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*Nonexistence and Existence Results for a Fourth-Order Discrete Mixed Boundary Value Problem**

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Abstract. This paper is concerned with a class of fourth-order nonlinear difference equations. By making use of the critical point theory, we establish various sets of sufficient conditions for the nonexistence and existence of solutions for mixed boundary value problem and give some new results. Results obtained generalize and complement the existing ones.

1. Introduction

Below **N**, **Z** and **R** denote the sets of all natural numbers, integers and real numbers respectively and k is a positive integer. For any a, $b \in \mathbf{Z}$, define $\mathbf{Z}(a) = \{a, a+1, \cdots\}$, $\mathbf{Z}(a, b) = \{a, a+1, \cdots, b\}$ when $a < b$. Also, * denotes the transpose of a vector.

Recently, difference equations have attracted the interest of many researchers since they provide a natural description of several discrete models. Such discrete models are often investigated in various fields of science and technology such as computer science, economics, neural networks, ecology, cybernetics, biological systems, optimal control, and population dynamics. These studies cover many of the branches of difference equations, such as stability, attractivity, periodicity, oscillation, and boundary value problems, see [9,10,12-15,23-25,31,32,34] and the references therein.

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The present paper considers the fourth-order nonlinear difference equation

$$\begin{aligned}(1.1)\qquad \Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n\left(\Delta u_{n-1}\right)^{\mu}\right) + r_n u_n^{\nu} \\ = f(n, u_{n+1}, u_n, u_{n-1}),\ n \in \mathbf{Z}(1,k),\end{aligned}$$

with boundary value conditions

$$(1.2)\qquad \Delta u_{-1} = \Delta u_0 = 0,\ u_{k+1} = u_{k+2} = 0,$$

where Δ is the forward difference operator $\Delta u_n = u_{n+1} - u_n$, $\Delta^2 u_n = \Delta(\Delta u_n)$, p_n is nonzero and real valued for each $n \in \mathbf{Z}(0, k+1)$, $\{q_n\}_{n\in\mathbf{Z}(1,k+1)}$ and $\{r_n\}_{n\in\mathbf{Z}(1,k)}$ are real sequences, μ and ν are the ratio of odd positive integers, $f \in C(\mathbf{R}^4, \mathbf{R})$.

We may think of (1.1) with (1.2) as being a discrete analogue of the following fourth-order nonlinear differential equation

$$(1.3)\quad \left[p(t)u''(t)\right]'' - \left[q(t)u'(t)\right]' = f(t, u(t+1), u(t), u(t-1)),\ t \in [a,b],$$

with boundary value conditions

$$(1.4)\qquad u(a) = u'(a) = 0,\ u(b) = u'(b) = 0.$$

Eq. (1.3) includes the following equation

$$(1.5)\qquad u^{(4)}(t) = f(t, u(t)),\ t \in \mathbf{R},$$

which is used to describe the bending of an elastic beam; see, for example, [1,6,26,28,30,43] and the references therein. Equations similar in structure to (1.3) arise in the study of the existence of solitary waves [39] of lattice differential equations and periodic solutions [20,22] of functional differential equations. Owing to its importance in physics, many methods are applied to study fourth-order boundary value problems by many authors.

In recent years, the study of boundary value problems for differential equations developed at a relatively rapid rate. By using various methods and techniques, such as fixed point theory, topological degree theory, coincidence degree theory, a series of existence results of nontrivial solutions for differential equations have been obtained in literatures, we refer to [3-5,8,27,41]. Critical point theory is also an important tool to deal with problems on differential equations [17,21,33,37,47]. Only since 2003, critical point theory has been employed to establish sufficient conditions on the existence of periodic solutions of difference equations. By using the critical point theory, Guo and Yu [23-25] and Shi *et al.* [38] have successfully proved the existence of periodic solutions of second-order nonlinear difference equations. We also refer to [44,45] for the discrete boundary value problems. Compared to first-order or second-order difference equations, the study of higher-order equations, and in particular, fourth-order equations, has received considerably less attention (see, for example, [10-16,19,36,40,42] and the references contained therein). Yan, Liu [42] in 1997 and Thandapani, Arockiasamy [40] in 2001 studied the following fourth-order difference equation of form,

$$\Delta^2\left(p_n\Delta^2 u_n\right) + f(n,u_n) = 0,\ n\in\mathbf{Z}, \tag{1.6}$$

and obtained criteria for the oscillation and nonoscillation of solutions for equation (1.6). In 2005, Cai, Yu and Guo [7] have obtained some criteria for the existence of periodic solutions of the fourth-order difference equation

$$\Delta^2\left(p_{n-2}\Delta^2 u_{n-2}\right) + f(n,u_n) = 0,\ n\in\mathbf{Z}. \tag{1.7}$$

In 1995, Peterson and Ridenhour considered the disconjugacy of equation (1.7) when $p_n \equiv 1$ and $f(n,u_n) = q_n u_n$ (see [36]).

The boundary value problem (BVP) for determining the existence of solutions of difference equations has been a very active area of research in the last twenty years, and for surveys of recent results, we refer the reader to the monographs by Agarwal *et al.* [2,18,29,34]. As far as we know results obtained in the literature for the BVP (1.1) with (1.2) are very scarce. Since f in (1.1) depends on u_{n+1} and u_{n-1}, the traditional ways of establishing the functional in [23-25,44-46] are inapplicable to our case. As a result, the goal of this paper is to fill the gap in this area.

