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# *A Mechanical Model of Brownian Motion with Uniform Motion Area*

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**Abstract.** We consider a system of plural massive particles interacting with an ideal gas, evolved according to non-random mechanical principles, via interaction potentials. We first prove the weak convergence of the (position, velocity)-process of the massive particles until certain time, under a certain scaling limit, and give the precise expression of the limiting process, a diffusion process. In the second half, we consider a special case which includes the case of "two same type massive particles" as a concrete example, and prove the convergence of the process of the massive particles until any time. The precise description of the limit process, a combination of a "diffusion phase" and a "uniform motion phase", is also given.

## 1. Introduction

Brownian motion, first observed accidentally by Brown in 1827, is a well-known physical phenomenon concerning the dynamics of a small particle put into a fluid in equilibrium, *e.g.*, a grain of pollen in a glass of water [14]. Its first physical explanation was given by Einstein: it is due to the collisions of the particle with the numerous much lighter fluid atoms. In more mathematical terms, the following rough explanation is well-used nowadays: since the massive particle is collided by a big number of very light water atoms, if we could assume that the interactions from each atom at each time are independent, then by central limit theorem for the sum of *i.i.d.* random variables, this will give in a suitable limit the Brownian motion.

However, this "independent-interacting-atoms assumption" could not be the case indeed, since the interactions affect not only the massive particle(s),

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but also the atoms. So even in a model where there is only one massive particle and only the interactions through collisions are considered, the possibility of re-collisions makes it hopeless to get the mentioned independence of the atoms. This becomes a more evident and significant drawback when there are more than one massive particles, or when considering the model with interactions caused by potentials. Indeed, the actual motion of the massive particles depends also on the past events, hence is not even a Markov process.

So in order to study this phenomenon more precisely, one needs to consider some model that consists with the mentioned dependence on the past. In such a model, a finite number of massive particles interact with a gas of infinitely many light particles, with the dynamics fully deterministic, Hamiltonian, as long as the initial condition is given. The only source of randomness is from the initial configuration of the light particles. The problem we will be concerned with is to describe the motion of the massive particles in the Brownian limit, where the mass $m$ of the light particles goes to 0, while the density and the velocities of them have order $m^{-1/2}$.

This type of model, called a mechanical model of Brownian motion, was first introduced and studied by Holley [8], for the case of only one massive particle, with the whole system in dimension $d = 1$, and the interactions given by collisions. This model was later extended by, *e.g.*, Dürr-Goldstein-Lebowitz [5], [6], [7], Calderoni-Dürr-Kusuoka [2], to the case of higher dimensional spaces. Szász-Tóth [16] also considered some related problem. But in all these papers, the numbers of massive objects consisted in their models were all 1, and the interactions were collisions. [11] considered this type of problem with plural massive particles and interactions given by potentials under some conditions, especially, if we want a convergence until any time, when there are two massive particles, they need to be "different types" (the explanation of this terminology will be given later). Since this paper is along the same line as [11], we will come back to some more detailed description of [11] later.

There are a lot of papers related to our topic, in the sense of "deriving Brownian motion from dynamics consisting the dependence on the past" (or "re-collision" for the collisional interactions). For example, Chernov-Dolgopyat [4] considered a model with only one heavy particle and one light particle but with full re-collisions, Caprino-Marchioro-Pulvirenti [3] considered a model with the mean-field approximation from the beginning, and with a different scaling. See also the references therein.

However, in the literature, to the best knowledge of the author, there are not so many papers concerning with our problem of "deriving Brownian motion from a Hamiltonian dynamics consisting of massive particle(s) with infinitely many ideal gas light particles", except the ones [8], [5], [6], [7], [2], [11] quoted before. Especially, when there are more than one massive particles, [11] is the only paper that we know. We notice that the Markov approximation method used in [5] and [16], *etc.*, is not available anymore when there are more than one massive particles. We tackle this problem with the help of martingale problem theory. This framework of proof was also used in [2] and [11]. Precisely, we first prove the tightness of the considered family of probabilities, and then prove that any cluster point of it must be the unique probability that satisfies certain conditions.

Also, we remark that the model with interactions given by potentials, which is discussed by this paper and [11], when compared with the hard core model, although has the advantage that the system could be expressed by ordinary differential equations (ODEs), has its own difficulty: the system is strong non-Markovian, due to the extensions in time of the interactions. On the contrary, for the hard core model, although still non-Markovian (caused by the possibly re-collisions), each interaction happens in an instant. Also, these two models have the following obvious difference: in the hard core model, after each collision, the gas particle changes its velocity a lot –almost reflecting in a certain direction–, since the masses of the light particles and the massive particles are too different; whereas in our model, the interaction is not strong to stop the light particle, and each light particle just "almost passes through" (see Propositions 3.9). We also want to remark that, as explained in Remark 1 of Section 2 below, for the case where there is only one massive particle, our limit process coincides with the one for the hard core model, which was given by [5]. (For the case where there are at least two massive particles, we could not make the comparison since the limit process for the hard core model is unknown).

In the rest of this section, we would like to give some heuristic explanation to our model and results. Let us start with a careful look at [11]. [11] considered a deterministic system consisting of $N$ massive particles and infinitely many light particles evolving in $\mathbf{R}^d$, with its Hamiltonian given by

$\sum_{i=1}^{N} M_i|V_i|^2 + \sum_{(x,v)} m|v|^2 + \sum_{i=1}^{N}\sum_{(x,v)} U_i(X_i - x)$, where $M_i$, $X_i$, $V_i$ are the mass, the position and the velocity of the $i$-th massive particles, respectively; the summation with respect to $(x, v)$ is for all of the light particles, with $x$ for the position and $v$ for the velocity; and $U_i$ are the potentials, which are supposed to be $C_0^\infty(\mathbf{R}^d)$. (See [11, (2.1)] for the corresponding infinite system of ODEs, or (2.1) below for the ODEs of our paper, a modification of [11, (2.1)]). As same as in our present paper, the density and the velocities of the light particles have order $m^{-1/2}$, and the initial velocities of all of the light particles are fast enough (precisely, $\geq Cm^{-1/2}$ for some proper constant $C$). Heuristically, since the initial velocities of the light particles are fast enough, the interactions with the massive particles are not strong enough to "stop" them, so they will leave the interaction region very quickly, and will never be seen by the massive particles again. In this sense, the incoming light particles are always "almost" fresh. However, we would like to emphasize that this does not mean that we have the independence of the incoming light particles at different times, since the interactions are given by potentials, hence last for a certain period of time. Again, our system is non-Markovian.

One of the main ideas of [11] was the following: Since the initial velocities of the light particles are very fast, as long as the massive particles are not too fast (for example, are bounded by a constant), when describing the motion of the light particles, we could use the approximation that the massive particles are "frozen". With the help of this "freezing" approximation, [11] proved the following decomposition of $V_i(t)$: let $\sigma_n$ be the first time that the speed of some massive particle is greater than $n$, then with a proper function $\widetilde{U}$, called "new potential" by [11], we have that

$$M_i V_i(t \wedge \sigma_n) \approx \text{ martingale } + \text{ smooth part} \\ - m^{-1/2} \int_0^{t\wedge\sigma_n} \nabla_i \widetilde{U}(X_1(s), \cdots, X_N(s))ds. \tag{1.1}$$

(See [11, Lemma 3.5.1] for the detailed expression). Notice that this new expression does not contain the motion of any light particles explicitly. Heuristically, this could be understood as the macroscopically observable interactions of the massive particles which are mediated by the light particles.

The heuristic meaning of the re-expression (1.1) is quite clear: The last term $-m^{-1/2}\int_0^{t\wedge\sigma_n}\nabla_i\widetilde{U}(X_1(s),\cdots,X_N(s))ds$ and the martingale term are approximately the mean and the variance of the forces after "freezing-approximation", respectively, and the smooth term is given approximately by the first order error of the approximation.

We have that $\nabla\widetilde{U}(X_1,\cdots,X_N)$ is equal to 0 as long as the "interaction ranges" of each massive particles do not overlap, *i.e.*, $|X_i-X_j|>R_{U_i}+R_{U_j}$ for any $i,j\in\{1,\cdots,N\}$ with $i\neq j$, where $R_{U_i}$ are given by $U_i(x)=0(|x|\geq R_{U_i})$. However, as soon as the interaction ranges of any pairs of the massive particles overlap, we have that $\nabla\widetilde{U}(X_1,\cdots,X_N)\neq 0$, therefore, since the coefficient $m^{-1/2}$ diverges to infinity as $m\to 0$, this last term in the new expression (1.1) of $V_i$ will give us an extremely strong force in the limit $m\to 0$, even if the overlap is very tiny.

It might not be so clear to the readers why does this phenomenon appear, so we would like to explain it here with a few words. Notice that the contribution to $\frac{dV_t}{dt}$ in [11, (2.1)] (respectively, $\frac{dP_t}{dt}$ in (2.1) of this paper) from each light particle is of order 1, and the total number of interacting light particles at any time is of order $m^{-1/2}$ in average, so it might be thought in a glance that $\frac{dV_t}{dt}$ (or $\frac{dP_t}{dt}$) is always with order $m^{-1/2}$ in average. This is not the case until the overlap of the interacting ranges happens, because of the cancellation of the effects from light particles in different direction. However, as soon as the overlap happens, no matter how tiny this overlap is, there will exist some light particles, with a total number of order $m^{-1/2}$ in mean, that interact with more than one massive particles at the same time, so this balance will be destroyed, which results in a force with order $m^{-1/2}$ immediately.

So in order to find the motion of the massive particles in the limit $m\to 0$, we have to look more precisely at the new potential $\widetilde{U}$, given by the average of the force after freezing-approximation. Consider the simplest case where there are exactly two massive particles with potentials $U_1$ and $U_2$ spherical-symmetric. (As will be explained later at the end of this section, our method in the second half of this paper is valid for the case with only two massive particles). It is trivial, since the system is invariant to parallel shift, that $\widetilde{U}(X_1,X_2)$ depends only on $X_1-X_2$. With a little bit abuse of notations, write it as $\widetilde{U}(X_1-X_2)$. Moreover, the spherical-symmetry of the potentials $U_i$ ensures the spherical-symmetry of $\widetilde{U}$ (this is also heuristically clear, since all of the quantities are now independent of the directions), which means that the extremely strong force $-m^{-1/2}\nabla_i \widetilde{U}(X_1, X_2)$ in (1.1) is parallel to $X_1 - X_2$. (For the benefit of this property, see the limiting "reflecting diffusion" (for "different-type-case") and "uniform motion phase" (for "same-type-case") described below).

If these two potentials are of "different types", *i.e.*, the signs of $U_1$ and $U_2$ in some neighborhoods $|x| \in (R_{U_i} - \varepsilon, R_{U_i})$, with $\varepsilon > 0$ small enough, are different (in words, one is positive and one is negative), then our new potential $\widetilde{U}$ will give us a reflecting force, which, after taking limit $m \to 0$, results in a reflecting diffusion process. This is discussed in [11, Section 6].

However, when these two potentials are of "same type", *i.e.*, in some neighborhoods $|x| \in (R_{U_i} - \varepsilon, R_{U_i})$, with $\varepsilon > 0$ small enough, $U_1$ and $U_2$ have the same sign, then our new potential $\widetilde{U}$ will give us an absorbing force, which means that (if we consider the same dynamics as in [11]), after taking limit $m \to 0$, the velocities of the massive particles will become infinity, and unfortunately, this situation with infinitely fast massive particles will last for a period of time.

To avoid the difficulty of describing the limit process with infinitely fast massive particles, we modify our model in the following way so that there exists a constant $c > 0$ such that the speed of the massive particles could not exceed $c$. In our new model, hinted by the relative efficacy, the Hamiltonian is given by

$$\sum_{i=1}^{N} \sqrt{M_i^2 c^4 + c^2 |P_i|^2} + \sum_{(x,v)} m|v|^2 + \sum_{i=1}^{N} \sum_{(x,v)} U_i(Q_i - x), \tag{1.2}$$

here $P_i$ is the momentum of the $i$-th massive particle, and $Q_i$ stands for the position of it. For example, when $c$ is equal to the speed of light, our model is such that the massive particles evolve relativistically. We remark that the energies of light particles in our model are not modified, so that their speeds can go to infinity as $m \to 0$. This setting is essential in our proof, so our method does not adapt to the model with both the massive particles and the light particles relativistic. However, we admit that we could not justify the physical relevance of this setting with "relativistic massive particles and non-relativistic light particles". See (2.1) for the corresponding infinite system of ODEs of our model.

In our present model, the velocities of the massive particles are bounded and the velocities of the light particles go to infinity, so the "freezing-massive-particles-approximation" that we described before is still valid. By following the same method as in [11], we can get a re-expression of $P_i(t)$ that is similar to (1.1) (See (3.4)). This, with the help of the mentioned martingale problem theory, gives us that until the first time of overlap of the potential ranges in any pair, the process of the states (*i.e.*, positions and momenta) of the massive particles converges to a (stopped) diffusion process. See (2.4) for the generator $L_1$ of the limit diffusion and see Theorem 2.1 for the statement of this result. As same as in the non-relativistic case, the heuristic meanings of the coefficients in $L_1$ are as follows: the diffusion term is approximately the variance of the 0-order of our "freezing-massive-particles approximation", *i.e.*, the variance of the quantity that substituted the position of the light particles by the "freezing" approximations. (The average of this 0-order is expressed approximately by the "new potential" term $-m^{-1/2}\nabla\widetilde{U}$, which is equal to 0 until overlap). Also, the drift term corresponds to the first order of our approximation, or in other words, is a result of the approximation error, which, by Taylor's expression, consists a derivative of the force.

Let us come back to our main purpose of this paper: consider the model with two massive particles, with the new potential $\widetilde{U}$, given by the average of force after approximation, resulting in an absorbing force right after the overlapping. We want to know the limit motion after this overlapping. Let us make some more observation before describing our main difficulty of this part. As we explained, in the limit $m \to 0$, the absorbing force $-m^{-1/2}\nabla_i\widetilde{U}$ is extremely strong, so $|P_i|$ becomes infinity in an instant, which means that $|V_i| = c$. Indeed, since $\nabla_i\widetilde{U}$ is parallel to $Q_2 - Q_1$, we have that the massive particles keep uniform motion in the area $\widetilde{U}(Q_1, Q_2) < 0$ with $V_i = \pm c\frac{Q_2-Q_1}{|Q_2-Q_1|}$. In summary, it is not difficult to be seen heuristically that the limit process (if exists) should have two phases: one is diffusion phase (for $|Q_2 - Q_1| > R_{U_1} + R_{U_2}$), and the other is uniform motion phase (an inner neighborhood of $|Q_2 - Q_1| = R_{U_1} + R_{U_2}$). Now we are able to describe our problem: we have to determine the motion of the massive particles when they reach the boundary $|Q_2 - Q_1| = R_{U_1} + R_{U_2}$ from the uniform motion phase: They could either stay in the uniform motion phase by reflecting or re-enter the diffusion phase. Also, if they re-enter the diffusion phase, we have to determine the new initial velocity. The answers to these questions are not so easy. Indeed, as claimed, we have that $|P_i| = \infty$ (equivalently,

$|V_i| = c$) in the uniform motion phase, and $|P_i| < \infty$ (equivalently, $|V_i| < c$) in the diffusion phase, so $P_i$ and $V_i$ are not continuous, they lose all of their informations at the instant that the massive particles entered the uniform motion phase. Therefore, we have to find some method to tackle this loss of information.

We solve this problem in the following way. This is also one of our main ideas of this paper. Notice that by spherical symmetry (or the reformulation (3.4) of the system), the most difficult situation that the momenta $P_1$ and $P_2$ become infinite, actually happens only in the $\pm(Q_2 - Q_1)$-direction, with same "size" and opposite directions. So the following quantities should be kept finite, continuous and trackable for any time: (1) the components $R_i(t)$ of $P_i(t)$ that are perpendicular to $Q_2(t) - Q_1(t)$; (2) the total momentum $Y(t)$ (because of the cancellation of infinities), and (3) the total energy $H(t)$ (because the infinite of the total kinetic energy is cancelled by the infinite value of the effective potential). (See (2.6) for the definitions of these quantities, and see (2.11) for the precise expression of the generator corresponding to these processes). We remark that these cancellations do not depend on the precise shape of the "new" potential $\widetilde{U}$, in particular, have no relation to the condition (T1) given in Section 2. On the other hand, when the massive particles enter the diffusion phase from the uniform motion phase (which means that $\widetilde{U} = 0$ and $(V_2(t) - V_1(t)) \cdot (Q_2(t) - Q_1(t)) > 0$), we have that $(R_1(t), R_2(t), Y(t), H(t))$ determines $P_1(t)$ and $P_2(t)$ uniquely. The detailed calculation is given in Section 7. (See [12] for a similar problem for SDE).

In short, the total energy and the total momentum balance, and together with the finite parts of the momenta, suffice to resolve the behavior of the massive particles at the boundary of the two phases. In this way, we are able to "reserve information" during the period of uniform motion. Since we have only two tractable quantities –the total energy $H(t)$ and the total momentum $Y(t)$– in order to determine the discontinuous part of the momenta, our method is valid for the model with only two massive particles, as claimed.

The rest of this paper is organized as follows. In Section 2, we give the precise formulation of our model and result. In Section 3, we give the re-expression of $P_i(t)$ similar to (1.1), which is used essentially in the rest of this paper (see Lemma 3.13). The main idea is the "freezing-approximation" that we explained. In Section 4, we give the proof of the convergence until overlap (Theorem 2.1), with the help of Lemma 3.13. From Section 5 on, we concentrate on the mentioned special case with two massive particles. We prove in Section 5 the tightness of the considered distribution. In Section 6, we prove with the help of the results of Section 5, that several terms in the re-expression are actually negligible. These calculations are necessary for the precise expression of the limit process. In Section 7, we give the precise formulation of determining $(V_1, V_2)$ by $(R_1, R_2, Y, H)$ when the massive particles arrive at the boundary of the two phases. In Section 8, we give the proof of the convergence until any time in this special case (Theorem 2.2). Finally, in Section 9, we proof that the examples of "same type" potentials $(U_1, U_2)$ given by Example 1 of Section 2 satisfy our assumption (T1) described in the same section.

## 2. Description of the Model and Statement of the Result

Let us give the precise formulation of our model and result in this section.

We consider a dynamical system consists of $N$ ($N \in \mathbf{N}$) massive particles with masses $M_1, \cdots, M_N > 0$, respectively, put into an environment of infinitely many light particles with mass $m > 0$, (we will take the limit $m \to 0$ later on). Describe the initial condition of the environment by $\widetilde{\omega} \in Conf(\mathbf{R}^d \times \mathbf{R}^d)$. Here $Conf(*)$ stands for the set of all non-empty closed subsets of $*$ which have not cluster point. Also, $(x, v) \in \widetilde{\omega}$ means that there exists an environmental particle with position $x$ and velocity $v$ at time 0. The distribution of $\widetilde{\omega}$ will be given later. As soon as the initial condition of the system is given, our system is totally deterministic, Hamiltonian, with the Hamilton given by (1.2). So we are assuming that there is no interaction between any two environmental light particles, and the interactions between the $i$-th massive particle and the light particles are given by a potential function $U_i \in C_0^\infty(\mathbf{R}^d)$, $i = 1, \cdots, N$. For $k = 1, \cdots, N$, let $Q_k^{(m)}(t, \widetilde{\omega})$, $V_k^{(m)}(t, \widetilde{\omega})$ and $P_k^{(m)}(t, \widetilde{\omega})$ denote the position, the velocity and the momentum of the $k$-th massive particle at time $t$, respectively. So our dynamical system is given by the following infinite system of ordinary differential equations:

$$
(2.1)\quad \begin{cases}
\dfrac{d}{dt}Q_k^{(m)}(t,\widetilde{\omega}) = V_k^{(m)}(t,\widetilde{\omega}) = \dfrac{P_k^{(m)}(t,\widetilde{\omega})}{M_k\sqrt{1+M_k^{-2}c^{-2}|P_k^{(m)}(t,\widetilde{\omega})|^2}}, \\
\dfrac{d}{dt}P_k^{(m)}(t,\widetilde{\omega}) \\
\qquad = -\displaystyle\int_{\mathbf{R}^d\times\mathbf{R}^d} \nabla U_k(Q_k^{(m)}(t,\widetilde{\omega}) - x^{(m)}(t,x,v,\widetilde{\omega}))\mu_{\widetilde{\omega}}(dx,dv), \\
(Q_k^{(m)}(0,\widetilde{\omega}), P_k^{(m)}(0,\widetilde{\omega})) = (Q_{k,0}, P_{k,0}), \qquad k=1,\cdots,N, \\
\\
\dfrac{d}{dt}x^{(m)}(t,x,v,\widetilde{\omega}) = v^{(m)}(t,x,v,\widetilde{\omega}), \\
m\dfrac{d}{dt}v^{(m)}(t,x,v,\widetilde{\omega}) = -\displaystyle\sum_{i=1}^N \nabla U_i(x^{(m)}(t,x,v,\widetilde{\omega}) - Q_i^{(m)}(t,\widetilde{\omega})), \\
(x^{(m)}(0,x,v,\widetilde{\omega}), v^{(m)}(0,x,v,\widetilde{\omega})) = (x,v), \qquad (x,v)\in\widetilde{\omega}.
\end{cases}
$$

Here $\mu_{\widetilde{\omega}}(\,\cdot\,)$ is the counting measure determined by $\widetilde{\omega}$: $\mu_{\widetilde{\omega}}(A) = \sharp(\widetilde{\omega}\cap A)$ for any $A\in\mathcal{B}(\mathbf{R}^d\times\mathbf{R}^d)$. ($\sharp(\,\cdot\,)$ thus denoting the number of points in the argument). Here $c$ is a constant. (If $c$ is equal to the speed of light, then our massive particles evolves relativistically).

The only randomness of our model comes from the distribution of the environmental initial condition $\widetilde{\omega}$, which is given by the following. Let $\rho:\mathbf{R}\to[0,\infty)$ be a continuous function such that $\rho(s)\to 0$ rapidly as $s\to\infty$ (see conditions A1, A2 below for details). Let $\widetilde{\lambda_m}$ be the non-atomic Radon measure on $\mathbf{R}^d\times\mathbf{R}^d$ given by

$$
\widetilde{\lambda_m}(dx,dv) = m^{\frac{d-1}{2}}\rho\Big(\frac{m}{2}|v|^2 + \sum_{i=1}^N U_i(x-Q_{i,0})\Big)dxdv,
$$

and let $\widetilde{\kappa_m}(d\widetilde{\omega})$ be the Poisson point process with the intensity measure $\widetilde{\lambda_m}$. So $\widetilde{\kappa_m}$ is a probability measure on $\widetilde{\Omega}(=Conf(\mathbf{R}^d\times\mathbf{R}^d))$. We assume that the distribution of $\widetilde{\omega}$ is given by $\widetilde{\kappa_m}$. (See, *e.g.*, [9] for more details about Poisson point processes).

Let $R_{U_i}$ be constants such that $U_i(x)=0$ if $|x|\ge R_{U_i}$, and define the constants $C_0 = \Big(2\sum_{i=1}^N R_{U_i}\|\nabla U_i\|_\infty\Big)^{1/2}$, $e_0 = \frac{1}{2}(2C_0+1)^2 + \sum_{i=1}^N \|U_i\|_\infty$. Assume that $\rho:\mathbf{R}\to[0,\infty)$ is a measurable function satisfying the following.

