a complex torus on V/Λ', we must give a complex structure on V which makes the action of $G' \cong \overline{G}_1$ complex linear.

In order to study the moduli spaces of the associated complex manifolds, we introduce therefore a further invariant, called the Hodge type, according to the following definition.

Definition 5.6. Given a faithful representation $G \to \mathrm{Aut}(\Lambda)$, where Λ is a free Abelian group of even rank $2n$, a *G-Hodge decomposition* is a G-invariant decomposition

$$\Lambda \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}, \ H^{0,1} = \overline{H^{1,0}}.$$

Write $\Lambda \otimes \mathbb{C}$ as the sum of isotypical components

$$\Lambda \otimes \mathbb{C} = \oplus_{\chi \in \mathrm{Irr}(G)} U_\chi.$$

Write also $U_\chi = W_\chi \otimes M_\chi$, where W_χ is the given irreducible representation, and M_χ is a trivial representation of dimension n_χ.

Then $V := H^{1,0} = \oplus_{\chi \in \mathrm{Irr}(G)} V_\chi$, where $V_\chi = W_\chi \otimes M_\chi^{1,0}$ and $M_\chi^{1,0}$ is a subspace of M_χ. The *Hodge type* of the decomposition is the datum of the dimensions

$$\nu(\chi) := \dim_{\mathbb{C}} M_\chi^{1,0}$$

corresponding to the Hodge summands for non real representations (observe in fact that one must have: $\nu(\chi) + \nu(\bar{\chi}) = \dim(M_\chi)$).

Remark 5.7. Given a faithful representation $G \to \mathrm{Aut}(\Lambda)$, where Λ is a free Abelian group of even rank $2n$, all the G-Hodge decompositions of a fixed Hodge type are parametrized by an open set in a product of Grassmannians. Since, for a non real irreducible representation χ one may simply choose $M_\chi^{1,0}$ to be a complex subspace of dimension $\nu(\chi)$ of M_χ, and for $M_\chi = \overline{(M_\chi)}$, one simply chooses a complex subspace $M_\chi^{1,0}$ of half dimension. Then the open condition is just that (since $M_\chi^{0,1} := \overline{M_\chi^{1,0}}$) we want $M_\chi = (M_\chi^{1,0}) \oplus (M_\chi^{0,1})$, or, equivalently, $M_\chi = (M_\chi^{1,0}) \oplus \overline{(M_{\bar{\chi}}^{1,0})}$.

5.2. Bagnera-de Franchis varieties of small dimension

We have shown that a Bagnera-de Franchis variety $X = A/G$ can be seen as the quotient of one of product type $(A_1 \times A_2)/G$ by a finite subgroup T of $A_1 \times A_2$, satisfying the properties stated in Proposition 5.4.

Dealing with appropriate choices of T is the easy part, since, as we saw, the points t_2 of T_2 satisfy, by property (2), $\alpha_2(t_2) = t_2$.

It suffices then to choose $T_2 \subset A_2[*] := \ker(\alpha_2 - \mathrm{Id}_{A_2})$, which is a finite subgroup of A_2, and then to pick an isomorphism $\psi\colon T_2 \to T_1 \subset A_1$, such that $T_1 := \mathrm{im}(\psi) \cap \langle\langle \beta_1 \rangle\rangle = \{0\}$.

We therefore restrict ourselves from now on to *Bagnera-de Franchis varieties of product type* and we show now how to further reduce to the investigation of *primary Bagnera-de Franchis varieties*.

In fact, in the case of a BdF variety of product type, Λ_2 is a G-module, hence a module over the group ring

$$R := R(m) := \mathbb{Z}[G] \cong \mathbb{Z}[x]/(x^m - 1).$$

The ring R is in general far from being an integral domain, since indeed it can be written as a direct sum of cyclotomic rings, which are the integral domains defined as $R_k := \mathbb{Z}[x]/(P_k(x))$. Here $P_k(x)$ is the k-th cyclotomic polynomial

$$P_k(x) = \prod_{0<j<k,\ (k,j)=1} (x - \epsilon^j)\,,$$

where $\epsilon = \exp(2\pi i/k)$. Then

$$R(m) = \bigoplus\nolimits_{k|m} R_k\,.$$

The following elementary lemma, together with the splitting of the vector space V as a direct sum of eigenspaces for g, yields a decomposition of A_2 as a direct product $A_2 = \oplus_{k|m} A_{2,k}$ of G -invariant Abelian subvarieties $A_{2,k}$ on which g acts with eigenvalues of order precisely k.

Lemma 5.8. *Assume that M is a module over a ring $R = \oplus_k R_k$. Then there is a unique direct sum decomposition*

$$M = \bigoplus\nolimits_k M_k,$$

such that

- *M_k is an R_k-module, and*
- *the R-module structure of M is obtained through the projections $R \to R_k$.*

PROOF. We can write the identity in R as a sum of idempotents $1 = \Sigma_k e_k$, where e_k is the identity of R_k, and $e_k e_j = 0$ for $j \neq k$.

