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On Coisotropic Deformations of Holomorphic Submanifolds

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Dedicated to the memory of Kunihiko Kodaira

Abstract. We describe the differential graded Lie algebras governing Poisson deformations of a holomorphic Poisson manifold and coisotropic embedded deformations of a coisotropic holomorphic submanifold. In both cases, under some mild additional assumption, we show that the infinitesimal first order deformations induced by the anchor map are unobstructed. Applications include the analog of Kodaira stability theorem for coisotropic deformation and a generalization of McLean-Voisin's theorem about the local moduli space of Lagrangian submanifold. Finally it is shown that our construction is homotopy equivalent to the homotopy Lie algebroid of Oh, Park, Cattaneo and Felder, in the cases where this is defined.

1. Introduction

The classical notion of coisotropic submanifold of a symplectic manifold extends immediately to the setup of Poisson geometry. More precisely, a closed submanifold $Z \subset X$ is called coisotropic if the ideal of Z is stable under the Poisson bracket. In recent years coisotropic submanifolds, and their cohomology, have received a lot of attention in view of their importance in mathematics and physics (see e.g. [4]). The definition of coisotropic submanifolds extends literally to the complex holomorphic case and more generally to every algebraic Poisson variety over a field.

The goal of this paper is to study deformation theory of holomorphic coisotropic submanifolds. The existing approach to deformations of differentiable coisotropic submanifolds, based on the notion of homotopy Lie

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algebroid of Oh, Park, Cattaneo and Felder [5, 6, 37], does not extend immediately to the holomorphic case since it relies on the identification of a neigbourhood with the total space of the normal bundle.

In the attempt of looking for a different and more general argument, we soon realized that the classical Kodaira's approach to deformation theory of manifolds and submanifolds [22, 23], enriched with the formalism of differential graded Lie algebras and Hinich's theorem on descent of Deligne groupoids [14], was very powerful and perfectly suitable to address the study of deformations of holomorphic Poisson manifolds and coisotropic submanifolds. Moreover, this approach is in great part purely algebraic and therefore most of the results of this paper also works for algebraic Poisson manifolds over a field of characteristic 0.

Given a holomorphic Poisson bivector π on a holomorphic manifold X, the Lichnerowicz-Poisson differential $d_\pi = [\pi, \cdot]_{SN}$ is a square zero operator on the sheaf of holomorphic polyvector fields on X. Then a closed submanifold $Z \subset X$ is coisotropic if and only if d_π factors to a differential on the exterior algebra of the normal sheaf $\mathcal{N}_{Z|X}$ of Z in X.

In the first part of the paper we show that the functor $\mathrm{Hilb}^{co}_{Z|X}$ of infinitesimal embedded coisotropic deformations of Z in X is governed by a differential graded Lie algebra K, explicitly described, having cohomology isomorphic to the hypercohomology of the complex of sheaves:

$$\bigwedge^{\geq 1}\mathcal{N}_{Z|X}\colon \qquad \mathcal{N}_{Z|X} \xrightarrow{d_\pi} \bigwedge^2 \mathcal{N}_{Z|X} \xrightarrow{d_\pi} \cdots ,$$

where $\bigwedge^i \mathcal{N}_{Z|X}$ is considered in degree i. In particular, the space of infinitesimal first order deformations is isomorphic to

$$\mathbb{H}^1(Z, \bigwedge^{\geq 1}\mathcal{N}_{Z|X}) = \ker(d_\pi\colon H^0(Z, \mathcal{N}_{Z|X}) \to H^0(Z, \bigwedge^2 \mathcal{N}_{Z|X}))$$

and there exists a complete obstruction theory with values in $\mathbb{H}^2(Z, \bigwedge^{\geq 1}\mathcal{N}_{Z|X})$. As a byproduct we also obtained the explicit description of two differential graded Lie algebras governing respectively the deformations of the pair (X, π), i.e., the deformations of X as a Poisson manifold and the deformations of the triple (X, Z, π).

In the second part we consider the effect of the anchor map $\pi^{\#}\colon \Omega^*_X \to \bigwedge^* \Theta_X$ on deformations of coisotropic submanifolds: by definition $\pi^{\#}$ is the unique morphism of sheaves of graded $\mathcal{O}_X$-algebras such that $\pi^{\#}(df) =$

$d_\pi(f) \in \Theta_X$ for every $f \in \mathcal{O}_X$. Very recently, N. Hitchin [16] has proved that, if X is a compact Kähler Poisson manifold, then every element in the image of $\pi^\#\colon H^1(X, \Omega^1_X) \to H^1(X, \Theta_X)$ is the Kodaira-Spencer class of a deformation of the pair (X, π) over a germ of smooth curve. Here we prove a similar statement for embedded coisotropic deformations: given a coisotropic submanifold $Z \subset X$, the anchor map factors to a morphism of sheaves of graded $\mathcal{O}_Z$-algebras $\pi^\#\colon \Omega^*_Z \to \bigwedge^* \mathcal{N}_{Z|X}$. If the Hodge to de Rham spectral sequence of Z degenerate at E_1, then every element in the image of $\pi^\#\colon H^0(Z, \Omega^1_Z) \to H^0(Z, \mathcal{N}_{Z|X})$ is the Kodaira-Spencer class of a coisotropic embedded deformation of Z in (X, π) over a germ of smooth curve. As a particular case, for a compact Kähler Lagrangian submanifold Z of a holomorphic symplectic manifold X we get that every small deformation in X of Z is Lagrangian and the Hilbert scheme of X is smooth at Z; when X is compact Kähler we recover in this way a classical result by Voisin and McLean [34, 40].

The underlying idea of proof, borrowed from [10], is to show that the anchor map is equivalent, in the homotopy category of differential graded Lie algebras, to a morphism $\pi^\#\colon J \to K$, where the cohomology of J is isomorphic to the hypercohomology of the complex of sheaves on Z

$$\Omega^{\geq 1}_Z\colon \qquad \Omega^1_Z \xrightarrow{d} \Omega^2_Z \xrightarrow{d} \cdots$$

where Ω^i_Z is considered in degree i. The formality criterion of [10] applies and, whenever the Hodge to de Rham spectral sequence of Z degenerates at E_1, the differential graded Lie algebra J is homotopy abelian, hence governing unobstructed deformations.

In the last part of the paper we compare our construction with the homotopy Lie algebroid in the situation where the latter may be defined. As expected, the two constructions are homotopy equivalent and then, according to the general principles of derived deformation theory, they define the same deformation problem, both classical and extended. It is worth to mention here that our construction provides the basic data for the local study of the extended moduli space of coisotropic submanifolds, extending the Lagrangian case carried out by Merkulov [35].

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2. Review of Deligne Groupoids and Totalization

Denote by **Set** the category of sets (in a fixed universe) and by **Grpd** the category of groupoids; we shall consider, in the obvious way, **Set** as a full subcategory of **Grpd**.

Given a field $\mathbb{K}$ we shall denote by $\mathbf{Art}_{\mathbb{K}}$ the category of local Artin $\mathbb{K}$-algebras with residue field $\mathbb{K}$. Unless otherwise specified, for every object $A \in \mathbf{Art}_{\mathbb{K}}$, we denote by $\mathfrak{m}_A$ its maximal ideal.

In order to simplify the terminology, by a formal pointed groupoid we shall mean a covariant functor $\mathcal{F}\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grpd}$ such that $\mathcal{F}(\mathbb{K}) = *$ is the one-point set. Similarly a formal pointed set is a functor $F\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}$ such that $F(\mathbb{K}) = *$, also called a functor of Artin rings. A morphism of formal pointed groupoid $\eta\colon \mathcal{F} \to \mathcal{G}$ is called an equivalence if $\mathcal{F}(A) \to \mathcal{G}(A)$ is an equivalence of groupoids for every $A \in \mathbf{Art}_{\mathbb{K}}$.

It is a nowadays standard to consider the pout of view that every deformation problem over a field of characteristic 0 is controlled by a differential graded Lie algebra (DGLA); we refer to the existing literature and in particular to [13, 25, 33] for the definition and main properties of differential graded Lie algebras, L_∞-algebras, Maurer-Cartan equation and gauge action. Later in this paper we also need to work with $L_\infty[1]$-algebras, i.e., desuspension of L_∞-algebras: we refer to [11] for a nice and clear introduction to these structures.

The Deligne groupoid of a differential graded Lie algebra L over a field $\mathbb{K}$ of characteristic 0 is the formal pointed groupoid

$$\mathrm{Del}_L\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Grpd}$$

defined in the following way [7, 13]: given $A \in \mathbf{Art}_{\mathbb{K}}$ the objects of $\mathrm{Del}_L(A)$ are the solutions to the Maurer-Cartan equation in $L \otimes \mathfrak{m}_A$:

$$\mathrm{Objects}(\mathrm{Del}_L(A)) = \left\{ x \in L^1 \otimes \mathfrak{m}_A \;\middle|\; dx + \frac{1}{2}[x,x] = 0 \right\}.$$

Given two objects x, y of $\mathrm{Del}_L(A)$, the morphisms between them are

$$\mathrm{Mor}_{\mathrm{Del}_L(A)}(x,y) = \{e^a \in \exp(L^0 \otimes \mathfrak{m}_A) \mid e^a * x = y\},$$

where $*$ is the gauge action.

The deformation functor associated to a differential graded Lie algebra L is the π_0 of the Deligne groupoid:

$$\operatorname{Def}_L\colon \mathbf{Art}_{\mathbb{K}} \to \mathbf{Set}, \qquad \operatorname{Def}_L(A) = \pi_0(\operatorname{Del}_L(A)).$$

The tangent space $T^1 \operatorname{Def}_L$ of the functor Def_L is isomorphic to the cohomology group $H^1(L)$. The homotopy invariance of Del and Def is summarized by the following result.

THEOREM 2.1. *Let $L \to M$ be a quasi-isomorphism of differential graded Lie algebras. Then:*

(1) *the induced natural transformation $\operatorname{Def}_L \to \operatorname{Def}_M$ is an isomorphism;*

(2) *if L and M are positively graded, i.e., $L^i = M^i = 0$ for every $i < 0$, then the induced natural transformation $\operatorname{Del}_L \to \operatorname{Del}_M$ is an equivalence.*

PROOF. The second item is one of the main results of [13]. The first item is proved in [25] via homotopy classification of L_∞-algebras, in [24, 29] via reduced Deligne groupoid and in [30] via extended deformation functors. □

As usual, we consider on the category of differential graded Lie algebras the homotopy theory induced by the standard model structure, where weak equivalences are the quasi-isomorphisms and fibrations are the surjective maps. In particular a differential graded Lie algebra is homotopy abelian if it is quasi-isomorphic to an abelian DGLA.

The Theorem 2.1 immediately implies that the functor Def_L is unobstructed whenever L is homotopy abelian. It is plain that if L is homotopy abelian, then its cohomology $H^*(L)$ is an abelian graded Lie algebra, while the converse is generally false. It is not difficult to give examples where $H^*(L)$ is abelian and the functor Def_L is obstructed, for instance by taking the Kodaira-Spencer DGLA of a surface of general type whose Kuranishi space is not defined by quadratic equations.

Every morphism $f\colon L \to M$ of differential graded Lie algebras has a canonical representative for its homotopy fiber: is defined as the DGLA

$$K(f) = \{(l, m(t)) \in L \times M[t, dt] \mid e_0(m(t)) = 0,\ e_1(m(t)) = f(l)\}.$$

Notice that the projection $K(f) \to L$ is a morphism of DGLA. Notice that $K(f)$ is also a homotopy fiber in the category of differential graded vector spaces and then it is quasi-isomorphic, as a complex, to the mapping cocone of f.

Let $\Delta_{\rm mon}$ be the category of finite ordinal, with strictly increasing maps. Following the terminology of [43], a semicosimplicial object in a category $\mathbf{C}$ is a covariant functor $A_\bullet \colon \Delta_{\rm mon} \to \mathbf{C}$. Equivalently, a semicosimplicial object is a diagram in $\mathbf{C}$:

$$A_\bullet: \qquad A_0 \rightrightarrows A_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} A_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots,$$

where each A_i is in $\mathbf{C}$, and, for each $i > 0$, there are $i+1$ morphisms

$$\partial_k \colon A_{i-1} \to A_i, \qquad k = 0, \dots, i,$$

such that $\partial_l \partial_k = \partial_{k+1}\partial_l$, for any $l \le k$.