Motivated by the above results, we use the critical point theory to give some sufficient conditions for the nonexistence and existence of solutions for the BVP (1.1) with (1.2). We shall study the superlinear and sublinear cases. The main idea in this paper is to transfer the existence of the BVP (1.1) with (1.2) into the existence of the critical points of some functional. The proof is based on the notable Mountain Pass Lemma in combination with variational technique. The purpose of this paper is two-folded. On one hand, we shall further demonstrate the powerfulness of critical point theory in the study of solutions for boundary value problems of difference equations. On the other hand, we shall complement existing results. The motivation for the present work stems from the recent paper [17].

Let

$$\begin{aligned}\bar{p} &= \max\{p_n : n \in \mathbf{Z}(1, k+1)\},\ \underline{p} = \min\{p_n : n \in \mathbf{Z}(1, k+1)\},\\ \bar{q} &= \max\{q_n : n \in \mathbf{Z}(2, k+1)\},\ \underline{q} = \min\{q_n : n \in \mathbf{Z}(2, k+1)\},\\ \bar{r} &= \max\{r_n : n \in \mathbf{Z}(1, k)\},\ \underline{r} = \min\{r_n : n \in \mathbf{Z}(1, k)\}.\end{aligned}$$

Our main results are as follows.

THEOREM 1.1. *Assume that the following hypotheses are satisfied:*

(p) *for any* $n \in \boldsymbol{Z}(1, k+1)$, $p_n < 0$;

(q) *for any* $n \in \boldsymbol{Z}(2, k+1)$, $q_n \leq 0$;

(r) *for any* $n \in \boldsymbol{Z}(1, k)$, $r_n \leq 0$;

(F_1) *there exists a functional* $F(n, \cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *with* $F(0, \cdot) = 0$ *such that*

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),\ \forall n \in \boldsymbol{Z}(1, k);$$

(F_2) *there exists a constant* $M_0 > 0$ *for all* $(n, v_1, v_2) \in \boldsymbol{Z}(1, k) \times \boldsymbol{R}^2$ *such that*

$$\left|\frac{\partial F(n, v_1, v_2)}{\partial v_1}\right| \leq M_0,\ \left|\frac{\partial F(n, v_1, v_2)}{\partial v_2}\right| \leq M_0.$$

Then the BVP (1.1) with (1.2) possesses at least one solution.

REMARK 1.1. Assumption (F_2) implies that there exists a constant $M_1 > 0$ such that

(F_2') $|F(n, v_1, v_2)| \leq M_1 + M_0(|v_1| + |v_2|),\ \forall (n, v_1, v_2) \in \mathbf{Z}(1, k) \times \mathbf{R}^2.$

THEOREM 1.2. *Suppose that* (F_1) *and the following hypotheses are satisfied:*

(p') *for any* $n \in \boldsymbol{Z}(1, k+1)$, $p_n > 0$;

(q') *for any* $n \in \boldsymbol{Z}(2, k+1)$, $q_n \geq 0$;

(r') *for any* $n \in \boldsymbol{Z}(1, k)$, $r_n \geq 0$;

(F_3) *there exists a functional* $F(n,\cdot) \in C^1(\boldsymbol{Z} \times \boldsymbol{R}^2, \boldsymbol{R})$ *such that*

$$\lim_{r\to 0} \frac{F(n,v_1,v_2)}{r^2} = 0,\ r = \sqrt{v_1^2+v_2^2},\ \forall n \in \boldsymbol{Z}(1,k);$$

(F_4) *there exists a constant* $\beta > \max\{2, \mu+1, \nu+1\}$ *such that for any* $n \in \boldsymbol{Z}(1,k)$,

$$0 < \frac{\partial F(n,v_1,v_2)}{\partial v_1} v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2} v_2 < \beta F(n,v_1,v_2),\ \forall (v_1,v_2) \neq 0.$$

Then the BVP (1.1) with (1.2) possesses at least two nontrivial solutions.

Remark 1.2. Assumption (F_4) implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

(F_4') $F(n,v_1,v_2) > a_1 \left(\sqrt{v_1^2+v_2^2}\right)^{\beta} - a_2,\ \forall n \in \mathbf{Z}(1,k).$

Theorem 1.3. *Suppose that* (p'), (q'), (r'), (F_1) *and the following assumption are satisfied:*

(F_5) *there exist constants* $R > 0$ *and* $1 < \alpha < 2$ *such that for* $n \in \boldsymbol{Z}(1,k)$ *and* $\sqrt{v_1^2+v_2^2} \geq R$,

$$0 < \frac{\partial F(n,v_1,v_2)}{\partial v_1} v_1 + \frac{\partial F(n,v_1,v_2)}{\partial v_2} v_2 \leq \alpha F(n,v_1,v_2).$$

Then the BVP (1.1) with (1.2) possesses at least one solution.

Remark 1.3. Assumption (F_5) implies that for each $n \in \mathbf{Z}(1,k)$ there exist constants $a_3 > 0$ and $a_4 > 0$ such that

(F_5') $F(n,v_1,v_2) \leq a_3 \left(\sqrt{v_1^2+v_2^2}\right)^{\alpha} + a_4,\ \forall (n,v_1,v_2) \in \mathbf{Z}(1,k) \times \mathbf{R}^2.$

Theorem 1.4. *Suppose that* (p), (q), (r), (F_1) *and the following assumption is satisfied:*

(F_6) $v_2 f(n,v_1,v_2,v_3) > 0$, *for* $v_2 \neq 0$, $\forall n \in \boldsymbol{Z}(1,k)$.

Then the BVP (1.1) with (1.2) has no nontrivial solutions.

Remark 1.4. In the existing literature, results on the nonexistence of solutions of discrete boundary value problems are scarce. Hence, Theorem 1.4 complements existing ones.