Then each element $w \in M$ can be written as

$$w = 1w = (\Sigma_k e_k)w = \Sigma_k e_k w =: \Sigma_k w_k.$$

Hence M_k is defined as $e_k M$. □

REMARK 5.9. 1) If we have a primary Bagnera-de Franchis variety, then Λ_2 is a module over the integral domain $R := R_m := \mathbb{Z}[x]/(P_m(x))$.

Since Λ_2 is a projective R-module, Λ_2 splits as the direct sum $\Lambda_2 = R^r \oplus I$ of a free module with an ideal $I \subset R$ (see [Mil71, Lemmas 1.5 and 1.6]), and Λ_2 is indeed free if the class number $h(R) = 1$. The integers m for which this occurs are listed in the table on [Was97, page 353].

2) To give a complex structure to $A_2 := (\Lambda_2 \otimes_{\mathbb{Z}} \mathbb{R})/\Lambda_2$ it suffices to give a decomposition $\Lambda_2 \otimes_{\mathbb{Z}} \mathbb{C} = V \oplus \overline{V}$, such that the action of x is holomorphic. This is equivalent to asking that V is a direct sum of eigenspaces V_λ, for $\lambda = \epsilon^j$ a primitive m-th root of unity.

Writing $U := \Lambda_2 \otimes_{\mathbb{Z}} \mathbb{C} = \oplus U_\lambda$, the desired decomposition is obtained by choosing, for each eigenvalue λ, a decomposition $U_\lambda = U_\lambda^{1,0} \oplus U_\lambda^{0,1}$ such that $\overline{U_\lambda^{1,0}} = U_{\overline{\lambda}}^{0,1}$.

The simplest case (see [CC93] for more details) is the one where $I = 0, r = 1$, hence $\dim(U_\lambda) = 1$. Therefore we have only a finite number of complex structures, depending on the choice of the $\frac{\varphi(m)}{2}$ indices j such that $U_{\epsilon^j} = U_{\epsilon^j}^{1,0}$ (here $\varphi(m)$ is the *Euler function*).

Observe that the classification of BdF varieties in small dimension is possible thanks to the observation that the $\mathbb{Z}$-rank of R (or of any ideal $I \subset R$) cannot exceed the real dimension of A_2: in other words we have

$$\varphi(m) \leq 2(n-1),$$

where $\varphi(m)$ is the Euler function, which is multiplicative for relatively prime numbers, and satisfies $\varphi(p^r) = (p-1)p^{r-1}$, if p is a prime number.

For instance, if $n \leq 3$, then $\varphi(m) \leq 4$. Observe that $\varphi(p^r) \leq 4$ iff

- $p = 3$, 5 and $r = 1$, or
- $p = 2$, $r \leq 3$.

Hence, for $n \leq 3$, the only possibilities for m are

- $\varphi(m) = 1$: $m = 2$;
- $\varphi(m) = 2$: $m = 3, 4, 6$;
- $\varphi(m) = 4$: $m = 5, 8, 10, 12$.

The classification is then also made easier by the fact that, in the above range for m, R_m is a P.I.D., hence every torsion free module is free. In particular Λ_2 is a free R-module.

The classification for $n = 4$, since we must have $\varphi(m) \leq 6$, is going to include also the case $m = 7, 9$.

We state now a result which will be useful in Section 6.

Proposition 5.10. *The Albanese variety of a Bagnera-de Franchis variety $X = A/G$ is the quotient $A_1/(T_1 + \langle\langle\beta_1\rangle\rangle)$.*

Proof. Observe that the Albanese variety $H^0(\Omega^1_X)^\vee / \operatorname{im}(H_1(X, \mathbb{Z}))$ of $X = A/G$ is a quotient of the vector space V_1 by the image of the fundamental group of X (actually of its abelianization, the first homology group $H_1(X, \mathbb{Z})$): since the dual of V_1 is the space of G-invariant forms on A, $H^0(\Omega^1_A)^G \cong H^0(\Omega^1_X)$.

We also observe that there is a well defined map $X \to A_1/(T_1 + \langle\langle\beta_1\rangle\rangle)$, since T_1 is the first projection of T. The image of the fundamental group of X contains the image of Λ, which is precisely the extension of Λ_1 by the image of T, namely T_1. Since we have the exact sequence

$$1 \longrightarrow \Lambda = \pi_1(A) \longrightarrow \pi_1(X) \longrightarrow G \longrightarrow 1$$

the image of the fundamental group of X is generated by the image of Λ and by the image of the transformation g, which however acts on A_1 by translation by $\beta_1 = [b_1]$. □

Remark 5.11. Unlike the case of complex dimension $n = 2$, there are Bagnera-de Franchis varieties $X = A/G$ with trivial canonical divisor, for instance an elementary example is given by any BdF variety which is standard (i.e., has $m = 2$) and is such that A_2 has even dimension.

5.3. Line bundles on quotients and linearizations

Recall the following well known result (see Mumford's books [Mum70], [Mum65]).

PROPOSITION 5.12. *Let $Y = X/G$ be a quotient algebraic variety and let $p\colon X \to Y$ be the quotient map. Then:*

(1) *there is a functor between*

- *line bundles $\mathcal{L}'$ on Y and*
- *G-linearized line bundles $\mathcal{L}$,*

associating to $\mathcal{L}'$ its pull back $p^*(\mathcal{L}')$.