In this paper we need to consider semicosimplicial groupoids and semicosimplicial differential graded Lie algebras. In both cases we can perform the totalization construction; let's first consider the case of groupoids.

Given a semicosimplicial groupoid

$$\mathcal{G}_\bullet: \qquad \mathcal{G}_0 \rightrightarrows \mathcal{G}_1 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \mathcal{G}_2 \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow} \cdots$$

the groupoid $\operatorname{Tot}(\mathcal{G}_\bullet)$, also called *groupoid of descent data*, is defined in the following way [14, 19]:

(1) The objects of $\operatorname{Tot}(\mathcal{G}_\bullet)$ are the pairs (l, m) with l an object in $\mathcal{G}_0$ and m a morphism in $\mathcal{G}_1$ between $\partial_0 l$ and $\partial_1 l$; we require that the three images of m via the maps ∂_i are the edges of a 2-simplex in the nerve of $\mathcal{G}_2$, i.e.,

$$(\partial_0 m)(\partial_1 m)^{-1}(\partial_2 m) = 1 \text{ in } \operatorname{Mor}_{\mathcal{G}_2}(\partial_2\partial_0 l, \partial_2\partial_0 l).$$

(2) The morphisms between (l_0, m_0) and (l_1, m_1) are the morphisms $a \in \operatorname{Mor}_{\mathcal{G}_0}(l_0, l_1)$ making the diagram

$$\begin{array}{ccc} \partial_0 l_0 & \xrightarrow{m_0} & \partial_1 l_0 \\ {\scriptstyle \partial_0 a}\downarrow & & \downarrow{\scriptstyle \partial_1 a} \\ \partial_0 l_1 & \xrightarrow{m_1} & \partial_1 l_1 \end{array}$$

commutative in $\mathcal{G}_1$.

The totalization of semicosimplicial groupoids is functorial and commutes with equivalences: more precisely, if $\gamma\colon \mathcal{F}_\bullet \to \mathcal{G}_\bullet$ is a morphism of semicosimplicial groupoids, then $\operatorname{Tot}(\gamma)\colon \operatorname{Tot}(\mathcal{F}_\bullet) \to \operatorname{Tot}(\mathcal{G}_\bullet)$ is a morphism of groupoids and, if every $\gamma_n\colon \mathcal{F}_n \to \mathcal{G}_n$ is an equivalence of groupoids, then also $\operatorname{Tot}(\gamma)$ is an equivalence of groupoids.

Let

$$V_\bullet\colon \qquad V_0 \rightrightarrows V_1 \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} V_2 \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \cdots,$$

be a semicosimplicial DG-vector space, with face operators $\partial_i\colon V_n \to V_{n+1}$. Then the graded vector space $C(V_\bullet) = \prod_{n\geq 0} V_n[-n]$ carries the two differentials

$$d = \prod_n d_{V_n[-n]} = \prod_n (-1)^n d_{V_n} \quad \text{and} \quad \partial = \sum_i (-1)^i \partial_i \,.$$

More explicitly, if $v \in V_n^i$, then $d(v) = (-1)^n d_{V_n}(v) \in V_n^{i+1}$ and $\partial(v) = \partial_0(v) - \partial_1(v) + \cdots + (-1)^{n+1}\partial_{n+1}(v) \in V_{n+1}^i$. Since $d^2 = \partial^2 = d\partial + \partial d = 0$ we may define the cochain complex of $V_\bullet$ as the differential graded vector space $C(V_\bullet)$ equipped with the differential $d + \partial$.

Let Ω_n be the polynomial de Rham algebra of the standard n-dimensional simplex. In other words, Ω_n is the polynomial DG-algebra generated by $t_0, \dots, t_n$ of degree zero and $dt_0, \dots, dt_n$ of degree one subject to the relations $t_0 + \cdots + t_n = 1$ and $dt_0 + \cdots + dt_n = 0$. The entire collection $\{\Omega_n\}$, $n \geq 0$, has a natural structure of simplicial DG-algebras, where the face map $\partial_i\colon [n-1] \to [n]$ induces the morphism of DG-algebras

$$\partial_i^*\colon \Omega_n \to \Omega_{n-1}, \qquad \partial_i^* t_k = \begin{cases} t_k & \text{for } k < i \\ 0 & \text{for } k = i \\ t_{k-1} & \text{for } k > i\,, \end{cases} \qquad \partial_i^*(dt_k) = d(\partial_i^* t_k)\,.$$

Definition 2.2. The (Thom-Whitney-Sullivan) totalization of a semicosimplicial DG-vector space

$$V_\bullet\colon \qquad V_0 \rightrightarrows V_1 \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} V_2 \substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow} \cdots,$$

is defined as

$$\operatorname{Tot}(V_\bullet)$$
$$=\left\{(x_n)\in\prod_{n\geq 0}\Omega_n\otimes V_n \;\middle|\; (\partial_k^*\otimes Id)x_n=(Id\otimes\partial_k)x_{n-1}\,,\quad \forall\, 0\leq k\leq n\right\}.$$

The functor Tot is exact: given a sequence $0\to V_\bullet\to W_\bullet\to U_\bullet\to 0$ of semicosimplicial DG-vector spaces such that $0\to V_n\to W_n\to U_n\to 0$ is exact for every n, then also $0\to \operatorname{Tot}(V_\bullet)\to\operatorname{Tot}(W_\bullet)\to\operatorname{Tot}(U_\bullet)\to 0$ is exact. This follows immediately from the definition and the simplicial contractibility of the simplicial DG-algebra $\Omega_\bullet$ [2, Prop. 1.1].

By Stokes formula, the *Whitney integration map*

$$I\colon\ \operatorname{Tot}(V_\bullet)\to C(V_\bullet)\,,$$

defined componentwise as

$$\operatorname{Tot}(V_\bullet)^p\xrightarrow{\text{inclusion}}\prod_{n\geq 0}\Big(\bigoplus\nolimits_i\Omega_n^{n-i}\otimes V_n^{p-n+i}\Big)\xrightarrow{\prod_n\int_{\Delta^n}\otimes Id_{V_n}}\prod_n V_n^{p-n}$$
$$=C(V_\bullet)^p\,,$$

is a surjective morphism of DG-vector spaces and, by a theorem of Whitney, Thom, Sullivan and Dupont (see e.g. [12, 36] for a proof), it is also a quasi-isomorphism.

Example 2.3. Let $\mathcal{F}^*$ be a bounded below complex of coherent sheaves on a complex manifold X. Given a open Stein covering $\mathcal{U}=\{U_i\}$ of X we can consider the semicosimplicial DG-vector space

$$\mathcal{F}^*(\mathcal{U})_\bullet\colon\qquad \prod_i\mathcal{F}^*(U_i)\rightrightarrows\prod_{i,j}\mathcal{F}^*(U_{ij})\;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\;\prod_{i,j,k}\mathcal{F}^*(U_{ijk})\cdots$$

and then for every integer p we have an isomorphism

$$H^p(\operatorname{Tot}(\mathcal{F}^*(\mathcal{U})_\bullet))\simeq H^p(C(\mathcal{F}^*(\mathcal{U})_\bullet))=\mathbb{H}^p(X,\mathcal{F}^*),$$

where $\mathbb{H}^*$ denotes the hypercohomology groups.

When $A_\bullet$ is a semicosimplicial algebra (either associative or Lie), then $\mathrm{Tot}(A_\bullet)$ inherits a natural structure of algebra and, via the quasi-isomorphism I, this structure induces in the cohomology of $C(A_\bullet)$ not only the cup products, but also the higher Massey products. Morally, the totalization is the smallest natural differential graded multiplicative structure giving the correct cohomology of a semicosimplicial algebra [38, pag. 300].

We are now ready to recall Hinich's theorem on descent of Deligne groupoids.

THEOREM 2.4 (Descent of Deligne groupoids, [14]). *Let*

$$L_\bullet : \qquad L_0 \rightrightarrows L_1 \rightrightarrows\!\!\!\!\rightarrow L_2 \rightrightarrows\!\!\!\!\rightrightarrows \cdots ,$$

be a semicosimplicial differential graded Lie algebra concentrated in positive degrees, i.e., $L_i^j = 0$ *for every* i *and every* $j < 0$*. Then there exists a natural equivalence of formal pointed groupoids*

$$\mathrm{Del}_{\mathrm{Tot}(L_\bullet)} \to \mathrm{Tot}(\mathrm{Del}_{L_\bullet}) ,$$

where $\mathrm{Del}_{L_\bullet}$ *is the semicosimplicial formal groupoid*

$$\mathrm{Del}_{L_\bullet} : \qquad \mathrm{Del}_{L_0} \rightrightarrows \mathrm{Del}_{L_1} \rightrightarrows\!\!\!\!\rightarrow \mathrm{Del}_{L_2} \rightrightarrows\!\!\!\!\rightrightarrows \cdots \quad .$$

Example 2.5. The simplest non trivial application of Theorem 2.4 describes a "small" model for the Deligne groupoid of the homotopy fiber of a morphism $\chi \colon L \to M$ of positively graded differential graded Lie algebras. In fact, the morphism χ gives the semicosimplicial DGLA

$$\chi_\bullet : \qquad L \underset{\partial_1 = 0}{\overset{\partial_0 = \chi}{\rightrightarrows}} M \rightrightarrows\!\!\!\!\rightarrow 0 \rightrightarrows\!\!\!\!\rightrightarrows \cdots ,$$

and, since $\Omega_1 \simeq \mathbb{K}[t, dt]$ we have $K(\chi) \simeq \mathrm{Tot}(\chi_\bullet)$; therefore there exists a natural equivalence of formal groupoids

$$\mathrm{Del}_{K(\chi)} \to \mathrm{Tot}\left(\mathrm{Del}_L \underset{0}{\overset{\chi}{\rightrightarrows}} \mathrm{Del}_M \right) .$$

Here, for every $A \in \mathbf{Art}_{\mathbb{K}}$, the objects of $\mathrm{Tot}\left(\mathrm{Del}_L \underset{0}{\overset{\chi}{\rightrightarrows}} \mathrm{Del}_M \right)(A)$ are the pairs (x, e^a), where x is a solution to Maurer-Cartan equation in $L \otimes \mathfrak{m}_A$ and $e^a \in \exp(M^0 \otimes \mathfrak{m}_A)$ satisfies the equation $e^a * \chi(x) = 0$.

A morphism between two objects (x, e^a) and (y, e^b) is an element $e^\alpha \in \exp(L^0 \otimes \mathfrak{m}_A)$ such that $e^\alpha * x = y$ and $e^b = e^a e^{-\chi(\alpha)}$.

If L is a differential graded Lie subalgebra of M and χ is the inclusion map, then the objects of the total groupoid are in bijection with the elements $e^a \in \exp(M^0 \otimes \mathfrak{m}_A)$ such that $e^{-a} * 0 \in L \otimes \mathfrak{m}_A$; moreover, in this particular case the natural transformation

$$\mathrm{Tot}\left(\mathrm{Del}_L \underset{0}{\overset{\chi}{\rightrightarrows}} \mathrm{Del}_M \right) \to \pi_0 \left(\mathrm{Tot}\left(\mathrm{Del}_L \underset{0}{\overset{\chi}{\rightrightarrows}} \mathrm{Del}_M \right)\right)$$

is an equivalence of formal groupoids.

We conclude this section with some remarks that will be useful in this paper.

Remark 2.6. Let $f\colon L_\bullet \to M_\bullet$ be a morphism of semicosimplicial differential graded Lie algebras: this is given by a sequence of morphisms of DGLA $f_n\colon L_n \to M_n$ commuting with face operators. Taking homotopy fibers we get a semicosimplicial DGLA $K(f)_\bullet$, where $K(f)_n = K(f_n)$ is the homotopy fiber of f_n.

It is easy to see that the homotopy fiber of the morphism $f\colon \mathrm{Tot}(L_\bullet) \to \mathrm{Tot}(M_\bullet)$ is naturally isomorphic to $\mathrm{Tot}(K(f)_\bullet)$; we refer to [18] for a detailed proof.