The rest of this paper is organized as follows. First, in Section 2, we shall establish the variational framework for the BVP (1.1) with (1.2) and transfer the problem of the existence of solutions the BVP (1.1) with (1.2) into that of the existence of critical points of the corresponding functional. Some related fundamental results will also be recalled. Then, in Section 3, we shall complete the proof of the results by using the critical point method. Finally, in Section 4, we shall give three examples to illustrate the main results.

For the basic knowledge of variational methods, the reader is referred to [33,35,37,47].

2. Variational Structure and Some Lemmas

In order to apply the critical point theory, we shall establish the corresponding variational framework for the BVP (1.1) with (1.2) and give some lemmas which will be of fundamental importance in proving our main results. First, we give some basic notation.

Let $\mathbf{R}^k$ be the real Euclidean space with dimension k. Define the inner product on $\mathbf{R}^k$ as follows:

$$\langle u, v\rangle = \sum_{j=1}^{k} u_j v_j, \ \forall u, v \in \mathbf{R}^k, \tag{2.1}$$

by which the norm $\|\cdot\|$ can be induced by

$$\|u\| = \left(\sum_{j=1}^{k} u_j^2\right)^{\frac{1}{2}}, \ \forall u \in \mathbf{R}^k. \tag{2.2}$$

On the other hand, we define the norm $\|\cdot\|_r$ on $\mathbf{R}^k$ as follows:

$$\|u\|_r = \left(\sum_{j=1}^{k} |u_j|^r\right)^{\frac{1}{r}}, \tag{2.3}$$

for all $u \in \mathbf{R}^k$ and $r > 1$.

Since $\|u\|_r$ and $\|u\|_2$ are equivalent, there exist constants $c_1,\ c_2$ such that $c_2 \geq c_1 > 0$, and

$$c_1\|u\|_2 \leq \|u\|_r \leq c_2\|u\|_2, \ \forall u \in \mathbf{R}^k. \tag{2.4}$$

Clearly, $\|u\| = \|u\|_2$. For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, for the BVP (1.1) with (1.2), consider the functional J defined on $\mathbf{R}^k$ as follows:

$$J(u) = \frac{1}{2}\sum_{n=1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{\mu+1}\sum_{n=1}^{k} q_{n+1}\left(\Delta u_n\right)^{\mu+1} + \frac{1}{\nu+1}\sum_{n=1}^{k} r_n u_n^{\nu+1} - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) + \frac{1}{2}p_1\left(\Delta u_1\right)^2, \tag{2.5}$$

where

$$\frac{\partial F(n-1, v_2, v_3)}{\partial v_2} + \frac{\partial F(n, v_1, v_2)}{\partial v_2} = f(n, v_1, v_2, v_3),$$
$$\Delta u_{-1} = \Delta u_0 = 0,\ u_{k+1} = u_{k+2} = 0.$$

Clearly, $J \in C^1(\mathbf{R}^k, \mathbf{R})$ and for any $u = \{u_n\}_{n=1}^{k} = (u_1, u_2, \ldots, u_k)^*$, by using $\Delta u_{-1} = \Delta u_0 = 0,\ u_{k+1} = u_{k+2} = 0$, we can compute the partial derivative as

$$\frac{\partial J}{\partial u_n} = \Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n\left(\Delta u_{n-1}\right)^{\mu}\right) + r_n u_n^{\nu} - f(n, u_{n+1}, u_n, u_{n-1}),\ \forall n \in \mathbf{Z}(1, k).$$

Thus, u is a critical point of J on $\mathbf{R}^k$ if and only if

$$\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n\left(\Delta u_{n-1}\right)^{\mu}\right) + r_n u_n^{\nu} = f(n, u_{n+1}, u_n, u_{n-1}),\ \forall n \in \mathbf{Z}(1, k).$$

We reduce the existence of the BVP (1.1) with (1.2) to the existence of critical points of J on $\mathbf{R}^k$. That is, the functional J is just the variational framework of the BVP (1.1) with (1.2).

Let P and Q be the $k \times k$ matrices defined by

$$P = \begin{pmatrix} 6 & -4 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & -4 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -4 & 6 \end{pmatrix},$$

$$Q = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Clearly, P and Q are positive definite. Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the eigenvalues of P, $\tilde{\lambda}_1, \tilde{\lambda}_2, \cdots, \tilde{\lambda}_k$ be the eigenvalues of Q. Applying matrix theory, we know $\lambda_j > 0$, $\tilde{\lambda}_j > 0$, $j = 1, 2, \cdots, k$. Without loss of generality, we may assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k, \tag{2.6}$$

$$0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_k. \tag{2.7}$$

Let E be a real Banach space, $J \in C^1(E, \mathbf{R})$, i.e., J is a continuously Fréchet-differentiable functional defined on E. J is said to satisfy the Palais-Smale condition (P.S. condition for short) if any sequence $\left\{u^{(l)}\right\} \subset E$ for which $\left\{J\left(u^{(l)}\right)\right\}$ is bounded and $J'\left(u^{(l)}\right) \to 0 (l \to \infty)$ possesses a convergent subsequence in E.

Let B_ρ denote the open ball in E about 0 of radius ρ and let ∂B_ρ denote its boundary.

Lemma 2.1 (Mountain Pass Lemma [37]). *Let E be a real Banach space and $J \in C^1(E, \boldsymbol{R})$ satisfy the P.S. condition. If $J(0) = 0$ and*

(J_1) there exist constants ρ, $a > 0$ such that $J|_{\partial B_\rho} \geq a$, and

(J_2) there exists $e \in E \setminus B_\rho$ such that $J(e) \leq 0$.