A1. $\rho(s) = 0$ if $s \leq e_0$,

A2. for any $a > 0$, there exists a $\widetilde{\rho_a} : \mathbf{R} \to [0, \infty)$ such that

$$\sup_{|b| \leq a} \rho(s+b) \leq \widetilde{\rho_a}(s), \qquad \text{for any } s \in \mathbf{R},$$

and

$$\int_{\mathbf{R}^d} (1 + |v|^3) \widetilde{\rho_a}(\frac{1}{2}|v|^2) dv < \infty.$$

The meaning of the assumption (A1) is that those environmental particles with their initial momenta less than a certain value are ignored. Notice that the velocities of the massive particles are bounded, so as claimed in Section 1, under this condition, (same as in the case with the massive particles "frozen", which we call the "classical case"), since the initial velocities of the light particles are fast enough, the interactions are not strong enough to "stop" the environmental particles, so they keep their velocities at a certain level for all time, hence they will leave the valid region for interaction very quickly (see Proposition 3.2 and Proposition 3.3 for the classical case, and Propositions 3.9 and 3.10 for the "non-classical" case). This helps us to keep the incoming light particles "almost fresh". We notice that (A1) could not be removed simply, since without this assumption, we would also have to take into account those light particles with their velocities less than $m^{-1/2}(2C_0 + 1)$, the total number of which is not small, (indeed, the total numbers of these "slow" light particles and the "fast" light particles, *i.e.*, those with velocities greater than $m^{-1/2}(2C_0 + 1)$, would have the same order); moreover, the effect of each slow light particle is not small at all, since it might stay in the valid region for interaction for a long time. (A2) is a assumption with respect to the rapidness of the decreasing of $\rho$.

We are interested in the limit behavior of the massive particles when $m \to 0$.

Our first main result is with respect to the process stopped at $\sigma_0$, defined as

$$\sigma_0(\widetilde{\omega}) = \inf \big\{ t > 0; \min_{i \neq j} \{ |Q_i(t; \widetilde{\omega}) - Q_j(t; \widetilde{\omega})| - (R_{U_i} + R_{U_j}) \} \leq 0 \big\},$$

the first time for which the distance between massive particles in some pair is less than the sum of the radii of their potentials. Write $\vec{Q} = (Q_1, \cdots, Q_N)$

and $\vec{P} = (P_1, \cdots, P_N)$. We prove that the distribution of $\{(\vec{Q}(t \wedge \sigma_0), \vec{P}(t \wedge \sigma_0))\}_{t \geq 0}$ converges to some diffusion stopped at $\sigma_0$ when $m \to 0$.

In order to formulate our limit diffusion precisely, we need to prepare several notations.

For any $\vec{Q} = (Q_1, \cdots, Q_N) \in \mathbf{R}^{dN}$, let $\varphi(t, x, v; \vec{Q}) = (\varphi^0(t, x, v; \vec{Q}), \varphi^1(t, x, v; \vec{Q}))$ denote the solution of the following system of standard differential equations:

$$\begin{cases} \dfrac{d}{dt}\varphi^0(t, x, v; \vec{Q}) = \varphi^1(t, x, v; \vec{Q}) \\ \dfrac{d}{dt}\varphi^1(t, x, v; \vec{Q}) = -\displaystyle\sum_{i=1}^{N} \nabla U_i(\varphi^0(t, x, v; \vec{Q}) - Q_i) \\ (\varphi^0(0, x, v; \vec{Q}), \varphi^1(0, x, v; \vec{Q})) = (x, v). \end{cases} \tag{2.2}$$

Compare (2.2) with the second half of (2.1) with $m = 1$, one finds that the only difference is that in (2.2), we have the massive particles fixed, whereas in (2.1), the massive particles are also evolving. We will use this $\varphi(t, x_0, v_0; \vec{Q})$ (with proper $\vec{Q}$) as an approximation of $(x(t, x_0, v_0), v(t, x_0, v_0))$.

Let

$$E = \big\{(x, v) \in \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\});\ x \cdot v = 0\big\},$$
$$E_v = \big\{x \in \mathbf{R}^d; x \cdot v = 0\big\}, \qquad v \in \mathbf{R}^d \setminus \{0\},$$

and let $\nu(dx, dv)$ be the measure on $E$ given by $\nu(dx, dv) = |v|\widetilde{\nu}(dx; v)dv$, where $\widetilde{\nu}(dx; v)$ is the Lebesgue measure on $E_v$. Define the ray representation $\Psi$ as follows:

$$\begin{aligned} \Psi : \quad & \mathbf{R} \times E \to \mathbf{R}^d \times (\mathbf{R}^d \setminus \{0\}), \\ & (s, (x, v)) \mapsto \Psi(s, (x, v)) = (\Psi^0(s, (x, v)), \Psi^1(s, (x, v))) = (x - sv, v), \end{aligned}$$

in other words, we decompose the position of each environmental particles into two parts: one parallel to its velocity and the other orthogonal to its velocity. We remark that in this new space $\mathbf{R} \times E$, $v$ is still the velocity of the light particle at time 0, while $x$ is not the position of it anymore: now $x$ is only the component of its initial position that is perpendicular to the velocity.

Then we have that

$$\psi(t,x,v;\vec{Q}) := \lim_{s\to\infty} \varphi(t+s,\Psi(s,x,v);\vec{Q})$$

is well-defined. Now we are ready to give the quadratic term of the limit diffusion until $\sigma_0$, which, as claimed in Section 1, corresponds to the variance after applying our freezing approximation: Let $D_k(\vec{Q},x,v) = {}^t\Big(D_k^1(\vec{Q},x,v), \cdots, D_k^d(\vec{Q},x,v)\Big)$, with

$$D_k^l(\vec{Q},x,v) = \int_{-\infty}^{\infty} \nabla_l U_k(Q_k - \psi^0(u,x,v;\vec{Q}))du,$$
$$k = 1,\cdots,N, l = 1,\cdots,d.$$

Also, for $k_1,k_2 = 1,\cdots,N$, $l_1,l_2 = 1,\cdots,d$, let

$$a_{k_1l_1;k_2l_2}(\vec{Q}) = \int_E D_{k_1}^{l_1}(\vec{Q},x,v)D_{k_2}^{l_2}(\vec{Q},x,v)\rho(\frac{1}{2}|v|^2)\nu(dx,dv).$$

Notice that the integrals above, although might look like infinite at a glance, are actually finite.

We next give the definition of the drift term of the limit diffusion until $\sigma_0$. As claimed in Section 1, this corresponds to the error of our freezing approximation. (See Lemma 3.12). For any $(x,v) \in E$, $\vec{Q},\vec{V} \in \mathbf{R}^{dN}$ and $a \in \mathbf{R}$, let $z(t;x,v,\vec{Q},\vec{V},a) \in \mathbf{R}^d$ denote the solution of the following standard differential equation.

$$(2.3)\quad \begin{cases} \frac{d^2}{dt^2}Z(t) = -\sum_{i=1}^N \nabla^2 U_i(\psi^0(t,x,v,\vec{Q}) - Q_i)\Big(Z(t) - (t+a)V_i\Big), \\ \lim_{t\to-\infty} Z(t) = \lim_{t\to-\infty}\frac{d}{dt}Z(t) = 0. \end{cases}$$

Then $z(t;x,v,\vec{Q},\vec{V},a)$ is a linear function of $\vec{V}$. Let $b_{ik;jl} : \mathbf{R}^{dN} \to \mathbf{R}$ be the $C^\infty$-functions determined by the following:

$$\int_E \Big(\int_{-\infty}^{\infty} \nabla^2 U_i(\psi^0(t,x,v,\vec{Q}) - Q_i)z(t,x,v,\vec{Q},\vec{V},-t)dt\Big)$$
$$\times \rho(\frac{1}{2}|v|^2)\nu(dx,dv)$$
$$= \sum_{\ell=1}^d \sum_{j=1}^N b_{i\cdot;j\ell}(\vec{Q})V_j^\ell =: b_i(\vec{Q})\vec{V},$$

or equivalently, $\int_E \Big( \int_{-\infty}^{\infty} \sum_{p=1}^{d} \nabla_k \nabla_p U_i(\psi^0(t,x,v,\vec{Q}) - Q_i) z_p(t,x,v,\vec{Q},\vec{V}, -t)dt \Big) \rho(\frac{1}{2}|v|^2)\nu(dx,dv) = \sum_{\ell=1}^{d}\sum_{j=1}^{N} b_{ik;j\ell}(\vec{Q})V_j^{\ell}, k = 1,\cdots,d$, where $z_p$ means the $p$-th element of the vector $z$ for $p = 1,\cdots,d$. By the same reason as that for the quadratic term, the integral on the left hand side above is finite.

Now we are in a position to give the definition of the generator $L_1$ on $\mathbf{R}^{2dN}$ of the limit diffusion stopped at $\sigma_0$:

$$
L_1 = \frac{1}{2}\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=1}^{d} a_{k_1l_1,k_2l_2}(\vec{Q})\frac{\partial^2}{\partial P_{k_1}^{l_1}\partial P_{k_2}^{l_2}} + \sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=1}^{d} b_{k_1l_1,k_2l_2}(\vec{Q})V_{k_2}^{l_2}\frac{\partial}{\partial P_{k_1}^{l_1}} + \sum_{k=1}^{N}\sum_{i=1}^{d} V_k^i \frac{\partial}{\partial Q_k^i}, \tag{2.4}
$$

with $V_k = \frac{P_k}{M_k\sqrt{1+M_k^{-2}c^{-2}|P_k|^2}}$.

REMARK 1. When $N = 1$, we have that $L_1$ is independent of $Q$, so our limit process coincides with that for the model with hard core (see [5]). Indeed, if $N = 1$, (we omit the index 1 in this case), then we have by the uniqueness of the solution of (2.2) that

$$
\varphi(t, x - Q, v; 0) = \varphi(t, x, v; Q) - Q.
$$

So

$$
\begin{aligned}
&\psi^0(t - (Q\cdot v/|v|^2), x - \pi_v^{\perp}Q, v; 0) \\
&= \lim_{s\to\infty} \varphi(t - (Q\cdot v/|v|^2) + s, x - \pi_v^{\perp}Q - sv, v; 0) \\
&= \lim_{s\to\infty} \varphi(t + s - (Q\cdot v/|v|^2), x - Q - (s - (Q\cdot v/|v|^2))v, v; 0) \\
&= \lim_{s\to\infty} \varphi(t + s - (Q\cdot v/|v|^2), x - (s - (Q\cdot v/|v|^2))v, v; Q) - Q \\
&= \psi^0(t, x, v; Q) - Q.
\end{aligned} \tag{2.5}
$$

Here "$\cdot$" stands for the inner product in $\mathbf{R}^d$, and $\pi_v^{\perp}Q := Q - (Q\cdot\frac{v}{|v|})\frac{v}{|v|}$ is the component of $Q$ that is perpendicular to $v$. Therefore,

$$
D^l(Q,x,v) = \int_{\mathbf{R}} \nabla_l U(-\psi^0(u - (Q\cdot v/|v|^2), x - \pi_v^{\perp}Q, v; 0))du
$$

$$= D^l(0, x - \pi_v^\perp Q, v),$$

hence

$$a_{l_1;l_2}(Q) = \int_E D^{l_1}(0, x - \pi_v^\perp Q, v) D^{l_2}(0, x - \pi_v^\perp Q, v)\rho(\frac{1}{2}|v|^2)\nu(dx, dv)$$
$$= a_{l_1;l_2}(0).$$

Similarly, by (2.5) and the uniqueness of the solution of (2.3), we have that

$$z(t; x, v, Q, V, a) = z(t - (Q \cdot v/|v|^2); x - \pi_v^\perp Q, v, 0, V, a + (Q, v/|v|^2)),$$

hence

$$b(Q)V = \int_E \Big( \int_{\mathbf{R}} \nabla^2 U(\psi^0(t - (Q \cdot v/|v|^2), x - \pi_v^\perp Q, v, 0)) \times z(t - (Q \cdot v/|v|^2); x - \pi_v^\perp Q, v, 0, V, -t + (Q \cdot v/|v|^2))dt \Big) \times \rho(\frac{1}{2}|v|^2)\nu(dx, dv)$$
$$= b(0)V.$$

This completes the proof of the fact that $L_1$ is independent of $Q$ when $N = 1$.

Notice that when $N \geq 2$, (we recall again that the limit process of the model with hard core and $N \geq 2$ is not known yet), this independence of $Q$ does not hold.

THEOREM 2.1. *Under our present setting, we have the following.*

(1) *Assume $N = 1$. Then as $m \to 0$, the distribution of $\{(Q_1^{(m)}(t), V_1^{(m)}(t)), t \geq 0\}$ under $\widetilde{\kappa_m}$ converges weakly to the diffusion process with generator $L_1$ in $C([0, \infty); \mathbf{R}^{2d})$ equipped with the Skorohod metric.*

(2) *Assume $N \geq 2$. Then as $m \to 0$, the distribution of $\{(\vec{Q}^{(m)}(t \wedge \sigma_0), \vec{V}^{(m)}(t \wedge \sigma_0)), t \geq 0\}$ converges weakly to the diffusion with generator $L_1$ stopped at $\sigma_0$ in $C([0, \infty); \mathbf{R}^{2dN})$ equipped with the Skorohod metric.*

As announced in Section 1, the second half of our paper is contributed to the problem of convergence without stopping at $\sigma_0$. Precisely, we consider the case where there are two massive particles with the "new potential" $\widetilde{U}$, given by the average of the freezing-approximated force, a spherical-symmetric absorbing force. As discussed in Section 1, this includes the model with the two massive particles the "same type". Write the "new" potential as $\widetilde{U}(Q_1, Q_2) = U(|Q_1 - Q_2|), Q_1, Q_2 \in \mathbf{R}^d$. We prove that when $m \to 0$, the distribution of the process that describes the behavior of the massive particles converges, and give the precise formulation of the limit process.

From Section 5 on, we consider this special case with two massive particles, and assume that our spherical-symmetric "new potential" $\widetilde{U}$, given by the average of the approximated force, satisfies the following condition.

(T1) There exists a constant $r_1 \in (0, R_{U_1} + R_{U_2})$ such that $U(r_1) = 0$ and $U(q) < 0$ if $q \in (r_1, R_{U_1} + R_{U_2})$. Also, $U'(r_1) < 0$ and $\lim_{q \to (R_{U_1}+R_{U_2})-} \frac{U'(q)}{U(q)} = -\infty$.

The essential part of (T1) is its first half, which implies that $\widetilde{U}$ gives us an absorbing force for a while after the valid ranges of the two massive particles overlap, as we declared. Also, the condition $U'(r_1) < 0$ ensures that the repulsive force around $|Q_1 - Q_2| = r_1$ is strong enough so that in the limit $m \to 0$, the two massive particles could never enter the domain $|Q_1 - Q_2| < r_1$. The last part is a condition with respect to the behavior of $U$ near to $R_{U_1} + R_{U_2}$.

Before going further, let us give a class of examples that satisfy (T1).

*Example* 1. We consider the special case that $U_1 = U_2$, with the common spherical-symmetric potential given by the following. Let $a_2 > a_1 > a_0 > 0$ be three positive numbers satisfying $2a_1 = a_0 + a_2$, and let $g_0, g_1 \in C_0^\infty([0, \infty))$ be two functions satisfying the following:

(1) $x \geq a_1^2 \Rightarrow g_0(x) = 0$,

(2) $x \in (a_0^2, a_1^2) \Rightarrow g_0(x) > 0, g_0'(x) < 0$,

(3) $g_1 \leq 0$ and $\mathrm{supp} g_1 = (a_1^2, a_2^2)$,

(4) $\lim_{x \to a_2^2-} \frac{g_1'(x)}{g_1(x)} = -\infty$.

Finally, for any $\lambda > 0$, let $g_\lambda(x) = g_0(x) + \lambda g_1(x)$. Our function $U_1 = U_2$ is given by $U_1(x) = U_2(x) = g_\lambda(|x|^2)$. (So we have that $R_{U_1} = R_{U_2} = a_2^2$).

Then (T1) is satisfied if $\lambda > 0$ is small enough. See Section 9 for the proof.

We make the observation here that by definition, our common interaction between the massive particles and the light particles are $0 \to$ attractive force $\to$ repulsive force when the distance of the particles decreases. Also, notice that $g_0$ is the "positive part" and $\lambda g_1$ is the "negative part" of our potential $U_1 = U_2$. So taking $\lambda$ small enough is equivalent to saying that the attractive force between the massive particles and the environmental particles are weak.

Write $r_2 := R_{U_1} + R_{U_2}$, and let

$$
\begin{aligned}
B_1 &:= \Big\{(q_1, q_2) \in \mathbf{R}^{2d} \Big| |q_1 - q_2| > r_2\Big\},\\
B_2 &:= \Big\{(q_1, q_2) \in \mathbf{R}^{2d} \Big| |q_1 - q_2| \in (r_1, r_2)\Big\}.
\end{aligned}
$$

As explained heuristically in Section 1, $B_1$ is our "diffusion phase" and $B_2$ is our "uniform motion phase", and the most difficulty of this part is to find the concrete formulation of the limit process when the massive particles arrive at the boundary of these two phases from the uniform motion phase, since the process $V_k(t)$ lose its information at the instant that the massive particles entered the uniform phase.

As suggested by the discussion in Section 1, we introduce the following notations. For any $a, b \in \mathbf{R}^d$ with $a \neq 0$, we use the notations $\pi_a b$ and $\pi_a^\perp b$ to denote the components of $b$ that are parallel and perpendicular to $a$, respectively: $\pi_a b = (b \cdot \frac{a}{|a|})\frac{a}{|a|}$, and $\pi_a^\perp b = b - (b \cdot \frac{a}{|a|})\frac{a}{|a|}$. Define

$$
\begin{aligned}
&R_k(t) = \pi^\perp_{Q_2(t)-Q_1(t)} P_k(t), \quad k = 1, 2,\\
&H(t) = \sum_{k=1}^{2} M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |P_k(t)|^2} + m^{-1/2} \widetilde{U}(Q_1(t), Q_2(t)),\\
(2.6)\quad &Y(t) = P_1(t) + P_2(t).
\end{aligned}
$$

Write

$$X := (Q_1, Q_2, V_1, V_2, R_1, R_2, Y, H).$$

Then as will be shown later, in the limit $m \to 0$, (1) $(R_1(t), R_2(t), Y(t), H(t))$ is continuous with respect to $t$, (2) in the instant that the massive particles reach the boundary from the uniform motion phase, $(R_1, R_2, Y, H)$ is sufficient to determine $(V_1, V_2)$, and (3) the process $\{X_t\}_{t\geq 0}$ is a diffusion with jumps, the precise formulation of which can be written concretely. (See Theorem 2.2 for the precise formulation of this limit stochastic process).

Let us first formulate how does $(R_1, R_2, Y, H)$ determine $(V_1, V_2)$ at the boundary (see Theorem 2.2 for details). In order to state our result, let us first prepare several more notations. Consider the following equation with respect to $x$:

$$\begin{cases} M_1c^2\sqrt{1+M_1^{-2}c^{-2}(|a_1|^2+x^2)} \\ \qquad +M_2c^2\sqrt{1+M_2^{-2}c^{-2}(|a_2|^2+(d-x)^2)} = b, \\ \frac{x}{M_1\sqrt{1+M_1^{-2}c^{-2}(|a_1|^2+x^2)}} - \frac{d-x}{M_2\sqrt{1+M_2^{-2}c^{-2}(|a_2|^2+(d-x)^2)}} < 0. \end{cases} \tag{2.7}$$

Here $a_1, a_2, b, d \in \mathbf{R}$ are constants. For any $a_1, a_2, d \in \mathbf{R}$, let

$$b_0(a_1, a_2, d) = \inf_{x\in\mathbf{R}} \Big( M_1c^2\sqrt{1+M_1^{-2}c^{-2}(|a_1|^2+x^2)} + M_2c^2\sqrt{1+M_2^{-2}c^{-2}(|a_2|^2+(d-x)^2)} \Big). \tag{2.8}$$

Then $b_0(a_1, a_2, d) \in (-\infty, +\infty)$, and for any $b \neq b_0(a_1, a_2, d)$, we have that (2.7) has a unique solution $x(a_1, a_2, b, d)$ if and only if $b > b_0(a_1, a_2, d)$, (see (7.1) for the precise expression of $x(a_1, a_2, b, d)$). For any $\vec{R} = (R_1, R_2)$, $\vec{Q} = (Q_1, Q_2) \in \mathbf{R}^{2d}$, $Y \in \mathbf{R}^d$ and $H \in \mathbf{R}$, let

$$\begin{aligned} &H_0(R_1, R_2, Y, Q_1, Q_2) = b_0(|R_1|, |R_2|, Y \cdot \frac{Q_2 - Q_1}{|Q_2 - Q_1|}), \\ &P_1(R_1, R_2, Y, H, Q_1, Q_2) \\ &\qquad = R_1 + x(|R_1|, |R_2|, H, Y \cdot \frac{Q_2 - Q_1}{|Q_2 - Q_1|})\frac{Q_2 - Q_1}{|Q_2 - Q_1|}, \\ &P_2(R_1, R_2, Y, H, Q_1, Q_2) = Y - P_1(R_1, R_2, Y, H, Q_1, Q_2). \end{aligned} \tag{2.9}$$

We use this to determine the bahavior of $V_1$ and $V_2$ at the boundary of the two phases.