(2) *The functor $\mathcal{L} \mapsto p_*(\mathcal{L})^G$ is a right inverse to the previous one, and $p_*(\mathcal{L})^G$ is invertible if the action is free, or if Y is smooth.*

(3) *Given a line bundle $\mathcal{L}$ on X, it admits a G-linearization if and only if there is a Cartier divisor D on X, which is G-invariant and such that $\mathcal{L} \cong \mathcal{O}_X(D) = \{f \in \mathbb{C}(X) | \operatorname{div}(f) + D \geq 0\}$.*

(4) *A necessary condition for the existence of a G-linearization on a line bundle $\mathcal{L}$ on X is that*

$$\forall g \in G,\ g^*(\mathcal{L}) \cong \mathcal{L}. \tag{5.1}$$

If condition (5.1) holds for $(\mathcal{L}, G)$, one defines the Theta group of $\mathcal{L}$ as:

$$\Theta(\mathcal{L}, G) := \{(\psi, g) | g \in G,\ \psi : g^*(\mathcal{L}) \to \mathcal{L} \text{ is an isomorphism}\},$$

and there is an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \Theta(\mathcal{L}, G) \longrightarrow G \longrightarrow 1. \tag{5.2}$$

- *The splittings of the above sequence correspond to the G-linearizations of $\mathcal{L}$.*
- *If the sequence splits, the linearizations are a principal homogeneous space over the dual group $\operatorname{Hom}(G, \mathbb{C}^*) =: G^*$ of G (namely, each linearization is obtained from a fixed one by multiplying with an arbitrary element in $\operatorname{Hom}(G, \mathbb{C}^*) =: G^*$).*

Thus, the question of the existence of a G-linearization on a line bundle $\mathcal{L}$ is reduced to the algebraic question of the splitting of the central extension (5.2) given by the Theta group. This question is addressed by group cohomology theory, as follows (for details see [Jac80]).

Corollary 5.13. *Let $\mathcal{L}$ be an invertible sheaf on X, whose class in* $\mathrm{Pic}(X)$ *is G-invariant. Then there exists a G-linearization of $\mathcal{L}$ if and only if the extension class $[\psi] \in H^2(G, \mathbb{C}^*)$ of the exact sequence (5.2) induced by the Theta group $\Theta(G, \mathcal{L})$ is trivial.*

The group $H^2(G, \mathbb{C}^*)$ is the group of *Schur multipliers* (see again [Jac80, page 369]).

Schur multipliers occur naturally when we have a projective representation of a group G. Since, if we have a homomorphism $\varphi\colon G \to \mathbb{P}\,\mathrm{GL}(r, \mathbb{C})$, we can pull back the central extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{GL}(r, \mathbb{C}) \longrightarrow \mathbb{P}\,\mathrm{GL}(r, \mathbb{C}) \longrightarrow 1$$

via φ, we obtain an exact sequence

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \hat{G} \longrightarrow G \longrightarrow 1,$$

and the extension class $[\psi] \in H^2(G, \mathbb{C}^*)$ is the obstruction to lifting the projective representation to a linear representation $G \to \mathrm{GL}(r, \mathbb{C})$.

It is an important remark that, if the group G is finite, and $n = \mathrm{ord}(G)$, then the cocycles take values in the group of roots of unity $\mu_n := \{z \in \mathbb{C}^* | z^n = 1\}$.

Remark 5.14. 1) Let E be an elliptic curve with origin O, and let G be the group of 2-torsion points $G := E[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$, acting by translations on E. The divisor class of $2O$ is never represented by a G-invariant divisor, since all the G-orbits consist of 4 points, and the degree of $2O$ is not divisible by 4. Hence, $\mathcal{L} := \mathcal{O}_E(2O)$ does not admit a G-linearization. However, we have a projective representation on $\mathbb{P}^1 = \mathbb{P}(H^0(\mathcal{O}_E(2O)))$, where each non zero element η_1 of the group fixes 2 divisors: the sum of the two points corresponding to $\pm\frac{\eta_1}{2}$, and its translate by another element $\eta_2 \in E[2]$.

The two group generators yield two linear transformations, which act on $V := H^0(\mathcal{O}_E(2O)) = \mathbb{C}x_0 \oplus \mathbb{C}x_1$ as follows:

$$\eta_1(x_0) = x_1,\ \eta_1(x_1) = x_0,\ \eta_2(x_j) = (-1)^j x_j.$$

The linear group generated is however $D_4 \neq G$, since

$$\eta_1\eta_2(x_0) = x_1,\ \eta_1\eta_2(x_1) = -x_0.$$

2) The previous example is indeed a special case of the *Heisenberg extension*, and V generalizes to the *Stone-von Neumann representation* associated to an Abelian group G.

This is simply the space $V := L^2(G, \mathbb{C})$ of square integrable functions on G (see [Igu72],[Mum70]):

- G acts on $V := L^2(G, \mathbb{C})$ by translation $f(x) \mapsto f(x - g)$,
- G^* acts on V by multiplication with the given character $f(x) \mapsto f(x) \cdot \chi(x)$, and
- the commutator $[g, \chi]$ acts on V by the scalar multiplication with the constant $\chi(g)$.