Then J possesses a critical value $c \geq a$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} J(g(s)), \tag{2.8}$$

where

$$\Gamma = \{g \in C([0,1], E) | g(0) = 0,\ g(1) = e\}. \tag{2.9}$$

Lemma 2.2. *Suppose that (p'), (q'), (r'), (F_1), (F_3) and (F_4) are satisfied. Then the functional J satisfies the P.S. condition.*

PROOF. Let $u^{(l)} \in \mathbf{R}^k,\ l \in \mathbf{Z}(1)$ be such that $\left\{J\left(u^{(l)}\right)\right\}$ is bounded. Then there exists a positive constant M_2 such that

$$-M_2 \leq J\left(u^{(l)}\right) \leq M_2,\ \forall l \in \mathbf{N}.$$

By (F_4'), we have

$$\begin{aligned}
-M_2 \leq J\left(u^{(l)}\right) &= \frac{1}{2}\sum_{n=1}^{k} p_{n+1}\left(\Delta^2 u_n^{(l)}\right)^2 + \frac{1}{\mu+1}\sum_{n=1}^{k} q_{n+1}\left(\Delta u_n^{(l)}\right)^{\mu+1} \\
&\quad + \frac{1}{\nu+1}\sum_{n=1}^{k} r_n\left(u_n^{(l)}\right)^{\nu+1} - \sum_{n=1}^{k} F\left(n, u_{n+1}^{(l)}, u_n^{(l)}\right) + \frac{1}{2}p_1\left(\Delta u_1^{(l)}\right)^2 \\
&\leq \frac{\bar{p}}{2}\sum_{n=1}^{k}\left(u_{n+2}^{(l)} - 2u_{n+1}^{(l)} + u_n^{(l)}\right)^2 \\
&\quad + \frac{\bar{q}c_2^{\mu+1}}{\mu+1}\left\{\left[\sum_{n=1}^{k}\left(u_{n+1}^{(l)} - u_n^{(l)}\right)^2\right]^{\frac{1}{2}}\right\}^{\mu+1} \\
&\quad + \frac{\bar{r}c_2^{\nu+1}}{\nu+1}\left\|u^{(l)}\right\|^{\nu+1} - a_1\sum_{n=1}^{k}\left[\sqrt{\left(u_{n+1}^{(l)}\right)^2 + \left(u_n^{(l)}\right)^2}\right]^{\beta} \\
&\quad + a_2 k + \bar{p}\left\|u^{(l)}\right\|^2 \\
&\leq \frac{\bar{p}}{2}\left(u^{(l)}\right)^* P u^{(l)} + \frac{\bar{q}c_2^{\mu+1}}{\mu+1}\left[\left(u^{(l)}\right)^* Q u^{(l)}\right]^{\frac{\mu+1}{2}} + \frac{\bar{r}c_2^{\nu+1}}{\nu+1}\left\|u^{(l)}\right\|^{\nu+1} \\
&\quad - a_1 c_1^{\beta}\left\|u^{(l)}\right\|^{\beta} + a_2 k + \bar{p}\left\|u^{(l)}\right\|^2 \\
&\leq \left(\frac{\lambda_k}{2} + 1\right)\bar{p}\left\|u^{(l)}\right\|^2 + \frac{\bar{q}c_2^{\mu+1}\tilde{\lambda}_k^{\frac{\mu+1}{2}}}{\mu+1}\left\|u^{(l)}\right\|^{\mu+1} + \frac{\bar{r}c_2^{\nu+1}}{\nu+1}\left\|u^{(l)}\right\|^{\nu+1} \\
&\quad - a_1 c_1^{\beta}\left\|u^{(l)}\right\|^{\beta} + a_2 k,
\end{aligned}$$

where $u^{(l)} = \left(u_1^{(l)}, u_2^{(l)}, \cdots, u_k^{(l)}\right)^*,\ u^{(l)} \in \mathbf{R}^k$. That is,

$$a_1 c_1^{\beta}\left\|u^{(l)}\right\|^{\beta} - \left(\frac{\lambda_k}{2} + 1\right)\bar{p}\left\|u^{(l)}\right\|^2 - \frac{\bar{q}c_2^{\mu+1}\tilde{\lambda}_k^{\frac{\mu+1}{2}}}{\mu+1}\left\|u^{(l)}\right\|^{\mu+1}$$

$$-\frac{\bar{r}c_2^{\nu+1}}{\nu+1}\left\|u^{(l)}\right\|^{\nu+1} \leq M_2 + a_2 k.$$

Since $\beta > \max\{2, \mu+1, \nu+1\}$, there exists a constant $M_3 > 0$ such that

$$\left\|u^{(l)}\right\| \leq M_3, \ \forall l \in \mathbf{N}.$$

Therefore, $\{u^{(l)}\}$ is bounded on $\mathbf{R}^k$. As a consequence, $\{u^{(l)}\}$ possesses a convergence subsequence in $\mathbf{R}^k$. Thus the P.S. condition is verified. □

3. Proof of the Main Results

In this Section, we shall prove our main results by using the critical point theory.