Remark 2.7. Let $L_\bullet$ be a semicosimplicial differential graded Lie algebra and denote by $H = \{x \in L_0 \mid \partial_0 x = \partial_1 x\}$ the equalizer of $\partial_0, \partial_1\colon L_0 \to L_1$. Then the map

$$e\colon H \to \mathrm{Tot}(L_\bullet), \qquad e(x) = (1 \otimes x, 1 \otimes \partial_0 x, 1 \otimes \partial_0^2 x, 1 \otimes \partial_0^3 x, \dots),$$

is a well defined morphism of differential graded Lie algebras. Moreover, the composition of e with the quasi-isomorphism $I\colon \mathrm{Tot}(L_\bullet) \to C(L_\bullet)$ is the natural inclusion $i\colon H \hookrightarrow C(L_\bullet)$, cf. [3, 19]. In particular e is a quasi-isomorphism if and only if i is a quasi-isomorphism.

3. Poisson Manifolds and Anchor Maps

Let X be a complex manifold and denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on X and by Θ_X the holomorphic tangent sheaf; also denote by

$\bigwedge^* \Theta_X = \bigoplus_{i\geq 0} \bigwedge^i_{\mathcal{O}_X} \Theta_X[-i]$ the sheaf of holomorphic polyvector fields and by (Ω^*_X, d) the sheaf of holomorphic differential forms on X, the last one with the usual structure of sheaf of differential graded algebras (DGAs for short). Following the notation of [10], given a polyvector field $\eta \in \bigwedge^i \Theta_X(U)$ we denote by

$$\boldsymbol{i}_\eta \colon \Omega^*_X(U) \to \Omega^{*-i}_X(U), \qquad \boldsymbol{i}_\eta(\alpha) = \eta \lrcorner \alpha,$$

the corresponding interior product operator; here we adopt the convention that $\boldsymbol{i}_{\alpha\wedge\beta} = \boldsymbol{i}_\alpha \circ \boldsymbol{i}_\beta$. Moreover we shall denote by $\boldsymbol{l}_\eta = [\boldsymbol{i}_\eta, d] \colon \Omega^*_X(U) \to \Omega^{*-i+1}_X(U)$ the holomorphic Lie derivative on differential forms.

The sheaf of graded algebras $\bigwedge^* \Theta_X$ carries also a structure of sheaf of Gerstenhaber algebras [31], equipped with the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN}$, see e.g. [39]. Recall that

$$[\cdot,\cdot]_{SN} : \bigwedge^i \Theta_X \otimes \bigwedge^j \Theta_X \to \bigwedge^{i+j-1} \Theta_X$$

is uniquely defined so that $[\eta,\xi]_{SN}$ is the usual bracket for $\eta, \xi \in \Theta_X$, while for $\eta \in \Theta_X$ and $f \in \bigwedge^0 \Theta_X = \mathcal{O}_X$ we have $[\eta, f]_{SN} = \eta(f) = \eta \lrcorner df$. Notice that for $\eta \in \Theta_X$ the operator $[\eta, \cdot]_{SN}$ is the Lie derivative with respect to η.

A holomorphic Poisson bivector on X is a section $\pi \in H^0(X, \bigwedge^2 \Theta_X)$ satisfying the integrability condition:

$$[\pi, \pi]_{SN} = 0\ . \tag{3.1}$$

A holomorphic Poisson manifold is a pair (X, π) consisting of a complex manifold X and a holomorphic Poisson bivector π on X.

For instance, if $A \subseteq H^0(X, \Theta_X)$ is an abelian Lie subalgebra, then every element in the image of $\bigwedge^2 A \to H^0(X, \bigwedge^2 \Theta_X)$ is a Poisson bivector; in particular every toric manifold of dimension $n \geq 2$ admits non trivial Poisson bivectors.

Every holomorphic symplectic manifold (X, ω) is a holomorphic Poisson manifold, where the Poisson bivector π is uniquely determined by the condition

$$\boldsymbol{i}_\pi(\boldsymbol{i}_\eta(\omega) \wedge \alpha) = \boldsymbol{i}_\eta(\alpha), \qquad \eta \in \Theta_X, \qquad \alpha \in \Omega^1_X.$$

The datum of a holomorphic Poisson bivector π on X induces several additional structures, cf. [27, 39]:

1) the *Lichnerowicz-Poisson differential* $d_\pi = [\pi, \cdot]_{SN} \colon \bigwedge^* \Theta_X \to \bigwedge^{*+1} \Theta_X$ inducing on $\bigwedge^* \Theta_X$ the structure of sheaf of differential Gerstenhaber algebras.

2) the *Poisson bracket* $\{\cdot,\cdot\}_\pi \colon \mathcal{O}_X \times \mathcal{O}_X \to \mathcal{O}_X$ given by

$$\{f, g\}_\pi = [[\pi, f]_{SN}, g]_{SN} = [d_\pi f, g]_{SN} = \boldsymbol{i}_\pi(df \wedge dg)\,.$$

It is well known that the integrability condition (3.1) is equivalent to the Jacobi identity for $\{\cdot,\cdot\}_\pi$.

3) the *Koszul bracket* $[\cdot,\cdot]_\pi \colon \Omega^i_X \otimes \Omega^j_X \to \Omega^{i+j-1}_X$, defined by the formula:

$$[\alpha, \beta]_\pi := (-1)^i(\boldsymbol{l}_\pi(\alpha \wedge \beta) - \boldsymbol{l}_\pi(\alpha) \wedge \beta) - \alpha \wedge \boldsymbol{l}_\pi(\beta), \qquad \alpha \in \Omega^i_X, \tag{3.2}$$

inducing on (Ω^*_X, d) the structure of a sheaf of differential Gerstenhaber algebras (this is shown for instance in [10, 28]).

4) the *anchor map* $\pi^\# \colon \Omega^*_X \to \bigwedge^* \Theta_X$: this is defined for $\alpha \in \Omega^1_X$ by the formula

$$\pi^\#(\alpha)(f) = \pi^\#(\alpha) \lrcorner df = \boldsymbol{i}_\pi(\alpha \wedge df), \qquad f \in \mathcal{O}_X, \tag{3.3}$$

and then uniquely extended to an $\mathcal{O}_X$-linear morphism of sheaves of graded algebras. It is well known [39], and in any case easy to prove, that

$$\pi^\# \colon (\Omega^*_X, \wedge, [\cdot,\cdot]_\pi, \partial) \to (\bigwedge^* \Theta_X, \wedge, [\cdot,\cdot]_{SN}, d_\pi)$$

is a morphism of sheaves of differential Gerstenhaber algebras.

4. Deformations of Holomorphic Poisson Manifolds

Let X be a complex manifold, a deformation of X over $A \in \mathbf{Art}_\mathbb{C}$ is a pull-back diagram of complex spaces:

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & \mathcal{X} \\ \downarrow & & \downarrow p \\ \operatorname{Spec} \mathbb{C} & \longrightarrow & \operatorname{Spec} A \end{array} \tag{4.1}$$

with p a smooth morphism; equivalently, it is the data of a sheaf $\mathcal{O}_\mathcal{X}$ of flat unitary A-algebras on X together with a morphism of sheaves of A-algebras $\mathcal{O}_\mathcal{X} \to \mathcal{O}_X$ which is locally isomorphic to the natural projection

$\mathcal{O}_X \otimes A \to \mathcal{O}_X$. Equivalences between deformations $\mathcal{X}_0$ and $\mathcal{X}_1$ of X over A are isomorphisms of sheaves of A-algebras $\mathcal{O}_{\mathcal{X}_0} \to \mathcal{O}_{\mathcal{X}_1}$ over $\mathcal{O}_X$; the self equivalences of the trivial deformation $\mathcal{O}_X \otimes A$ form a group canonically isomorphic to $\exp(H^0(X, \Theta_X) \otimes \mathfrak{m}_A)$, being the isomorphism the usual exponential of derivations.

Given $\{U_i\}$ an open covering of X by Stein open sets, every deformation $\mathcal{X}$ trivializes globally over each U_i, so that it can be reconstructed up to isomorphism by gluing the trivial deformations $\mathcal{O}_{U_i} \otimes A \to \mathcal{O}_{U_i}$ along double intersections via a family of transition automorphisms $e^{\eta_{ij}} \in \exp(\Theta_X(U_{ij}) \otimes \mathfrak{m}_A)$ satisfying the cocycle condition $e^{\eta_{ij}} e^{\eta_{jk}} = e^{\eta_{ik}} \in \exp(\Theta_X(U_{ijk}) \otimes \mathfrak{m}_A)$ on triple intersections (equivalently $\eta_{ij} \bullet \eta_{jk} = \eta_{ik}$, where $\bullet$ is the Baker-Campbell-Hausdorff product in the nilpotent Lie algebra $\Theta_X(U_{ijk}) \otimes \mathfrak{m}_A$).

Consider the sheaf of $\mathcal{O}_{\mathcal{X}}$-modules $\Theta_{\mathcal{X}/A}$ of A-linear derivations of $\mathcal{O}_{\mathcal{X}}$; as in the previous section the natural structure of sheaf of Lie algebras on $\Theta_{\mathcal{X}/A}$ extends to a sheaf of Gerstenhaber algebras structure on $\bigwedge^* \Theta_{\mathcal{X}/A} = \bigoplus_{i \geq 0} \bigwedge^i_{\mathcal{O}_{\mathcal{X}}} \Theta_{\mathcal{X}/A}[-i]$ via the Schouten-Nijenhuis bracket; notice that if $U \subset X$ is a Stein open subset, then $\bigwedge^* \Theta_{\mathcal{X}/A}(U) \cong \bigwedge^* \Theta_X(U) \otimes A$ with the Gersthenaber algebra structure given by scalar extension with A. Let as before $\{U_i\}$ be a covering of X by Stein open sets and let $\{e^{\eta_{ij}}\}$ be the set of transition automorphisms, where $\eta_{ij} \in \Theta_X(U_{ij}) \otimes \mathfrak{m}_A \subset \Theta_{\mathcal{X}/A}(U_{ij})$. The adjoint operators

$$\mathrm{ad}\ \eta_{ij} = [\eta_{ij}, \cdot]_{SN} \colon \bigwedge^* \Theta_{\mathcal{X}/A}(U_{ij}) \to \bigwedge^* \Theta_{\mathcal{X}/A}(U_{ij})$$

are degree zero nilpotent Gerstenhaber derivations and their exponentials $e^{\mathrm{ad}\ \eta_{ij}}$ are the transition automorphisms for the sheaf $\bigwedge^* \Theta_{\mathcal{X}/A}$: in fact it suffices to check this for $f \in \mathcal{O}_{\mathcal{X}}(U_{ij}) = \bigwedge^0 \Theta_{\mathcal{X}/A}(U_{ij})$ where $e^{\mathrm{ad}\ \eta_{ij}}(f) = e^{\eta_{ij}}(f)$ and for $\xi \in \Theta_{\mathcal{X}/A}(U_{ij})$ where $e^{\mathrm{ad}\ \eta_{ij}}(\xi) = e^{\eta_{ij}} \circ \xi \circ e^{-\eta_{ij}}$. In the same way an equivalence between deformations $\mathcal{X}_0$ and $\mathcal{X}_1$ of X over A induces an isomorphism $\bigwedge^* \Theta_{\mathcal{X}_0/A} \to \bigwedge^* \Theta_{\mathcal{X}_1/A}$; if the equivalence is locally given by the isomorphisms

$$\mathcal{O}_{\mathcal{X}_0}(U_i) \cong \mathcal{O}_X(U_i) \otimes A \xrightarrow{e^{\eta_i}} \mathcal{O}_X(U_i) \otimes A \cong \mathcal{O}_{\mathcal{X}_1}(U_i)\ ,$$

then the induced isomorphism is locally described by the maps

$$\bigwedge^* \Theta_{\mathcal{X}_0/A}(U_i) \cong \bigwedge^* \Theta_X(U_i) \otimes A \xrightarrow{e^{\mathrm{ad}\ \eta_i}} \bigwedge^* \Theta_X(U_i) \otimes A \cong \bigwedge^* \Theta_{\mathcal{X}_1/A}(U_i)\ .$$

DEFINITION 4.1. A deformation of a holomorphic Poisson manifold (X,π) over $A\in\mathbf{Art}_{\mathbb{C}}$ is the data of a deformation of X as in (4.1) $X\xrightarrow{i}\mathcal{X}\xrightarrow{p}\operatorname{Spec}A$ and of a section $\widetilde{\pi}\in H^0(X,\bigwedge^2\Theta_{\mathcal{X}/A})$ such that $[\widetilde{\pi},\widetilde{\pi}]_{SN}=0$ and such that $\widetilde{\pi}$ restricts to π under the natural projection $\bigwedge^*\Theta_{\mathcal{X}/A}\to\bigwedge^*\Theta_X$.