The Heisenberg group is the group of automorphisms of V generated by G, G^* and by $\mathbb{C}^*$ acting by scalar multiplication. Then there is a central extension

$$1 \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Heis}(G) \longrightarrow G \times G^* \longrightarrow 1,$$

whose class in $H^2(G \times G^*, \mathbb{C}^*)$ is given by the $\mathbb{C}^*$-valued bilinear form

$$\beta\colon (g, \chi) \mapsto \chi(g) \in \Lambda^2(\mathrm{Hom}(G \times G^*, \mathbb{C}^*)) \subset H^2(G \times G^*, \mathbb{C}^*).$$

The relation with Abelian varieties $A = V/\Lambda$ is through the Theta group associated to an ample divisor L.

In fact, by the theorem of Frobenius the alternating form $c_1(L) \in H^2(A, \mathbb{Z}) \cong \wedge^2(\mathrm{Hom}(\Lambda, \mathbb{Z}))$ admits, in a suitable basis of Λ, the normal form

$$D := \begin{pmatrix} 0 & D' \\ -D' & 0 \end{pmatrix}, \tag{5.3}$$

where $D' := \mathrm{diag}(d_1, d_2, \ldots, d_g)$, $d_1 \mid d_2 \mid \cdots \mid d_g$.

If one sets $G := \mathbb{Z}^g/D'\mathbb{Z}^g$, then L is invariant under $G \times G^* \cong G \times G \subset A$, acting by translation, and the Theta group of L is just isomorphic to the Heisenberg group $\mathrm{Heis}(G)$.

The nice part of the story is the following very useful result, which was used by Atiyah in the case of elliptic curves to study vector bundles on these (cf. [Ati57]). We give a proof even if the result is well known.

Proposition 5.15. *Let G be a finite Abelian group, and let $V := L^2(G, \mathbb{C})$ be the Stone-von Neumann representation. Then $V \otimes V^\vee$ is a representation of $G \times G^*$ and splits as the direct sum of all the characters of $G \times G^*$.*

Proof. Since the centre $\mathbb{C}^*$ of the Heisenberg group $\mathrm{Heis}(G)$ acts trivially on $V \otimes V^\vee$, we have that $V \otimes V^\vee$ is a representation of $G \times G^*$. Observe that $G \times G^*$ is equal to its group of characters, and its cardinality equals the dimension of $V \otimes V^\vee$, hence it suffices (and it will also be useful for applications) to write for each character of $G \times G^*$ an explicit eigenvector.

We shall use the letters g, h, k for elements of G, and the greek letters χ, η, ξ for elements in the dual group. Observe that V has two bases, one given by $\{g \in G\}$, and the other given by the characters $\{\chi \in G^*\}$. The *Fourier transform* $\mathcal{F}$ yields an isomorphism of the vector spaces $V := L^2(G, \mathbb{C})$ and $W := L^2(G^*, \mathbb{C})$:

$$\mathcal{F}(f) := \hat{f},\ \hat{f}(\chi) := \int f(g)(\chi, g)\ \mathrm{d}g.$$

The action of $h \in G$ on V sends $f(g) \mapsto f(g-h)$, hence for the characteristic functions in $\mathbb{C}[G]$, $h \in G$ acts as $g \mapsto g + h$. Instead $\eta \in G^*$ sends $f \mapsto f \cdot \eta$, hence $\chi \mapsto \chi + \eta$. Note that we use the additive notation also for the group of characters.

Restricting V to the *finite Heisenberg group*, which is a central extension of $G \times G^*$ by μ_n, we get a unitary representation, hence we identify $V^\vee$ with $\bar{V}$. Then a basis of $V \otimes \bar{V}$ is given by the set $\{g \otimes \bar{\chi}\}$.

Given a vector $w := \sum_{g,\chi} a_{g,\chi}(g \otimes \bar{\chi}) \in V \otimes \bar{V}$, then the action by $h \in G$ is given by

$$h(w) = \sum_{g,\chi} (\chi, h) a_{g-h,\chi}(g \otimes \bar{\chi}),$$

while the action by $\eta \in G^*$ is given by

$$\eta(w) = \sum_{g,\chi} (\eta, g) a_{g,\chi-\eta}(g \otimes \bar{\chi}).$$

Hence one verifies right away that

$$F_{k,\xi} := \sum_{g,\chi} (\chi - \xi, g - k)(g \otimes \bar{\chi})$$

is an eigenvector with character $(\xi, h)(\eta, k)$ for $(h, \eta) \in (G \times G^*)$. □

6. A Surface in a Bagnera-de Franchis Threefold

Let A_1 be an elliptic curve, and let A_2 be an Abelian surface together with a line bundle L_2 yielding a polarization of type $(1, 2)$.