Next, in order to present our limit distrubition of the process $\{X(t); t \geq 0\}$, let us define a new generator. For $k \in \{1, 2\}$, let $D_k$, $a_{kj,kl}$ and $b_k$ be as before. Let $I_d$ be the $d \times d$ identity matrix, and define

$$\gamma_k^V(\vec{Q}, \vec{V}, x, v) = M_k^{-1}\sqrt{1 - c^{-2}|V_k|^2}\ \Big(I_d - c^{-2}({}^tV_kV_k)\Big)D_k(\vec{Q}, x, v).$$

The diffusion coefficients of our new generator is given by

$$\alpha_{ij}(\vec{Q}, \vec{V}) = \int_E \gamma_i(\vec{Q}, \vec{V}, x, v)\gamma_j(\vec{Q}, \vec{V}, x, v)\rho(\frac{1}{2}|v|^2)\nu(dx, dv),$$

where

$$\gamma(\vec{Q}, \vec{V}, x, v) = \Big(\gamma_i(\vec{Q}, \vec{V}, x, v)\Big)_{i=1}^{7d+1}$$
$$= \begin{pmatrix} 0 \\ 0 \\ 1_{\{|Q_1-Q_2|>r_2\}}\gamma_1^V(x, v) \\ 1_{\{|Q_1-Q_2|>r_2\}}\gamma_2^V(x, v) \\ \pi_{Q_2-Q_1}^{\perp}(D_1(\vec{Q}, x, v)) \\ \pi_{Q_2-Q_1}^{\perp}(D_2(\vec{Q}, x, v)) \\ D_1(\vec{Q}, x, v) + D_2(\vec{Q}, x, v) \\ V_1 \cdot D_1(\vec{Q}, x, v) + V_2 \cdot D_2(\vec{Q}, x, v). \end{pmatrix}.$$

We next formulate the drift coefficients of our new generator. Define $F_{ijl}$ as

$$(2.10) \qquad F_{ijl}(V) = \begin{cases} \delta_{\{l \neq i\}}2c^{-2}(1 - c^{-2}|V|^2)V_l \\ \qquad -3c^{-2}(1 - c^{-2}|V|^2)^2V_l \\ \qquad -3c^{-2}(1 - c^{-2}|V|^2)V_l \\ \qquad +3c^{-4}(1 - c^{-2}|V|^2)V_i^2V_l, & \text{if } i = j, \\ 3c^{-4}(1 - c^{-2}|V|^2)V_iV_jV_l \\ \qquad -c^{-2}(1 - c^{-2}|V|^2) \\ \qquad \times\Big(\delta_{\{l=i\}}V_j + \delta_{\{l=j\}}V_i\Big), & \text{if } i \neq j. \end{cases}$$

Let

$$\beta_k^V(\vec{Q}, \vec{V}) = M_k^{-1}\sqrt{1 - c^{-2}|V_k|^2}\ \Big(I_d - c^{-2}({}^tV_kV_k)\Big)b_k(\vec{Q})\vec{V}$$
$$+ \frac{1}{2}M_k^{-2}\Big(\sum_{j,l=1}^{d} F_{ijl}(V_k)a_{kj,kl}(\vec{Q})\Big)_{i=1}^{d},$$

$$\beta_k^R(\vec{Q},\vec{V},\vec{R},Y) = \pi^{\perp}_{Q_2-Q_1}(b_1(\vec{Q})\vec{V}) - \frac{1}{|Q_2-Q_1|^2}\sum_{j=1}^{2}(V_j\cdot(Q_2-Q_1))R_j + \beta_k^{R,extra},$$

$$\beta^H(\vec{Q},\vec{V}) = \sum_{k=1}^{2} {}^tV_k b_k(\vec{Q})\vec{V} + \beta^{H,extra},$$

where $\beta_k^{R,extra} = \beta_k^{R,extra}(\vec{Q},\vec{V},\vec{R},Y)$ is given by

$$\begin{aligned}\beta_k^{R,extra} = & -\frac{1}{|Q_2-Q_1|^2}\sum_{j=1}^{2} M_j^{-1}\sqrt{1-c^{-2}|V_j|^2}|R_j|^2(Q_2-Q_1)\\ & -\Big[\Big(Y\cdot\frac{Q_2-Q_1}{|Q_2-Q_1|}\Big)\Big(V_{k^c}\cdot\frac{Q_2-Q_1}{|Q_2-Q_1|}\Big) - Y\cdot V_{k^c}\Big]\frac{Q_2-Q_1}{|Q_2-Q_1|^2}\\ & + \frac{1}{|Q_2-Q_1|^2}\Big(Y\cdot(Q_2-Q_1)\Big)M_{k^c}^{-1}\sqrt{1-c^{-2}|V_{k^c}|^2}R_{k^c},\end{aligned}$$

with $k^c$ given by $k^c=2$ if $k=1$, and $k^c=1$ if $k=2$; and $\beta^{H,extra} = \beta^{H,extra}(\vec{Q},\vec{V})$ is given by

$$\beta^{H,extra} = M_k^{-1}\sqrt{1-c^{-2}|V_k|^2}\ \sum_{i,j=1}^{d}(\delta_{ij}-c^{-2}V_k^iV_k^j)a_{ki,kj}\Big\}.$$

The drift coefficients of our new generator is given by

$$\beta(\vec{Q},\vec{V},\vec{R},Y) = (\beta_i(\vec{Q},\vec{V},\vec{R},Y))_{i=1}^{7d+1} = \begin{pmatrix} V_1\\ V_2\\ 1_{\{|Q_1-Q_2|>r_2\}}\beta_1^V(\vec{Q},\vec{V})\\ 1_{\{|Q_1-Q_2|>r_2\}}\beta_2^V(\vec{Q},\vec{V})\\ \beta_1^R(\vec{Q},\vec{V},\vec{R},Y)\\ \beta_2^R(\vec{Q},\vec{V},\vec{R},Y)\\ \sum_{k=1}^2 b_k(\vec{Q})\vec{V}\\ \beta^H(\vec{Q},\vec{V})\end{pmatrix}.$$

Our new generator is given by

$$L = \frac{1}{2}\sum_{i,j=1}^{7d+1}\alpha_{ij}(\vec{Q},\vec{V})\nabla_i\nabla_j + \sum_{i=1}^{7d+1}\beta_i(\vec{Q},\vec{V},(\vec{R},Y))\nabla_i,$$

with

$$\nabla_i = \begin{cases} \nabla_{Q_1^i}, & i = 1, \cdots, d, \\ \nabla_{Q_2^{i-d}}, & i = d+1, \cdots, 2d, \\ \nabla_{V_1^{i-2d}}, & i = 2d+1, \cdots, 3d, \\ \nabla_{V_2^{i-3d}}, & i = 3d+1, \cdots, 4d, \\ \nabla_{R_1^{i-4d}}, & i = 4d+1, \cdots, 5d, \\ \nabla_{R_2^{i-5d}}, & i = 5d+1, \cdots, 6d, \\ \nabla_{Y^{i-6d}}, & i = 6d+1, \cdots, 7d, \\ \nabla_H, & i = 7d+1. \end{cases}$$

Finally, let $D([0,\infty);\mathbf{R}^{2d})$ denote the Skorohod space, and let $\widetilde{W}^d := C([0,\infty);\mathbf{R}^{2d}) \times D([0,\infty);\mathbf{R}^{2d}) \times C([0,\infty);\mathbf{R}^{3d+1})$, with metric function $dist(\cdot,\cdot)$ given by

$$\begin{aligned} dist(x,\widetilde{x}) := \sum_{m=1}^{\infty} 2^{-n}\Big\{1 \wedge \sum_{i=1}^{2}\Big(&\max_{t\in[0,n]}|q_i(t)-\widetilde{q}_i(t)| \\ &+\Big(\int_0^n |v_i(t)-\widetilde{v}_i(t)|^n\Big)^{1/n} + \max_{t\in[0,n]}|r_i(t)-\widetilde{r}_i(t)|\Big) \\ &+\max_{t\in[0,n]}|y(t)-\widetilde{y}(t)| + \max_{t\in[0,n]}|h(t)-\widetilde{h}(t)|\Big\} \end{aligned}$$

for $x=(q_1,q_2,v_1,v_2,r_1,r_2,y,h)$ and $\widetilde{x}=(\widetilde{q}_1,\widetilde{q}_2,\widetilde{v}_1,\widetilde{v}_2,\widetilde{r}_1,\widetilde{r}_2,\widetilde{y},\widetilde{h})$.

Now we are ready to state our second main result of the present paper. Our second main result is the following.

Theorem 2.2. *Assume that $N = 2$ and that (T1) is satisfied. Then when $m \to 0$, the distribution of $\{X^{(m)}(t); t \in [0,\infty)\}$ converges to $\mu_0$ as probabilities on $\widetilde{W}^d$. Here $\mu_0$ is the unique probability on $\widetilde{W}^d$ that satisfies the followings.*

($\mu$1) $\mu_0\Big(Q_k(0) = Q_{k,0}, V_k(0) = \frac{P_{k,0}}{M_k\sqrt{1+M_k^{-2}c^{-2}|P_{k,0}|^2}}, R_k(0) = \pi^{\perp}_{Q_{2,0}-Q_{1,0}}P_{k,0}$ *for* $k \in \{1,2\}, Y(0) = P_{1,0}+P_{2,0}, H(0) = \sum_{j=1}^{2} M_k c^2\sqrt{1+M_k^{-2}c^{-2}|P_{j,0}|^2}\Big) = 1.$

($\mu$2) $\mu_0(|Q_1(t)-Q_2(t)| \geq r_1, |V_k(t)| \leq 1$ *for* $k=1,2, t\in[0,\infty)) = 1.$

($\mu$3) *For any* $f \in C_0^\infty(\mathbf{R}^{7d+1})$ *with supp*$(f) \subset \Big((B_1 \cup B_2) \times \mathbf{R}^{5d+1}\Big)$, *we have that* $\Big\{ f(\vec{Q}(t), \vec{V}(t), \vec{R}(t), Y(t), H(t)) - \int_0^t Lf(\vec{Q}(s), \vec{V}(s), \vec{R}(s), Y(s), H(s))ds; t \geq 0 \Big\}$ *is a continuous martingale under* $\mu_0$.

($\mu$4) *We have* $\mu_0$*-almost surely the following: For any* $t \in [0, \infty)$ *and* $k \in \{1, 2\}$, $|Q_1(t) - Q_2(t)| \in (r_1, r_2)$ *implies that* $V_k(t) = \pm c \frac{Q_2(t) - Q_1(t)}{|Q_2(t) - Q_1(t)|}$ *and that* $V_k(t) = V_k(t-)$, *also*, $|Q_1(t) - Q_2(t)| = r_1$ *implies that* $V_1(t) = -V_2(t) = -c \frac{Q_2(t) - Q_1(t)}{|Q_2(t) - Q_1(t)|}$.

($\mu$5) *We have* $\mu_0$*-almost surely that for* $t \in [0, \infty)$ *with* $|Q_1(t) - Q_2(t)| = r_2$,

(1) *if* $\Big(Q_1(t) - Q_2(t)\Big) \cdot \Big(V_1(t-) - V_2(t-)\Big) < 0$, *then* $V_1(t) = -V_2(t) = c \frac{Q_2(t) - Q_1(t)}{|Q_2(t) - Q_1(t)|}$;

(2) *if* $\Big(Q_1(t) - Q_2(t)\Big) \cdot \Big(V_1(t-) - V_2(t-)\Big) > 0$ *and* $H(t) < H_0(t)$, *then* $V_1(t) = -V_2(t) = c \frac{Q_2(t) - Q_1(t)}{|Q_2(t) - Q_1(t)|}$;

(3) *if* $\Big(Q_1(t) - Q_2(t)\Big) \cdot \Big(V_1(t-) - V_2(t-)\Big) > 0$ *and* $H(t) > H_0(t)$, *then* $V_k(t) = \frac{P_k(t)}{M_k \sqrt{1 + M_k^{-2} c^{-2} |P_k(t)|^2}}$ *with* $P_k(t) = P_k(R_1(t), R_2(t), Y(t), H(t), Q_1(t), Q_2(t))$, $k = 1, 2$.

*Here* $H_0(t) = H_0(R_1(t), R_2(t), Y(t), Q_1(t), Q_2(t))$, *with* $H_0(\cdot)$ *and* $P_k(\cdot)$ *given by (2.9).*

We remark that the condition $\Big(Q_1(t) - Q_2(t)\Big) \cdot \Big(V_1(t-) - V_2(t-)\Big) > 0$ in (2) of ($\mu$5) is "almost" redundant: in the domain $|Q_1 - Q_2| > r_1$, it is possible that $H(t) < H_0(t)$ only if $|Q_1 - Q_2| \leq r_2$ (see the proof of Lemma 8.4 for the proof of this fact).

REMARK 2. As claimed, our limit process can be understood as a combination of two diffusions. One is described by $L_1$ in the diffusion phase $B_1$, and the other phase occurs in $B_2$: the two massive particles evolve in uniform motion, while the "background evolving quantity" $(R_1, R_2, Y, H)$ (which, as claimed in Section 1, is always well-defined by the spherical symmetry, and is necessary when determining the behavior of these two massive particles when they reach the boundary of the two phases), evolves as a diffusion with generator $L_2$ given by

$$(2.11) \qquad L_2 f = \frac{1}{2} \sum_{i,j=4d+1}^{7d+1} \alpha_{ij}^{UI} \nabla_i \nabla_j + \sum_{i=4d+1}^{7d+1} \beta_i^{UI} \nabla_i,$$

with

$$\beta^{UI} = (\beta_i^{UI})_{i=4d+1}^{7d+1} = \begin{pmatrix} \pi_{Q_2-Q_1}^{\perp}(b_1(\vec{Q})\vec{V}) - \frac{1}{|Q_2-Q_1|^2} \sum_{j=1}^{2} (V_j \cdot (Q_2 - Q_1)) R_j \\ \pi_{Q_2-Q_1}^{\perp}(b_2(\vec{Q})\vec{V}) - \frac{1}{|Q_2-Q_1|^2} \sum_{j=1}^{2} (V_j \cdot (Q_2 - Q_1)) R_j \\ \sum_{k=1}^{2} b_k(\vec{Q})\vec{V} \\ \sum_{k=1}^{2} {}^t V_k b_k(\vec{Q})\vec{V}, \end{pmatrix},$$

and $\alpha_{ij}^{UI}(\vec{Q}, \vec{V}) = \alpha_{ij}(\vec{Q}, \vec{V})$ for $i, j = 3d+1, \cdots, 7d+1$.

Precisely, our limit process satisfies the following:

(1) the particles keep in the area $|Q_1(t) - Q_2(t)| \geq r_1$;

(2) when $|Q_1(t) - Q_2(t)| > r_2$, $(Q_1(t), Q_2(t), \frac{M_1}{\sqrt{1-c^{-2}|V_1(t)|^2}} V_1, \frac{M_2}{\sqrt{1-c^{-2}|V_2(t)|^2}} V_2)$ evolves according to the diffusion with generator $L_1$, and $(R_1(t), R_2(t), Y(t), H(t))$ is given by $R_k(t) = \frac{M_k}{\sqrt{1-c^{-2}|V_k(t)|^2}} \times \pi_{Q_2(t)-Q_1(t)}^{\perp} V_k(t)$, $Y(t) = \sum_{k=1}^{2} \frac{M_k}{\sqrt{1-c^{-2}|V_k(t)|^2}} V_k$ and $H_t = \sum_{k=1}^{2} M_k c^2 (1 - c^{-2}|V_k(t)|^2)^{-1/2}$.

(3) the two massive particles keep uniform motions in the area $|Q_1(t) - Q_2(t)| \in (r_1, r_2)$ with $V_k(t) = \pm c \frac{Q_1(t)-Q_2(t)}{|Q_1(t)-Q_2(t)|}$ and they reflect at $|Q_1(t) - Q_2(t)| = r_1$, and $(R_1(t), R_2(t), Y(t), H(t))$ is a diffusion with generator $L_2$,

(4) finally, the behavior of these two massive particles at the boundary $|Q_1(t) - Q_2(t)| = r_2$ of these two phases are determined as follows, when the massive particles reach $|Q_1(t) - Q_2(t)| = r_2$ from the diffusion phase, they simply enter the uniform motion phase by taking $V_1(t) = -V_2(t) = c \frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}$; when the massive particles reach $|Q_1(t) - Q_2(t)| = r_2$ from the uniform motion phase, they either keep in the

uniform motion phase by reflecting or re-enter the diffusion phase, depending on the value of $(R_1(t), R_2(t), Y(t), H(t))$ at that moment, according to ($\mu$5).

Indeed, suppose that a probability measure $\mu$ satisfies ($\mu$1) $\sim$ ($\mu$5). Notice that the term $1_{\{|Q_1(t)-Q_2(t)|>r_2\}}$ in $\beta$ is not 0 only if $|Q_1(t)-Q_2(t)| > r_2$, and in this domain, we get by a simple calculation that under $\mu$, the followings hold: (1) $|V_k(t)| < 1$ for $k \in \{1,2\}$, (2) $P_k(t) := \frac{cM_k}{\sqrt{c^2-|V_k(t)|^2}} V_k(t)$ is finite, and the distribution of $(Q_1(t), Q_2(t), P_1(t), P_2(t))$ is a solution of the martingale problem $L_1$, (3) $(R_1(t), R_2(t), Y(t), H(t))$ is actually completely determined by $(Q_1(t), Q_2(t), P_1(t), P_2(t))$: $R_k(t) = \pi^{\perp}_{Q_2(t)-Q_1(t)} P_k(t)$, $Y(t) = P_1(t) + P_2(t)$ and $H_t = \sum_{k=1}^{2} M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |P_k(t)|^2}$. Also, when $|Q_1(t) - Q_2(t)| \in (r_1, r_2)$, we have by ($\mu$4) that $|V_k(t)| = c$ and $V_1(t) \parallel V_2(t) \parallel (Q_2(t) - Q_1(t))$, hence $\beta_k^{R,extra} = \beta^{H,extra} = 0$, therefore,

$$\beta = \begin{pmatrix} V_1 \\ V_2 \\ 0 \\ 0 \\ \beta^{UI} \end{pmatrix}, \gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma^{UI} \end{pmatrix}.$$

In particular, in this domain, $(\vec{Q}(t), \vec{V}(t))$ is deterministic, and $(R_1(t), R_2(t), Y(t), H(t))$ is a diffusion with generator $L_2$.

The opposite is also true: if a probability satisfies all of the conditions stated here, it also satisfies ($\mu$1) $\sim$ ($\mu$5).

Throughout this paper, $C$ stands for positive constants that may be different in different places.

# Part I. General cases

## 3. Ray Representation and Decomposition of $P_k(t)$

The main aim of this section is to give the re-expression of $P_k(t)$ as claimed. (See Lemma 3.13). All of the results of this section are gotten by exact the same method as in [11], so we omit the proof here. (See [13] for the detailed proofs).

### 3.1. Ray representation

Let $\Omega = Conf(\mathbf{R} \times E)$, let $\lambda(ds, dx, dv)$ be the measure on $\Omega$ given by

$$\begin{aligned}\lambda(ds, dx, dv) &= \lambda_m(ds, dx, dv) \\ &= m^{-1}\rho\Big(\frac{1}{2}|v|^2 + \sum_{i=1}^{N} U_i(x - m^{-1/2}sv - Q_{i,0})\Big)ds\nu(dx, dv),\end{aligned}$$

and let $\kappa_m(d\omega) = \kappa_{\lambda_m}(d\omega)$ be the Poisson point process on $Conf(\mathbf{R} \times E)$ with intensity function $\lambda_m(ds, dx, dv)$. Then we can convert our problem with respect to $Conf(\mathbf{R}^d \times \mathbf{R}^d)$ to a problem with respect to $Conf(\mathbf{R} \times E)$. Our $\omega \in \Omega$ has distribution $\kappa_m$, and for each initial condition $\omega$, we are considering the following system of infinite ODEs (we omit the superscription $(m)$ for the sake of simplicity):

$$(3.1)\qquad \begin{cases} \dfrac{d}{dt}Q_i(t, \omega) = \dfrac{P_i(t, \omega)}{M_i\sqrt{1 + M_i^{-2}c^{-2}|P_i(t, \omega)|^2}}, \\ \dfrac{d}{dt}P_i(t, \omega) = -\displaystyle\int_{\mathbf{R}\times E} \nabla U_i(Q_i(t, \omega) - x(t, \Psi(s, x, m^{-\frac{1}{2}}v))) \\ \qquad\qquad \times\mu_\omega(ds, dx, dv), \\ (Q_i(0, \omega), P_i(0, \omega)) = (Q_{i,0}, P_{i,0}), \qquad i = 1, \cdots, N, \\ \\ \quad \dfrac{d}{dt}x(t, x, v, \omega) = v(t, x, v, \omega), \\ m\dfrac{d}{dt}v(t, x, v, \omega) = -\displaystyle\sum_{i=1}^{N} \nabla U_i(x(t, x, v, \omega) - Q_i(t, \omega)), \\ (x(0, x, v, \omega), v(0, x, v, \omega)) = (x, v), \qquad (x, v) \in \Psi(\omega). \end{cases}$$

We also use the notation $V_i(t) = V_i(t, \omega) = \frac{P_i(t,\omega)}{M_i\sqrt{1+M_i^{-2}c^{-2}|P_i(t,\omega)|^2}}$. We remark that $P_i = \frac{MV_i}{\sqrt{1-c^{-2}|V_i|^2}}$.

### 3.2. Classical scattering

Let $\varphi(t, x, v; \vec{Q}) = (\varphi^0(t, x, v; \vec{Q}), \varphi^1(t, x, v; \vec{Q}))$ and $\psi(t, x, v; \vec{Q})$ be as defined in Section 2. Notice that $\varphi(t, x, v; \vec{Q})$ is exactly the same as in [11], so all of the results in [11, Chapter 3] with respect to $\varphi(t, x, v; \vec{Q})$ and $\psi(t, x, v; \vec{Q})$ are valid in our case, too. In particular, we have the followings.

LEMMA 3.1. *For any* $(x,v) \in E$ *and* $t \in \mathbf{R}$*, we have that* $\psi(t,x,v;\vec{Q}) = \varphi(t+s,\Psi(s,x,v);\vec{Q})$ *for any* $s \geq \frac{\max\{R_{U_i}+|Q_i|;i=1,\cdots,N\}}{|v|}$.

PROPOSITION 3.2. *Suppose that* $(x,v) \in E$ *and* $|v| > 2C_0$*. Then*

$$\varphi^1(t,x,v;\vec{Q}) \cdot (|v|^{-1}v) > C_0, \qquad \text{for any } t \in \mathbf{R}.$$

PROPOSITION 3.3. *For any* $(x,v) \in E$ *with* $|v| > 2C_0$*, we have that*

$$\left|\psi^0(t,x,v;\vec{Q}) - Q_i\right| > R_{U_i}, \qquad i = 1,\cdots,N,$$

*if* $t \geq 2C_0^{-1}R(\vec{Q})$ *or* $t \leq -C_0^{-1}R(\vec{Q})$*. Here* $R(\vec{Q}) := \max\{R_{U_i} + |Q_i|; i = 1,\cdots,N\}$.

PROPOSITION 3.4. *For any measurable* $f : \mathbf{R}^{2d} \to [0,\infty)$ *such that at least one of the integrals below is finite, we have*

$$\begin{aligned}
&\int_{\mathbf{R}^{2d}} f(x,v)\rho(\frac{1}{2}|v|^2 + \sum_{i=1}^N U_i(x-Q_i))dxdv \\
&= \int_E \left(\int_{-\infty}^{\infty} f(\psi(t,x,v;\vec{Q}))dt\right)\rho(\frac{1}{2}|v|^2)\nu(dx,dv). \qquad (3.2)
\end{aligned}$$

LEMMA 3.5. *For any* $A > 0$ *and* $t_0 > 0$*, there exists a constant* $\widetilde{C}$ *(depending on* $\max_{i=1,\cdots,N} R_{U_i} + A$*,* $t_0$*,* $C_0$ *and* $\sum_{i=1}^N \|\nabla^2 U_i\|_\infty$*) such that*

$$\left|\psi^0(t,x,v;\vec{Q^1}) - \psi^0(t,x,v;\vec{Q^2})\right| \leq \widetilde{C}\|\vec{Q^1} - \vec{Q^2}\|_{\mathbf{R}^d},$$

*for any* $(x,v) \in E, |v| \geq 2C_0 + 1, |t| \leq t_0$ *and* $|\vec{Q^1}|, |\vec{Q^2}| \leq A$.

### 3.3. Basic lemmas for tightness

We prepare several basic facts for integrals with respect to tightness. These will be used in the following sections.

Let us first recall some basic facts with respect to the Skorohod space $(D([0,T];\mathbf{R}^d), d^0)$ and the tightness of the probability measures on it. (See Billingsley [1] for more details).

For any $T > 0$, $D([0,T];\mathbf{R}^d)$ denotes the Skorohod space:

$$D([0,T];\mathbf{R}^d) = \Big\{w : [0,T] \to \mathbf{R}^d; \quad w(t) = w(t+) := \lim_{s\downarrow t} w(s), t \in [0,T), \\ \text{and } w(t-) := \lim_{s\uparrow t} w(s) \text{ exists}, t \in (0,T]\Big\},$$

with the metric $d^0 = d^0_T$ given by

$$d^0(w,\widetilde{w}) = \inf_{\lambda\in\Lambda} \Big\{\|\lambda\|^0 \vee \|w - \widetilde{w}\circ\lambda\|_\infty\Big\}$$

for any $w, \widetilde{w} \in D([0,T];\mathbf{R}^d)$, where

$$\Lambda = \Big\{\lambda : [0,T] \to [0,T]; \text{ continuous, non-decreasing}, \lambda(0) = 0, \lambda(T) = T\Big\},$$

$\|w\|_\infty = \sup_{0\le t\le T} |w(t)|$, and $\|\lambda\|^0 = \sup_{0\le s<t\le T} \left|\log \frac{\lambda(t)-\lambda(s)}{t-s}\right|$ for any $\lambda \in \Lambda$.

It is well-known that $(D([0,T];\mathbf{R}^d), d^0)$ is a complete metric space. Also, $C([0,T];\mathbf{R}^d) = \{w : [0,T] \to \mathbf{R}^d;\ \text{continuous}\}$ is closed in $(D([0,T];\mathbf{R}^d), d^0)$, and the Skorohod topology relativized to $C([0,T];\mathbf{R}^d)$ coincides with the uniform topology there. (See, *e.g.*, [1]).

Our base for the proof of tightness is the following. We quote it here from [11]. Let $\wp(D([0,T];\mathbf{R}^d))$ denote the space of all probabilities on $D([0,T];\mathbf{R}^d)$.

Theorem 3.6 ([11]). *Let $(\Omega_n, \mathcal{F}_n, Q_n)$, $n \in \mathbf{N}$, be probability spaces, and let $X_n : \Omega_n \to D([0,T];\mathbf{R}^d)$, $n \in \mathbf{N}$, be measurable. Let $\mu_{X_n} = Q_n \circ X_n^{-1}$. Suppose that there exist constants $\varepsilon, \beta, \gamma, C > 0$ such that*

(1) $E^{Q_n}\big[\|X_n(\,\cdot\,)\|^\varepsilon_\infty\big] \le C$,

(2) $E^{Q_n}\Big[|X_n(r) - X_n(s)|^\beta |X_n(s) - X_n(t)|^\beta\Big] \le C|t-r|^{1+\varepsilon}$ *for any* $0 \le r \le s \le t \le 1$,

(3) $E^{Q_n}\Big[|X_n(s) - X_n(t)|^\varepsilon\Big] \le C|t-s|^\gamma$ *for any* $0 \le s \le t \le 1$,

*for any $n \in \mathbf{N}$. Then $\{\mu_{X_n}\}_{n=1}^\infty$ is tight in $\wp(D([0,T];\mathbf{R}^d))$.*

Lemmas 3.7 and 3.8 below are easy consequences of Theorem 3.6.