3.1. Proof of Theorem 1.1

PROOF. By (F_2'), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{n=1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{\mu+1}\sum_{n=1}^{k} q_{n+1}\left(\Delta u_n\right)^{\mu+1} + \frac{1}{\nu+1}\sum_{n=1}^{k} r_n u_n^{\nu+1} \\ &\quad - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) + \frac{1}{2}p_1\left(\Delta u_1\right)^2 \\ &\leq \frac{\bar{p}}{2}\sum_{n=1}^{k}\left(u_{n+2} - 2u_{n+1} + u_n\right)^2 + M_0\sum_{n=1}^{k}\left(|u_{n+1}| + |u_n|\right) + M_1 k \\ &\leq \frac{\bar{p}}{2}u^* P u + 2M_0\sum_{n=1}^{k}|u_n| + M_1 k \\ &\leq \frac{\bar{p}\lambda_1}{2}\|u\|^2 + 2M_0\sqrt{k}\|u\| + M_1 k \\ &\to -\infty \text{ as } \|u\| \to +\infty. \end{aligned}$$

The above inequality means that $-J(u)$ is coercive. By the continuity of $J(u)$, J attains its maximum at some point, and we denote it $\breve{u}$, that is,

$$J(\breve{u}) = \max\left\{J(u) | u \in \mathbf{R}^k\right\}.$$

Clearly, $\breve{u}$ is a critical point of the functional J. This completes the proof of Theorem 1.1. □

3.2. Proof of Theorem 1.2

Proof. By (F_3), for any $\epsilon = \frac{p\lambda_1}{8}$($\lambda_1$ can be referred to (2.6)), there exists $\rho > 0$, such that

$$|F(n, v_1, v_2)| \le \frac{p\lambda_1}{8}\left(v_1^2 + v_2^2\right), \forall n \in \mathbf{Z}(1, k),$$

for $\sqrt{v_1^2 + v_2^2} \le \sqrt{2}\rho$.

For any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$ and $\|u\| \le \rho$, we have $|u_n| \le \rho,\ n \in \mathbf{Z}(1, k)$.

For any $n \in \mathbf{Z}(1, k)$,

$$\begin{aligned} J(u) &= \frac{1}{2}\sum_{n=1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{\mu+1}\sum_{n=1}^{k} q_{n+1}\left(\Delta u_n\right)^{\mu+1} + \frac{1}{\nu+1}\sum_{n=1}^{k} r_n u_n^{\nu+1} \\ &\quad - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) + \frac{1}{2}p_1\left(\Delta u_1\right)^2 \\ &\ge \frac{p}{2}\sum_{n=1}^{k}\left(u_{n+2} - 2u_{n+1} + u_n\right)^2 - \frac{p\lambda_1}{8}\sum_{n=1}^{k}\left(u_{n+1}^2 + u_n^2\right) \\ &\ge \frac{p}{2}u^* P u - \frac{p\lambda_1}{4}\|u\|^2 \\ &\ge \frac{p\lambda_1}{2}\|u\|^2 - \frac{p\lambda_1}{4}\|u\|^2 \\ &= \frac{p\lambda_1}{4}\|u\|^2, \end{aligned}$$

where $u = (u_1, u_2, \cdots, u_k)^*,\ u \in \mathbf{R}^k$.

Take $a = \frac{p\lambda_1}{4}\rho^2 > 0$. Therefore,

$$J(u) \ge a > 0,\ \forall u \in \partial B_\rho.$$

At the same time, we have also proved that there exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho} \ge a$. That is to say, J satisfies the condition (J_1) of the Mountain Pass Lemma.

For our setting, clearly $J(0) = 0$. In order to exploit the Mountain Pass Lemma in critical point theory, we need to verify all other conditions of the Mountain Pass Lemma. By Lemma 2.2, J satisfies the P.S. condition. So it suffices to verify the condition (J_2).

From the proof of the P.S. condition in Lemma 2.2, we know

$$J(u) \leq \left(\frac{\lambda_k}{2}+1\right)\bar{p}\,\|u\|^2 + \frac{\bar{q}c_2^{\mu+1}\tilde{\lambda}_k^{\frac{\mu+1}{2}}}{\mu+1}\,\|u\|^{\mu+1} + \frac{\bar{r}c_2^{\nu+1}}{\nu+1}\,\|u\|^{\nu+1} - a_1c_1^{\beta}\|u\|^{\beta} + a_2k.$$

Since $\beta > \max\{2, \mu+1, \nu+1\}$, we can choose $\bar{u}$ large enough to ensure that $J(\bar{u}) < 0$.

By the Mountain Pass Lemma, J possesses a critical value $c \geq a > 0$, where

$$c = \inf_{h\in\Gamma}\sup_{s\in[0,1]} J(h(s)),$$

and

$$\Gamma = \{h \in C([0,1], \mathbf{R}^k) \mid h(0) = 0,\ h(1) = \bar{u}\}.$$

Let $\tilde{u} \in \mathbf{R}^k$ be a critical point associated to the critical value c of J, i.e., $J(\tilde{u}) = c$. Similar to the proof of the P.S. condition, we know that there exists $\hat{u} \in \mathbf{R}^k$ such that

$$J(\hat{u}) = c_{\max} = \max_{s\in[0,1]} J(h(s)).$$

Clearly, $\hat{u} \neq 0$. If $\tilde{u} \neq \hat{u}$, then the conclusion of Theorem 1.2 holds. Otherwise, $\tilde{u} = \hat{u}$. Then $c = J(\tilde{u}) = c_{\max} = \max\limits_{s\in[0,1]} J(h(s))$. That is,

$$\sup_{u\in\mathbf{R}^k} J(u) = \inf_{h\in\Gamma}\sup_{s\in[0,1]} J(h(s)).$$

Therefore,

$$c_{\max} = \max_{s\in[0,1]} J(h(s)),\ \forall h \in \Gamma.$$

By the continuity of $J(h(s))$ with respect to s, $J(0) = 0$ and $J(\bar{u}) < 0$ imply that there exists $s_0 \in (0,1)$ such that

$$J\left(h\left(s_0\right)\right) = c_{\max}.$$

Choose $h_1,\ h_2 \in \Gamma$ such that $\{h_1(s) \mid s \in (0,1)\} \cap \{h_2(s) \mid s \in (0,1)\}$ is empty, then there exists $s_1,\ s_2 \in (0,1)$ such that

$$J\left(h_1\left(s_1\right)\right) = J\left(h_2\left(s_2\right)\right) = c_{\max}.$$

Thus, we get two different critical points of J on $\mathbf{R}^k$ denoted by

$$u^1 = h_1(s_1), \ u^2 = h_2(s_2).$$

The above argument implies that the BVP (1.1) with (1.2) possesses at least two nontrivial solutions. The proof of Theorem 1.2 is finished. □

3.3. Proof of Theorem 1.3

PROOF. We only need to find at least one critical point of the functional J defined as in (2.5).