Take on A_1 the line bundle $L_1 = \mathcal{O}_{A_1}(2O)$, and let L be the line bundle on $A' := A_1 \times A_2$, obtained as the exterior tensor product of L_1 and L_2, so that

$$H^0(A', L) = H^0(A_1, L_1) \otimes H^0(A_2, L_2).$$

Moreover, we choose the origin in A_2 such that the space of sections $H^0(A_2, L_2)$ consists only of *even* sections (hence, we shall no longer be free to further change the origin by an arbitrary translation).

We want to construct a Bagnera-de Franchis variety $X := A/G$, where

- $A = (A_1 \times A_2)/T$, and $G \cong T \cong \mathbb{Z}/2\mathbb{Z}$, such that
- there is a $G \times T$ invariant divisor $D \in |L|$, whence we get a surface $S = D/(T \times G) \subset X$, with $K_S^2 = \frac{1}{4}K_D^2 = \frac{1}{4}D^3 = 6$.

Write as usual $A_1 = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$, and let $A_2 = \mathbb{C}^2/\Lambda_2$. Suppose moreover, that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ is a basis of Λ_2 such that with respect to this basis the Chern class of L_2 is in Frobenius normal form. Let then $G = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$ act on $A_1 \times A_2$ by

$$g(a_1, a_2) := \left(a_1 + \frac{\tau}{2}, -a_2 + \frac{\lambda_2}{2} \right), \tag{6.1}$$

and define $T := (\mathbb{Z}/2\mathbb{Z})(\frac{1}{2}, \frac{\lambda_4}{2})$.

Now, $G \times T$ surjects onto the group of two torsion points $A_1[2]$ of the elliptic curve, and also on the subgroup $(\mathbb{Z}/2\mathbb{Z})(\lambda_2/2) \oplus (\mathbb{Z}/2)(\lambda_4/2) \subset A_2[2]$. Moreover, both $H^0(A_1, L_1)$ and $H^0(A_2, L_2)$ are the Stone-von Neumann representation of the finite Heisenberg group of G, which is a central $\mathbb{Z}/2\mathbb{Z}$ extension of $G \times T$.

By Proposition 5.15, since in this case $V \cong \overline{V}$ (the only roots of unity occurring are just ± 1), we conclude that there are exactly 4 divisors in $|L|$, invariant by:

- $(a_1, a_2) \mapsto (a_1, -a_2)$ (since the sections of L_2 are even),
- $(a_1, a_2) \mapsto \left(a_1 + \frac{\tau}{2}, a_2 + \frac{\lambda_2}{2}\right)$, and
- $(a_1, a_2) \mapsto \left(a_1 + \frac{1}{2}, a_2 + \frac{\lambda_4}{2}\right)$.

Hence these four divisors descend to give four surfaces $S_i \subset X$, $i \in \{1, 2, 3, 4\}$.

Theorem 6.1. *Let S be a minimal surface of general type with invariants $K_S^2 = 6$, $p_g(S) = q(S) = 1$ such that*

- *there exists an unramified double cover $\hat{S} \to S$ with $q(\hat{S}) = 3$, and such that*
- *the Albanese morphism $\hat{\alpha}\colon \hat{S} \to A = \mathrm{Alb}(\hat{S})$ is birational onto its image Z, a divisor in A with $Z^3 = 12$.*

1) Then the canonical model of $\hat{S}$ is isomorphic to Z, and the canonical model of S is isomorphic to $Y = Z/(\mathbb{Z}/2\mathbb{Z})$, which is a divisor in a Bagnera-de Franchis threefold $X := A/G$, where $A = (A_1 \times A_2)/T$, $G \cong T \cong \mathbb{Z}/2\mathbb{Z}$, and where the action is as in (6.1).

2) These surfaces exist, have an irreducible four dimensional moduli space, and their Albanese map $\alpha\colon S \to A_1 = A_1/A_1[2]$ has general fibre a non hyperelliptic curve of genus $g = 3$.

Proof. By assumption the Albanese map $\hat{\alpha}\colon \hat{S} \to A$ is birational onto Z, and we have $K_{\hat{S}}^2 = 12 = K_Z^2$, since $\mathcal{O}_Z(Z)$ is the dualizing sheaf of Z.

We shall argue similarly to [BC12, Step 4 of Theorem 0.5, page 31]. Denote by W the canonical model of $\hat{S}$, and observe that by adjunction (see loc. cit.) we have $K_W = \hat{\alpha}^*(K_Z) - \mathfrak{A}$, where $\mathfrak{A}$ is an effective $\mathbb{Q}$-Cartier divisor.

We observe now that K_Z and K_W are ample, hence we have an inequality,

$$12 = K_W^2 = (\hat{\alpha}^*(K_Z) - \mathfrak{A})^2 = K_Z^2 - (\hat{\alpha}^*(K_Z) \cdot \mathfrak{A}) - (K_W \cdot \mathfrak{A}) \geq K_Z^2 = 12,$$

and since both terms are equal to 12, we conclude that $\mathfrak{A} = 0$, which means that K_Z pulls back to K_W, whence W is isomorphic to Z. We have a covering involution $\iota\colon \hat{S} \to \hat{S}$, such that $S = \hat{S}/\iota$. Since the action of $\mathbb{Z}/2\mathbb{Z}$ is free on $\hat{S}$, $\mathbb{Z}/2\mathbb{Z}$ also acts freely on Z.