Lemma 3.7 ([11]). *Let* $(\Omega_n, \{\mathcal{F}_n(t)\}_{t\in[0,T]}, Q_n)$, $n \in \mathbf{N}$, *be filterated probability spaces, and let* $f_n : [0,T] \times \Omega_n \to \mathbf{R}$ *be* $\{\mathcal{F}_n(t)\}_{t\in[0,T]}$*-adapted,* $n \in \mathbf{N}$. *If*

$$\sup_{n\in\mathbf{N}} \sup_{s\in[0,T]} E^{Q_n}\left[|f_n(s)|^2\right] < \infty,$$

*then* $\left\{ \text{the distribution of } \{\int_0^t f_n(s)ds\}_{t\in[0,T]} \text{ under } Q_n; n \in \mathbf{N}\right\}$ *is tight in* $\wp(C([0,T]; \mathbf{R}^d))$.

Lemma 3.8 ([11]). *Let* $(\Omega_n, \{\mathcal{F}_n(t)\}_{t\in[0,T]}, Q_n)$, $n \in \mathbf{N}$, *be filterated probability spaces, and let* $\{M_n(t)\}_t$ *be* $(\{\mathcal{F}_n(t)\}_{t\in[0,T]}, Q_n)$*-martingales. If there exists a constant* $C > 0$ *such that*

$$E^{Q_n}\left[|M_n(t) - M_n(s)|^2 \middle| \mathcal{F}_s\right] \le C(t-s), \qquad 0 \le s < t \le T,$$

*then* $\left\{ \text{the distribution of } \{M_n(t)\}_{t\in[0,T]} \text{ under } Q_n; n \in \mathbf{N}\right\}$ *is tight in* $\wp(D([0,T]; \mathbf{R}^d))$.

### 3.4. Some estimates

All of the results of this subsection are proved in exact the same way as that of [11], and we omit the proofs here. (See [13, Appendix] for the proofs).

First notice that $|Q_i(t,\omega)| \le |Q_{i,0}| + cT$ for any $t \in [0,T]$. Let $R_0 = \max_{i=1,\cdots,N}\{R_{U_i} + |Q_{i,0}| + cT + 1\}$, and let $\tau = C_0^{-1}R_0$. Here $R_{U_1}, \cdots, R_{U_N}$ and $C_0$ are constants defined in Section 2.

Proposition 3.9. *Suppose that* $(x,v) \in E$, $|v| > (2C_0+1)m^{-1/2}$ *and* $m \le \frac{1}{c^2}$. *Then*

$$\left(|v|^{-1}v\right) \cdot v(t,x,v;\omega) \ge m^{-1/2}(C_0+1), \qquad \textit{for any } t \in [0,T].$$

Proposition 3.10. *Suppose that* $(x,v) \in E$, $|v| > 2C_0+1$, $m \le \frac{1}{c^2}$, $0 \le m^{1/2}t + s \le T$ *and* $t \in (-\infty, -\tau) \cup (2\tau, \infty)$. *Then*

$$\nabla U_i(x(m^{1/2}t+s, \Psi(s,x,m^{-1/2}v);\omega) - Q_i(m^{1/2}t+s;\omega)) = 0.$$

Since we are interested in the bahavior as $m \to 0$, without loss of generality, we assume from now on that $m \le \frac{1}{c^2}$.

LEMMA 3.11. *Let $a \in \mathbf{R}$ be any constant and suppose that the following holds: $0 \le s - am^{1/2} \le T$, $0 \le s - m^{1/2}\tau \le T$ and $|v| > 2C_0 + 1$. Let*

$$y(t) := x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) - \psi^0(t, x, v; \vec{Q}(s - am^{1/2}, \omega)).$$

*Then we have the following.*

(1) $y(t) = 0$ *if* $0 \le m^{1/2}t + s \le T$ *and* $t \le -\tau$,

(2)

$$\begin{aligned} \frac{d^2}{dt^2}y(t) = \; -\sum_{i=1}^{N}\Big\{ & \nabla U_i(y(t) + \psi^0(t, x, v; \vec{Q}(s - am^{1/2}; \omega)) \\ & - Q_i(m^{1/2}t + s; \omega)) \\ & - \nabla U_i(\psi^0(t, x, v; \vec{Q}(s - am^{1/2}; \omega)) \\ & - Q_i(s - am^{1/2}; \omega))\Big\}, \end{aligned}$$

(3) *Let* $C_1 = 3\tau\Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty + 1\Big)\exp\Big(3\tau\Big(\sum_{i=1}^N \|\nabla^2 U_i\|_\infty + 1\Big)\Big)$. *Then*

$$|y(t)| + \left|\frac{d}{dt}y(t)\right| \le m^{1/2}(2\tau + |a|)C_1 c$$

*if* $|t| \le 2\tau$ *and* $0 \le m^{1/2}t + s \le T$.

Moreover, let $z(t; x, v, \vec{Q}, \vec{V}, a)$ be the solution of (2.3). Then we have the following.

LEMMA 3.12. *Let $a \in [-2\tau, \tau]$ be a constant and assume the following: $0 \le s - m^{1/2}\tau \le T$, $-\tau \le t \le 2\tau$ and $0 \le s - am^{1/2} \le s + m^{1/2}t \le T$. Also, assume that $|v| > 2C_0 + 1$. Then we have that*

$$\begin{aligned} &\Big| x(m^{1/2}t + s, \Psi(s, x, m^{-1/2}v)) - \psi^0(t, x, v; \vec{Q}(s - am^{1/2})) \\ &\qquad - m^{1/2} z(t; x, v, \vec{Q}(s - am^{1/2}), \vec{V}(s - am^{1/2}), a)\Big| \\ &\le C_3 m^{1/2}\Big\{(2\tau + |a|)^2 c^2 m^{1/2} + \int_{s-\tau m^{1/2}}^{s+2\tau m^{1/2}} |V_i(r) - V_i(s - am^{1/2})|dr\Big\}, \end{aligned}$$

*with* $C_3 := 3\tau C_2 e^{3\tau(C_2+1)}$, $C_2 := \sum_{i=1}^N \|\nabla^3 U_i\|_\infty (C_1 + 1)^2 \vee \sum_{i=1}^N \|\nabla^2 U_i\|_\infty$.

### 3.5. Decomposition of $P_k(t)$

Our main result of this section is the following decomposition of $P_k(t)$: $P_k(t)$ can be decomposed as the summation of a martingale part, a "smooth part", a negligible part which converges to 0 fast enough and the term $m^{-1/2}\int_0^t \nabla_k \widetilde{U}(\vec{Q}(s))ds$, which causes the "uniform motion phase" in the limit. We give the precise statement in the following.

Our new potential $\widetilde{U}$ is given by

$$\widetilde{U}(\vec{Q}) = \int_{\mathbf{R}^d} \Big(p\Big(\sum_{i=1}^N U_i(Q_i - x)\Big) - p(0)\Big)dx,$$

with

$$p(s) = \int_{\mathbf{R}^d} \widetilde{\rho}\Big(\frac{1}{2}|v|^2 + s\Big)dv,$$

$$\widetilde{\rho}(t) = -\int_t^\infty \rho(s)ds, \qquad t \in \mathbf{R}.$$

The concrete definitions of the martingale part and the "smooth part" are also needed in the following sections.

Let $\mathcal{F}_t = \mathcal{F}_t^{(m)} = \mathcal{F}_{(-\infty, 2m^{1/2}\tau + t]\times E} \vee \mathcal{N}$, here $\mathcal{N}$ denotes the set of all null sets. Also, let

$$N((0,t]\times A) = \mu_\omega((2m^{1/2}\tau, 2m^{1/2}\tau + t] \times A), \qquad A \in \mathcal{B}(E).$$

Then $N$ is a $\{\mathcal{F}_t\}$-adapted Poisson point process with density

$$\overline{\lambda}(dr,dx,dv) = \overline{\lambda}_m(dr,dx,dv) = m^{-1}\rho(\frac{1}{2}|v|^2)dr\nu(dx,dv).$$

Let

$$\overline{N}(dr,dx,dv) = N(dr,dx,dv) - \overline{\lambda}(dr,dx,dv).$$

Now, we are ready to give the definition of our martingale term:

$$M_k(t) = M_k^{(m)}(t) = -\int_{[0,t]\times E} \overline{N}(dr,dx,dv) m^{1/2} D_k(\vec{Q}(r), x, v).$$

Here $D_k(\cdot,\cdot,\cdot)$ is the one defined in Section 2.

Let us next give the definition of the "smooth" term. For any $r \geq 0$, let

$$\widetilde{r} = ((r - 2m^{1/2}\tau) \vee 0) \wedge T. \tag{3.3}$$

Let

$$J_{k1}(t) = -\int_0^t ds \int_{\mathbf{R}\times E} \nabla^2 U_k(Q_k(s) - \psi^0(u, x, v; \vec{Q}(s)))$$
$$\times z(u, x, v; \vec{Q}(s), \vec{V}(s), -u)\rho(\frac{1}{2}|v|^2) du\nu(dx, dv).$$

Also, let

$$J_{k2}(t) = -\int_0^t ds 1_{[4m^{1/2}\tau,\infty)}(s) \int_{\mathbf{R}\times E} g_{k2}(s, r, x, v)(\mu_\omega - \lambda)(dr, dx, dv)$$

$$J_{k3}(t) = \int_0^t ds 1_{[4m^{1/2}\tau,\infty)}(s) \int_{\mathbf{R}\times E} (\mu_\omega - \lambda)(dr, dx, dv)$$
$$\times \nabla^2 U_k\Big(Q_k(\widetilde{r}) - \psi(m^{-1/2}(s-r), x, v; \vec{Q}(\widetilde{r}))\Big)$$
$$\times \Big[\Big(Q_k(s) - Q_k(\widetilde{r}) - (s-\widetilde{r})V_k(\widetilde{r})\Big) - \psi^0(m^{-1/2}(s-r), x, v; \vec{Q}(s))$$
$$+ \psi^0(m^{-1/2}(s-r), x, v; \vec{Q}(\widetilde{r}) + (s-\widetilde{r})\vec{V}(\widetilde{r})\Big],$$

$$J_{k4}(t) = -\int_0^t ds 1_{[4m^{1/2}\tau,\infty)}(s)$$
$$\times \int_{\mathbf{R}\times E} \nabla^2 U_i(Q_i(s) - \psi^0(m^{-1/2}(s-r), x, v, \vec{Q}(s)))$$
$$\Big(x(s, \Psi(r, x, m^{-1/2}v)) - \psi^0(m^{-1/2}(s-r), x, v, \vec{Q}(s))$$
$$- m^{1/2} z(m^{-1/2}(s-r), x, v, \vec{Q}(s), \vec{V}(s), -m^{-1/2}(s-r))\Big)$$
$$\times \mu_\omega(dr, dx, dv),$$

with

$$g_{k2}(s, r, x, v) = \frac{1}{2}\nabla^2 U_k(Q_k(s) - \psi^0(m^{-1/2}(s-r), x, v; \vec{Q}(s)))$$
$$\times m^{1/2} z(m^{-1/2}(s-r), x, v; \vec{Q}(s), \vec{V}(s), -m^{-1/2}(s-r))$$
$$- \frac{1}{2}\nabla^2 U_k(Q_k(\widetilde{r}) - \psi^0(m^{-1/2}(s-r), x, v; \vec{Q}(\widetilde{r})))$$
$$\times m^{1/2} z(m^{-1/2}(s-r), x, v; \vec{Q}(\widetilde{r}), \vec{V}(\widetilde{r}), -m^{-1/2}(s-r)).$$

Our "smooth" term is given by

$$J_k(t) := J_k^{(m)}(t) = J_{k1}(t) + J_{k2}(t) + J_{k3}(t) + J_{k4}(t).$$

(Here we divide the term $J_k(t)$ into several parts because the terms $J_{k2}$, $J_{k3}$ and $J_{k4}(t)$, which will be shown to be "smooth" at first, will be proven later to be negligible (1) until $\sigma_0$ in general case (see Lemma 4.1) and (2) in the special case described in Theorem 2.2 (see Section 6).

Now, we are ready to state our decomposition of $P_k(t)$. Our main result of this section is the following:

Lemma 3.13. *For $k \in \{1, \cdots, N\}$, we have the following.*

(1) *There exists an $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-adapted process $\eta_k(t)$ such that*

$$P_k(t) - P_k(0) = M_k(t) + \eta_k(t) + J_k(t) - m^{-1/2} \int_0^t \nabla_k \widetilde{U}(\vec{Q}(s))ds, \tag{3.4}$$

(2) *$M_k(t)$ is an $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-martingale, and there exists a constant $C$ independent of $m$ such that for any $0 \le s \le t \le T$ and $m \in (0,1]$, we have*

$$E^{\kappa_m}\Big[\big|M_k(t) - M_k(s)\big|^2\Big|\mathcal{F}_s\Big] \le C|t-s|,$$

*and the jumps of $M_k(\cdot)$ satisfy $|\Delta M_k(t)| \le C m^{1/2}$,*

(3) *$J_k(t)$ is an $\mathbf{R}^d$-valued $(\mathcal{F}_t)_t$-adapted $C^1$-class (in $t$) process such that*

$$\sup_{m\in(0,1]} \sup_{t\in[0,T]} E^{\kappa_m}\Big[|\frac{d}{dt}J_k(t)|^2\Big] < \infty, \tag{3.5}$$

(4) *there exists a constant $C$ independent of $m$ such that*

$$y_m := E^{\kappa_m}\big[\sup_{t\in[0,T]} |\eta_k(t)|^2\big] \to 0, \qquad m \to 0, \tag{3.6}$$

*and*

$$E^{\kappa_m}\big[|\eta_k(t)|^2\big] \le C m^{1/2}, \qquad m \le 1, t \in [0,T]. \tag{3.7}$$

*In particular, $\Big\{$ the distributions of $\{M_k(t) + \eta_k(t); t \in [0,T]\}$ under $\kappa_m; m \in (0,1]\Big\}$ and $\Big\{$ the distributions of $\{J_k(t); t \in [0,T]\}$ under $\kappa_m; m \in (0,1]\Big\}$ are tight in $\wp(C([0,T];\mathbf{R}^d))$.*

The process $\eta_k(t)$ is the error part of our re-expression (3.4). We do not have much information on its detailed behavior, especially, we have almost no information with respect to its derivative. The only thing we know is that it converges to 0 fast enough (Lemma 3.13 (4)). However, by considering the corresponding quantities with $\eta_k(t)$ withdrew from the beginning, this is enough for our estimates. Precisely, later on, instead of $P_k(t)$, we will consider the quantity $P_k(t) - \eta_k(t)$ (see (3.8) below and the other related overlined quantities).

The following is an easy consequence of Theorem 3.6.

Lemma 3.14.

(1) *For any* $f^{(m)} : [0,T] \times \Omega \to \mathbf{R}^d$ *that are bounded and* $\{\mathcal{F}_t\}_t$*-adapted, and* $k \in \{1, \cdots, N\}$*, we have that*

$$\sup_{m\in(0,1]} E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\int_{0+}^{t} f^{(m)}(s)\cdot dM_k^{(m)}(s)\Big|^2\Big] < \infty,$$

*and that* $\Big\{$ *the distribution of* $\{\int_{0+}^{t} f^{(m)}(s) \cdot dM_k^{(m)}(s)\}_{t\in[0,T]}; m \in (0,1]\Big\}$ *is tight in* $\wp(D([0,T];\mathbf{R}^d))$*, with all of its cluster points as* $m \to 0$ *in* $\wp(C([0,T];\mathbf{R}^d))$.

(2) *For any* $f^{(m)} : [0,T] \times \Omega \to \mathbf{R}^d$ *that are bounded and* $\{\mathcal{F}_t\}_t$*-adapted, we have that*

$$\sup_{m\in(0,1]}\sup_{t\in[0,T]} E^{\kappa_m}\Big[\Big|\int_{0+}^{t} f^{(m)}(s)\cdot dJ_k^{(m)}(s)\Big|^2\Big] < \infty,$$

*and that* $\Big\{$ *the distribution of* $\{\int_{0+}^{t} f^{(m)}(s) \cdot dJ_k^{(m)}(s)\}_{t\in[0,T]}; m \in (0,1]\Big\}$ *is tight in* $\wp(C([0,T];\mathbf{R}^d))$.

Also, we have the following result with respect to the tightness in $L^p([0,T])$.

Lemma 3.15 ([11]). *For any* $f^{(m)} : [0,T] \times \Omega \to \mathbf{R}$ *that are* $\{\mathcal{F}_t\}_t$*-adapted, if*

$$\lim_{K\to\infty}\inf_{m\in(0,1]} P\Big(\int_0^T |f^{(m)}(s)|ds \le K\Big) = 1,$$

*then* $\Big\{$ *the distribution of* $\{\int_0^t f^{(m)}(s)ds\}_{t\in[0,T]}; m \in (0,1]\Big\}$ *is tight in* $\wp(L^p([0,T];\mathbf{R}^d))$ *for any* $p > 1$*, with all of its cluster points in* $\wp(D([0,T];\mathbf{R}^d))$*. In particulat, if*

$$\sup_{m\in(0,1]} E\Big[\int_0^T |f^{(m)}(s)|ds\Big] < \infty,$$

*then our assertion holds.*

The following is easy from the definition of $M_k(\,\cdot\,)$.

Lemma 3.16. *For any* $k_1, k_2 \in \{1,\cdots,N\}$ *and* $l_1, l_2 \in \{1,\cdots,d\}$*, the following holds:*

(1)

$$\begin{aligned}\sum_{s\in[0,t]} \Big(\Delta M_{k_1}^{l_1}(s)\Big)\Big(\Delta M_{k_2}^{l_2}(s)\Big) &= \Big[M_{k_1}^{l_1}, M_{k_2}^{l_2}\Big]_t \\ &= m\int_{[0,t]\times E} D_{k_1}^{l_1}(\vec{Q}(r),x,v) D_{k_2}^{l_2}(\vec{Q}(r),x,v) N(dr,dx,dv),\end{aligned}$$

(2)

$$\sup_{m\in(0,1]} E^{\kappa_m}\Big[\sum_{s\in[0,T]} \Big|\Delta M_{k_1}(s)\Big|^2\Big] < \infty,$$

(3) *for any* $f^{(m)} : [0,\infty)\times\Omega \to \mathbf{R}$ *that are bounded and* $\{\mathcal{F}_t\}_t$*-adapted, there exists a constant* $C > 0$ *such that*

$$\begin{aligned}\lim_{m\to 0} E^{\kappa_m}\Big[\sup_{t\in[0,T]} \Big| \int_{0+}^t f^{(m)}(s,\cdot) d[M_{k_1}^{l_1}, M_{k_2}^{l_2}]_s \\ - \int_{0+}^t f^{(m)}(s,\cdot) a_{k_1 l_1, k_2 l_2}(\vec{Q}(s)) ds\Big|^2\Big] \le Cm.\end{aligned}$$

As claimed, since we have almost no information with respect to the derivative of $\eta_k(\,\cdot\,)$, we will use the following useful approximation of $P_k(t)$:

$$\overline{P_k}(t) := P_k(t) - \eta_k(t) = M_k(t) + J_k(t) - \int_0^t m^{-1/2}\nabla_k \widetilde{U}(\vec{Q}(s))ds. \tag{3.8}$$

We have the following.

Lemma 3.17. *For any* $g : \mathbf{R}^{dN} \times \mathbf{R}^{dN} \to \mathbf{R}$ *with* $\|\nabla^3 g\|_\infty < \infty$, *there exists a constant* $C > 0$ *such that for any* $m \in (0, 1]$, *the following holds.*

$$E^{\kappa_m}\Big[\Big(\sum_{s\in(0,T]}\Big|g(\vec{Q}(s),\vec{\overline{P}}(s)) - g(\vec{Q}(s),\vec{\overline{P}}(s-)) - \sum_{j=1}^{N}\nabla_{p_j} g(\vec{Q}(s),\vec{\overline{P}}(s-))\cdot\Delta\overline{P_j}(s) - \frac{1}{2}\sum_{j_1,j_2=1}^{N}\sum_{l_1,l_2=1}^{d}\nabla_{p_{i_1}^{l_1}}\nabla_{p_{j_2}^{l_2}} g(\vec{Q}(s),\vec{\overline{P}}(s-))\Delta\overline{P_{j_1}^{l_1}}(s)\Delta\overline{P_{j_2}^{l_2}}(s)\Big|^2\Big] \le Cm.$$

## 4. Convergence until $\sigma_0$

We give the proof of Theorem 2.1 in this section. Let $\sigma_0(\omega) = \inf\big\{t > 0; \min_{i\neq j}\{|Q_i(t;\omega) - Q_j(t;\omega)| - (R_i + R_j)\} \le 0\big\}$ if $N \ge 2$ as in Theorem 2.1.

We first prove that $J_{k2}$ and $J_{k3}$ are negligible until $\sigma_0$ for $k \in \{1, \cdots, N\}$ in the following sense.

Lemma 4.1. *There exists a constant* $C > 0$ *such that for* $l \in \{2, 3, 4\}$, *we have that*

$$E^{\kappa_m}\Big[\Big|\frac{d}{dt}J_{kl}(t\wedge\sigma_0)\Big|\Big] \le Cm^{1/4}, \qquad m \in (0,1], t \in [0,T].$$

Before giving the proof of Lemma 4.1, let us first prepare the following.