By (F_5'), for any $u = (u_1, u_2, \cdots, u_k)^* \in \mathbf{R}^k$, we have

$$\begin{aligned}
J(u) &= \frac{1}{2}\sum_{n=1}^{k} p_{n+1}\left(\Delta^2 u_n\right)^2 + \frac{1}{\mu+1}\sum_{n=1}^{k} q_{n+1}\left(\Delta u_n\right)^{\mu+1} + \frac{1}{\nu+1}\sum_{n=1}^{k} r_n u_n^{\nu+1} \\
&\quad - \sum_{n=1}^{k} F(n, u_{n+1}, u_n) + \frac{1}{2}p_1\left(\Delta u_1\right)^2 \\
&\geq \frac{p}{2}\sum_{n=1}^{k}\left(u_{n+2} - 2u_{n+1} + u_n\right)^2 - a_3\sum_{n=1}^{k}\left(\sqrt{u_{n+1}^2 + u_n^2}\right)^{\alpha} - a_4 k \\
&= \frac{p}{2}u^* P u - a_3\left\{\left[\sum_{n=1}^{k}\left(\sqrt{u_{n+1}^2 + u_n^2}\right)^{\alpha}\right]^{\frac{1}{\alpha}}\right\}^{\alpha} - a_4 k \\
&\geq \frac{p\lambda_1}{2}\|u\|^2 - a_3 c_2^{\alpha}\left\{\left[\sum_{n=1}^{k}\left(u_{n+1}^2 + u_n^2\right)\right]^{\frac{1}{2}}\right\}^{\alpha} - a_4 k \\
&\geq \frac{p\lambda_1}{2}\|u\|^2 - 2^{\alpha} a_3 c_2^{\alpha}\|u\|^{\alpha} - a_4 k \\
&\to +\infty \text{ as } \|u\| \to +\infty.
\end{aligned}$$

By the continuity of J, we know from the above inequality that there exist lower bounds of values of the functional. And this means that J attains its minimal value at some point which is just the critical point of J with the finite norm. □

3.4. Proof of Theorem 1.4

PROOF. Assume, for the sake of contradiction, that the BVP (1.1) with

(1.2) has a nontrivial solution. Then J has a nonzero critical point $u^{\star}$. Since

$$\frac{\partial J}{\partial u_n} = \Delta^2\left(p_{n-1}\Delta^2 u_{n-2}\right) - \Delta\left(q_n\left(\Delta u_{n-1}\right)^{\mu}\right) + r_n u_n^{\nu} - f(n, u_{n+1}, u_n, u_{n-1}),$$

we get

$$\begin{aligned}
(3.1) \qquad & \sum_{n=1}^{k} f(n, u_{n+1}^{\star}, u_n^{\star}, u_{n-1}^{\star}) u_n^{\star} \\
&= \sum_{n=1}^{k} \left[\Delta^2\left(p_{n-1}\Delta^2 u_{n-2}^{\star}\right) - \Delta\left(q_n\left(\Delta u_{n-1}^{\star}\right)^{\mu}\right) + r_n\left(u_n^{\star}\right)^{\nu}\right] u_n^{\star} \\
&= \sum_{n=1}^{k} p_{n+1}\left(\Delta^2 u_n^{\star}\right)^2 + \sum_{n=1}^{k} q_{n+1}\left(\Delta u_n^{\star}\right)^{\mu+1} \\
&\quad + \sum_{n=1}^{k} r_n\left(u_n^{\star}\right)^{\nu+1} + p_1\left(\Delta u_1^{\star}\right)^2 \leq 0.
\end{aligned}$$

On the other hand, it follows from (F_6) that

$$\sum_{n=1}^{k} f(n, u_{n+1}^{\star}, u_n^{\star}, u_{n-1}^{\star}) u_n^{\star} > 0. \tag{3.2}$$

This contradicts (3.1) and hence the proof is complete. □

4. Examples

As an application of Theorems 1.2, 1.3 and 1.4, we give three examples to illustrate our main results.

Example 4.1. For $n \in \mathbf{Z}(1,k)$, assume that

$$\begin{aligned}
(4.1) \qquad & \Delta^4 u_{n-2} - \Delta\left(2^n\left(\Delta u_{n-1}\right)^{\mu}\right) + 8^n u_n^{\nu} \\
&= \beta u_n\left[\varphi(n)\left(u_{n+1}^2 + u_n^2\right)^{\frac{\beta}{2}-1} + \varphi(n-1)\left(u_n^2 + u_{n-1}^2\right)^{\frac{\beta}{2}-1}\right],
\end{aligned}$$

with boundary value conditions (1.2), where μ and ν are the ratio of odd positive integers, $\beta > \max\{2, \mu+1, \nu+1\}$, φ is continuously differentiable and $\varphi(n) > 0,\ n \in \mathbf{Z}(1,k)$ with $\varphi(0) = 0$. We have

$$p_n \equiv 1,\ q_n = 2^n,\ r_n = 8^n,$$

$$f(n,v_1,v_2,v_3)=\beta v_2\left[\varphi(n)\left(v_1^2+v_2^2\right)^{\frac{\beta}{2}-1}+\varphi(n-1)\left(v_2^2+v_3^2\right)^{\frac{\beta}{2}-1}\right]$$

and

$$F(n,v_1,v_2)=\varphi(n)\left(v_1^2+v_2^2\right)^{\frac{\beta}{2}}.$$

It is easy to verify all the assumptions of Theorem 1.2 are satisfied and then the BVP (4.1) with (1.2) possesses at least two nontrivial solutions.