Given two deformations $(\mathcal{X}_0,\widetilde{\pi}_0)$, $(\mathcal{X}_1,\widetilde{\pi}_1)$ an isomorphism between them is an isomorphism between $\mathcal{X}_0$ and $\mathcal{X}_1$ such that the induced map $\bigwedge^*\Theta_{\mathcal{X}_0/A}\to\bigwedge^*\Theta_{\mathcal{X}_1/A}$ sends $\widetilde{\pi}_0$ to $\widetilde{\pi}_1$. Thus, every holomorphic Poisson manifold (X,π) defines a formal pointed groupoid

$$\operatorname{Del}_{(X,\pi)}:\mathbf{Art}_{\mathbb{C}}\to\mathbf{Grpd},$$

sending A to the groupoid whose objects are deformations of (X,π) over A and whose arrows are isomorphisms between them; at the same time it defines a formal set $\operatorname{Def}_{(X,\pi)}:\mathbf{Art}_{\mathbb{C}}\to\mathbf{Set}$, $A\mapsto\pi_0(\operatorname{Del}_{(X,\pi)}(A))$, sending A to the set of isomorphism classes of deformations of (X,π) over A.

REMARK 4.2. Equivalently a deformation of (X,π) over A could be defined as a sheaf $\mathcal{O}_{\mathcal{X}}$ of flat Poisson A-algebras on X (by that we mean flat A-algebras equipped with an A-bilinear Poisson bracket) and a sheaves of Poisson A-algebras morphism $\mathcal{O}_{\mathcal{X}}\to\mathcal{O}_X$ locally isomorphic to the projection $\mathcal{O}_X\otimes A\to\mathcal{O}_X$ (with the Poisson structure on $\mathcal{O}_X\otimes A$ given by scalar extension).

In order to exhibit a differential graded Lie algebra governing the above deformation problem we are going to apply descent of Deligne groupoids (Theorem 2.4); to this end we consider the sheaf $\bigwedge^{\geq 1}\Theta_X[1]$ of sub DGLAs of $\bigwedge^*\Theta_X[1]$. Fixing an open Stein covering $\mathcal{U}=\{U_i\}$, consider the non-negatively graded semicosimplicial differential graded Lie algebra

$$\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet:\ \prod_i\bigwedge^{\geq 1}\Theta_X[1](U_i)\rightrightarrows\prod_{i,j}\bigwedge^{\geq 1}\Theta_X[1](U_{ij})\ \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\ \prod_{i,j,k}\bigwedge^{\geq 1}\Theta_X[1](U_{ijk})\cdots$$

with the usual Čech face operators given by restriction.

THEOREM 4.3. *The totalization* $\operatorname{Tot}(\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet)$ *governs the deformations of* (X,π)*; more precisely there exists an equivalence of formal pointed groupoids:*

$$\operatorname{Del}_{\operatorname{Tot}(\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet)} \simeq \operatorname{Del}_{(X,\pi)} \ .$$

PROOF. Given $A \in \mathbf{Art}_{\mathbb{C}}$ and a deformation $(\mathcal{X},\widetilde{\pi})$ of (X,π) over A, since $\mathcal{X}$ trivializes over each U_i we have $\bigwedge^2\Theta_{\mathcal{X}/A}(U_i) \cong \bigwedge^2\Theta_X(U_i)\otimes A$ and then we can write $\widetilde{\pi}_{|U_i} = \pi_{|U_i} + \sigma_i$ with $\sigma_i \in \bigwedge^2\Theta_X(U_i)\otimes\mathfrak{m}_A$. Then $[\widetilde{\pi}_{|U_i},\widetilde{\pi}_{|U_i}]_{SN} = 0$ is equivalent to $[\pi_{|U_i},\sigma_i]_{SN} + \frac{1}{2}[\sigma_i,\sigma_i]_{SN} = 0$, i.e., σ_i is a solution to Maurer-Cartan equation in the differential graded Lie algebra $\bigwedge^{\geq 1}\Theta_X[1](U_i)\otimes\mathfrak{m}_A$. If the $e^{\eta_{ij}}$'s are the transition automorphisms for $\mathcal{X}$ on double intersections we have the equality

$$e^{\operatorname{ad}\eta_{ij}}(\pi_{|U_{ij}} + \sigma_{j|U_{ij}}) = \pi_{|U_{ij}} + \sigma_{i|U_{ij}} \ ,$$

which in terms of gauge action reads as $e^{\eta_{ij}} * \sigma_{j|U_{ij}} = \sigma_{i|U_{ij}}$ in the differential graded Lie algebra $\bigwedge^{\geq 1}\Theta_X[1](U_{ij})\otimes\mathfrak{m}_A$.

On the other hand, by Theorem 2.4 the groupoid $\operatorname{Del}_{\operatorname{Tot}(\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet)}(A)$ is naturally equivalent to the totalization of the semicosimplicial groupoid:

$$\prod_i \operatorname{Del}_{\bigwedge^{\geq 1}\Theta_X[1](U_i)}(A) \rightrightarrows \prod_{i,j} \operatorname{Del}_{\bigwedge^{\geq 1}\Theta_X[1](U_{ij})}(A) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots$$

Objects of this groupoid are precisely the collections

$$\sigma_i \in \bigwedge^2\Theta_X(U_i)\otimes\mathfrak{m}_A, \qquad e^{\eta_{ij}} \in \exp(\Theta_X(U_{ij})\otimes\mathfrak{m}_A),$$

such that for every i,j,k:

(1) every σ_i is a solution to the Maurer-Cartan equation in $\bigwedge^{\geq 1}\Theta_X[1](U_i)\otimes\mathfrak{m}_A$;

(2) $e^{\eta_{ij}} * \sigma_{j|U_{ij}} = \sigma_{i|U_{ij}}$ in $\bigwedge^{\geq 1}\Theta_X[1](U_{ij})\otimes\mathfrak{m}_A$;

(3) $e^{\eta_{ij}}e^{\eta_{jk}} = e^{\eta_{ik}}$ in $\exp(\Theta_X(U_{ijk})\otimes\mathfrak{m}_A)$.

The last condition ensures that we can glue via the $e^{\eta_{ij}}$ the trivial deformations over each U_i to a global deformation $\mathcal{X}$ of X over A, the second ensures that the local sections $\widetilde{\pi}_i = \pi_{|U_i} + \sigma_i$ paste to a global section $\widetilde{\pi}$ of $\bigwedge^2 \Theta_{\mathcal{X}/A}$, restricting to π, and the first one ensures that $[\widetilde{\pi}, \widetilde{\pi}]_{SN} = 0$, i.e., $(\mathcal{X}, \widetilde{\pi})$ is a deformation of (X, π) over A, canonically associated to the descent data. Conversely, by the previous discussion, every deformation of (X, π) over A determines descent data from which it can be reconstructed up to isomorphism.

A similar argument shows that equivalences of descent data correspond to equivalences of the associated deformations, and the other way around, so that it is defined this way a fully faithful, essentially surjective, functor

$$\mathrm{Del}_{\mathrm{Tot}(\wedge^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet)}(A) \to \mathrm{Del}_{(X,\pi)}(A)\,.$$

Naturality of this construction in A is clear. □

Some of the results of this section are also obtained recently and independently by C. Kim in his thesis [21].

5. Coisotropic Deformations

DEFINITION 5.1. Let (X, π) be a holomorphic Poisson manifold. A holomorphic closed submanifold $Z \subset X$ is called coisotropic if its ideal sheaf $\mathcal{I}_Z$ is closed under the Poisson bracket $\{\cdot,\cdot\}_\pi$.

For a closed submanifold Z of a complex manifold X we denote by $\mathcal{N}_{Z|X}$ the normal sheaf of Z in X and by $\bigwedge^* \mathcal{N}_{Z|X} := \bigoplus_{i\geq 0} \bigwedge^i_{\mathcal{O}_Z} \mathcal{N}_{Z|X}[-i]$ its graded exterior algebra. By a little abuse of notation we also denote by $\bigwedge^* \mathcal{N}_{Z|X}$ its direct image under the inclusion $Z \subset X$.

PROPOSITION 5.2. *The kernel $\mathcal{L}^*_Z$ of the natural epimorphism $\bigwedge^* \Theta_X \to \bigwedge^* \mathcal{N}_{Z|X}$ is a sheaf of Gerstenhaber subalgebras of $\bigwedge^* \Theta_X$. Moreover the following conditions are equivalent:*

(1) *Z is coisotropic;*

(2) *$\pi \in H^0(X; \mathcal{L}^2_Z)$;*

(3) *$d_\pi(\mathcal{L}^*_Z) \subseteq \mathcal{L}^*_Z$.*

Proof. This is a straightforward computations in local holomorphic coordinates. For later use we point out that $\mathcal{L}_Z^0 = \mathcal{I}_Z$ and $\mathcal{L}_Z^1 = \Theta_X(-\log Z)$. $\square$

We shall first study coisotropic deformations of (X, Z, π); recall that for $A \in \mathbf{Art}_{\mathbb{C}}$ a deformation of the pair (X, Z) over A can be described as a deformation $\mathcal{X}$ of X together with a sheaf $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{X}}$ of A-flat ideals and a commutative diagram

$$\begin{array}{ccc} \mathcal{I}_{\mathcal{Z}} & \hookrightarrow & \mathcal{O}_{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathcal{I}_Z & \hookrightarrow & \mathcal{O}_X \end{array}$$

of sheaves of A-algebras locally isomorphic to

$$\begin{array}{ccc} \mathcal{I}_Z \otimes A & \hookrightarrow & \mathcal{O}_X \otimes A \\ \downarrow & & \downarrow \\ \mathcal{I}_Z & \hookrightarrow & \mathcal{O}_X\ . \end{array}$$

Equivalences are isomorphisms of pairs over $\mathcal{I}_Z \hookrightarrow \mathcal{O}_X$, the group of self-equivalences of the trivial deformation is naturally isomorphic to $\exp(H^0(X, \Theta_X(-\log Z)) \otimes \mathfrak{m}_A)$, where

$$\Theta_X(-\log Z) = \{\eta \in \Theta_X \mid \eta(\mathcal{I}_Z) \subset \mathcal{I}_Z\}$$

is the sheaf of vector fields tangent everywhere to Z. Again given an open Stein covering $\{U_i\}$ of X, every deformation of (X, Z) trivializes over each U_i, so that it can be reconstructed up to isomorphism by the family of transition automorphisms $e^{\eta_{ij}} \in \exp(\Theta_X(-\log Z)(U_{ij}) \otimes \mathfrak{m}_A)$ satisfying the cocycle condition on triple intersections.

Definition 5.3. Given a holomorphic Poisson manifold (X, π) and a coisotropic submanifold $Z \subset X$, a coisotropic deformation of the triple (X, Z, π) is a deformation $(\mathcal{X}, \widetilde{\pi})$ of (X, π) equipped with a sheaf of coisotropic ideals $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{X}}$ such that $\mathcal{I}_{\mathcal{Z}} \hookrightarrow \mathcal{O}_{\mathcal{X}}$ is a deformation of the pair (X, Z). Together with the obvious notion of isomorphism, the deformations over a fixed basis have a natural groupoid structure. We denote by

$$\mathrm{Del}^{co}_{(X,Z,\pi)} \colon \mathbf{Art}_{\mathbb{C}} \to \mathbf{Grpd}, \qquad \mathrm{Def}^{co}_{(X,Z,\pi)} \colon \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}\,,$$

the associated formal pointed groupoid and functor of Artin rings.

Fixing an open Stein covering $\mathcal{U} = \{U_i\}$ we can proceed as in the proof of Theorem 4.3, considering the non negatively graded semicosimplicial DGLA $\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_\bullet$ of Čech cochains of the graded sheaf $\mathcal{L}_Z^{\geq 1}[1]$.