Lemma 4.2. *There exists a constant* $C > 0$ *such that for any* $s \in [0, T]$, *we have*

$$\Big|\frac{d}{ds}J_{k2}(s)\Big| \le C\int_{\mathbf{R}\times E} m^{1/2}\Big(m^{1/2} + \sum_{j=1}^{d}|V_j(s) - V_j(\widetilde{r})|\Big) \times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\lambda(dr,dx,dv).$$

Proof. First notice that by Propositions 3.3 and 3.10, we have that $g_{k2}(s,r,x,v)$ is not 0 only if $|x| \le R_0$ and $m^{-1/2}(s-r) \in [-\tau, 2\tau]$, and in this domain, we have that $|\vec{Q}(s) - \vec{Q}(\widetilde{r})| \le Nc|s-\widetilde{r}| \le 4Ncm^{1/2}\tau$. Therefore,

$$\begin{aligned}
&\Big|g_{k2}(s,r,x,v)\Big| \\
&= \frac{1}{2}m^{1/2}\Big|\nabla^2 U_k(Q_k(s) - \psi^0(m^{-1/2}(s-r),x,v;\vec{Q}(s))) \times \\
&\qquad \times \Big\{z(m^{-1/2}(s-r),x,v;\vec{Q}(s),\vec{V}(s),-m^{-1/2}(s-r)) \\
&\qquad\qquad - z(m^{-1/2}(s-r),x,v;\vec{Q}(\widetilde{r}),\vec{V}(\widetilde{r}),-m^{-1/2}(s-r))\Big\} \\
&\quad + \Big\{\nabla^2 U_k(Q_k(s) - \psi^0(m^{-1/2}(s-r),x,v;\vec{Q}(s))) \\
&\qquad\qquad - \nabla^2 U_k(Q_k(\widetilde{r}) - \psi^0(m^{-1/2}(s-r),x,v;\vec{Q}(\widetilde{r})))\Big\} \\
&\qquad \times z(m^{-1/2}(s-r),x,v;\vec{Q}(\widetilde{r}),\vec{V}(\widetilde{r}),-m^{-1/2}(s-r))\Big| \\
&\le \frac{1}{2}m^{1/2}1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r) \\
&\qquad \times \Big[\|\nabla^2 U_k\|_\infty C(|\vec{Q}(s)-\vec{Q}(\widetilde{r})| + |\vec{V}(s)-\vec{V}(\widetilde{r})|) \\
&\qquad\qquad + \|\nabla^3 U_k\|_\infty(|Q_k(s)-Q_k(\widetilde{r})| + \widetilde{C}|\vec{V}(s)-\vec{V}(\widetilde{r}|))C\Big] \\
&\le Cm^{1/2}\Big(m^{1/2} + \sum_{j=1}^d |V_j(s)-V_j(\widetilde{r})|\Big)1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r). \quad \square
\end{aligned}$$

Proof of Lemma 4.1 with $l = 2$. Notice that by the definition of $V_j$, we have that $|V_j(s_1) - V_j(s_2)| \le M_j^{-1}|P_j(s_1) - P_j(s_2)|$ for any $s_1, s_2 \ge 0$ and $j \in \{1, \cdots, N\}$. Therefore, by Lemma 4.2, we have that

$$\begin{aligned}
\Big|\frac{d}{ds}J_{k2}(s)\Big| \le C\int_{\mathbf{R}\times E} m^{1/2}\Big(m^{1/2} + \sum_{j=1}^d M_j^{-1}|P_j(s) - P_j(\widetilde{r})|\Big) \\
\times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)(\mu_\omega + \lambda)(dr,dx,dv). \qquad (4.1)
\end{aligned}$$

Notice that $s \le \sigma_0$ combined with $|s-r| \le 2m^{1/2}\tau$ implies $\widetilde{r} \le \sigma_0$. Therefore, by (4.1), we have that

$$E^{\kappa_m}\Big[\Big|\frac{d}{ds}J_{k2}(s\wedge\sigma_0)\Big|\Big]$$

$$\leq 2CE^{\kappa_m}\Big[\int_{\mathbf{R}\times E} m^{1/2}\Big(m^{1/2}+\sum_{j=1}^{d} M_j^{-1}|P_j(s\wedge\sigma_0)-P_j(\widetilde{r}\wedge\sigma_0)|\Big)$$

$$\times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\lambda(dr,dx,dv)\Big]. \tag{4.2}$$

Since $\nabla\widetilde{U}(Q_1(t),\cdots,Q_N(t)) = 0$ for any $t \leq \sigma_0$, we have by Lemma 3.13 (1) that

$$P_j(t\wedge\sigma_0)-P_j(0) = M_j(t)+J_j(t)+\eta_j(t), \qquad t\in[0,T].$$

Therefore, by Lemma 3.13 (2), (3), (4), we get that there exists a constant $C > 0$ such that

$$E^{\kappa_m}\Big[\Big|P_j(s_1\wedge\sigma_0)-P_j(s_2\wedge\sigma_0)\Big|\Big]$$

$$\leq E^{\kappa_m}\Big[\Big|P_j(s_1\wedge\sigma_0)-P_j(s_2\wedge\sigma_0)\Big|^2\Big]^{1/2}$$

$$\leq C\Big(|s_1-s_2|+m^{1/2}\Big)^{1/2}, \qquad s_1,s_2\in[0,T]. \tag{4.3}$$

This combined with (4.2) implies our assertion. □

Proof of Lemma 4.1 with $l = 3$. The basic idea is the same at that for $l = 2$.

First, we have by Lemma 3.5 that there exists a constant $C > 0$ such that

$$\Big|\frac{d}{ds}J_{k3}(s)\Big| \leq C\int_{\mathbf{R}\times E}(\mu_\omega+\lambda)(dr,dx,dv)1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)$$

$$\times\sum_{j=1}^{N}\Big|Q_j(s)-Q_j(\widetilde{r})-(s-\widetilde{r})V_j(\widetilde{r})\Big|. \tag{4.4}$$

This combined with

$$\Big|Q_j(s)-Q_j(\widetilde{r})-(s-\widetilde{r})V_j(\widetilde{r})\Big| = \Big|\int_{\widetilde{r}}^{s}\Big(V_j(u)-V_j(\widetilde{r})\Big)du\Big|$$

$$\leq M_j^{-1}\int_{\widetilde{r}}^{s}\Big|P_j(u)-P_j(\widetilde{r})\Big|du$$

implies that

$$
\begin{aligned}
&E^{\kappa_m}\Big[\Big|\frac{d}{ds}J_{k3}(s\wedge\sigma_0)\Big|\Big] \\
&\le 2C\int_{\mathbf{R}\times E}\lambda(dr,dx,dv)1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r) \\
&\qquad\times\sum_{j=1}^{N}\int_{\tilde r}^{s}E^{\kappa_m}\Big[\Big|P_j(u\wedge\sigma_0)-P_j(\tilde r\wedge\sigma_0)\Big|\Big]du. \qquad (4.5)
\end{aligned}
$$

This combined with (4.3) implies our assertion for $l=3$. □

Proof of Lemma 4.1 with $l=4$. With the help of Lemma 3.12, the proof is similar to that of $l=2,3$. We omit the details. □

Let $\overline{P_k}(t)=P_k(t)-\eta_k(t)$ as before. The following is trivial.

Lemma 4.3 *For any $f\in C_0^\infty(B_1\times\mathbf{R}^{dN})$, we have that when $m\to 0$, $\{f(\vec{Q}(t\wedge\sigma_0),\vec{P}(t\wedge\sigma_0))\}_t$ and $\{f(\vec{Q}(t\wedge\sigma_0),\vec{\overline{P_k}}(t\wedge\sigma_0))\}_t$ converge or not at the same time, and when they converge, they have the same limit.*

Proof of Theorem 2.1. For any $f\in C_0^\infty(B_1\times\mathbf{R}^{dN})$, we have by (3.4) that

$$
\begin{aligned}
&f(\vec{Q}(t\wedge\sigma_0),\vec{\overline{P}}(t\wedge\sigma_0)-f(\vec{Q_0},\vec{P_0}) \\
&=\int_0^{t\wedge\sigma_0}f_q(\vec{Q}(s),\vec{\overline{P}}(s-))\cdot\vec{V}(s)ds+\sum_{j=1}^{N}\int_0^{t\wedge\sigma_0}f_{p_j}(\vec{Q}(s),\vec{\overline{P}}(s-))\cdot dM_j(s) \\
&\quad+\sum_{j=1}^{N}\int_0^{t\wedge\sigma_0}f_{p_j}(\vec{Q}(s),\vec{\overline{P}}(s-))\cdot dJ_{j1}(s) \\
&\quad+\sum_{j=1}^{N}\int_0^{t\wedge\sigma_0}f_{p_j}(\vec{Q}(s),\vec{\overline{P}}(s-))\cdot(\sum_{l=2}^{4}dJ_{jl}(s)) \\
&\quad+\sum_{k_1,k_2=1}^{N}\sum_{l_1,l_2=1}^{d}\int_{0+}^{t\wedge\sigma_0}f_{p_{k_1}^{l_1}p_{k_2}^{l_2}}(\vec{Q}(s),\vec{\overline{P}}(s-))d[M_{k_1}^{l_1},M_{k_2}^{l_2}]_s \\
&\quad+\sum_{s\in[0,t\wedge\sigma_0]}\Big\{f(\vec{Q}(s),\vec{\overline{P}}(s))-f(\vec{Q}(s),\vec{\overline{P}}(s-))
\end{aligned}
$$

$$- \sum_{k=1}^{N} f_{p_k}(\vec{Q}(s), \vec{\vec{P}}(s-)) \cdot \Delta M_k(s)$$
$$- \frac{1}{2} \sum_{k_1,k_2=1}^{N} f_{p_{k_1} p_{k_2}}(\vec{Q}(s), \vec{\vec{P}}(s-))(\Delta M_{k_1}(s))(\Delta M_{k_2}(s))\Big\}.$$

The second term on the right hand side above is a martingale. The first term on the right hand side above gives us the terms of $\frac{\partial}{\partial q_j}$ in $L_1$. The third term on the right hand side above gives us the terms of $\frac{\partial}{\partial v_j}$ in $L_1$ by the definition of $J_{j1}$. By Lemma 4.1, the forth term on the right hand side above converges to 0 as $m \to 0$. For the fifth term on the right hand side above, we have by Lemma 3.16 (3) that

$$\lim_{m\to 0} E^{\kappa_m}\Big[\sup_{t\in 0, T\wedge\sigma_0]} \Big| \int_{0+}^{t\wedge\sigma_0} f_{p_{k_1}^{l_1} v_{k_2}^{l_2}}(\vec{Q}(s), \vec{\vec{P}}(s-)) d[M_{k_1}^{l_1}, M_{k_2}^{l_2}]_s$$
$$- \int_{0+}^{t\wedge\sigma_0} f_{p_{k_1}^{l_1} v_{k_2}^{l_2}}(\vec{Q}(s), \vec{\vec{P}}(s-)) a_{k_1 l_1, k_2 l_2}(\vec{Q}(s)) ds \Big|\Big] = 0.$$

Finally, for the last term on the right hand side above, we have by Lemma 3.13 (2) that

$$\lim_{m\to 0} E^{\kappa_m}\Big[\sum_{s\in[0,T\wedge\sigma_0]} \Big| f(\vec{Q}(s), \vec{\vec{P}}(s)) - f(\vec{Q}(s), \vec{\vec{P}}(s-))$$
$$- \sum_{k=1}^{N} f_{p_k}(\vec{Q}(s), \vec{\vec{P}}(s-)) \cdot \Delta M_l(s)$$
$$- \frac{1}{2} \sum_{k_1,k_2=1}^{N} f_{p_{k_1} v_{k_2}}(\vec{Q}(s), \vec{\vec{P}}(s-))(\Delta M_{l_1}(s))(\Delta M_{l_2}(s)) \Big|\Big] = 0.$$

This combined with Lemma 4.3 completes the proof of Theorem 2.1. □

## Part II. Special case: $N = 2$ with (T1) satisfied

In the second half of this paper, we restrict ourselves to the special case that $N = 2$, $U_1$ and $U_2$ are spherical-symmetric, and that (T1) is satisfied.

## 5. Tightness

It is a trivial consequence of (T1) that there exists a constant $\varepsilon_0 \in (0, r_1)$ such that $U(Y) > 0$ if $Y \in (r_1 - \varepsilon_0, r_1)$. Let

$$\sigma = \inf\{t > 0; |Q_2(t) - Q_1(t)| \le r_1 - \frac{\varepsilon_0}{2}\}.$$

We prove in the following (see Corollary 5.5) that $\sigma$ converges to $\infty$ as $m \to 0$, precisely, $\lim_{m\to 0} \kappa_m(\sigma > K) = 1$ for any $K > 0$.

In this section, we use Lemma 3.13 to prove that $\Big\{$ the distribution of $\{(\vec{Q}(t\wedge\sigma), \vec{V}(t\wedge\sigma), \vec{R}(t\wedge\sigma), Y(t\wedge\sigma), H(t\wedge\sigma)), t \in [0,T]\}$ under $\kappa_m; m \le 1\Big\}$ is tight in Theorem 2.2's sense.

Since the derivative of $Q_k(t)$ with respect to $t$ is bounded by $c$, a finite constant, we have the tightness of the distribution of $\vec{Q}(t)$.

For $Y(t)$, we have the following:

Lemma 5.1.

(1)
$$Y(t) = \sum_{k=1}^{2} (M_k(t) + \eta_k(t) + J_k(t)).$$

(2) $\Big\{$ *the distribution of* $\{Y(t)\}_{t\in[0,T]}$ *under* $\kappa_m; m \le 1\Big\}$ *is tight in* $\wp(C([0,T]; \mathbf{R}^d))$,

(3) $\sup_{m\in(0,1]} E^{\kappa_m}\Big[\sup_{t\in[0,T]} |Y(t)|^2\Big] < \infty$.

Proof. Since $\nabla_1 \widetilde{U} = -\nabla_2 \widetilde{U}$, we have our first assertion by Lemma 3.13 and the definition of $Y_t$. The others are now easy. □

In order to prove the tightness for $(V_1(t), V_2(t), R_1(t), R_2(t), H_t)$, let us first define several notations. Let

$$\overline{V_k} = \frac{\overline{P_k}(t)}{M_k\sqrt{1 + M_k^{-2} c^{-2} |\overline{P_k}(t)|^2}},$$
$$\overline{R_k}(t) = \pi^{\perp}_{Q_2(t)-Q_1(t)} \overline{P_k}(t),$$

$$\overline{H}(t) = \sum_{k=1}^{2} M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |\overline{P_k}(t)|^2} + m^{-1/2} \widetilde{U}(Q_1(t), Q_2(t)).$$

Here $\overline{P_k}(t) = P_k(t) - \eta_k(t)$ as defined at the end of Subsection 3.5. Then we have the following.

LEMMA 5.2.

(1) *There exists a constant $C > 0$ such that*

$$E^{\kappa_m}\Big\{ \sup_{t\in[0,T]} \Big|V_k(t) - \overline{V_k}(t)\Big|^2 + \Big|R_k(t) - \overline{R_k}(t)\Big|^2 + \Big|H_k(t) - \overline{H_k}(t)\Big|^2 \Big\}$$
$$\le Cy(m).$$

*Here $y(\cdot)$ is the one defined in Lemma 3.13 (4).*

(2) *The tightness for $V_k(\cdot)$ (respectively, $R_k(\cdot)$, $H(\cdot)$) is equivalent to the tightness for $\overline{V_k}(\cdot)$ (respectively, $\overline{R_k}(\cdot)$, $\overline{H}(\cdot)$), and when they do converge as $m \to 0$, they have the same limits.*

PROOF. Since $\Big|\frac{x}{\sqrt{1+|x|^2}} - \frac{y}{\sqrt{1+|y|^2}}\Big| \le d|x-y|$ for any $x, y \in \mathbf{R}^d$, we have by (3.6) that

$$E\Big\{ \sup_{t\in[0,T]} \Big|\frac{P_k(t)}{M_k\sqrt{1+M_k^{-2}c^{-2}|P_k(t)|^2}} - \frac{\overline{P_k}(t)}{M_k\sqrt{1+M_k^{-2}c^{-2}|\overline{P_k}(t)|^2}}\Big|^2\Big\}$$
$$\le dcE\Big[ \sup_{t\in[0,T]} \Big|P_k(t) - \overline{P_k}(t)\Big|^2\Big]$$
$$= dcE\Big[ \sup_{t\in[0,T]} \Big|\eta_k(t)\Big|^2\Big].$$

Similarly,

$$E\Big[ \sup_{t\in[0,T]} |R_k(t) - \overline{R_k}(t)|^2\Big] \le E\Big[ \sup_{t\in[0,T]} |\eta_k(t)|^2\Big],$$

and

$$E\Big[ \sup_{t\in[0,T]} |H(t) - \overline{H}(t)|^2\Big] \le c\sum_{k=1}^{2} E\Big[ \sup_{t\in[0,T]} |\eta_k(t)|^2\Big].$$

These imply our first assertion by (3.6). Also, the second assertion is a direct consequence of the first assertion. □

We prove the tightnesses for $R_1(t)$ and $R_2(t)$ in Subsection 5.3, and prove the tightnesses for $V_1(t)$, $V_2(t)$ and $H(t)$ in the subsection after it.

## 5.1. Notations

Let us prepare some notations. For $k \in \{1,2\}$ and $x = (x_1, \cdots, x_d) \in \mathbf{R}^d$, let

$$f_{k,i}(x) := \frac{x_i}{M_k\sqrt{1 + M_k^{-2}c^{-2}|x|^2}},$$

and $f_{k,ij}(x) := \frac{\partial}{\partial x_j} f_i(x)$, $f_{k,ijl}(x) := \frac{\partial^2}{\partial x_j \partial x_l} f_i(x)$, $f_{k,ij_1j_2j_3}(x) := \frac{\partial^3}{\partial x_{j_1}\partial x_{j_2}\partial x_{j_3}}\left(\frac{x_i}{M_k\sqrt{1+M_k^{-2}c^{-2}|x|^2}}\right)$. So

$$f_{k,ij}(x) = \frac{\delta_{ij}(1 + M_k^{-2}c^{-2}|x|^2) - M_k^{-2}c^{-2}x_ix_j}{M_k(1 + M_k^{-2}c^{-2}|x|^2)^{3/2}}, \tag{5.1}$$

and

$$f_{k,ijl}(x) = \begin{cases} \frac{\delta_{l\neq i}2M_k^{-2}c^{-2}x_l(1+M_k^{-2}c^{-2}|x|^2) - 3M_k^{-2}c^{-2}(1+M_k^{-2}c^{-2}\sum_{r\neq i}|x_r|^2)x_l}{M_k(1+M_k^{-2}c^{-2}|x|^2)^{5/2}}, & \text{if } i = j, \\ \frac{3M_k^{-4}c^{-4}x_ix_jx_l - M_k^{-2}c^{-2}(\delta_{l=i}x_j + \delta_{l=j}x_i)(1+M_k^{-2}c^{-2}|x|^2)}{M_k(1+M_k^{-2}c^{-2}|x|^2)^{5/2}}, & \text{if } i \neq j. \end{cases}$$

So there exists a constant $C > 0$ such that

$$|f_{k,ij}(x)| \le \frac{C}{\sqrt{1 + M_k^{-2}c^{-2}|x|^2}}, \quad |f_{k,ijl}(x)| \le \frac{C}{1 + M_k^{-2}c^{-2}|x|^2}$$

for any $i, j, l \in \{1, \cdots, d\}$. Also, by calculating the third partial derivatives of $f_{k,i}$, we get that $\|f_{k,ijlr}\|_\infty < \infty$.

Also, let us prepare one more notation: for any $v \in \mathbf{R}^d$ with $|v| \le c$, define $F^2(v) := \Big(F_{ij}(v)\Big)_{i,j=1,\cdots,d}$ with

$$F_{ij}(v) = \sqrt{1 - c^{-2}|v|^2}\ (\delta_{ij} - c^{-2}v_iv_j) \tag{5.2}$$

and let $F_{ijl}$ be as defined in (2.10). Then by a simple calculation, we have that

$$v = \Big(f_{k,i}(\frac{M_k}{\sqrt{1-c^{-2}|v|^2}}v)\Big)_{i=1}^{d}, \tag{5.3}$$

$$M_k^{-1}F_{ij}(v) = f_{k,ij}(\frac{M_k}{\sqrt{1-c^{-2}|v|^2}}v), \tag{5.4}$$

$$M_k^{-2}F_{ijl}(v) = f_{k,ijl}(\frac{M_k}{\sqrt{1-c^{-2}|v|^2}}v). \tag{5.5}$$

### 5.2. A decomposition and an estimate of $H(t)$

We first prove the following decomposition of $H(t)$.

LEMMA 5.3. *There exists a stochastic process* $\eta_{H1}(\,\cdot\,)$ *such that*

$$\begin{aligned} H(t) = H(0) + \eta_{H1}(t) + \sum_{i=1}^{2}\Big(\int_{0+}^{t} V_k(s)\cdot dM_k(s) + \int_{0+}^{t} V_k(s)\cdot dJ_k(s) \\ + \sum_{i,j=1}^{d}\int_{0+}^{t} f_{k,ij}(P_k(s))a_{ki,kj}(\vec{Q}(s))ds\Big) \\ + \sum_{k=1}^{2}\int_{0+}^{t} m^{-1/2}\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot(\overline{V_k}(s-)-V_k(s))ds, \end{aligned} \tag{5.6}$$

*and there exists a constant* $C>0$ *such that*

$$E^{\kappa_m}\Big[\sup_{t\in[0,T]}|\eta_{H1}(t)|\Big] \le C\Big(m^{1/4}+y(m)^{1/2}\Big), \qquad \textit{for all } m\in(0,1].$$

In Lemma 5.21, we prove that the term $\sum_{k=1}^{2}\int_{0+}^{t} m^{-1/2}\nabla_k\widetilde{U}(Q_1(s), Q_2(s))\cdot(\overline{V_k}(s)-V_k(s))ds$ in the above decomposition of $H(t)$ is also negligible, by using the estimates in Subsection 5.4. We first prove Lemma 5.3 in the rest of this subsection.

PROOF OF LEMMA 5.3. First, by the definition of $\overline{H}(t)$ and Ito's formula, we have that

$$\overline{H}(t) - \overline{H}(0)$$

$$= \sum_{k=1}^{2} \int_{0+}^{t} \Big( \overline{V_k}(s-) \cdot dM_k(s) + \int_{0+}^{t} \overline{V_k}(s-) \cdot dJ_k(s) + \sum_{i,j=1}^{d} \int_{0+}^{t} f_{k,ij}(\overline{P_k}(s-)) d[M_k^i, M_k^j]_s + \sum_{s \in (0,t]} \Big\{ M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |\overline{P_k}(s)|^2} - M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |\overline{P_k}(s-)|^2} - \sum_{i=1}^{d} f_{k,i}(\overline{P_k}(s-)) \Delta \overline{P_k^i}(s) - \frac{1}{2} \sum_{i,j=1}^{d} f_{k,ij}(\overline{P_k}(s-)) \Delta \overline{P_k^i}(s) \Delta \overline{P_k^j}(s) \Big\} \Big)$$

$$+ \sum_{k=1}^{2} \int_{0+}^{t} m^{-1/2} \nabla_k \widetilde{U}(Q_1(s), Q_2(s)) \cdot (\overline{V_k}(s-) - V_k(s)) ds. \tag{5.7}$$

So our assertion follows from the following: There exists a constant $C > 0$ such that

$$E^{\kappa_m} \Big[ \sup_{t \in [0,T]} \Big| \int_{0+}^{t} \Big( \overline{V_k}(s-) - V_k(s) \Big) \cdot dM_k(s) \Big| \Big] \le C m^{1/4}, \tag{5.8}$$

$$E^{\kappa_m} \Big[ \sup_{t \in [0,T]} \Big| \int_{0+}^{t} \Big( \overline{V_k}(s-) - V_k(s) \Big) \cdot dJ_k(s) \Big| \Big] \le C y(m)^{1/2}, \tag{5.9}$$

$$E^{\kappa_m} \Big[ \sup_{t \in [0,T]} \Big| \int_{0+}^{t} f_{k,ij}(\overline{P_k}(s-)) d[M_k^i, M_k^j]_s - \int_{0+}^{t} f_{k,ij}(P_k(s)) a_{ki,kj}(\vec{Q}(s)) ds \Big| \Big] \le C m^{1/4}, \tag{5.10}$$

$$E^{\kappa_m} \Big[ \sup_{t \in [0,T]} \Big| \sum_{s \in (0,t]} \Big\{ M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |\overline{P_k}(s)|^2} - M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |\overline{P_k}(s-)|^2} - \sum_{i=1}^{d} f_{k,i}(\overline{P_k}(s-)) \Delta \overline{P_k^i}(s)$$

$$-\frac{1}{2}\sum_{i,j=1}^{d} f_{k,ij}(\overline{P_k}(s-))\Delta\overline{P_k^i}(s)\Delta\overline{P_k^j}(s)\Big\}\Big|\Big] \le Cm^{1/2}. \tag{5.11}$$

(5.11) is a direct consequence of Lemma 3.14 (3).

Next let us prove (5.10). Since $P_k(s) = P_k(s-)$, we have that

$$\Big| f_{k,ij}(\overline{P_k}(s-)) - f_{k,ij}(P_k(s))\Big| \le C|\eta_k(s-)|.$$

Also, with $C_1 := \Big(3\tau\|\nabla U_k\|_\infty\Big)^2 (2R_0)^{d-1}\int_{\mathbf{R}^d}\rho(\frac{1}{2}|v|^2)|v|dv < \infty$, we have that

$$\int_E |D_k^i(\vec{Q}(s),x,v)||D_k^j(\vec{Q}(s),x,v)|\rho(\frac{1}{2}|v|^2)\nu(dx,dv) \le C_1 \tag{5.12}$$

for any $s\in[0,T]$, $k\in\{1,2\}$ and $i,j\in\{1,\cdots,d\}$. So by (3.7), we have that

$$\begin{aligned} & E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\int_{0+}^t \Big(f_{k,ij}(\overline{P_k}(s-)) - f_{k,ij}(P_k(s))\Big) \\ &\qquad \times\Big(\int_E D_k^i(\vec{Q}(s),x,v)D_k^j(\vec{Q}(s),x,v)\rho(\frac{1}{2}|v|^2)\nu(dxdv)\Big)ds\Big|\Big] \\ &\le CC_1\int_0^T E^{\kappa_m}\Big[|\eta_k(s-)|\Big]ds \le Cm^{1/4}. \qquad (5.13)\end{aligned}$$

Also, by Lemma 3.16 (3), we have that

$$E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\int_{0+}^t f_{k,ij}(\overline{P_k}(s-))\Big(d[M_k^i,M_k^j]_s - a_{ki,kj}(\vec{Q}(s))ds\Big)\Big|\Big] \le Cm^{1/2}.$$

This combined with (5.13) implies (5.10).