Example 4.2. For $n\in\mathbf{Z}(1,k)$, assume that

$$\begin{aligned}(4.2)\qquad \Delta^2\left(3^{n-1}\Delta^2u_{n-2}\right)-\Delta\left(9^n\left(\Delta u_{n-1}\right)^{\mu}\right)+6^nu_n^{\nu}&\\ =\alpha u_n\left[\psi(n)\left(u_{n+1}^2+u_n^2\right)^{\frac{\alpha}{2}-1}+\psi(n-1)\left(u_n^2+u_{n-1}^2\right)^{\frac{\alpha}{2}-1}\right],&\end{aligned}$$

with boundary value conditions (1.2), where μ and ν are the ratio of odd positive integers, $1<\alpha<2$, ψ is continuously differentiable and $\psi(n)>0$, $n\in\mathbf{Z}(1,k)$ with $\psi(0)=0$. We have

$$\begin{aligned}&p_n=3^n,\ q_n=9^n,\ r_n=6^n,\\ &f(n,v_1,v_2,v_3)=\alpha v_2\left[\psi(n)\left(v_1^2+v_2^2\right)^{\frac{\alpha}{2}-1}+\psi(n-1)\left(v_2^2+v_3^2\right)^{\frac{\alpha}{2}-1}\right]\end{aligned}$$

and

$$F(n,v_1,v_2)=\psi(n)\left(v_1^2+v_2^2\right)^{\frac{\alpha}{2}}.$$

It is easy to verify all the assumptions of Theorem 1.3 are satisfied and then the BVP (4.2) with (1.2) possesses at least one solution.

Example 4.3. For $n\in\mathbf{Z}(1,k)$, assume that

$$\begin{aligned}(4.3)\qquad -\Delta^4u_{n-2}+\Delta\left(5^n\left(\Delta u_{n-1}\right)^{\mu}\right)-7^nu_n^{\nu}&\\ =\frac{4}{3}u_n\left[\left(u_{n+1}^2+u_n^2\right)^{-\frac{1}{3}}+\left(u_n^2+u_{n-1}^2\right)^{-\frac{1}{3}}\right],&\end{aligned}$$

with boundary value conditions (1.2), where μ and ν are the ratio of odd positive integers. We have

$$\begin{aligned}&p_n\equiv-1,\ q_n=-5^n,\ r_n=-7^n,\\ &f(n,v_1,v_2,v_3)=\frac{4}{3}v_2\left[\left(v_1^2+v_2^2\right)^{-\frac{1}{3}}+\left(v_2^2+v_3^2\right)^{-\frac{1}{3}}\right]\end{aligned}$$

and

$$F(n, v_1, v_2) = \left(v_1^2 + v_2^2\right)^{\frac{2}{3}}.$$

It is easy to verify all the assumptions of Theorem 1.4 are satisfied and then the BVP (4.3) with (1.2) has no nontrivial solutions.

References

[1] Aftabizadeh, A. R., Existence and uniqueness theorems for fourth-order boundary value problems, J. Math. Anal. Appl. **116**(2) (1986), 415–426.

[2] Agarwal, R. P., *Difference Equations and Inequalities: Theory, Methods and Applications*, Marcel Dekker, New York (2000).

[3] Ahlbrandt, C. D., Dominant and recessive solutions of symmetric three term recurrences, J. Differential Equations **107**(2) (1994), 238–258.

[4] Anuradha, V., Maya, C. and R. Shivaji, Positive solutions for a class of nonlinear boundary value problems with Neumann-Robin boundary conditions, J. Math. Anal. Appl. **236**(1) (1999), 94–124.

[5] Arcoya, D., Positive solutions for semilinear Dirichlet problems in an annulus, J. Differential Equations **94**(2) (1991), 217–227.

[6] Bai, Z. and H. Wang, On positive solutions of some nonlinear fourth-order beam equations, J. Math. Anal. Appl. **270**(2) (2002), 357–368.

[7] Cai, X. C., Yu, J. S. and Z. M. Guo, Existence of periodic solutions for fourth-order difference equations, Comput. Math. Appl. **50**(1–2) (2005), 49–55.

[8] Cecchi, M., Marini, M. and G. Villari, On the monotonicity property for a certain class of second order differential equations, J. Differential Equations **82**(1) (1989), 15–27.

[9] Chen, S. Z., Disconjugacy, disfocality, and oscillation of second order difference equations, J. Differential Equations **107**(2) (1994), 383–394.

[10] Chen, P., Existence of homoclinic orbits in discrete Hamilton systems without Palais-Smale condition, J. Difference Equ. Appl. **19**(11) (2013), 1781–1794.

[11] Chen, P. and H. Fang, Existence of periodic and subharmonic solutions for second-order p-Laplacian difference equations, Adv. Difference Equ. **2007** (2007), 1–9.

[12] Chen, P. and X. H. Tang, Infinitely many homoclinic solutions for the second-order discrete p-Laplacian systems, Bull. Belg. Math. Soc. **20**(2) (2013), 193–212.

[13] Chen, P. and X. H. Tang, Existence of homoclinic solutions for some second-order discrete Hamiltonian systems, J. Difference Equ. Appl. **19**(4) (2013), 633–648.

[14] Chen, P. and X. H. Tang, Existence and multiplicity of homoclinic orbits for $2n$th-order nonlinear difference equations containing both many advances and retardations, J. Math. Anal. Appl. **381**(2) (2011), 485–505.