THEOREM 5.4. *There is an equivalence of formal groupoids:*

$$\mathrm{Del}_{\mathrm{Tot}(\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_\bullet)} \simeq \mathrm{Del}^{co}_{(X,Z,\pi)} \ .$$

PROOF. This is proved in the same way as in Theorem 4.3: the descent data for the semicosimplicial groupoid

$$\prod_i \mathrm{Del}_{\mathcal{L}_Z^{\geq 1}[1](U_i)}(A) \rightrightarrows \prod_{i,j} \mathrm{Del}_{\mathcal{L}_Z^{\geq 1}[1](U_{ij})}(A) \overset{\longrightarrow}{\rightrightarrows} \prod_{i,j.k} \mathrm{Del}_{\mathcal{L}_Z^{\geq 1}[1](U_{ijk})}(A) \ \cdots$$

can be glued to a deformation $\mathcal{I}_{\mathcal{Z}} \hookrightarrow \mathcal{O}_{\mathcal{X}}$ of (X,Z) and a deformation $\widetilde{\pi}$ of π; conversely every deformation determines descent data from which it can be reconstructed up to isomorphism. The only non trivial fact to point out is that, according to Proposition 5.2, for a given solution σ_i to Maurer-Cartan equation in $\bigwedge^{\geq 1}\Theta_X(U_i)\otimes\mathfrak{m}_A$, the condition that $\mathcal{I}_Z(U_i)\otimes A\subset\mathcal{O}_X(U_i)\otimes A$ is a coisotropic ideal with respect to the bracket induced by $\widetilde{\pi}_i = \pi_{|U_i} + \sigma_i$, is equivalent to $\sigma_i \in \mathcal{L}^2_Z(U_i)\otimes\mathfrak{m}_A$. □

As an application of the above result we are able to give the analog of Kodaira's stability theorem for coisotropic submanifolds.

COROLLARY 5.5 (Stability of coisotropic submanifolds). *Let (X,π) be a compact holomorphic Poisson manifold and let Z be a coisotropic submanifold. Consider the complex of sheaves*

$$\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1]\colon \qquad \mathcal{N}_{Z|X} \xrightarrow{d_\pi} \bigwedge^2 \mathcal{N}_{Z|X} \xrightarrow{d_\pi} \cdots$$

where $\bigwedge^i \mathcal{N}_{Z|X}$ *is considered in degree* $i-1$*. Let* $\mathcal{X} \to (B,0)$ *be a Poisson deformation of* (X,π) *over a germ of complex space* $(B,0)$*. If* $\mathbb{H}^1(Z, \bigwedge^{\geq 1} \mathcal{N}_{Z|X}[1]) = 0$ *then, after a possible shrinking of* B*, there exists a family of coisotropic submanifolds* $\mathcal{Z} \subset \mathcal{X}$ *which is smooth over* B *and such that* $\mathcal{Z}_0 = Z$*.*

PROOF. Following the same standard argument used in the proof of [17, Thm. 8.1], involving relative Douady space and Artin's theorem on the solution of analytic equations, it is not restrictive to assume B a fat point, i.e., $B = \operatorname{Spec}(A)$ for some $A \in \mathbf{Art}_{\mathbb{C}}$. Thus the stability theorem is proved whenever we show that the natural transformation of functors of Artin rings

$$\mathrm{Def}^{co}_{(X,Z,\pi)} \xrightarrow{\eta} \mathrm{Def}_{(X,\pi)}$$

is smooth; fixing an open Stein covering $\mathcal{U} = \{U_i\}$ of X, the above natural transformation is induced by the inclusion of differential graded Lie algebras

$$\mathrm{Tot}(\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_\bullet) \xrightarrow{i} \mathrm{Tot}(\bigwedge^{\geq 1} \Theta_X[1](\mathcal{U})_\bullet)\,.$$

According to standard smoothness criterion, see e.g. [31], the morphism η is smooth whenever i is surjective on H^1 and injective on H^2. By the definition of $\mathcal{L}_Z^{\geq 1}$ we have an exact sequence of complexes of coherent sheaves

$$0 \to \mathcal{L}_Z^{\geq 1}[1] \to \bigwedge^{\geq 1} \Theta_X[1] \to \bigwedge^{\geq 1} \mathcal{N}_{Z|X}[1] \to 0,$$

and the conclusion follows from Example 2.3 and hypercohomology long exact sequence. □

Finally recall that for a complex manifold X and a complex submanifold $Z \subset X$ the local Hilbert functor $\mathrm{Hilb}_{Z|X} \colon \mathbf{Art}_{\mathbb{C}} \to \mathbf{Set}$ sends A to the set of sheaves of A-flat ideals $\mathcal{I}_{\mathcal{Z}} \subset \mathcal{O}_X \otimes A$ such that $\mathcal{I}_{\mathcal{Z}} \otimes_A \mathbb{C} = \mathcal{I}_Z$; in other terms, $\mathrm{Hilb}_{Z|X}$ is the functor of formal embedded deformations of Z in X. If X is Stein then every $\mathcal{I}_{\mathcal{Z}} \hookrightarrow \mathcal{O}_X \otimes A$ is isomorphic as a pair to $\mathcal{I}_Z \otimes A \hookrightarrow \mathcal{O}_X \otimes A$, i.e., there exists $\eta \in H^0(X; \Theta_X) \otimes \mathfrak{m}_A$ for which $\mathcal{I}_{\mathcal{Z}} = e^{\eta}(\mathcal{I}_Z \otimes A)$. It has been already proved in [32] that the homotopy fiber of the morphisms of sheaves of differential graded Lie algebras $\mathcal{L}_Z^1 \hookrightarrow \Theta_X$ controls the functor $\mathrm{Hilb}_{Z|X}$; here we consider the case of embedded coisotropic deformations.

DEFINITION 5.6. Given a holomorphic Poisson manifold (X,π) and a coisotropic submanifold $Z\subset X$, the local coisotropic Hilbert functor of X in Z is the functor of Artin rings $\mathrm{Hilb}^{co}_{Z|X}\colon \mathbf{Art}_{\mathbb{C}}\to\mathbf{Set}$ sending A to the set of sheaves of A-flat coisotropic ideals $\mathcal{I}_{\mathcal{Z}}\subset\mathcal{O}_X\otimes A$ such that $\mathcal{I}_{\mathcal{Z}}\otimes_A\mathbb{C}=\mathcal{I}_Z$.

Let $\mathcal{K}_Z^{\geq 1}$ be the homotopy fiber of the inclusion of sheaves of (non negatively graded) differential graded Lie algebras $\mathcal{L}_Z^{\geq 1}[1]\hookrightarrow\bigwedge^{\geq 1}\Theta_X[1]$ (the notation is a little ambiguous, this is not the non negatively graded part of the homotopy fiber $\mathcal{K}_Z^*$ of the inclusion $\mathcal{L}_Z^*[1]\hookrightarrow\bigwedge^*\Theta_X[1]$); for an open covering $\mathcal{U}$ let $\mathcal{K}_Z^{\geq 1}(\mathcal{U})_\bullet$ be the associated semicosimplicial differential graded Lie algebra.

THEOREM 5.7. *For every open Stein covering $\mathcal{U}$ of X there exists an equivalence of formal pointed groupoids:*

$$\mathrm{Del}_{\mathrm{Tot}(\mathcal{K}_Z^{\geq 1}(\mathcal{U})_\bullet)}\simeq\mathrm{Hilb}^{co}_{Z|X}\,,$$

where $\mathrm{Hilb}^{co}_{Z|X}$ is regarded via the natural inclusion $\mathbf{Set}\to\mathbf{Grpd}$.

PROOF. We shall first show that $\mathrm{Del}_{\mathcal{K}_Z^{\geq 1}(U)}\simeq\mathrm{Hilb}^{co}_{U\cap\, Z|U}$ for a Stein open subset $U\subset X$, then the theorem will follow from descent of Deligne groupoids as in the previous cases. We saw in Example 2.5 that the groupoid $\mathrm{Del}_{\mathcal{K}_Z^{\geq 1}(U)}(A)$ admits the following description: objects are $e^\eta\in\exp(\Theta_X(U)\otimes\mathfrak{m}_A)$ such that $e^{-\eta}*0\in\mathcal{L}_Z^2(U)\otimes\mathfrak{m}_A$, morphisms between objects e^η, e^ξ are $e^\alpha\in\exp(\Theta_X(-\log Z)(U)\otimes\mathfrak{m}_A)$ such that $e^\eta=e^\xi e^\alpha$; moreover, the natural transformation $\mathrm{Del}_{\mathrm{Tot}(\mathcal{K}_Z^{\geq 1}(\mathcal{U})_\bullet)}(A)\to\mathrm{Def}_{\mathrm{Tot}(\mathcal{K}_Z^{\geq 1}(\mathcal{U})_\bullet)}(A)$ is an equivalence of groupoids.

The equivalence $\mathrm{Del}_{\mathcal{K}_Z^{\geq 1}(U)}(A)\simeq\mathrm{Hilb}^{co}_{U\cap\, Z|U}(A)$ is given on the set of objects sending e^η to the ideal $e^\eta(\mathcal{I}_{Z\cap U}\otimes A)\subset\mathcal{O}_U\otimes A$. This takes values in $\mathrm{Hilb}^{co}_{U\cap\, Z|U}(A)$, in fact applying the Gerstenhaber automorphism $e^{-\operatorname{ad}\eta}$ we see that $e^\eta(\mathcal{I}_{Z\cap U}\otimes A)$ is coisotropic if and only if $\mathcal{I}_{Z\cap U}\otimes A$ is coisotropic with respect to the bracket induced by $e^{-\operatorname{ad}\eta}(\pi_{|U})=\pi_{|U}+e^{-\eta}*0$ and, as in Proposition 5.2, this is equivalent to $e^{-\eta}*0\in\mathcal{L}^2_{Z\cap U}\otimes\mathfrak{m}_A$; it is a morphism of groupoids as it factors through $\mathrm{Del}_{\mathrm{Tot}(\mathcal{K}_Z^{\geq 1}(\mathcal{U})_\bullet)}(A)\to\mathrm{Def}_{\mathrm{Tot}(\mathcal{K}_Z^{\geq 1}(\mathcal{U})_\bullet)}(A)$, since this is an equivalence it remains to show bijectivity of the induced

$\mathrm{Def}_{\mathcal{K}^{\geq 1}_{\overline{Z}}(U)}(A) \to \mathrm{Hilb}^{co}_{U\cap Z|U}(A)$: injectivity is plain, surjectivity follows from U being Stein.

It follows from the definitions that $U \to \mathrm{Hilb}^{co}_{U\cap Z|U}(A)$ is a sheaf of (pointed) sets on X, in particular this means that $\mathrm{Hilb}^{co}_{Z|X}(A)$ is in canonical bijective correspondence with the totalization of the semicosimplicial set:

$$\prod_i \mathrm{Hilb}^{co}_{U_i\cap Z|U_i}(A) \rightrightarrows \prod_{i,j} \mathrm{Hilb}^{co}_{U_{ij}\cap Z|U_{ij}}(A) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \prod_{i,j,k} \mathrm{Hilb}^{co}_{U_{ijk}\cap Z|U_{ijk}}(A) \cdots$$

and the first part of the proof gives a natural equivalence between this and the semicosimplicial groupoid:

$$\prod_i \mathrm{Del}_{\mathcal{K}^{\geq 1}_{\overline{Z}}(U_i)}(A) \rightrightarrows \prod_{i,j} \mathrm{Del}_{\mathcal{K}^{\geq 1}_{\overline{Z}}(U_{ij})}(A) \substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow} \prod_{i,j,k} \mathrm{Del}_{\mathcal{K}^{\geq 1}_{\overline{Z}}(U_{ijk})}(A) \cdots$$

thus a natural equivalence of the corresponding totalizations. Now Theorem 2.4 gives the desired natural equivalence $\mathrm{Del}_{\mathrm{Tot}(\mathcal{K}^{\geq 1}_{\overline{Z}}(\mathcal{U})_\bullet)}(A) \simeq \mathrm{Hilb}^{co}_{Z|X}(A)$. $\square$

6. Coisotropic Deformations Induced by the Anchor Map

In order to show that the coisotropic deformations induced by the anchor map are unobstructed, both in classical and derived sense, we need an algebraic criterion for the homotopy abelianity of certain homotopy fibers, which we think of independent interest.