We have (5.9) with the help of Lemma 3.13 (3) (4), since

$$\begin{aligned} & E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\int_{0+}^t \Big(\overline{V_k}(s-) - V_k(s)\Big)\cdot dJ_k(s)\Big|\Big] \\ &\le M_k^{-1}TE^{\kappa_m}\Big[\sup_{u\in[0,T]}|\eta_k(u)|^2\Big]^{1/2}\sup_{s\in[0,T]}E^{\kappa_m}\Big[\Big|\frac{d}{ds}J_k(s)\Big|^2\Big]^{1/2}.\end{aligned}$$

Finally, let us prove (5.8). Since $\int_{0+}^t\Big(\overline{V_k}(s-) - V_k(s)\Big)\cdot dM_k(s)$ is a martingale, we have by Doob's inequality and Lemma 3.16 (1) that

$$E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\int_{0+}^t \Big(\overline{V_k}(s-) - V_k(s)\Big)\cdot dM_k(s)\Big|^2\Big]$$

$$
\begin{aligned}
&\le 4E^{\kappa_m}\Big[\Big|\int_{0+}^{T}\Big(\overline{V_k}(s-)-V_k(s)\Big)\cdot dM_k(s)\Big|^2\Big]\\
&=4\sum_{i,j=1}^{d}E^{\kappa_m}\Big[\int_{0+}^{T}\Big(\overline{V_k}^i(s-)-V_k^i(s)\Big)\Big(\overline{V_k}^j(s-)-V_k^j(s)\Big)d[M_k^i,K_k^j]_s\Big]\\
&=4\sum_{i,j=1}^{d}E^{\kappa_m}\Big[\int_{0+}^{T}\Big(\overline{V_k}^i(s-)-V_k^i(s)\Big)\Big(\overline{V_k}^j(s-)-V_k^j(s)\Big)\\
&\qquad\qquad\times m\int_E D_k^i(\vec{Q}(s),x,v)D_k^j(\vec{Q}(s),x,v)N(ds,dx,dv)\Big]\\
&=4\sum_{i,j=1}^{d}\int_{(0,T]\times E}E^{\kappa_m}\Big[\Big(\overline{V_k}^i(s-)-V_k^i(s)\Big)\Big(\overline{V_k}^j(s-)-V_k^j(s)\Big)\\
&\qquad\qquad\times D_k^i(\vec{Q}(s),x,v)D_k^j(\vec{Q}(s),x,v)\Big]\rho(\frac{1}{2}|v|^2)\nu(dx,dv)ds\\
&\le 4M_k^{-2}d^2C_1\int_{0+}^{T}E^{\kappa_m}\Big[\Big|\eta_k(s-)\Big|^2\Big]ds,
\end{aligned}
$$

where we used (5.12) when passing to the last line. This combined with (3.7) implies (5.8), and completes the proof of our assertion. □

We prepare the following estimate with respect to $H(t)$ and $\overline{H}(t)$. This is used in the prove of the tightness of $V_k$.

LEMMA 5.4. *There exists a constant $C>0$ such that*

$$
\begin{aligned}
&E^{\kappa_m}\Big[\sup_{t\in[0,T]}|H_t|^2\Big]\le Cm^{-1/2},\\
&E^{\kappa_m}\Big[\sup_{t\in[0,T]}|\overline{H}_t|^2\Big]\le Cm^{-1/2},\qquad m\in(0,1].
\end{aligned}
$$

PROOF OF LEMMA 5.4. We use Lemma 5.3 to prove our assertion.

By Lemma 3.14 (1) (2), all of the terms except the last one on the right hand side of (5.6) are fine. We prove in the following that the last term on the right hand side of (5.6) is also fine. Since $\Big|\overline{V_k}(s)-V_k(s)\Big|\le M_k^{-1}\Big|\overline{P_k}(s)-P_k(s)\Big|\le M_k^{-1}|\eta(s)|$, we have by (3.7) that

$$
E^{\kappa_m}\Big[\sup_{s\in[0,T]}\Big|\int_{0+}^{t}m^{-1/2}\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot\Big(\overline{V_k}(s)-V_k(s)\Big)ds\Big|^2\Big]
$$

$$
\begin{aligned}
&\leq m^{-1} T E^{\kappa_m}\Big[\int_0^T (\|\nabla_k \widetilde{U}\|_\infty M_k^{-1}|\eta(s)|)^2 ds\Big] \\
&\leq C m^{-1}\int_0^T E^{\kappa_m}\Big[|\eta_k(s)|^2\Big] ds \\
&\leq C m^{-1/2}.
\end{aligned}
$$

This completes the proof of our assertion. □

As an easy corollary of Lemma 5.4, we get the following. In particular, this implies that $\sigma$ converges to $\infty$ as $m \to 0$.

COROLLARY 5.5. *For any $\varepsilon > 0$, we have that*

$$
\lim_{m\to 0} \kappa_m\Big(\inf_{t\in[0,T]} |Q_1(t) - Q_2(t)| \leq r_1 - \varepsilon\Big) = 0.
$$

PROOF. Choose any $\varepsilon \in (0, \frac{\varepsilon_0}{2})$ and fix it. Then there exists a $\delta_0 > 0$ such that $\widetilde{U}(Q_1, Q_2) > \delta_0$ as long as $|Q_1 - Q_2| \in (r_1 - \varepsilon_0 + \varepsilon, r_1 - \varepsilon]$. Therefore, if $|Q_1(t) - Q_2(t)| \in (r_1 - \varepsilon_0 + \varepsilon, r_1 - \varepsilon]$, then by the definition of $\overline{H}_t$, we get that $\overline{H}_t \geq m^{-1/2}\widetilde{U}(Q_1(t), Q_2(t)) \geq m^{-1/2}\delta_0$. Therefore,

$$
\begin{aligned}
&\kappa_m\Big(\inf_{t\in[0,T]} |Q_1(t) - Q_2(t)| \leq r_1 - \varepsilon\Big) \\
&\leq \kappa_m\Big(\sup_{t\in[0,T]} \overline{H}_t \geq m^{-1/2}\delta_0\Big) \\
&\leq (m^{-1/2}\delta_0)^{-2} E^{\kappa_m}\Big[\sup_{t\in[0,T]} |\overline{H}_t|^2\Big] \\
&\leq C m^{1/2} \to 0, \qquad m \to 0. \ \square
\end{aligned}
$$

### 5.3. Tightness for $R_k(t)$

In this section, we first make some calculation with respect to $\overline{R_k}(t)$ (Lemma 5.6), then get an estimate of $R_k(t)$ (Lemma 5.7), which is also one of the essential idea of our present paper. We finally use these two results to prove the tightness for $R_k(t)$ (Lemma 5.8).

For any $s \in [0, T]$ and $k \in \{1, 2\}$, let

$$
A_{Rk1}(t) = \int_0^t \Big[dM_k(s) - \Big(\frac{Q_2(s) - Q_1(s)}{|Q_2(s) - Q_1(s)|} \cdot dM_k(s)\Big)\frac{Q_2(s) - Q_1(s)}{|Q_2(s) - Q_1(s)|}\Big],
$$

$$
\begin{aligned}
Y_{Rk}(t) = \Big[&\Big(Y(t)\cdot\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}\Big)\Big(\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}\cdot V_k(t)\Big) \\
&\qquad - Y(t)\cdot V_k(t)\Big]\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|^2} \\
&- \frac{1}{|Q_2(t)-Q_1(t)|^2}\Big(Y(t)\cdot(Q_2(t)-Q_1(t))\Big) \\
&\times M_k^{-1}\sqrt{1-c^{-2}|V_k(t)|^2}R_k(t), \\
f_R(t) = &\frac{1}{|Q_2(t)-Q_1(s)|^2} \\
&\times \sum_{j=1}^{2}\Big(M_j^{-1}\sqrt{1-c^{-2}|V_j(t)|^2}|R_j(t)|^2(Q_2(t)-Q_1(t)) \\
&\qquad + (V_j(t)\cdot(Q_2(t)-Q_1(t))R_j(t)\Big).
\end{aligned}
$$

Also, define $k^c$ as $k^c = 1$ if $k = 2$, and $k^c = 2$ if $k = 1$.

Our main results of this subsection are the following three lemmas.

Lemma 5.6. *For $k \in \{1,2\}$, there exists a stochastic process $\eta_{\overline{R}k}(\,\cdot\,)$ such that for any $s \in [0,T]$, we have*

$$
\begin{aligned}
\overline{R_k}(t) - \overline{R_k}(0) = A_{Rk1}(t) + \int_0^t \Big[&\pi^{\perp}_{Q_2(s)-Q_1(s)}\Big(\frac{d}{ds}J_k(s)\Big) \\
&- f_R(s) - Y_{Rk^c}(s) - \eta_{\overline{R}k}(s)\Big]ds,
\end{aligned} \tag{5.14}
$$

*and*

$$
|\eta_{\overline{R}k}(t)| \le 4c|Q_2(t)-Q_1(t)|^{-1}|\eta_k(t)|. \tag{5.15}
$$

Lemma 5.7. *There exists a constant $C > 0$ such that*

$$
\begin{aligned}
&E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]}(|R_1(t)|^2+|R_2(t)|^2)\Big] \le C, \\
&E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]}(|\overline{R_1}(t)|^2+|\overline{R_2}(t)|^2)\Big] \le C, \qquad m \in (0,1].
\end{aligned}
$$

Lemma 5.8.

(1) *We have that*

$$R_k(t) - R_k(0) = A_{Rk1}(t) + \int_0^t \Big[\pi^{\perp}_{Q_2(s)-Q_1(s)}\Big(\frac{d}{ds}J_k(s)\Big) \tag{5.16}$$
$$- f_R(s) - Y_{Rk^c}(s)\Big]ds + \eta_{Rk}(t),$$

*and there exists a constant* $C > 0$ *such that*

$$E^{\kappa_m}\Big[\sup_{t\in T\sigma}|\eta_{Rk}(t)|^2\Big] \le Cy(m), \qquad m \in (0,1]. \tag{5.17}$$

(2) $\Big\{$ *the distribution of* $\{R_k(t\wedge\sigma)\}_{t\in[0,T]}$ *under* $\kappa_m; m \le 1\Big\}$ *is tight in* $\wp(D([0,T];\mathbf{R}^d))$ *for* $k \in \{1,2\}$.

Proof of Lemma 5.6. We have by definition that

$$\begin{aligned}\overline{R_k}(t) &= \pi^{\perp}_{Q_2(t)-Q_1(t)}(\overline{P_k}(t))\\ &= \overline{P_k}(t) - \Big(\overline{P_k}(t)\cdot Q_2(t) - Q_1(t)\Big)\frac{1}{|Q_2(t)-Q_1(t)|^2}(Q_2(t)-Q_1(t)),\end{aligned}$$

and

$$\overline{P_k}(t) = M_k(t) + J_k(t) - m^{-1/2}\int_0^t \nabla_k\widetilde{U}(Q_1(s),Q_2(s))ds.$$

Since $\nabla_2\widetilde{U}(Q_1(t),Q_2(t))$ is parallel to $Q_2(t)-Q_1(t)$, we have by Ito's formula that

$$\overline{R_k}(t) - \overline{R_k}(0) = A_{Rk1}(t) + \int_0^t \pi^{\perp}_{Q_2(s)-Q_1(s)}\Big(\frac{d}{ds}J_2(s)\Big)ds - \int_0^t g_{Rk}(s)ds,$$

with

$$\begin{aligned}g_{Rk}(s) = &\Big(\overline{P_k}(s)\cdot(V_2(s)-V_1(s)\Big)\frac{1}{|Q_2(s)-Q_1(s)|^2}(Q_2(s)-Q_1(s))\\ &+\Big(\overline{P_k}(s)\cdot(Q_2(s)-Q_1(s)\Big)\frac{1}{|Q_2(s)-Q_1(s)|^2}(V_2(s)-V_1(s))\\ &-\Big(\overline{P_k}(s)\cdot(Q_2(s)-Q_1(s)\Big)\end{aligned}$$

$$\times \frac{2(Q_2(s)-Q_1(s))\cdot(V_2(s)-V_1(s))}{|Q_2(s)-Q_1(s)|^4}(Q_2(s)-Q_1(s)). \tag{5.18}$$

So it suffices to prove that

$$g_{Rk}(s) = f_R(s) + Y_{Rk^c}(s) + \eta_{\overline{R}k}(s) \tag{5.19}$$

with some $\eta_{\overline{R}k}(s)$ satisfying (5.15). We prove (5.19) for $k=2$, the assertion for $k=1$ is gotten in the same way.

Notice that for any $p, q \in \mathbf{R}^d$, we have that

$$|p|^2 q - (p\cdot q)^2 \frac{1}{|q|^2} q = \Big(|p|^2 - (p\cdot \frac{q}{|q|})^2\Big) q = \left|\pi_q^\perp p\right|^2 q,$$

hence

$$\begin{aligned} & |p|^2 q + (p\cdot q)p - 2(p\cdot q)^2 \frac{1}{|q|^2} q \\ = & \left|\pi_q^\perp p\right|^2 q + (p\cdot q)p - (p\cdot q)^2 \frac{1}{|q|^2} q \\ = & \left|\pi_q^\perp p\right|^2 q + (p\cdot q)\pi_q^\perp p. \end{aligned}$$

Therefore, for $j \in \{1,2\}$, we have that

$$\begin{aligned} & \frac{1}{M_j\sqrt{1+M_j^{-2}c^{-2}|P_j(s)|^2}}\Big(|P_j(s)|^2(Q_2(s)-Q_1(s)) \\ & \qquad + (P_j(s)\cdot(Q_2(s)-Q_1(s)))P_j(s) \\ & \qquad - 2(P_j(s)\cdot(Q_2(s)-Q_1(s)))^2 \\ & \qquad \times \frac{1}{|Q_2(s)-Q_1(s)|^2}(Q_2(s)-Q_1(s))\Big) \\ = & M_j^{-1}\sqrt{1-c^{-2}|V_j(s)|^2}|R_j(s)|^2(Q_2(s)-Q_1(s)) \\ & \qquad + (V_j(s)\cdot(Q_2(s)-Q_1(s)))R_j(s). \end{aligned} \tag{5.20}$$

Re-write $\overline{P_2}(s)$ in (5.18) as $P_2(s)-\eta_2(s)$, and re-write $P_2(s)$ as $Y(s) - P_1(s)$ in further in the terms that include $V_1(s)$, then by (5.20) and a simply calculation, we get (5.19) with

$$\eta_{\overline{R}2}(s) = |Q_2(s)-Q_1(s)|^{-2}\Big[(\eta_2(s)\cdot(V_1(s)-V_2(s)))(Q_2(s)-Q_1(s))$$

$$
\begin{aligned}
&+\Big(\eta_2(s)\cdot(Q_2(s)-Q_1(s))(V_1(s)-V_2(s))\Big)\\
&+2\Big(\eta_2(s)\cdot(Q_2(s)-Q_1(s))\Big)\\
&\times\Big(\frac{Q_2(s)-Q_1(s)}{|Q_2(s)-Q_1(s)|}\cdot(V_2(s)-V_1(s))\Big)\frac{Q_2(s)-Q_1(s)}{|Q_2(s)-Q_1(s)|}\Big].
\end{aligned}
$$

So (5.15) is also satisfied. □

Proof of Lemma 5.7. Since

$$\Big|M_k^{-1}\sqrt{1-c^{-2}|V_k(t)|^2}R_k(t)\Big|\le c,$$

we have by the definition of $Y_{Rj}(s)$ that

$$|Y_{Rj}(s)|\le 3c|Q_2(s)-Q_1(s)|^{-1}|Y(s)|,\qquad j=1,2.$$

Also, we have

$$|Q_2(s)-Q_1(s)|^{-1}\le(r_1-\frac{\varepsilon_0}{2})^{-1},\qquad \text{if } s\le\sigma.$$

Therefore, we get from Lemma 5.6 that there exists a constant $C>0$ such that for any $t\in[0,T\wedge\sigma]$ and $k\in\{1,2\}$, we have that

$$
\begin{aligned}
|R_k(t)|\le C\Big(&1+A_{Rk1}(t)+\sup_{s\in[0,T]}|\eta_k(s)|+\int_0^t|Y(s)|\\
&+\int_0^t\Big|\frac{d}{ds}J_k(s)\Big|ds+\int_0^t\Big(|R_1(s)|+|R_2(s)|\Big)ds\Big).
\end{aligned}
\tag{5.21}
$$

Let

$$h(r):=E^{\kappa_m}\Big[\sup_{t\in[0,r\wedge\sigma]}(|R_1(t)|+|R_2(t)|)^2\Big].$$

Then $h$ is continuous and by (5.21), there exists a constant $C>0$ such that

$$
\begin{aligned}
h(t)\le C\Big(1+\sum_{j=1}^2\Big\{&E^{\kappa_m}\Big[\sup_{u\in[0,T]}|\eta_j(u)|^2\Big]+E^{\kappa_m}\Big[\sup_{u\in[0,T]}|A_{Rj1}(u)|^2\Big]\\
&+\sup_{s\in[0,T]}E^{\kappa_m}\Big[\Big|\frac{d}{ds}J_j(s)\Big|^2\Big]\Big\}
\end{aligned}
$$

$$+E^{\kappa_m}\Big[\Big(\int_0^T |Y(u)|du\Big)^2\Big] + \int_0^t h(r)dr\Big). \tag{5.22}$$

By Lemma 3.14, we have that $E^{\kappa_m}\Big[\sup_{u\in[0,T]} |A_{Rj1}(u)|^2\Big]$ is bounded for $m \in (0,1]$. Also, $E^{\kappa_m}\Big[\Big(\int_0^T |Y(u)|du\Big)^2\Big] \le T^2 \sup_{s\in[0,T]} E^{\kappa_m}[|Y(s)|^2]$, which is also bounded for $m \in (0,1]$ by Lemma 5.1. Finally, by Lemma 3.13 (3) (4), $E^{\kappa_m}\Big[\sup_{u\in[0,T]} |\eta_j(u)|^2\Big]$ and $\sup_{s\in[0,T]} E^{\kappa_m}\Big[\Big|\frac{d}{ds}J_j(s)\Big|^2\Big]$ ( $j = 1,2$) are also bounded for $m \in (0,1]$. Therefore, (5.22) implies that there exists a constant $C > 0$ such that

$$h(t) \le C + C\int_0^t h(s)ds, \qquad t \in [0,T].$$

So by Gronwall's inequality, we have that $h(t) \le Ce^{Ct}$ for any $t \in [0,T]$ and any $m \in (0,1]$. In particular,

$$E^{\kappa_m}\Big[\sup_{s\in[0,T\wedge\sigma]} (|R_1(s)| + |R_2(s)|)^2\Big] \le Ce^{CT}, \qquad m \in (0,1]. \ \square$$

Proof of Lemma 5.8. The first assertion is a direct corollary of Lemma 5.6.

The second assertion is an easy consequence of Lemmas 5.6 and 5.7. Indeed, by Lemma 3.14 (1), we have that the distributions of $\{A_{Rk1}; t \in [0, T\wedge\sigma]\}$ under $\kappa_m$ with $m \in (0,1]$ is tight in $\wp(D([0,T];\mathbf{R}^d))$, with all of its cluster points in $\wp(C([0,T];\mathbf{R}^d))$. Also, by Lemmas 5.7, 3.13 (3) and 5.1, the integrand of the second term on the right hand side of (5.14) satisfies $\sup_{m\in(0,1]}\sup_{s\in[0,T]} E\Big[\Big| * (s\wedge\sigma)\Big|^2\Big] < \infty$, which, by Lemma 3.7, implies that the distributions of the second term on the right hand side of (5.14) ($t \in [0, T\wedge\sigma]$) with $m \in (0,1]$ is tight in $\wp(C([0,T];\mathbf{R}^d))$. Therefore, $\Big\{$ the distribution of $\{R_k(t\wedge\sigma); t \in [0,T]\}$ under $\kappa_m; m \in (0,1]\Big\}$ is tight in $\wp(D([0,T],\mathbf{R}^d)$. Since $R_k^{(m)}(\cdot)$ is continuous for any $m \in (0,1]$, and $C([0,T];\mathbf{R}^d)$ is closed in $D([0,T];\mathbf{R}^d)$, we get our assertion. $\square$

### 5.4. Tightness for $V_k(t)$

We prove the following result in this subsection.

Lemma 5.9. *For any $p > 1$ and $k \in \{1, 2\}$, we have that* $\Big\{$ *the distribution of* $\{V_k(t \wedge \sigma); t \in [0, T]\}$ *under* $\kappa_m; m \in (0, 1]\Big\}$ *is tight as probabilities on* $L^p([0, T]; \mathbf{R}^d)$*, with all of its cluster points as* $m \to 0$ *in* $\wp(D([0, T]; \mathbf{R}^d))$.

In §5.4.1, we give a basic decomposition of $V_k$. Especially, it decompose the "singular" part of $V_k$ into a parallel part and a perpendicular part, tightnesses for which will be proven in §5.4.2 and §5.4.3, respectively.

Choose $k \in \{1, 2\}$ and fix it through this subsection.

*5.4.1 Some calculation for* $V_k(t)$

For any $k \in \{1, 2\}$, let $A_{\overline{V}k1}(t) = \left(A^1_{\overline{V}k1}(t), \cdots, A^d_{\overline{V}k1}(t)\right)$, with

$$\begin{aligned} A^i_{\overline{V}k1}(t) = & \int_{0+}^t \sum_{j=1}^d f_{k,ij}(\overline{P_k}(s-))\left(dM^j_k(s) + dJ^j_k(s)\right) \\ & + \frac{1}{2}\int_{0+}^t f_{k,ijl}(\overline{P_k}(s-))a_{kj,kl}(\vec{Q}(s))ds \end{aligned} \tag{5.23}$$

for $i = 1, \cdots, d$. Then we have the following.

Lemma 5.10.

(1) $\sup_{m \in (0,1]} E^{\kappa_m}\left[\sup_{t \in [0,T]} |A_{\overline{V}k1}(t)|^2\right] < \infty$,

(2) $\Big\{$ *The distribution of* $\{A_{\overline{V}k1}(t); t \in [0, T]\}$ *under* $\kappa_m; m \in (0, 1]\Big\}$ *is tight in* $\wp(D([0, T]; \mathbf{R}^d))$*, with all of its cluster points as* $m \to 0$ *in* $\wp(C([0, T]; \mathbf{R}^d))$.

(3) *There exists a constant* $C > 0$ *such that*

$$E^{\kappa_m}\left[\left|A_{\overline{V}k1}(t_1) - A_{\overline{V}k1}(t_2)\right|\right] \le C|t_1 - t_2|^{1/2}$$

*for any* $t_1, t_2 \in [0, T]$.

(4) *There exists a stochastic process* $\eta_{\overline{V}k}(\cdot)$ *such that*

$$\begin{aligned} \overline{V_k}(t) = & \overline{V_k}(0) + A_{\overline{V}k1}(t) + \eta_{\overline{V}k}(t) \\ & - m^{-1/2}\int_{0+}^t \sum_{j=1}^d f_{k,\cdot j}(\overline{P_k}(s-))\nabla_{kj}\widetilde{U}(\vec{Q}(s))ds, \end{aligned} \tag{5.24}$$

*and there exists a constant $C > 0$ such that*

$$E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\eta_{\overline{V}k}(t)\Big|^2\Big] \le Cm. \tag{5.25}$$

PROOF. Since $|f_{k,ijl}(\overline{P_k}(s-))| \le C$, the first three assertions are easy by Lemma 3.14 (1) (2) and Lemma 3.7. We prove the last assertion in the following.