Since $Z^3 = 12$, Z is a divisor of type $(1,1,2)$ in A. The covering involution $\iota\colon \hat{S} \to \hat{S}$ can be lifted to an involution g of A, which we write as an affine transformation $g(a) = \alpha a + \beta$.

We have now Abelian subvarieties $A_1 = \ker(\alpha - \mathrm{Id})$, $A_2 = \ker(\alpha + \mathrm{Id})$, and since the irregularity of S equals 1, A_1 has dimension 1, and A_2 has dimension 2.

We observe preliminarly that g is fixed point free: since otherwise the fixed point locus would be non empty of dimension one (as there is exactly one eigenvalue equal to 1), so it would intersect the ample divisor Z, contradicting that $\iota\colon Z \to Z$ acts freely.

Therefore, $Y = Z/\iota$ is a divisor in the Bagnera-de Franchis threefold $X = A/G$, where G is the group of order two generated by g.

We can then write the Abelian threefold A as $(A_1 \times A_2)/T$, and since $\beta_1 \notin T_1$ (cf. Proposition 5.4) we have only two possible cases:

0) $T = 0$, or

1) $T \cong \mathbb{Z}/2\mathbb{Z}$.

We further observe that, since the divisor Z is g-invariant, its polarization is α invariant, in particular its Chern class $c \in \wedge^2(\mathrm{Hom}(\Lambda, \mathbb{Z}))$, where $A = V/\Lambda$. Since $T = \Lambda/(\Lambda_1 \oplus \Lambda_2)$, c pulls back to

$$c' \in \wedge^2(\mathrm{Hom}(\Lambda_1 \oplus \Lambda_2, \mathbb{Z})) = \wedge^2(\Lambda_1^\vee) \oplus \wedge^2(\Lambda_2^\vee) \oplus (\Lambda_1^\vee) \otimes (\Lambda_2^\vee),$$

and by invariance $c' = (c_1' \oplus c_2') \in \wedge^2(\Lambda_1^\vee) \oplus \wedge^2(\Lambda_2^\vee)$. So Case 0) bifurcates in the following cases:

0-I) c_1' is of type (1), c_2' is of type $(1,2)$;

0-II) c_1' is of type (2), c_2' is of type $(1,1)$.

Both cases can be discarded, since they lead to the same contradiction. Setting $D := Z$, then D is the divisor of zeros on $A = A_1 \times A_2$ of a section of a line bundle L which is an exterior tensor product of L_1 and L_2. Since

$$H^0(A, L) = H^0(A_1, L_1) \otimes H^0(A_2, L_2),$$

and $H^0(A_1, L_1)$ has dimension one in case 0-I), while $H^0(A_2, L_2)$ has dimension one in case 0-II), we conclude that D is a reducible divisor, a contradiction, since D is smooth and connected.

In case 1), we denote $A' := A_1 \times A_2$, and we let D be the inverse image of Z inside A'. Again D is smooth and connected, since $\pi_1(\hat{S})$ surjects onto Λ. Now $D^2 = 24$, so the Pfaffian of c' equals 4, and there are a priori several possibilities:

1-I) c'_1 is of type (1);

1-II) c'_2 is of type $(1, 1)$;

1-III) c'_1 is of type (2), c'_2 is of type $(1, 2)$.

The cases 1-I) and 1-II) can be excluded as case 0), since D would then be reducible.

We are then left only with case 1-III), and we may, without loss of generality, assume that $H^0(A_1, L_1) = H^0(A_1, \mathcal{O}_{A_1}(2O))$. Moreover, we have already assumed that we have chosen the origin so that all the sections of $H^0(A_2, L_2)$ are even.

We have $A = A'/T$, and we may write the generator of T as $t_1 \oplus t_2$, and write $g(a_1 \oplus a_2) = (a_1 + \beta_1) \oplus (a_2 - \beta_2)$.

By the description of Bagnera-de Franchis varieties (cf. Proposition 5.4) we have that t_1 and β_1 are a basis of the group of 2 torsion points of the elliptic curve A_1.

Since all sections of L_2 are even, the divisor D is $G \times T$-invariant if and only if it is invariant under T and under translation by β.

This condition however implies that translation of L_2 by β_2 is isomorphic to L_2, and similarly for t_2. It follows that β_2, t_2 form a basis of $K_2 := \ker(\phi_{L_2} \colon A_2 \to \mathrm{Pic}^0(A_2))$, where $\phi(y) = t_y L_2 \otimes L_1^{-1}$. The isomorphism of $G \times T$ with both $K_1 := A_1[2]$ and K_2 allows to identify both $H^0(A_1, L_1)$ and $H^0(A_2, L_2)$ with the Stone-von Neumann representation $L^2(T, \mathbb{C})$: observe in fact that there is only one alternating function $(G \times T) \to \mathbb{Z}/2\mathbb{Z}$, independent of the chosen basis.