According to (generalized) Quillen's construction [15, Prop. 3.3.2], two DGLAs are quasi-isomorphic if and only if they are weak equivalent as L_∞-algebras; in particular a differential graded Lie algebra is homotopy abelian if and only if every bracket on its L_∞ minimal model [25] vanishes.

Let $(L, d, [\cdot,\cdot])$ be a differential graded Lie algebra and assume that there exists an L_∞-morphism

$$f_\infty \colon (L, d, [\cdot,\cdot]) \dashrightarrow (L, d, 0), \quad f_\infty = \{f_n\}, \quad f_n \colon L^{\wedge n} \to L,$$

with linear part f_1 equal to the identity, then $(L, d, [\cdot,\cdot])$ is homotopy abelian. Notice that, according to homotopy classification of L_∞-algebras [25], the converse is also true.

THEOREM 6.1. *Let $L = (L, d, [\cdot,\cdot])$ be a differential graded Lie algebra over a field of characteristic* 0. *If there exists a bilinear map $h\colon L \times L \to L$ of degree -1 such that*

(1) $h(a,b) = -(-1)^{\overline{a}\,\overline{b}}h(b,a)$,

(2) $[a,b] = dh(a,b) + h(da,b) + (-1)^{\overline{a}}h(a,db)$,

(3) $\oint [h(a,b),c] + \oint h([a,b],c) = 0$, *where $\oint$ denotes the sum over the cyclic permutations, taking care of Koszul signs.*

Then L is homotopy abelian. If $M \subset L$ is a differential graded Lie subalgebra such that $h(M \times M) \subset M$, then both M and the homotopy fiber of the inclusion $M \subset L$ are homotopy abelian.

PROOF. The proof given here is a slight generalization of [10, Theorem 3.2]; as pointed out above, the triviality of the bracket up to homotopy, described by conditions (1) and (2), is not sufficient to ensure homotopic abelianity.

It is convenient to consider the $L_\infty[1]$-algebra $(V, q_1, q_2, 0, \dots)$ corresponding to L via standard décalage isomorphism:

$$V = L[1], \qquad q_1 = -d, \qquad q_2(a,b) = (-1)^{\overline{a}}[a,b],$$

where $\overline{a}$ denotes the degree of a in L. The homotopy h corresponds to a map

$$r \in \mathrm{Hom}^0_{\mathbb{K}}(V^{\odot 2}, V), \qquad r(a,b) = (-1)^{\overline{a}}h(a,b),$$

and it is straightforward to check that the above conditions (2) and (3) become

$$[r, q_1]_{NR} = q_2, \qquad [r, q_2]_{NR} = 0,$$

where $[\cdot,\cdot]_{NR}$ is the Nijenhuis-Richardson bracket on $\mathrm{Hom}^*_{\mathbb{K}}(\overline{S^c}(V), V)$, i.e., the transfer of the graded commutator bracket on the Lie algebra of coderivations of the reduced symmetric coalgebra $\overline{S^c}(V)$ via the corestriction isomorphism

$$\mathrm{Hom}^*_{\mathbb{K}}(\overline{S^c}(V), V) \simeq \mathrm{Coder}^*_{\mathbb{K}}(\overline{S^c}(V), \overline{S^c}(V))\,.$$

Let's denote by $R, Q_1, Q_2 \in \mathrm{Coder}^*_{\mathbb{K}}(\overline{S^c}(V), \overline{S^c}(V))$ the coderivations with corestrictions r, q_1, q_2 respectively; we have that $R(V^{\odot n}) \subset V^{\odot n-1}$ and then it is well defined an isomorphism of graded coalgebras $e^R \colon \overline{S^c}(V) \to \overline{S^c}(V)$. To conclude the proof it is sufficient to prove that

$$Q_1 + Q_2 = e^R \circ Q_1 \circ e^{-R} = e^{[R,\cdot]}(Q_1).$$

Taking corestrictions the above equality becomes

$$q_1 + q_2 = e^{[r,\cdot]_{NR}}(q_1) = q_1 + [r, q_1]_{NR} + \frac{1}{2}[r, [r, q_1]]_{NR} + \cdots$$

which is clearly equivalent to $[r, q_1]_{NR} = q_2$ and $[r, q_2]_{NR} = 0$. If P is a differential graded commutative algebra we can extend the operator h to $P \otimes L$ in the obvious way:

$$h(u \otimes a, v \otimes b) = (-1)^{\overline{u}+\overline{v}+\overline{v}\,\overline{a}}\; uv \otimes h(a, b)\ .$$

The validity of the properties (2) and (3) for the operator h in $P \otimes L$ is clear.

In particular h extends to $L[t, dt]$ and commutes with the evaluation maps $L[t, dt] \xrightarrow{t\mapsto 0} L$ and $L[t, dt] \xrightarrow{t\mapsto 1} L$. Therefore h extends to the homotopy fiber of the inclusion. It is worth to notice here that, in general, the homotopy fiber of a morphism of homotopy abelian differential graded Lie algebras is not homotopy abelian. □

Corollary 6.2. *Let*

$$L_\bullet : \qquad L_0 \rightrightarrows L_1 \;\substack{\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow}\; L_2 \;\substack{\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow\\[-3pt]\longrightarrow}\; \cdots$$

be a semicosimplicial differential graded Lie algebra. Assume it is given a sequence of bilinear maps $h_n \colon L_n \times L_n \to L_n$ of degree -1 such that:

(1) *for every n the map h_n satisfies the conditions of Theorem 6.1;*

(2) *the sequence $\{h_n\}$ gives a semicosimplicial map, i.e., for every n and every face operator $\partial_k \colon L_n \to L_{n+1}$ we have $h_{n+1}(\partial_k a, \partial_k b) = \partial_k h_n(a, b)$.*

Then the totalization $\operatorname{Tot}(L_\bullet)$ *is a homotopy abelian differential graded Lie algebra.*

PROOF. By scalar extension, every map h_n extends to a bilinear map on $\Omega_n \otimes L_n$ and then we have a bilinear map

$$h\colon (\prod_n \Omega_n \otimes L_n) \times (\prod_n \Omega_n \otimes L_n) \to \prod_n \Omega_n \otimes L_n \ ,$$
$$h((a_1, a_2, \dots), (b_1, b_2, \dots)) = (h_1(a_1, b_1), h_2(a_2, b_2), \dots),$$

which satisfies the conditions of Theorem 6.1. Since the morphisms h_n commute with face operators, the totalization $\operatorname{Tot}(L_\bullet) \subset \prod_n \Omega_n \otimes L_n$ is stable under h. □

LEMMA 6.3. *Let* (X, π) *be a holomorphic Poisson manifold. Then the bilinear morphism*

$$h\colon \Omega^i_X \otimes \Omega^j_X \to \Omega^{i+j-2}_X,$$
$$h(\alpha, \beta) = (-1)^i (\boldsymbol{i}_\pi(\alpha \wedge \beta) - \boldsymbol{i}_\pi(\alpha) \wedge \beta - \alpha \wedge \boldsymbol{i}_\pi(\beta)),$$

defined on the sheaf of DGLAs $(\Omega^*_X[1], [\cdot,\cdot]_\pi, \partial)$ *verify the assumptions of Theorem 6.1. Given an open covering* $\mathcal{U}$ *of* X*, there is induced a sequence of linear maps on the semicosimplicial differential graded Lie algebra* $\Omega^*_X[1](\mathcal{U})_\bullet$ *as in the hypotheses of Corollary 6.2.*

PROOF. The first part is a direct straightforward verification, substantially made in [10]; the last part is clear. □

For any integer $k \geq 0$, let $\Omega^{\geq k}_X$ denote the truncation in degree $\geq k$ of Ω^*_X, in particular $\Omega^{\geq k}_X = \Omega^*_X$ for $k = 0$. For every $k \neq 1$ the sheaf of DGLAs $\Omega^{\geq k}_X[1]$ is stable with respect to the operator h defined in the previous Lemma 6.3; this and Theorem 6.1 immediately imply the following proposition.

PROPOSITION 6.4. *Let* (X, π) *be a holomorphic Poisson manifold. Then, for every nonnegative integer* $k \neq 1$, $(\Omega^{\geq k}_X[1], [\cdot,\cdot]_\pi, \partial)$ *is a sheaf of homotopy abelian DGLAs on* X*. Moreover, for every open covering* $\mathcal{U}$ *of* X *the totalization* $\operatorname{Tot}(\Omega^{\geq k}_X[1](\mathcal{U})_\bullet)$ *is homotopy abelian.*

Next, for a closed submanifold $Z \subset X$, let $\mathcal{I}^*_Z \subset \Omega^*_X$ denote the ideal of forms vanishing along Z, i.e., the kernel of the natural restriction morphism $\Omega^*_X \to \Omega^*_Z$, also recall the sheaf $\mathcal{L}^*_Z \subset \bigwedge^* \Theta_X$ defined in the previous section.

PROPOSITION 6.5. *Let (X, π) be a holomorphic Poisson manifold and $Z \subset X$ a coisotropic submanifold. Then $\mathcal{I}^*_Z \subset \Omega^*_X$ is a sheaf of differential graded Lie subalgebras, it is closed with respect to the operator h introduced in Lemma 6.3 and $\pi^\#(\mathcal{I}^*_Z) \subset \mathcal{L}^*_Z$.*

PROOF. The question is purely local, we fix a system of local holomorphic coordinates $z_1, \dots, z_n$ such that $Z = \{z_1 = \cdots = z_p = 0\}$ and the coisotropy of Z means that $\{z_i, z_j\}_\pi \in \mathcal{I}_Z$ for $1 \le i, j \le p$. Therefore for $1 \le i, j \le p$ we also have

$$\{z_i, z_j\}_\pi = \boldsymbol{i}_\pi(dz_i \wedge dz_j) = -h(dz_i, dz_j) = \pi^\#(dz_i)(z_j) = [dz_i, z_j]_\pi \in \mathcal{I}_Z$$
$$[dz_i, dz_j]_\pi = d[dz_i, z_j]_\pi \in \mathcal{I}^*_Z \ .$$

Since $\mathcal{I}^*_Z \subset \Omega^*_X$ is the multiplicative ideal generated locally by $S = \{z_j,\ dz_i\}_{1 \le i,j \le p}$ the above computation shows that $\mathcal{I}^*_Z \subset \Omega^*_X$ is a sheaf of differential graded Lie subalgebras and $\pi^\#(\mathcal{I}^*_Z) \subset \mathcal{L}^*_Z$.

Now the h-closedness of the ideal sheaf $\mathcal{I}^*_Z$ is equivalent to $\boldsymbol{i}_\pi(\alpha \wedge \beta) \in \mathcal{I}^*_Z$ for $\alpha, \beta \in \mathcal{I}^*_Z$; to this end it is not restrictive to assume α an element of S. If $\alpha = z_i$ with $1 \le j \le p$ the claim follows by $\mathcal{O}_X$-linearity of $\boldsymbol{i}_\pi$, while for $\alpha = dz_i$ the statement is equivalent to the fact that $\mathcal{I}^*_Z$ is $(\boldsymbol{i}_\pi \circ dz_j \wedge \cdot)$-closed for $1 \le j \le p$: this is true if and only if the ideal is closed with respect to the operator

$$[\boldsymbol{i}_\pi, dz_j \wedge \cdot] = [\boldsymbol{i}_\pi, [d, z_j \wedge \cdot]] = [\boldsymbol{l}_\pi, z_j \wedge \cdot] = [z_j, \cdot]_\pi \,,$$

where the last equality follows from (3.2) and the conclusion follows from the fact that $\mathcal{I}^*_Z$ is closed under the Koszul bracket.

Notice that for a coisotropic submanifold Z the ideal $\mathcal{I}^*_Z$ is not $\boldsymbol{i}_\pi$-closed in general. □

Denote by $\mathcal{J}^{\ge k}_Z$ the homotopy fiber of the inclusion of sheaves of DGLAs $\mathcal{I}^{\ge k}_Z[1] \hookrightarrow \Omega^{\ge k}_X[1]$.