First, we have by definition that $\overline{V_k}^i(t) = f_{k,i}(\overline{P_k}(t))$, where $f_{k,i}$ is as defined in Section 5.1. So by Ito's formula, we have the decomposition

$$\begin{aligned}
&\overline{V_k}^i(t) - \overline{V_k}^i(0) = f_{k,i}(\overline{V_k}(t)) - f_{k,i}(\overline{V_k}(0)) \\
&= \int_{0+}^t \sum_{j=1}^d f_{k,ij}(\overline{P_k}(s-))\Big(dM_k^j(s) + dJ_k^j(s)\Big) \\
&\qquad + \frac{1}{2}\sum_{j,l=1}^d \int_{0+}^t f_{k,ijl}(\overline{P_k}(s-))d[M_k^j, M_k^l]_s \\
&\qquad + \sum_{s\in(0,t]} \Big\{f_{k,i}(\overline{P_k}(s)) - f_{k,i}(\overline{P_k}(s-)) - \sum_{j=1}^d f_{k,ij}(\overline{P_k}(s-))\Delta\overline{P_k}^j(s) \\
&\qquad\qquad - \frac{1}{2}\sum_{j,l=1}^d f_{k,ijl}(\overline{P_k}(s-))\Delta\overline{P_k}^j(s)\Delta\overline{P_k}^l(s)\Big\} \\
&\quad - m^{-1/2}\int_{0+}^t \sum_{j=1}^d f_{k,ij}(\overline{P_k}(s-))\nabla_{kj}\widetilde{U}(\vec{Q}(s))ds.
\end{aligned}$$

Therefore, (5.24) holds with $\eta_{\overline{V}k}^i$ given by

$$\begin{aligned}
\eta_{\overline{V}k}^i(t) = \frac{1}{2}\sum_{j,l=1}^d \int_{0+}^t f_{k,ijl}(\overline{P_k}(s-))\Big(d[M_k^j, M_k^l]_s - a_{kj,kl}(\vec{Q}(s))ds\Big) \\
+ \sum_{s\in(0,t]} \Big\{f_{k,i}(\overline{P_k}(s)) - f_{k,i}(\overline{P_k}(s-)) \\
- \sum_{j=1}^d f_{k,ij}(\overline{P_k}(s-))\Delta\overline{P_k}^j(s)
\end{aligned}$$

$$-\frac{1}{2}\sum_{j,l=1}^{d} f_{k,ijl}(\overline{P_k}(s-))\Delta\overline{P_k}^j(s)\Delta\overline{P_k}^l(s)\Big\}.$$

(5.25) is now a direct consequence of Lemma 3.17 and Lemma 3.16 (3). □

As an easy consequence of Lemma 5.10, we get the following decomposition of $V_k(t)$. This gives us the coefficients $\gamma_k^V$ and $\beta_k^V$ in our limiting generator $L$.

LEMMA 5.11. *For $k \in \{1,2\}$, there exists a stochastic process $\eta_{Vk}(\,\cdot\,)$ such that for any $s \in [0,T]$ and $i = 1,\cdots,d$, we have*

$$\begin{aligned} V_k^i(t) - V_k^i(0) = &\int_0^t \sum_{j=1}^{d} M_k^{-1} F_{ij}(V_k(s)) dM_k^j(s) \\ &+ \int_0^t \sum_{j=1}^{d} M_k^{-1} F_{ij}(V_k(s)) dJ_k^j(s) \\ &+ \frac{1}{2}\int_0^t \sum_{j,l=1}^{d} M_k^{-2} F_{ijl}(V_k(s)) a_{kj,kl}(\vec{Q}(s))ds + \eta_{Vk}^i(t) \\ &- m^{-1/2}\int_0^t \sum_{j=1}^{d} f_{k,jl}(\overline{P}_k(s))\nabla_k^j \widetilde{U}(\vec{Q}(s))ds, \end{aligned} \tag{5.26}$$

*and there exists a constant $C > 0$ such that*

$$E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]} |\eta_{Vk}(t)|\Big] \le C\Big(y(m)^{1/2} + m^{1/2}\Big), \quad m \in (0,1]. \tag{5.27}$$

*Here $F_{ij}$ and $F_{ijl}$ are as defined in (5.2) and (2.10), respectively.*

PROOF. There exists a constant $C > 0$ such that $\Big|f_{k,ij}(P_k(s)) - f_{k,ij}(\overline{P_k}(s))\Big| \le C\Big|P_k(s) - \overline{P_k}(s)\Big| = C|\eta_k(s)|$, and similarly, $\Big|f_{k,ijl}(P_k(s)) - f_{k,ijl}(\overline{P_k}(s))\Big| \le C|\eta_k(s)|$. Therefore, by the same argument as we used to prove (5.8) $\sim$ (5.10), we get that

$$E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\int_{0+}^t \Big(f_{k,ij}(P_k(s-)) - f_{k,ij}(\overline{P_k}(s-))\Big)dM_k^j(s)\Big|\Big] \le Cm^{1/4},$$

$$E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\int_{0+}^{t}\Big(f_{k,ij}(P_k(s-))-f_{k,ij}(\overline{P_k}(s-))\Big)dJ_k^j(s)\Big|\Big]\le Cy(m)^{1/2},$$
$$E^{\kappa_m}\Big[\sup_{t\in[0,T]}\Big|\int_{0+}^{t}\Big(f_{k,ijl}(P_k(s))-f_{k,ijl}(\overline{P_k}(s-))\Big)a_{ki,kj}(\vec{Q}(s))ds\Big|\Big]\le Cm^{1/4}.$$

This combined with (5.24), (5.23) and Lemma 5.2 (1) implies that

$$\begin{aligned}V_k^i(t)&=V_k^i(0)+\eta_{Vk}^i(t)+\int_{0+}^{t}\sum_{j=1}^{d}f_{k,ij}(P_k(s-))\Big(dM_k^j(s)+dJ_k^j(s)\Big)\\&\quad+\frac{1}{2}\int_{0+}^{t}\sum_{j,l=1}^{d}f_{k,ijl}(P_k(s-))a_{kj,kl}(\vec{Q}(s))ds\\&\quad-m^{-1/2}\int_{0+}^{t}\sum_{j=1}^{d}f_{k,ij}(\overline{P_k}(s-))\nabla_{kj}\widetilde{U}(\vec{Q}(s))ds,\end{aligned}$$

with some properly defined $\eta_{Vk}(t)$ satisfying (5.27). This combined with (5.4) and (5.5) completes the proof of our assertion. □

By Lemma 5.10, we get that the tightness for $\overline{V_k}(\cdot)$ is equivalent to that of $\int_{0+}^{\cdot}m^{-1/2}f_{ij}(\overline{P_k}(s-))\nabla_{kj}\widetilde{U}(\vec{Q}(s))ds$. We prove the tightness for it in the rest of §5.4.

Before closing this subsection, let us make some more observation. Notice that for any $x,a\in\mathbf{R}^d$, we have that

$$\begin{aligned}&\Big(\sum_{j=1}^{d}\Big(\delta_{ij}(1+|x|^2)-x_ix_j\Big)a_j\Big)_{i=1,\cdots,d}\\&=(1+|x|^2)a-(x,a)x\\&=\Big(1+|\pi_a^{\perp}x|^2\Big)a-(a,x)\pi_a^{\perp}x.\end{aligned}$$

So by (5.1), we have that

$$\begin{aligned}&\Big(\sum_{j=1}^{d}f_{ij}(p)a_j\Big)_{i=1,\cdots,d}\\&=\frac{1}{M_k(1+M_k^{-2}c^{-2}|p|^2)^{3/2}}\Big[\Big(1+M_k^{-2}c^{-2}|\pi_a^{\perp}p|^2\Big)a-M_k^{-2}c^{-2}(a,p)\pi_a^{\perp}p\Big].\end{aligned}$$

Therefore, since $\nabla_k \widetilde{U}(Q_1, Q_2)$ is parallel to $Q_2 - Q_1$, and $\pi^{\perp}_{Q_2(s)-Q_1(s)} \overline{P_k}(s-) = \overline{R_k}(s-)$, we get from (5.24) that

$$(5.28)\quad \overline{V_k}(t) - \overline{V_k}(0) = A_{\overline{V}k1}(t) + \eta_{\overline{V}k}(t) - \int_{0+}^{t} m^{-1/2}\Big(g_{k1}(s) + g_{k2}(s)\Big) ds,$$

where $g_{k1}(s)$ and $g_{k2}(s)$ are given by

$$\begin{aligned} g_{k1}(s) = &\ \frac{1}{M_k(1 + M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\ &\times \Big(1 + M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2\Big) \nabla_k \widetilde{U}(Q_1(s), Q_2(s)), \\ g_{k2}(s) = &\ -\frac{1}{M_k(1 + M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\ &\times M_k^{-2}c^{-2}(\nabla_k \widetilde{U}(Q_1(s), Q_2(s)), \overline{P_k}(s-))\overline{R_k}(s-). \end{aligned}$$

Notice that $g_{k1}(s)$ is parallel to $Q_2(s) - Q_1(s)$, and $g_{k2}(s)$ is perpendicular to $Q_2(s) - Q_1(s)$.

*5.4.2 Tightness for* $\pi_{Q_2(t)-Q_1(t)}(\overline{V_k}(t))$

Our main results of this subsubsection are the following two lemmas.

LEMMA 5.12.

$$\sup_{m \in (0,1]} E^{\kappa_m}\Big[m^{-1/2} \int_{0+}^{T \wedge \sigma} \frac{1 + M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1 + M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \times |\nabla_k \widetilde{U}(Q_1(s), Q_2(s))| ds\Big] < \infty.$$

LEMMA 5.13. *For any $p > 1$, we have that $\Big\{$ the distribution of $\{\pi_{Q_2(t\wedge\sigma)-Q_1(t\wedge\sigma)} V_k(t \wedge \sigma); t \in [0, T]\}$ under $\kappa_m; m \in (0, 1]\Big\}$ is tight as probabilities on $L^p([0, T]; \mathbf{R}^d)$, with all of its cluster points as $m \to 0$ in $\wp(D([0, T]; \mathbf{R}^d))$.*

Before proving Lemmas 5.12 and 5.13, let us first prepare a decomposition of $\pi_{Q_2(t)-Q_1(t)}(\overline{V_k}(t))$. Let

$$A_{\overline{V}k2}(t) := \pi_{Q_2(t)-Q_1(t)}(\overline{V_k}(0)) + \pi_{Q_2(t)-Q_1(t)}(A_{\overline{V}k1}(t) + \eta_{\overline{V}k}(t))$$

$$
\begin{aligned}
&- \int_0^t \Big( A_{\overline{V}k1}(t) + \eta_{\overline{V}k}(t) - \overline{V_k}(s) + \overline{V_k}(0), V_2(s) - V_1(s) \Big) \\
&\qquad \times \frac{Q_2(s) - Q_1(s)}{|Q_2(s) - Q_1(s)|^2} ds \\
&+ \int_0^t \Big( A_{\overline{V}k1}(t) + \eta_{\overline{V}k}(t) - \overline{V_k}(s) + \overline{V_k}(0), Q_2(s) - Q_1(s) \Big) \\
&\qquad \times \frac{2(Q_2(s) - Q_1(s), V_2(s) - V_1(s))}{|Q_2(s) - Q_1(s)|^2} (Q_2(s) - Q_1(s)) ds \\
&- \int_0^t \Big( A_{\overline{V}k1}(t) + \eta_{\overline{V}k}(t) - \overline{V_k}(s) + \overline{V_k}(0), Q_2(s) - Q_1(s) \Big) \\
&\qquad \times \frac{V_2(s) - V_1(s)}{|Q_2(s) - Q_1(s)|^2} ds.
\end{aligned}
$$

Then we have the following two lemmas.

Lemma 5.14.

(1) $\sup_{m \in (0,1]} E^{\kappa_m}\Big[\sup_{t \in [0, T \wedge \sigma]} |A_{\overline{V}k2}(t)|^2\Big] < \infty$,

(2) $\Big\{$*the distribution of* $\{A_{\overline{V}k2}(t \wedge \sigma); t \in [0,T]\}$ *under* $\kappa_m; m \in (0,1]\Big\}$ *is tight in* $\wp(D([0,T]; \mathbf{R}^d))$.

Lemma 5.15. *We have that*

$$
\pi_{Q_2(t)-Q_1(t)}(\overline{V_k}(t)) = A_{\overline{V}k2}(t) - \int_0^t m^{-1/2} g_{k1}(s) ds. \tag{5.29}
$$

Proof. By (5.28), we have that

$$
\begin{aligned}
&\pi_{Q_2(t)-Q_1(t)}(\overline{V_k}(t)) - \pi_{Q_2(t)-Q_1(t)}(\overline{V_k}(0)) \\
&= \pi_{Q_2(t)-Q_1(t)}(A_{\overline{V}k1}(t) + \eta_{\overline{V}k1}(t)) \\
&\quad - \pi_{Q_2(t)-Q_1(t)}\Big( \int_0^t m^{-1/2} \Big( g_{k1}(s) + g_{k2}(s) \Big) ds \Big).
\end{aligned} \tag{5.30}
$$

Notice that $g_{k2}(t) \cdot (Q_2(t) - Q_1(t)) = 0$ and $g_{k1}(t) \cdot (Q_2(t) - Q_1(t)) \cdot \frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|^2} = g_{k1}(t)$. So

$$
\frac{d}{dt}\Big( \pi_{Q_2(t)-Q_1(t)}\Big( \int_0^t m^{-1/2} \Big( g_{k1}(s) + g_{k2}(s) \Big) ds \Big) \Big)
$$

$$
\begin{aligned}
&= \frac{d}{dt}\Big(\Big(\int_0^t m^{-1/2}\Big(g_{k1}(s) + g_{k2}(s)\Big)ds, Q_2(t) - Q_1(t)\Big)\frac{Q_2(t) - Q_1(t)}{|Q_2(t) - Q_1(t)|^2}\Big) \\
&= m^{-1/2}g_{k1}(t) + \Big(\int_0^t m^{-1/2}\Big(g_{k1}(s) + g_{k2}(s)\Big)ds, V_2(t) - V_1(t)\Big) \\
&\qquad\qquad \times \frac{Q_2(t) - Q_1(t)}{|Q_2(t) - Q_1(t)|^2} \\
&\qquad - \Big(\int_0^t m^{-1/2}\Big(g_{k1}(s) + g_{k2}(s)\Big)ds, Q_2(t) - Q_1(t)\Big) \\
&\qquad\qquad \times \frac{2(Q_2(t) - Q_1(t), V_2(t) - V_1(t))}{|Q_2(t) - Q_1(t)|^4}(Q_2(t) - Q_1(t)) \\
&\qquad + \Big(\int_0^t m^{-1/2}\Big(g_{k1}(s) + g_{k2}(s)\Big)ds, Q_2(t) - Q_1(t)\Big)\frac{V_2(t) - V_1(t)}{|Q_2(t) - Q_1(t)|^2}.
\end{aligned}
$$

Notice that $\int_0^t m^{-1/2}\Big(g_{k1}(s)+g_{k2}(s)\Big)ds = A_{\overline{V}k1}(t) + \eta_{\overline{V}k}(t) - \overline{V_k}(t) + \overline{V_k}(0)$. So if we integrate the both sides of the equation above, and substitute it into (5.30), we get our decomposition (5.29). □

The following is a direct consequence of Lemmas 5.14 (1), 5.15 and the definition of $g_{k1}$.

LEMMA 5.16.

$$
\begin{aligned}
&\sup_{m\in(0,1]} E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]}\Big(m^{-1/2}\int_0^t \frac{1 + M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1 + M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\
&\qquad\qquad \times \nabla_k\widetilde{U}(Q_1(s), Q_2(s))ds\Big)^2\Big] < \infty. \qquad (5.31)
\end{aligned}
$$

The following is also needed in the proof of Lemma 5.12.

LEMMA 5.17. *For any $\varepsilon > 0$, we have that*

$$
\begin{aligned}
&\sup_{m\in(0,1]} m^{-1/2}E^{\kappa_m}\Big[\int_{0+}^{T\wedge\sigma} \frac{1}{M_k(1 + M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{1/2}} \\
&\qquad\qquad \times 1_{\{|Q_2(s)-Q_1(s)|\in(r_1+\varepsilon, r_2-\varepsilon)\}}ds\Big] < \infty. \qquad (5.32)
\end{aligned}
$$

Proof. Without loss of generality, we assume that $\varepsilon \in (0, \varepsilon_0)$.

We have by assumption that there exists a $\delta > 0$ such that if $|Q_2(s) - Q_1(s)| \in (r_1 + \varepsilon, r_2 - \varepsilon)$, then $\widetilde{U}(Q_1(s), Q_2(s)) < -\delta$. If we assume in addition that $\overline{H}(s) > -\frac{\delta}{2} m^{-1/2}$, then

$$\begin{aligned}\sum_{l=1}^{2} M_l c^2 \sqrt{1 + M_l^{-2} c^{-2} |P_l(s)|^2} &= \overline{H}(s) - m^{-1/2} \widetilde{U}(Q_1(s), Q_2(s)) \\ &> \frac{\delta}{2} m^{-1/2}. \end{aligned} \tag{5.33}$$

Let

$$\overline{Y}(s) := \overline{P_1}(s) + \overline{P_2}(s).$$

Then $\overline{Y}(s) = \sum_{l=1}^{2} (M_l(s) + J_l(s))$, hence

$$\sup_{m \in (0,1]} E^{\kappa_m} \Big[ \sup_{s \in [0,T]} |\overline{Y}(s)|^2 \Big] < \infty.$$

Notice that $\sqrt{1 + M_k^{-2} c^{-2} |\overline{P_k}(s)|^2} \leq 1 + M_k^{-1} c^{-1} |\overline{P_k}(s)|$. So if $m \leq 1 \wedge \Big( \frac{\delta}{4} c^{-2} (M_1 + M_2)^{-1} \Big)^2$, equivalently, if $M_1 + M_2 \leq \frac{\delta}{4} m^{-1/2} c^{-2}$, then (5.33) implies that $|\overline{P_1}(s)| + |\overline{P_2}(s)| > \frac{\delta}{4} m^{-1/2} c^{-1}$. So if $|\overline{Y}(s)| < \frac{\delta}{8} m^{-1/2} c^{-1}$ in addition, then we get that

$$|\overline{P_k}(s)| \geq \frac{1}{2} \Big( |\overline{P_1}(s)| + |\overline{P_2}(s)| - |\overline{P_1}(s) + \overline{P_2}(s)| \Big) \geq \frac{\delta}{16} m^{-1/2} c^{-1},$$

hence

$$M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |P_k(s)|^2} \geq c |\overline{P_k}(s)| \geq \frac{\delta}{16} m^{-1/2}.$$

Therefore, there exists a constant $C > 0$ such that

$$\begin{aligned} m^{-1/2} E^{\kappa_m} \Big[ \int_{0+}^{T \wedge \sigma} & \frac{1}{M_k (1 + M_k^{-2} c^{-2} |\overline{P_k}(s-)|^2)^{1/2}} \\ & \times 1_{\{|Q_2(s) - Q_1(s)| \in (r_1 + \varepsilon, r_2 - \varepsilon)\}} ds \Big] \\ \leq m^{-1/2} T \Big\{ & \frac{16}{\delta} m^{1/2} + \frac{1}{M_k} \Big( \frac{\delta}{2} m^{-1/2} \Big)^{-2} E^{\kappa_m} \Big[ \sup_{s \in [0,T]} |\overline{H}(s)|^2 \Big] \\ & + \frac{1}{M_k} \Big( \frac{\delta}{8} m^{-1/2} c^{-1} \Big)^{-2} E^{\kappa_m} \Big[ \sup_{s \in [0,T]} |\overline{Y}(s)|^2 \Big] \Big\} \end{aligned}$$

$$\le Cm^{-1/2}(m^{1/2}+m^{1/2}+m),$$

which is bounded for $m \in (0, 1\wedge \Big(\frac{\delta}{4}c^{-2}(M_1+M_2)\Big)^2)$. Here we used Lemma 5.4. The boundedness for $m \in [1\wedge \Big(\frac{\delta}{4}c^{-2}(M_1+M_2)\Big)^2, 1]$ is trivial. This completes the proof of our assertion. □

We use Lemmas 5.16, 5.14 and 5.17 to prove Lemma 5.12.

Proof of Lemma 5.12. By assumption, there exist functions $g_1, g_2 \in C_b^1(\mathbf{R}^{2d};\mathbf{R}^d)$ such that

$$|\nabla_k \widetilde{U}(x_1,x_2)| = \nabla_k \widetilde{U}(x_1,x_2)\cdot g_k(x_1,x_2), \qquad \text{if } |x_1-x_2| \in (r_1-\varepsilon_0, r_1+\varepsilon_0)\cup(r_2-\varepsilon_0,\infty).$$

Let $A_{\overline{V}k2}$ be as defined before. Then by Lemma 5.16, we have that

$$m^{-1/2}\int_0^s \frac{1+M_k^{-2}c^{-2}|\overline{R_k}(u-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(u-)|^2)^{3/2}}\nabla_k\widetilde{U}(Q_1(u),Q_2(u))du$$
$$= A_{\overline{V}k2}(s) - \pi_{Q_2(s)-Q_1(s)}\overline{V_k}(s).$$

Therefore,

$$\begin{aligned}
&m^{-1/2}\int_{0+}^{T\wedge\sigma} \frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\
&\qquad \times \nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot g_k(Q_1(s),Q_2(s))ds \\
&= \int_{0+}^{T\wedge\sigma} g_k(Q_1(s),Q_2(s)) \\
&\qquad \cdot \frac{d}{ds}\Big(m^{-1/2}\int_0^s \frac{1+M_k^{-2}c^{-2}|\overline{R_k}(u-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(u-)|^2)^{3/2}} \\
&\qquad\qquad \times \nabla_k\widetilde{U}(Q_1(u),Q_2(u))du\Big)ds \\
&= g_k(Q_1(T\wedge\sigma),Q_2(T\wedge\sigma))\cdot(A_{\overline{V}k2}(T\wedge\sigma) - \pi_{Q_2(T\wedge\sigma)-Q_1(T\wedge\sigma)}\overline{V_k}(T\wedge\sigma)) \\
&\qquad - \int_{0+}^{T\wedge\sigma}(A_{\overline{V}k2}(s)-\pi_{Q_2(s)-Q_1(s)}\overline{V_k}(s)) \\
&\qquad \cdot\Big(\sum_{i=1}^2 \nabla_i g_k(Q_1(s),Q_2(s))\cdot V_i(s)\Big)ds.
\end{aligned}$$

Therefore, since $g_k$ and $\nabla g_k$ are bounded, we have by Lemma 5.14 that

$$\sup_{m\in(0,1]} E^{\kappa_m}\Big[\Big(m^{-1/2}\int_{0+}^{T\wedge\sigma}\frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \times \nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot g_k(Q_1(s),Q_2(s))ds\Big)^2\Big]<\infty.$$

For any $s\le\sigma$, we have that

$$\Big|g_k(Q_1(s),Q_2(s))\cdot\nabla_k\widetilde{U}(Q_1(s),Q_2(s))-|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|\Big| \le 2|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|\cdot 1_{\{|Q_2(s)-Q_1(s)|\in(r_1+\varepsilon_0,r_2-\varepsilon_0)\}}.$$

Therefore, in order to prove our assertion, it suffices to prove that

$$\sup_{m\in(0,1]} E^{\kappa_m}\Big[m^{-1/2}\int_{0+}^{T\wedge\sigma}\frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \times|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))| \times 1_{\{|Q_2(s)-Q_1(s)|\in(r_1+\varepsilon_0,r_2-\varepsilon_0)\}}ds\Big]<\infty. \tag{5.34}$$

On the other hand, (5.34) is a direct corollary of Lemma 5.17. □

REMARK 3. We proved in Lemma 5.12 an estimate with respect to the $L^1$-norm. For the $L^2$-norm, we can only get the following result: There exists a constant $C>0$ such that

$$E^{\kappa_m}\Big[\Big\{m^{-1/2}\int_{0+}^{T}\frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}}|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))| \times 1_{\{|Q_2(s)-Q_1(s)|\in(r_1+\varepsilon_0,r_2-\varepsilon_0)\}}ds\Big\}^2\Big]\le Cm^{-1/2},$$

hence

$$E^{\kappa_m}\Big[\Big\{m^{-1/2}\int_{0+}^{T\wedge\sigma}\frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \times|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|ds\Big\}^2\Big]\le Cm^{-1/2}.$$

Proof of Lemma 5.13. By Lemma 3.15, Lemma 5.12 implies that $\Big\{$ the distribution of $\{\int_0^{t\wedge\sigma} m^{-1/2}\frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}}\nabla_k\widetilde{U}(Q_1(s), Q_2(s))ds; t\in[0,T]\}$ under $\kappa_m; m\in(0,1]\Big\}$ is tight as probabilities on $L^p$. This combined with Lemma 5.14 (2) and (5.29) gives us the tightness for $\pi_{Q_2(t)-Q_1(t)}(\overline{V_k}(t))$, which is equivalent to the tightness of $\pi_{Q_2(t)-Q_1(t)}(V_k(t))$. □

*5.4.3 Tightness for $\pi^{\perp}_{Q_2(t)-Q_1(t)}\overline{V_k}(t)$*

In this subsection, we prove the tightness for $\pi^{\perp}_{Q_2(t)-Q_1(t)}V_k(t)$. This combined with Lemma 5.13 completes the proof of the tightness for $V_k(t)$ (Lemma 5.9).