Therefore, there are exactly 4 invariant divisors in the linear system $|L|$. Explicitly, if $H^0(A_1, L_1)$ has basis x_0, x_1 and $H^0(A_2, L_2)$ has basis y_0, y_1, then the invariant divisors correspond to the four eigenvectors

$$x_0y_0 + x_1y_1\,, \quad x_0y_0 - x_1y_1\,, \quad x_0y_1 + x_1y_0\,, \quad x_0y_1 - x_1y_0\,.$$

To prove irreducibility of the above family of surfaces, it suffices to show that all the four invariant divisors occur in the same connected family.

To this purpose, we just observe that the monodromy of the family of elliptic curves $E_\tau := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$ on the upper half plane has the effect that a transformation in $\mathrm{SL}(2,\mathbb{Z})$ acts on the subgroup $E_\tau[2]$ of points of 2-torsion by its image matrix in $\mathrm{GL}(2,\mathbb{Z}/2\mathbb{Z})$, and in turn the effect on the Stone-von Neumann representation is the one of twisting it by a character of $E_\tau[2]$.

This concludes the proof that the moduli space is irreducible of dimension 4, since the moduli space of elliptic curves, respectively the moduli space of Abelian surfaces with a polarization of type $(1,2)$, are irreducible, of respective dimensions 1, 3.

The final assertion is a consequence of the fact that $\mathrm{Alb}(S) = A_1/(T_1 + \langle\langle\beta_1\rangle\rangle)$, so that the fibres of the Albanese map are just divisors in A_2 of type $(1,2)$. Their self intersection equals $4 = 2(g-1)$, hence $g = 3$.

In order to establish that the general curve is non hyperelliptic, it suffices to prove the following lemma.

LEMMA 6.2. *Let A_2 be an Abelian surface, endowed with a divisor L of type $(1,2)$, so that there is an isogeny of degree two $f\colon A_2 \to A'$ onto a principally polarised Abelian surface, and $L = f^*(\Theta)$. Then the only curves $C \in |L|$ which are hyperelliptic are contained in the pull backs of a translate of Θ by a point of order 2 for a suitable such isogeny $f'\colon A_2 \to A''$. In particular, the general curve $C \in |L|$ is not hyperelliptic.*

PROOF. Note that A' is the quotient of A by an involution, given by translation with a two torsion element $t \in A[2]$. Let $C \in |L|$, and consider $D := f_*(C) \in |2\Theta|$. There are two cases:

I) $C + t = C$;

II) $C + t \neq C$.

In case I) $D = 2B$, where B has genus 2, so that $C = f^*(B)$, hence, since $2B \equiv 2\Theta$, B is a translate of Θ by a point of order 2. There are exactly two such curves, and for them $C \to B$ is étale.

In case II) the map $C \to D$ is birational, $f^*(D) = C \cup (C + t)$. Now, $C + t$ is also linearly equivalent to L, hence C and $C + t$ intersect in the 4

base points of the pencil $|L|$. Hence D has two double points and geometric genus equal to 3. These double points are the intersection points of Θ and a translate of Θ by a point of order 2, and are points of 2-torsion.

The sections of $H^0(\mathcal{O}_{A'}(2\Theta))$ are all even and $|2\Theta|$ is the pull-back of the space of hyperplane sections of the Kummer surface $\mathcal{K} \subset \mathbb{P}^3$, the quotient $\mathcal{K} = A'/\{\pm 1\}$.

Therefore, the image E' of each such curve D lies in the pencil of planes through 2 nodes of $\mathcal{K}$.

E' is a plane quartic, hence E' has geometric genus 1, and we conclude that C admits an involution σ with quotient an elliptic curve E (normalization of E'), and the double cover is branched in 4 points.

Assume that C is hyperelliptic, and denote by h the hyperelliptic involution, which lies in the centre of $\mathrm{Aut}(C)$. Hence we have $(\mathbb{Z}/2\mathbb{Z})^2$ acting on C, with quotient $\mathbb{P}^1$. We easily see that there are exactly six branch points, two being the branch points of $C/h \to \mathbb{P}^1$, four being the branch points of $E \to \mathbb{P}^1$. It follows that there is an étale quotient $C \to B$, where B is the genus 2 curve, double cover of $\mathbb{P}^1$ branched on the six points.

Now, the inclusion $C \subset A_2$ and the degree 2 map $C \to B$ induce a degree two isogeny $A_2 \to J(B)$, and C is the pull back of the Theta divisor of $J(B)$, thus it cannot be a general curve. □

This ends the proof of Theorem 6.1. □

We shall give the surfaces of Theorem 6.1 a name.

Definition 6.3. A minimal surface S of general type with invariants $K_S^2 = 6$, $p_g(S) = q(S) = 1$ such that

- there exists an unramified double cover $\hat{S} \to S$ with $q(\hat{S}) = 3$, and such that
- the Albanese morphism $\hat{\alpha}\colon \hat{S} \to A = \mathrm{Alb}(\hat{S})$ is birational onto its image Z, a divisor in A with $Z^3 = 12$,

is called a *Sicilian surface with* $q(S) = p_g(S) = 1$.