PROPOSITION 6.6. *Let (X, π) be a holomorphic Poisson manifold, $Z \subset X$ a coisotropic submanifold and k a nonnegative integer. If $k \neq 1$, then*

$\mathcal{I}_Z^{\geq k}[1]$, $\mathcal{J}_Z^{\geq k}$ *are sheaves of homotopy abelian differential graded Lie algebras on* X*. Moreover, for every open covering* $\mathcal{U}$ *of* X *the totalizations* $\mathrm{Tot}(\mathcal{I}_Z^{\geq k}[1](\mathcal{U})_\bullet)$, $\mathrm{Tot}(\mathcal{J}_Z^{\geq k}(\mathcal{U})_\bullet)$ *are homotopy abelian.*

Proof. Immediate from Proposition 6.5 and Theorem 6.1. □

In the situation of Proposition 6.6 we have a commutative diagram of DGLAs:

$$\begin{array}{ccccc} \mathrm{Tot}(\mathcal{J}_Z^{*}(\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(\mathcal{I}_Z^{*}[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(\Omega_X^{*}[1](\mathcal{U})_\bullet) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Tot}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(\mathcal{I}_Z^{\geq 1}[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet) \end{array} \tag{6.1}$$

with the DGLAs in the top row homotopy abelian and the inclusions as vertical arrows.

Lemma 6.7. *Let* $\mathcal{U}$ *be an open Stein covering of* X*.*

(1) *if the Hodge to de Rham spectral sequence of* X *degenerates at* E_1*, then the differential graded Lie algebra* $\mathrm{Tot}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet)$ *is homotopy abelian.*

(2) *if the Hodge to de Rham spectral sequence of* Z *degenerates at* E_1*, then the differential graded Lie algebra* $\mathrm{Tot}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet)$ *is homotopy abelian.*

Proof. Recall that the Hodge to de Rham spectral sequence of a smooth complex manifold X may be defined as the spectral sequence associated to the filtration of Čech (double) complexes $F^k = C(\mathcal{U}, \Omega_X^{\geq k})$ (see e.g. [8]). Since the Whitney maps

$$I\colon\ \mathrm{Tot}(\Omega_X^{\geq k}(\mathcal{U})_\bullet) \to C(\mathcal{U}, \Omega_X^{\geq k})$$

are quasi-isomorphisms of complexes, the first item is an immediate consequence of the well known fact that if $f\colon L \to M$ is a morphism of differential graded Lie algebras which is injective in cohomology and M is homotopy abelian, then also L is homotopy abelian (for a proof see e.g. [20, Proposition 4.11] or [10, Lemma 6.1]).

The second item is proved in the same way, pointing out that for every Stein open subset $U \subset X$ and every $k \geq 0$ the complexes $\mathcal{J}_Z^{\geq k}(U)$ and $\Omega_Z^{\geq k}(U)$ are quasi-isomorphic. □

Next, fix an open Stein covering $\mathcal{U}$ and consider the following commutative diagram of anchor maps

$$\begin{array}{ccccc} \mathrm{Tot}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(\mathcal{I}_Z^{\geq 1}[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet) \\ \downarrow{\scriptstyle \pi^\#} & & \downarrow{\scriptstyle \pi^\#} & & \downarrow{\scriptstyle \pi^\#} \\ \mathrm{Tot}(\mathcal{K}_Z^{\geq 1}(\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(\mathcal{L}_Z^{\geq 1}[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet)\,. \end{array} \tag{6.2}$$

In particular the anchor maps induce morphisms of deformation functors

$$\pi^\#\colon\ \mathrm{Def}_{\mathrm{Tot}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet)} \to \mathrm{Hilb}^{co}_{Z|X}\,, \qquad \pi^\#\colon\ \mathrm{Def}_{\mathrm{Tot}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet)} \to \mathrm{Def}_{(X,\pi)}\,,$$

which at first order reduce to the anchor maps in cohomology:

$$\pi^\#\colon H^1(\mathrm{Tot}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet)) = \mathbb{H}^1(Z, \Omega_Z^{\geq 1}) \to T^1\,\mathrm{Hilb}^{co}_{Z|X},$$
$$\pi^\#\colon H^1(\mathrm{Tot}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet)) = \mathbb{H}^2(X, \Omega_X^{\geq 1}) \to T^1\,\mathrm{Def}_{(X,\pi)}\,.$$

Whenever the Hodge to de Rham spectral sequence of Z (resp.: X) degenerates at E_1 we have an isomorphism $\mathbb{H}^1(Z, \Omega_Z^{\geq 1}) \simeq H^0(Z, \Omega_Z^1)$ (resp.: $\mathbb{H}^2(X, \Omega_X^{\geq 1}) \simeq H^0(X, \Omega_X^2) \oplus H^1(X, \Omega_X^1)$).

THEOREM 6.8. *In the notation above, if the Hodge to de Rham spectral sequence of Z degenerates at E_1, then for every $\omega \in H^0(Z, \Omega_Z^1)$ the first order embedded coisotropic deformation $\pi^\#(\omega)$ extends to an embedded coisotropic deformation of Z over* $\mathrm{Spec}(\mathbb{C}[[t]])$.

PROOF. Clear, since for every open Stein covering $\mathcal{U}$ the DGLA $\mathrm{Tot}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet)$ is homotopy abelian and then the functor $\mathrm{Def}_{\mathrm{Tot}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet)}$ is unobstructed. □

THEOREM 6.9. *In the notation above, if the Hodge to de Rham spectral sequence of X degenerates at E_1 then for every $\omega \in H^0(X, \Omega_X^2) \oplus H^1(X, \Omega_X^1)$ the first order deformation $\pi^\#(\omega)$ extends to a deformation of (X, π) over* $\mathrm{Spec}(\mathbb{C}[[t]])$.

PROOF. As above, for every open Stein covering $\mathcal{U}$ the differential graded Lie algebra $\mathrm{Tot}(\Omega_X^{\geq 1}[1](\mathcal{U})_\bullet)$ is homotopy abelian. □

The above Theorem 6.9 has been proved is a different way by Hitchin [16] under the additional assumption that X is compact Kähler. As a further application we can generalize to coisotropic submanifolds part of classical results by McLean and Voisin about deformations of Lagrangian submanifolds [34, 40].

COROLLARY 6.10. *Let Z be a compact coisotropic submanifold of a holomorphic Poisson manifold (X, π). Assume that the Hodge to de Rham spectral sequence of Z degenerates at E_1 and the anchor map*

$$\pi^{\#}\colon H^0(Z, \Omega^1_Z) \to \ker(d_\pi\colon H^0(Z, \mathcal{N}_{Z|X}) \to H^0(Z, \bigwedge^2 \mathcal{N}_{Z|X}))$$

is surjective; then the functor $\mathrm{Hilb}^{co}_{Z|X}$ is unobstructed. If moreover

$$\pi^{\#}\colon H^0(Z, \Omega^1_Z) \to H^0(Z, \mathcal{N}_{Z|X})$$

is surjective, then every small embedded deformation of Z is coisotropic and the Hilbert functor $\mathrm{Hilb}_{Z|X} = \mathrm{Hilb}^{co}_{Z|X}$ is unobstructed.

PROOF. Since Z is compact, by the argument used in Corollary 5.5 it is sufficient to consider infinitesimal deformations. It is now sufficient to apply Theorem 6.8. □

Obviously the above corollary fails without the assumption about the anchor map. For instance, if $Z = p$ is a point, then Z is coisotropic if and only π vanishes at p; this shows that in general $\mathrm{Hilb}^{co}_{Z|X}$ is obstructed and strictly contained in $\mathrm{Hilb}_{Z|X}$. Corollary 6.10 holds in particular for Lagrangian submanifolds of a holomorphic symplectic manifold; a different proof of this case, based on Ran-Kawamata's T^1-lifting theorem, is given in [26]. As pointed out by the referee, the T^1-lifting argument can also be used for a different proof of Theorem 6.8, where the smoothness of the functor $\mathrm{Def}_{\mathrm{Tot}(\mathcal{J}_Z^{\geq 1}(\mathcal{U})_\bullet)}$ is replaced by the deformation invariance of Hodge numbers of Z.

7. Dolbeault Resolutions

In the previous sections we described the differential graded Lie algebras controlling Poisson deformations and embedded coisotropic deformations using a purely algebraic construction, namely the Thom-Whitney-Sullivan totalization. Therefore all the above results can be easily extended to every algebraic Poisson manifold defined over a field of characteristic 0: roughly speaking, it is sufficient to replace holomorphic with algebraic and Stein with affine and everything still works.

In this section we shall use Dolbeault's resolutions in order to give another description of the differential graded Lie algebra governing embedded coisotropic deformations; clearly this new interpretation only works in the complex analytic setting.

Given a locally free sheaf $\mathcal{E}$ on a complex manifold X we shall denote by $\mathcal{A}_X^{0,j}(\mathcal{E})$ the sheaf of differentiable forms of type $(0,j)$ with values in $\mathcal{E}$. The Dolbeault resolution of a bounded below complex

$$(\mathcal{E}^*, \delta)\colon \qquad 0 \to \mathcal{E}^i \xrightarrow{\delta} \mathcal{E}^{i+1} \xrightarrow{\delta} \cdots$$

of locally free sheaves on a complex manifold is the sheaf of DG-vector spaces $\mathcal{A}_X^{0,*}(\mathcal{E}^*)$, where

$$\mathcal{A}_X^{0,*}(\mathcal{E})^i = \bigoplus_{j+h=i} \mathcal{A}_X^{0,j}(\mathcal{E}^h),$$

and the differential $\overline{\partial}_{\mathcal{E}^*}$ is defined by the formula

$$\overline{\partial}_{\mathcal{E}^*}\colon \mathcal{A}_X^{0,j}(\mathcal{E}^h) \to \mathcal{A}_X^{0,j+1}(\mathcal{E}^h) \oplus \mathcal{A}_X^{0,j}(\mathcal{E}^{h+1}),$$
$$\overline{\partial}_{\mathcal{E}^*}(\phi \otimes e) = \overline{\partial}\phi \otimes e + (-1)^j \phi \otimes \delta e\,.$$

According to Dolbeault's lemma, the natural inclusion $\mathcal{E}^* \to \mathcal{A}_X^{0,*}(\mathcal{E}^*)$ is a quasi-isomorphism.

Similarly we denote by $A_X^{0,*}(\mathcal{E}^*)$ the DG-vector space of global sections of the Dolbeault resolution; more generally, for every open subset $U \subset X$ we shall denote by $A_U^{0,*}(\mathcal{E}^*)$ the DG-vector space of sections of $\mathcal{A}_X^{0,*}(\mathcal{E}^*)$ over U. Notice that, by Dolbeault theorem, the cohomology $A_X^{0,*}(\mathcal{E}^*)$ is isomorphic to the hypercohomology of $\mathcal{E}^*$.

Let $(\mathcal{E}^*,\delta)$ be a bounded below complex of locally free sheaves on a complex manifold X and let $\mathcal{U}=\{U_i\}$ be an open Stein covering of X. Thus we have a natural morphism of semicosimplicial DG-vector spaces:

$$\begin{array}{ccccccc}
\mathcal{E}^*(\mathcal{U})_\bullet\colon & \prod_i \mathcal{E}^*(U_i) & \rightrightarrows & \prod_{i,j}\mathcal{E}^*(U_{ij}) & \rightrightarrows\!\!\!\!\to & \prod_{i,j,k}\mathcal{E}^*(U_{ijk})\cdots \\
\downarrow & \downarrow & & \downarrow & & \downarrow \\
A^{0,*}_{\mathcal{U}}(\mathcal{E}^*)_\bullet\colon & \prod_i A^{0,*}_{U_i}(\mathcal{E}^*) & \rightrightarrows & \prod_{i,j} A^{0,*}_{U_{ij}}(\mathcal{E}^*) & \rightrightarrows\!\!\!\!\to & \prod_{i,j,k} A^{0,*}_{U_{ijk}}(\mathcal{E}^*)\cdots
\end{array}$$

Since $A^{0,*}_X(\mathcal{E}^*)$ is the equalizer of $\partial_0,\partial_1\colon A^{0,*}_{\mathcal{U}}(\mathcal{E}^*)_0\to A^{0,*}_{\mathcal{U}}(\mathcal{E}^*)_1$ and every map

$$\mathcal{E}^*(U_{i_1\cdots i_k})\to A^{0,*}_{U_{i_1\cdots i_k}}(\mathcal{E}^*)$$

is a quasi-isomorphism, according to Remark 2.7 there exists a diagram of quasi-isomorphisms

$$\begin{array}{ccc}
 & & \mathrm{Tot}(\mathcal{E}^*(\mathcal{U})_\bullet) \\
 & & \downarrow \\
A^{0,*}_X(\mathcal{E}^*) & \xrightarrow{e} & \mathrm{Tot}(A^{0,*}_{\mathcal{U}}(\mathcal{E}^*)_\bullet)
\end{array}\quad, \qquad e(x)=(1\otimes x,1\otimes\partial_0x,1\otimes\partial_0^2x,\dots).$$

Here we apply the above general construction in two particular cases, both of them related with a coisotropic submanifold Z of a holomorphic Poisson manifold (X,π). In the first case we consider the complex of locally free sheaves on X

$$\bigwedge^{\ge1}\Theta_X[1]:\qquad 0\to\Theta_X\xrightarrow{d_\pi}\bigwedge^2\Theta_X\xrightarrow{d_\pi}\bigwedge^3\Theta_X\cdots\ ,$$

while in the second we consider the complex of locally free sheaves on Z

$$\bigwedge^{\ge1}\mathcal{N}_{Z|X}[1]:\qquad 0\to\mathcal{N}_{Z|X}\xrightarrow{d_\pi}\bigwedge^2\mathcal{N}_{Z|X}\xrightarrow{d_\pi}\bigwedge^3\mathcal{N}_{Z|X}\cdots\ .$$

The complex $A^{0,*}_X(\bigwedge^{\ge1}\Theta_X[1])$ admits a natural structure of differential graded Lie algebra, where the bracket is the antiholomorphic extension of the Schouten-Nijenhuis bracket on $\mathcal{A}^{0,0}_X(\bigwedge^{\ge1}\Theta_X[1])$.