[15] Chen, P. and X. H. Tang, Existence of homoclinic solutions for the second-order discrete p-Laplacian systems, Taiwanese J. Math. **15**(5) (2011), 2123–2143.

[16] Chen, P. and X. H. Tang, Existence of infinitely many homoclinic orbits for fourth-order difference systems containing both advance and retardation, Appl. Math. Comput. **217**(9) (2011), 4408–4415.

[17] Chen, P. and X. H. Tang, New existence and multiplicity of solutions for some Dirichlet problems with impulsive effects, Math. Comput. Modelling **55**(3–4) (2012), 723–739.

[18] Elaydi, S., *An Introduction to Difference Equations*, Springer, New York (2005).

[19] Fang, H. and D. P. Zhao, Existence of nontrivial homoclinic orbits for fourth-order difference equations, Appl. Math. Comput. **214**(1) (2009), 163–170.

[20] Guo, C. J., O'Regan, D. and R. P. Agarwal, Existence of multiple periodic solutions for a class of first-order neutral differential equations, Appl. Anal. Discrete Math. **5**(1) (2011), 147–158.

[21] Guo, C. J., O'Regan, D., Xu, Y. T. and R. P. Agarwal, Existence and multiplicity of homoclinic orbits of a second-order differential difference equation via variational methods, Appl. Math. Inform. Mech. **4**(1) (2012), 1–15.

[22] Guo, C. J. and Y. T. Xu, Existence of periodic solutions for a class of second order differential equation with deviating argument, J. Appl. Math. Comput. **28**(1–2) (2008), 425–433.

[23] Guo, Z. M. and J. S. Yu, Applications of critical point theory to difference equations, Fields Inst. Commun. **42** (2004), 187–200.

[24] Guo, Z. M. and J. S. Yu, Existence of periodic and subharmonic solutions for second-order superlinear difference equations, Sci. China Math. **46**(4) (2003), 506–515.

[25] Guo, Z. M. and J. S. Yu, The existence of periodic and subharmonic solutions of subquadratic second order difference equations, J. London Math. Soc. **68**(2) (2003), 419–430.

[26] Gupta, C. P., Existence and uniqueness theorems for the bending of an elastic beam equation, Appl. Anal. **26**(4) (1988), 289–304.

[27] Hale, J. K. and J. Mawhin, Coincidence degree and periodic solutions of neutral equations, J. Differential Equations **15**(2) (1974), 295–307.

[28] Han, G. and Z. Xu, Multiple solutions of some nonlinear fourth-order beam equations, Nonlinear Anal. **68**(12) (2008), 3646–3656.

[29] Kocic, V. L. and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications*, Kluwer Academic Publishers, Dordrecht (1993).

[30] Li, F., Zhang, Q. and Z. Liang, Existence and multiplicity of solutions of a kind of fourth-order boundary value problem, Nonlinear Anal. **62**(5) (2005), 803–816.

[31] Liu, Y. J. and W. G. Ge, Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator, J. Math. Anal. Appl. **278**(2) (2003), 551–561.

[32] Matsunaga, H., Hara, T. and S. Sakata, Global attractivity for a nonlinear difference equation with variable delay, Computers Math. Appl. **41**(5–6) (2001), 543–551.

[33] Mawhin, J. and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York (1989).

[34] Mickens, R. E., *Difference Equations: Theory and Application*, Van Nostrand Reinhold, New York (1990).

[35] Pankov, A. and N. Zakhrchenko, On some discrete variational problems, Acta Appl. Math. **65**(1–3) (2001), 295–303.

[36] Peterson, A. and J. Ridenhour, The (2,2)-disconjugacy of a fourth order difference equation, J. Difference Equ. Appl. **1**(1) (1995), 87–93.

[37] Rabinowitz, P. H., *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, Amer. Math. Soc., Providence, RI, New York (1986).

[38] Shi, H. P., Ling, W. P., Long, Y. H. and H. Q. Zhang, Periodic and subharmonic solutions for second order nonlinear functional difference equations, Commun. Math. Anal. **5**(2) (2008), 50–59.

[39] Smets, D. and M. Willem, Solitary waves with prescribed speed on infinite lattices, J. Funct. Anal. **149**(1) (1997), 266–275.

[40] Thandapani, E. and I. M. Arockiasamy, Fourth-order nonlinear oscillations of difference equations, Comput. Math. Appl. **42**(3–5) (2001), 357–368.

[41] Wang, H. Y., On the existence of positive solutions for semilinear elliptic equations in the annulus, J. Differential Equations **109**(1) (1994), 1–7.

[42] Yan, J. and B. Liu, Oscillatory and asymptotic behavior of fourth order nonlinear difference equations, Acta. Math. Sinica **13**(1) (1997), 105–115.

[43] Yao, Q., Positive solutions of a nonlinear elastic beam equation rigidly fastened on the left and simply supported on the right, Nonlinear Anal. **69**(5–6) (2008), 1570–1580.

[44] Yu, J. S. and Z. M. Guo, Boundary value problems of discrete generalized Emden-Fowler equation, Sci. China Math. **49**(10) (2006), 1303–1314.

[45] Yu, J. S. and Z. M. Guo, On boundary value problems for a discrete generalized Emden-Fowler equation, J. Differential Equations **231**(1) (2006), 18–31.

[46] Zhou, Z., Yu, J. S. and Y. M. Chen, Periodic solutions of a $2n$th-order nonlinear difference equation, Sci. China Math. **53**(1) (2010), 41–50.

[47] Zou, W. M. and M. Schechter, *Critical Point Theory and Its Applications*, Springer, New York (2006).

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