Remark 6.4. We have seen that the canonical model of a Sicilian surface S is an ample divisor in a Bagnera-de Franchis threefold $X = A/G$,

where $G = \langle g \rangle \cong \mathbb{Z}/2\mathbb{Z}$. Hence the fundamental group of S is isomorphic to the fundamental group Γ of X. Moreover, Γ fits into the exact sequence

$$1 \longrightarrow \Lambda \longrightarrow \Gamma \longrightarrow G = \mathbb{Z}/2\mathbb{Z} \longrightarrow 1\,,$$

and is generated by the union of the set $\{g, t\}$ with the set of translations by the elements of a basis $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of Λ_2, where

$$g(v_1 \oplus v_2) = \left(v_1 + \frac{\tau}{2}\right) \oplus \left(-v_2 + \frac{\lambda_2}{2}\right)$$
$$t(v_1 \oplus v_2) = \left(v_1 + \frac{1}{2}\right) \oplus \left(v_2 + \frac{\lambda_4}{2}\right).$$

Γ is therefore a semidirect product of $\mathbb{Z}^5 = \Lambda_2 \oplus \mathbb{Z}t$ with the infinite cyclic group generated by g: conjugation by g acts as -1 on Λ_2, and it sends $t \mapsto t - \lambda_4$ (hence $2t - \lambda_4$ is an eigenvector for the eigenvalue 1).

We shall now give a topological characterization of Sicilian surfaces with $q = p_g = 1$, following the lines of [BC12].

Observe in this respect that X is a $K(\Gamma, 1)$-space, so that its cohomology and homology are just group cohomology, respectively homology, of the group Γ.

Corollary 6.5. *A Sicilian surface S with $q(S) = p_g(S) = 1$ is characterized by the following properties:*

(1) $K_S^2 = 6$,

(2) $\chi(S) = 1$,

(3) $\pi_1(S) \cong \Gamma$, *where Γ is as above,*

(4) *the classifying map $f\colon S \to X$, where X is the Bagnera-de Franchis threefold which is a classifying space for Γ, has the property that $f_*[S] =: B$ satisfies $B^3 = 6$.*

In particular, any surface homotopically equivalent to a Sicilian surface is a Sicilian surface, and we get a connected component of the moduli space of surfaces of general type which is stable under the action of the absolute Galois group $Gal(\bar{\mathbb{Q}}, \mathbb{Q})$.

Proof. Since $\pi_1(S) \cong \Gamma$, first of all $q(S) = 1$, hence also $p_g(S) = 1$. By the same token there is a double étale cover $\hat{S} \to S$ such that $q(\hat{S}) = 3$, and the Albanese image of $\hat{S}$, counted with multiplicity, is the inverse image of B, therefore $Z^3 = 12$. From this, it follows that $\hat{S} \to Z$ is birational, since the class of Z is indivisible.

We may now apply the previous Theorem 6.1 in order to obtain the classification.

Observe finally that the condition $(\hat{\alpha}_* \hat{S})^3 = 12$ is not only a topological condition, it is also invariant under $Gal(\bar{\mathbb{Q}}, \mathbb{Q})$. □

7. Proof of the Main Theorems

We conclude in this last short section the proofs of Main Theorem 1 and Main Theorem 2.

Proof of Main Theorem 1. Statements 1), 2) and 3) summarize the contents of Proposition 3.4 and Theorem 3.6.

4) We observe preliminarly that our fundamental groups are virtually Abelian of rank 6 (i.e., they have a normal subgroup of finite index $\cong \mathbb{Z}^6$). By the results of [BCGP12], the fundamental group of (the minimal resolution of) a product-quotient surface has a finite index normal subgroup which is the product of at most two fundamental groups of curves. Therefore if it is virtually Abelian it has rank 2 or 4.

This argument excludes rightaway that our fundamental groups may be isomorphic to the fundamental groups of some product-quotient surfaces.

The only remaining case for $p_g = 0$ is the Kulikov surface, whose first homology group has 3-torsion.[1]

The known surfaces with $p_g = q = 1$ and $K^2 = 6$ are either product-quotient surfaces (cf. [Pol09]) or mixed quasi-étale surfaces, which are constructed in [FP15]. Comparing Table 2 from loc. cit with our Table 4, we see that they have different homology groups from ours.

5) is proved in Theorem 4.7. □

Proof of Main Theorem 2. The assertions 1) and 2) are contained in Theorem 6.1.

[1]Disclaimer: the fundamental group of the Inoue surface with $p_g = 0$, $K^2 = 6$ has not yet been calculated and we do not claim it is different from ours.

4) is contained in Corollary 6.5.

3) Observe that in cases $\mathcal{S}_{11}$ and $\mathcal{S}_{12}$ of Table 4 there is a subgroup $H \cong (\mathbb{Z}/2\mathbb{Z})^2$ acting by translations on $E_1 \times E_2 \times E_3$. Denote by $\hat{S}$ the quotient of the Burniat hypersurface by H. Then $\hat{S}$ is an étale double cover of the GBT S, which satisfies the defining property of Sicilian surfaces.

There remains to show that the other GBT surfaces (with $p_g = q = 1$) are not Sicilian surfaces. This is now obvious since they have fundamental groups non-isomorphic to $\pi_1(S_{11})$, where S_{11} belongs to the family $\mathcal{S}_{11}$. □

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