It is straightforward to check that there exists a short exact sequence

$$0\to L_{Z|X}\xrightarrow{\chi}A^{0,*}_X(\bigwedge^{\ge1}\Theta_X[1])\xrightarrow{P}A^{0,*}_Z(\bigwedge^{\ge1}\mathcal{N}_{Z|X}[1])\to0, \tag{7.1}$$

where P is the natural projection map, $L_{Z|X}$ is a differential graded subalgebra of $A^{0,*}_X(\bigwedge^{\geq 1}\Theta_X[1])$ and χ is the inclusion. By Proposition 5.2 $L_{Z|X} \subset A^{0,*}_X(\bigwedge^{\geq 1}\Theta_X[1])$ is a also a differential graded Lie subalgebra.

THEOREM 7.1. *In the notation of Section 5, for every open Stein covering $\mathcal{U}$ of X, the homotopy fiber of the inclusion $L_{Z|X} \xrightarrow{\chi} A^{0,*}_X(\bigwedge^{\geq 1}\Theta_X[1])$ is quasi-isomorphic to* $\mathrm{Tot}(\mathcal{K}^{\geq 1}_Z(\mathcal{U})_\bullet)$ *and then governs the functor of infinitesimal embedded coisotropic deformations of Z in X.*

PROOF. Since Tot is an exact functor we have a short exact sequence

$$0 \to \mathrm{Tot}(\mathcal{L}^{\geq 1}_Z[1](\mathcal{U})_\bullet) \xrightarrow{\alpha} \mathrm{Tot}(\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet) \to \mathrm{Tot}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1](\mathcal{U})_\bullet) \to 0$$

and $\mathrm{Tot}(\mathcal{K}^{\geq 1}_Z(\mathcal{U})_\bullet)$ is isomorphic to the homotopy fiber of α. The above sequence is part of a 5×3 diagram with exact columns, with the first two rows made by morphisms of differential graded Lie algebras and where every horizontal map is a quasi-isomorphism:

$$\begin{array}{ccccc}
0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Tot}(\mathcal{L}^{\geq 1}_Z[1](\mathcal{U})_\bullet) & \longrightarrow & K & \longleftarrow & L_{Z|X} \\
\downarrow \alpha & & \downarrow & & \downarrow \chi \\
\mathrm{Tot}(\bigwedge^{\geq 1}\Theta_X[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(A^{0,*}_{\mathcal{U}}(\bigwedge^{\geq 1}\Theta_X[1])_\bullet) & \xleftarrow{e} & A^{0,*}_X(\bigwedge^{\geq 1}\Theta_X[1]) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Tot}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1](\mathcal{U})_\bullet) & \longrightarrow & \mathrm{Tot}(A^{0,*}_{\mathcal{U}}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1])_\bullet) & \xleftarrow{e} & A^{0,*}_Z(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1]) \\
\downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0
\end{array}$$

This diagram induces a quasi-isomorphism between the homotopy fibers of α and χ. □

8. Relation with the Homotopy Lie Algebroid

Coisotropic deformations have been studied, in the differentiable setting, by using L_∞-algebras together with Voronov's construction of higher derived brackets [5, 6, 11, 37, 41].

THEOREM 8.1 (Th. Voronov [42]). *Let* $(M,[\cdot,\cdot])$ *be a graded Lie algebra, splitting, as a graded vector space, in the direct sum* $M=L\oplus A$, *where* L, A *are graded Lie subalgebras of* M *and* A *is abelian; denote by* $P\colon M\to A$ *the projection with kernel* L. *For every derivation* $D\in \operatorname{Der}^*(M)$ *such that* $D(L)\subset L$, *its higher derived brackets* $\{\cdots\}_D^n\colon A^{\odot n}\to A$ *are defined as*

$$\{a_1,\dots,a_n\}_D^n = P[[\cdots[[Da_1,a_2],a_3],\dots],a_n]\,,\quad n\geq 1\,. \tag{8.1}$$

Then, for every degree one derivation $D\in\operatorname{Der}^1(M)$ *such that* $D^2=0$ *and* $D(L)\subset L$, *the higher derived brackets of* D *give a structure of* L_∞*-algebra on* $A[-1]$.

Notice that $\{a\}_D^1=PDa$ and the graded symmetry follows easily from Leibniz rule and the abelianity of A: for instance

$$\begin{aligned}\{a,b\}_D^2 &= P[Da,b]=P(D[a,b]-(-1)^{\overline{a}\overline{D}}[a,Db])\\ &=(-1)^{\overline{a}(\overline{D}+\overline{Db})}P[Db,a]=(-1)^{\overline{a}\,\overline{b}}\{b,a\}_D^2\,.\end{aligned}$$

If $D(A)\subset A$, then $\{a\}_D^1=Da$ and $\{\cdots\}_D^n=0$ for every $n>1$.

The link between higher derived brackets and homotopy fibers is given by the following result.

THEOREM 8.2 (Th. Voronov). *In the same setup of the above Theorem 8.1, if* D *has degree 1 and* $D^2=0$, *then the* L_∞*-algebra* $(A[-1],\{\cdot\}_D^1,\{\cdot,\cdot\}_D^2,\dots)$ *is weakly equivalent to the homotopy fiber of the inclusion of DGLAs*

$$(L,D,[\cdot,\cdot])\hookrightarrow(M,D,[\cdot,\cdot])\,.$$

PROOF. See either [42, Cor. 4.1] or [1, Thm. 1.1]. □

In order to apply the higher derived brackets construction to the study of holomorphic coisotropic deformations, we look for a splitting $A_Z^{0,*}(\bigwedge^{\geq 1}\mathcal{N}_{Z|X}[1])\to A_X^{0,*}(\bigwedge^{\geq 1}\Theta_X[1])$ of the exact sequence (7.1), such that the image is an abelian graded Lie subalgebra of $A_X^{0,*}(\bigwedge^{\geq 1}\Theta_X[1])$. In the differentiable setting something similar is accomplished after restricting to a tubular neighborhood of Z in X; in the complex analytic setting, however, one has to work from the outset in the rather restrictive hypothesis that $X = E$ is the total space of a holomorphic vector bundle $p\colon E \to Z$ over Z, which is embedded in E as the zero section.

Denoting by $N_{Z|E}$ the normal bundle, there is a canonical identification $N_{Z|E} \cong E$, in this way the pull-back bundle $p^*N_{Z|E} \to E$ is canonically identified with the subbundle $p^*E \subset TE$ of vertical tangent vectors. The above induces a morphism $\mathcal{N}_{Z|E} \to p_*\Theta_E$ of sheaves on Z, sending a section ξ of $N_{Z|E}$ to the vector field constantly ξ_p along the fiber E_p; by multiplicative extension we also get $\bigwedge^{\geq 1} \mathcal{N}_{Z|E} \to p_* \bigwedge^{\geq 1} \Theta_E$. Thus, for every open subset $U \subset Z$ we have a morphism

$$\bigwedge^{\geq 1} \mathcal{N}_{Z|E}[1](U) \to \bigwedge^{\geq 1} \Theta_E[1](p^{-1}(U))$$

whose image is an abelian graded Lie subalgebra. Acting via pull-back on differential forms, we obtain a splitting:

$$\sigma\colon A_Z^{0,*}(\bigwedge^{\geq 1} \mathcal{N}_{Z|E}[1]) \longrightarrow A_E^{0,*}(\bigwedge^{\geq 1} \Theta_E[1]) \tag{8.2}$$

of the exact sequence (7.1) with the required properties.

Suppose given a Poisson bivector π on E such that $Z \subset E$ is a coisotropic submanifold: we look at π as a section of $A_E^{0,0}(\bigwedge^2 \Theta_E) \subset \left(A_E^{0,*}(\bigwedge^{\geq 1} \Theta_E[1])\right)^1$, then it follows from Proposition 5.2 that $\pi \in L^1_{Z|E}$. Since π is holomorphic Poisson, putting $D = \overline{\partial} + d_\pi$ we are in the algebraic setup of Theorem 8.1.

Denote by $P\colon A_E^{0,*}(\bigwedge^{\geq 1} \Theta_E[1]) \to A_Z^{0,*}(\bigwedge^{\geq 1} \mathcal{N}_{Z|E}[1])$ the projection as in the exact sequence (7.1), and take $\sigma : A_Z^{0,*}(\bigwedge^{\geq 1} \mathcal{N}_{Z|E}[1]) \to A_E^{0,*}(\bigwedge^{\geq 1} \Theta_E[1])$ as in (8.2). Since the image of σ is $\overline{\partial}$-closed, the higher derived brackets

$$\{\cdots\}_D^n\colon A_Z^{0,*}(\bigwedge^{\geq 1} \mathcal{N}_{Z|E}[1])^{\odot n} \to A_Z^{0,*}(\bigwedge^{\geq 1} \mathcal{N}_{Z|E}[1])$$

are equal to

$$\{\xi\}_D^1 = P(\overline{\partial}\sigma(\xi) + d_\pi(\sigma(\xi))) = (\overline{\partial} + d_\pi)\xi,$$
$$\{\xi_1, \dots, \xi_n\}_D^n = P([[\cdots[d_\pi(\sigma(\xi_1)), \sigma(\xi_2)]_{SN}, \cdots], \sigma(\xi_n)]_{SN}), \qquad n \geq 2,$$

and define an L_∞ structure on $A_Z^{0,*}(\bigwedge^{\geq 1}\mathcal{N}_{Z|E})$.

Thus, according to Theorem 8.2, the corresponding L_∞ structure on $A_Z^{0,*}(\bigwedge^{\geq 1}\mathcal{N}_{Z|E})$ is weakly equivalent to the homotopy fiber of $\chi\colon L_{Z|E} \to A_E^{0,*}(\bigwedge^{\geq 1}\Theta_E[1])$; therefore Theorems 7.1 and 5.7 immediately imply the following:

Corollary 8.3. *In the above hypotheses there is an isomorphism of functors of Artin rings*

$$\mathrm{Def}_{A_Z^{0,*}(\wedge^{\geq 1}\mathcal{N}_{Z|E})} \cong \mathrm{Hilb}^{co}_{Z|E}\ .$$

Remark 8.4. The L_∞-algebra $A_Z^{0,*}(\bigwedge^{\geq 1}\mathcal{N}_{Z|E})$ is concentrated in degrees ≥ 1; in particular the elements of $\mathrm{Def}_{A_Z^{0,*}(\wedge^{\geq 1}\mathcal{N}_{Z|E})}(A)$ correspond in a bijective way to solutions $\xi \in A_Z^{0,0}(\mathcal{N}_{Z|E})\otimes\mathfrak{m}_A$ of the Maurer-Cartan equation:

$$\sum_{n\geq 1}\frac{\{\xi,\dots,\xi\}_D^n}{n!} = 0\,.$$

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