Our main result of this subsubsection is the following.

Lemma 5.18. *For any $p>1$, we have that $\Big\{$ the distribution of $\{\pi^{\perp}_{Q_2(t\wedge\sigma)-Q_1(t\wedge\sigma)}V_k(t\wedge\sigma); t\in[0,T]\}$ under $\kappa_m; m\in(0,1]\Big\}$ is tight as probabilities on $L^p([0,T];\mathbf{R}^d)$, with all of its cluster points as $m\to 0$ in $\wp(D([0,T];\mathbf{R}^d))$.*

We prove Lemma 5.18 in the rest of Subsubsection 5.4.3.

Let $A_{\overline{V}k3}(t)=\overline{V_k}(0)+A_{\overline{V}k1}(t)-A_{\overline{V}k2}(t)$. Then as a result of Lemma 5.10 (2) and Lemma 5.14 (2), we have that $\Big\{$the distribution of $\{A_{\overline{V}k3}(t\wedge\sigma); t\in[0,T]\}$ under $\kappa_m$; $m\in(0,1]\Big\}$ is tight in $\wp(D([0,T];\mathbf{R}^d))$. Also, by (5.28) and Lemma 5.15, we get that the following holds.

Lemma 5.19.

$$\pi^{\perp}_{Q_2(t)-Q_1(t)}\overline{V_k}(t)=A_{\overline{V}k3}+\int_0^t m^{-1/2}g_{k2}(s)ds.$$

In order to prove the tightness of $\int_0^t m^{-1/2}g_{k2}(s)ds$, let us first prepare the following.

Lemma 5.20. *We have the following.*

$$\sup_{m\in(0,1]} E^{\kappa_m}\Big[\int_0^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|}{1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2}ds\Big]<\infty. \tag{5.35}$$

Proof. First, for any $k \in \{1,2\}$, let us consider the decomposition of $\frac{\overline{P_k}(t)}{1+M_k^{-2}c^{-2}|\overline{P_k}(t)|^2}$. For any $i,j \in \{1,\cdots,d\}$, we have that $\frac{\partial}{\partial x_j}\left(\frac{x_i}{1+M_k^{-2}c^{-2}|x|^2}\right) = \frac{\delta_{ij}}{1+M_k^{-2}c^{-2}|x|^2} - \frac{2M_k^{-2}c^{-2}x_ix_j}{(1+M_k^{-2}c^{-2}|x|^2)^2}$. Notice that for any $a \in \mathbf{R}$ and any $p,q \in \mathbf{R}^d$, we have that

$$\begin{aligned}
&\frac{q}{1+a|p|^2} - \frac{2a(p\cdot q)p}{(1+a|p|^2)^2}\\
=&\ -\frac{q}{1+a|p|^2} + \frac{2(1+a|p|^2)q - 2a(p\cdot q)p}{(1+a|p|^2)^2}\\
=&\ -\frac{q}{1+a|p|^2} + 2\frac{q}{(1+a|p|^2)^2} + \frac{2a}{(1+a|p|^2)^2}\left[|p|^2q - (p,q)p\right]\\
=&\ -\frac{q}{1+a|p|^2} + 2\frac{q}{(1+a|p|^2)^2} + \frac{2a}{(1+a|p|^2)^2}|p|^2\pi_p^{\perp}q.
\end{aligned}$$

So by the same method as in the proof of Lemma 5.10, we get that there exists a stochastic process $A_{\overline{V}k6}$ (which corresponds to $A_{\overline{V}k1} + \eta_{\overline{V}k}$ there) such that

$$\sup_{m\in(0,1]} E^{\kappa_m}\left[\sup_{t\in[0,T]} |A_{\overline{V}k6}(t)|^2\right] < \infty,$$

and

$$\begin{aligned}
&\frac{\overline{P_k}(t)}{1+M_k^{-2}c^{-2}|\overline{P_k}(t)|^2} - \frac{\overline{P_k}(0)}{1+M_k^{-2}c^{-2}|\overline{P_k}(0)|^2}\\
=&\int_{0+}^t m^{-1/2}\frac{\nabla_k\widetilde{U}(Q_1(s),Q_2(s))}{1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2}ds\\
&\quad - 2\int_{0+}^t m^{-1/2}\frac{\nabla_k\widetilde{U}(Q_1(s),Q_2(s))}{(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^2}ds\\
&\quad - 2M_k^{-2}c^{-2}\int_{0+}^t m^{-1/2}\frac{|\overline{P_k}(s-)|^2\pi^{\perp}_{\overline{P_k}(s-)}\nabla_k\widetilde{U}(Q_1(s),Q_2(s))}{(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^2}ds\\
&\quad + A_{\overline{V}k6}(t).
\end{aligned} \tag{5.36}$$

We use this to prove that

$$\sup_{m\in(0,1]} E^{\kappa_m}\left[\sup_{t\in[0,T\wedge\sigma]}\left|\int_0^t m^{-1/2}\frac{\nabla_k\widetilde{U}(Q_1(s),Q_2(s))}{1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2}ds\right|\right] < \infty. \tag{5.37}$$

First, we have that

$$E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]}\Big|\int_{0+}^{t} m^{-1/2}\frac{\nabla_k\widetilde{U}(Q_1(s),Q_2(s))}{(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^2}ds\Big|\Big]$$

$$\le E^{\kappa_m}\Big[\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|(1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2)}{(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}}ds\Big], \tag{5.38}$$

which is bounded for $m\in(0,1]$ by Lemma 5.12. Also, since

$$\Big||\overline{P_k}(s-)|^2\pi^{\perp}_{\overline{P_k}(s-)}\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\Big|$$
$$=|\overline{P_k}(s-)|\cdot|\overline{R_k}(s-)|\cdot|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|,$$

we have that

$$E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]}\Big|\int_{0+}^{t} m^{-1/2}\frac{|\overline{P_k}(s-)|^2\pi^{\perp}_{\overline{P_k}(s-)}\nabla_k\widetilde{U}(Q_1(s),Q_2(s))}{(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^2}ds\Big|\Big]$$
$$\le E^{\kappa_m}\Big[\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{|\overline{R_k}(s-)|}{(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}}$$
$$\times|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|ds\Big], \tag{5.39}$$

which is also bounded for $m\in(0,1]$ by Lemma 5.12. Combining (5.36), (5.38) and (5.39), we get (5.37).

We next use (5.37) to prove (5.35). The idea is the same as that we used in the proof of Lemma 5.12. Let $g_1,g_2\in C_b^1(\mathbf{R}^{2d};\mathbf{R}^d)$ be as there, *i.e.*,

$$|\nabla_k\widetilde{U}(x_1,x_2)|=\nabla_k\widetilde{U}(x_1,x_2)\cdot g_k(x_1,x_2),$$
$$\text{if } |x_1-x_2|\in(r_1-\varepsilon_0,r_1+\varepsilon_0)\cup(r_2-\varepsilon_0,\infty).$$

Notice that

$$\Big|\int_0^t m^{-1/2}\frac{g_k(Q_2(s)-Q_1(s))\cdot\nabla_k\widetilde{U}(Q_1(s),Q_2(s))}{1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2}ds\Big|$$
$$=\Big|g_k(Q_2(t)-Q_1(t))\cdot\int_0^t m^{-1/2}\frac{\nabla_k\widetilde{U}(Q_1(r),Q_2(r))}{1+M_k^{-2}c^{-2}|\overline{P_k}(r-)|^2}dr$$
$$-\int_0^t\Big(\int_0^s m^{-1/2}\frac{\nabla_k\widetilde{U}(Q_1(r),Q_2(r))}{1+M_k^{-2}c^{-2}|\overline{P_k}(r-)|^2}dr\Big)$$

$$\cdot \nabla g_k(Q_2(s) - Q_1(s))\Big(V_2(s) - V_1(s)\Big)ds\Big|$$
$$\leq \Big(\|g_k\|_\infty + 2\|\nabla g_k\|_\infty\Big) \sup_{s\in[0,t]} \Big| \int_0^s m^{-1/2} \frac{\nabla_k \widetilde{U}(Q_1(r), Q_2(r))}{1 + M_k^{-2}c^{-2}|\overline{P_k}(r-)|^2} dr\Big|.$$

Therefore,

$$E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]}\Big|\int_0^t m^{-1/2}\frac{g_k(Q_2(s)-Q_1(s))\cdot\nabla_k\widetilde{U}(Q_1(s),Q_2(s))}{1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2}ds\Big|\Big]$$
$$\leq \Big(\|g_k\|_\infty + 2\|\nabla g_k\|_\infty\Big)$$
$$\times E^{\kappa_m}\Big[\sup_{s\in[0,T\wedge\sigma]}\Big|\int_0^s m^{-1/2}\frac{\nabla_k\widetilde{U}(Q_1(r),Q_2(r))}{1+M_k^{-2}c^{-2}|\overline{P_k}(r-)|^2}dr\Big|\Big],$$

which is bounded for $m \in (0,1]$ by (5.37). This combined with

$$\sup_{m\in(0,1]} E^{\kappa_m}\Big[\int_0^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|}{1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2}1_{\{|Q_s|\in(r_1+\varepsilon_0,r_2-\varepsilon_0)\}}ds\Big]$$
$$< \infty,$$

which is a direct consequence of (5.32), completes the proof of our assertion. □

Proof of Lemma 5.18. By Lemma 5.19 and the tightness of $A_{\overline{V}k3}$, it suffices to prove that the distribution of $\{\int_0^{t\wedge\sigma} m^{-1/2} g_{k2}(s)ds\}_t$ under $\kappa_m$ with $m \in (0,1]$ is tight in $\wp(L^p([0,T];\mathbf{R}^d))$, with all of its cluster points as $m \to 0$ in $\wp(D([0,T];\mathbf{R}^d))$. By Lemma 3.15 and the definition of $g_{k2}$, it suffices in turn to prove that

$$\lim_{K_1\to\infty}\inf_{m\in(0,1]}\kappa_m\Big(\int_0^{T\wedge\sigma} m^{-1/2}\frac{M_k^{-2}c^{-2}|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot\overline{P_k}(s-)|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}}$$
$$\times|\overline{R_k}(s-)|ds \leq K_1\Big) = 1. \tag{5.40}$$

We prove (5.40) in the following. For any $K_1, K_2 > 0$, we have that

$$\kappa_m\Big(\int_0^{T\wedge\sigma} m^{-1/2}\frac{M_k^{-2}c^{-2}|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot\overline{P_k}(s-)|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}}$$

$$
\begin{aligned}
&\quad\times |\overline{R_k}(s-)|ds \le K_1\Big) \\
&= 1 - \kappa_m\Big(\int_0^{T\wedge\sigma} m^{-1/2}\frac{M_k^{-2}c^{-2}|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot\overline{P_k}(s-)|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\
&\quad\times |\overline{R_k}(s-)|ds > K_1\Big) \\
&\ge 1 - \kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]}|\overline{R_k}(t)| > K_2\Big) \\
&\quad - \kappa_m\Big(\int_0^{T\wedge\sigma} m^{-1/2}\frac{M_k^{-2}c^{-2}|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot\overline{P_k}(s-)|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}}ds \\
&\quad > \frac{K_1}{K_2}) \\
&\ge 1 - \frac{1}{K_2^2}E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]}|\overline{R_k}(t)|^2\Big] \\
&\quad - \frac{K_2}{K_1}E^{\kappa_m}\Big[\int_0^{T\wedge\sigma} m^{-1/2}\frac{M_k^{-2}c^{-2}|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot\overline{P_k}(s-)|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}}ds\Big].
\end{aligned}
$$

Taking first $K_2 > 0$ large enough then $K_1 > 0$ large enough, we get (5.40), which combined with Lemma 3.15 implies the tightness of $\{\int_0^{t\wedge\sigma} m^{-1/2}g_{k2}(s)ds\}_t$ under $\kappa_m$ with $m \in (0,1]$ in $\wp(L^p([0,T];\mathbf{R}^d))$, and completes the proof of our assertion. □

## 5.5. Tightness for $H(t)$

In this subsection, we give a decomposition of $H(t)$ and prove the tightness of it.

LEMMA 5.21.

(1) *There exists a stochastic process* $\eta_H(\,\cdot\,)$ *such that*

$$
H(t) = H(0) + \eta_H(t) + \sum_{k=1}^{2}\Big(\int_{0+}^{t} V_k(s)\cdot dM_k(s) + \int_{0+}^{t} V_k(s)\cdot dJ_k(s) + \sum_{i,j=1}^{d}\int_{0+}^{t} f_{k,ij}(P_k(s))a_{ki,kj}(\vec{Q}(s))ds\Big),
$$

*and*

$$
\lim_{m\to 0}\kappa_m(\sup_{t\in[0,T\wedge\sigma]}|\eta_H(t)| > \varepsilon) = 0, \quad \forall\varepsilon > 0.
$$

(2) *We have that* $\Big\{$ *the distribution of* $\{H(t\wedge\sigma); t \in [0,T]\}$ *under* $\kappa_m$; $m\in(0,1]\Big\}$ *is tight in* $\wp(C([0,T];\mathbf{R}^d))$.

Proof. By Lemma 5.3, in order to get our first assertion, it suffices to prove that

$$\lim_{m\to 0}\kappa_m\Big[\sup_{t\in[0,T\wedge\sigma]}\Big|\int_{0+}^t m^{-1/2}\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot(V_k(s)-\overline{V_k}(s-))ds\Big|>\varepsilon\Big]=0$$

for any $\varepsilon>0$ and $k\in\{1,2\}$. We prove it in the following.

In general, for any $h\in C_b^3(\mathbf{R}^d,\mathbf{R})$, we have that

$$\begin{aligned}
&h(y)-h(x)\\
&=\nabla h(x)\cdot(y-x)+\frac{1}{2}(y-x)\cdot\nabla^2 h(x)(y-x)\\
&\quad+\int_0^1 dr_1\int_0^1 dr_2\int_0^1 dr_3\sum_{j_1,j_2,j_3=1}^d\nabla_{j_1}\nabla_{j_2}\nabla_{j_3}h(x+r_1r_2r_3(y-x))\\
&\qquad\times r_1r_2^2(y_{j_1}-x_{j_1})(y_{j_2}-x_{j_2})(y_{j_3}-x_{j_3}).
\end{aligned}$$

Let $f_{k,ij}(x)$, $f_{k,ijl}(x)$ and $f_{k,ij_1j_2j_3}(x)$ be as defined before, then we have that

$$\begin{aligned}
&\int_{0+}^t m^{-1/2}\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot(V_k(s)-\overline{V_k}(s-))ds\\
&=\int_{0+}^t m^{-1/2}\sum_{i=1}^d\nabla_{ki}\widetilde{U}(Q_1(s),Q_2(s))\sum_{j=1}^d f_{k,ij}(\overline{P_k}(s-))\eta_k^j(s-)ds\\
&\quad+\frac{1}{2}\int_{0+}^t m^{-1/2}\sum_{i=1}^d\nabla_{ki}\widetilde{U}(Q_1(s),Q_2(s))\\
&\qquad\times\sum_{j_1j_2=1}^d f_{k,ij_1j_2}(\overline{P_k}(s-))\eta_k^{j_1}(s-)\eta_k^{j_2}(s-)ds\\
&\quad+\int_{0+}^t m^{-1/2}\sum_{i=1}^d\nabla_{ki}\widetilde{U}(Q_1(s),Q_2(s))\int_0^1 dr_1\int_0^1 dr_2\int_0^1 dr_3\sum_{j_1j_2,j_3=1}^d
\end{aligned}$$

$$\times f_{k,ij_1j_2j_3}(\overline{P_k}(s-)+r_1r_2r_3\eta_k(s))r_1r_2^2\eta_k^{j_1}(s-)\eta_k^{j_2}(s-)\eta_k^{j_3}(s-)ds.$$

Since

$$\begin{aligned}
&\sup_{t\in[0,T\wedge\sigma]}\Big|\int_{0+}^{t} m^{-1/2}\sum_{i=1}^{d}\nabla_{ki}\widetilde{U}(Q_1(s),Q_2(s))\sum_{j=1}^{d} f_{k,ij}(\overline{P_k}(s-))\eta_k^{j}(s-)ds\Big| \\
&= \sup_{t\in[0,T\wedge\sigma]}\Big|\int_{0+}^{t} m^{-1/2}\eta_k(s-)\cdot\frac{1}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\
&\qquad\times\Big[\Big(1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2\Big)\nabla_k\widetilde{U}(Q_1(s),Q_2(s)) \\
&\qquad\quad - M_k^{-2}c^{-2}\nabla_k\widetilde{U}(Q_1(s),Q_2(s))\cdot\overline{P_k}(s-)\overline{R_k}(s-)\Big]ds\Big| \\
&\le \sup_{s\in[0,T\wedge\sigma]}|\eta_k(s)|\cdot\Big(\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\
&\qquad\times|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|ds \\
&\qquad+\sup_{t\in[0,T\wedge\sigma]}|\overline{R_k}(t)|\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)}ds\Big),
\end{aligned}$$

we have for any $K>0$ that

$$\begin{aligned}
&\kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]}\Big|\int_{0+}^{t} m^{-1/2}\sum_{i=1}^{d}\nabla_{ki}\widetilde{U}(Q_1(s),Q_2(s)) \\
&\qquad\times\sum_{j=1}^{d} f_{k,ij}(\overline{P_k}(s-))\eta_k^{j}(s-)ds\Big|>\varepsilon\Big) \\
&\le \kappa_m\Big(\sup_{s\in[0,T\wedge\sigma]}|\eta_k(s)|>\frac{\varepsilon}{K}\Big)+\kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]}|\overline{R_k}(t)|>\sqrt{\frac{K}{2}}\Big) \\
&\qquad+\kappa_m\Big(\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\
&\qquad\times|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|ds>\frac{K}{2}\Big) \\
&\qquad+\kappa_m\Big(\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)}ds>\sqrt{\frac{K}{2}}\Big) \\
&\le \kappa_m\Big(\sup_{s\in[0,T\wedge\sigma]}|\eta_k(s)|>\frac{\varepsilon}{K}\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{K}E\Big[\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{1+M_k^{-2}c^{-2}|\overline{R_k}(s-)|^2}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)^{3/2}} \\
&\times|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|ds\Big] \\
&+\sqrt{\frac{2}{K}}E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]}|\overline{R_k}(t)|\Big] \\
&+\sqrt{\frac{2}{K}}E^{\kappa_m}\Big[\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)}ds\Big].
\end{aligned}
$$

The expectations on the right hand side above are all bounded for $m\in(0,1]$, and $\lim_{m\to 0}\kappa_m\Big(\sup_{s\in[0,T\wedge\sigma]}|\eta_k(s)|>\frac{\varepsilon}{K}\Big)=0$ for any $K,\varepsilon>0$. Therefore, we get that

$$
\begin{aligned}
\lim_{m\to 0}\kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]}\Big|\int_{0+}^{t} m^{-1/2}\sum_{i=1}^{d}\nabla_{ki}\widetilde{U}(Q_1(s),Q_2(s)) \\
\times\sum_{j=1}^{d}f_{k,ij}(\overline{P_k}(s-))\eta_k^j(s-)ds\Big|>\varepsilon\Big)=0.
\end{aligned}
$$

Similarly, for any $K,\varepsilon>0$, we have that

$$
\begin{aligned}
&\kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]}\Big|\int_{0+}^{t} m^{-1/2}\sum_{i=1}^{d}\nabla_{ki}\widetilde{U}(Q_1(s),Q_2(s)) \\
&\qquad\times\sum_{j_1,j_2=1}^{d}f_{k,ij_1j_2}(\overline{P_k}(s-))\eta_k^{j_1}(s-)\eta_k^{j_2}(s-)ds\Big|>\varepsilon\Big) \\
&\le\kappa_m\Big(C\int_{0}^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)}\cdot|\eta_k(s-)|^2ds>\varepsilon) \\
&\le\kappa_m\Big(\int_{0}^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)}ds>K\Big) \\
&\quad+\kappa_m\Big(\sup_{s\in[0,T\wedge\sigma]}|\eta_k(s)|>\sqrt{\frac{\varepsilon}{CK}}\Big) \\
&\le\frac{1}{K}E^{\kappa_m}\Big[\int_{0}^{T\wedge\sigma} m^{-1/2}\frac{|\nabla_k\widetilde{U}(Q_1(s),Q_2(s))|}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(s-)|^2)}ds\Big]
\end{aligned}
$$

$$+ \kappa_m\Big( \sup_{s\in[0,T\wedge\sigma]} |\eta_k(s)| > \sqrt{\frac{\varepsilon}{CK}}\Big).$$

The expectation on the right hand side above is bounded for $m \in (0,1]$, and the probability on the right hand side above converges to 0 as $m \to 0$ for any $K, \varepsilon > 0$. Therefore, we have that

$$\kappa_m\Big( \sup_{t\in[0,T\wedge\sigma]} \Big| \int_{0+}^{t} m^{-1/2} \sum_{i=1}^{d} \nabla_{ki}\widetilde{U}(Q_1(s), Q_2(s)) \\ \times \sum_{j_1,j_2=1}^{d} f_{k,ij_1j_2}(\overline{P_k}(s-))\eta_k^{j_1}(s-)\eta_k^{j_2}(s-)ds\Big| > \varepsilon\Big) \to 0, \qquad m \to 0.$$

Similarly, since $\|f_{k,ij_1j_2j_3}\|_\infty \le 1$, we have that

$$\kappa_m\Big( \sup_{t\in[0,T\wedge\sigma]} \Big| \int_{0+}^{t} m^{-1/2} \sum_{i=1}^{d} \nabla_{ki}\widetilde{U}(Q_1(s), Q_2(s)) \\ \times \int_0^1 dr_1 \int_0^1 dr_2 \int_0^1 dr_3 \sum_{j_1,j_2,j_3=1}^{d} \\ \times f_{k,ij_1j_2j_3}(\overline{P_k}(s-) + r_1r_2r_3\eta_k(s))r_1r_2^2\eta_k^{j_1}(s-)\eta_k^{j_2}(s-)\eta_k^{j_3}(s-)ds\Big| > \varepsilon\Big) \\ \le \kappa_m\Big( \int_{0+}^{T\wedge\sigma} m^{-1/2}|\eta_k(s-)|^3 ds > C\varepsilon\Big) \\ \le K^{-1}m^{-1/2}E^{\kappa_m}\Big[\int_0^T |\eta_k(s)|^2 ds\Big] + \kappa_m\Big( \sup_{s\in[0,T\wedge\sigma]} |\eta_k(s)| > \frac{C\varepsilon}{K}\Big).$$

Since $m^{-1/2}E^{\kappa_m}\Big[\int_0^T |\eta_k(s)|^2 ds\Big] = \int_0^T m^{-1/2}E^{\kappa_m}\Big[|\eta_k(s)|^2\Big] ds$ is bounded for $m \in (0,1]$, and $\kappa_m\Big(\sup_{s\in[0,T\wedge\sigma]} |\eta_k(s)| > \frac{C\varepsilon}{K}\Big) \to 0$ as $m \to 0$ for any $K, \varepsilon > 0$, we get that the left hand side above converges to 0 as $m \to 0$, too.

This completes the proof of Lemma 5.21. □

## 6. $J_{k2}$, $J_{k3}$ and $J_{k4}$ are Actually Also Negligible

In this section, we use the results of Section 5 to prove that $J_{k2}$, $J_{k3}$ and $J_{k4}$ are actually also negligible when $m \to 0$. Our main result of this section is the following.

Lemma 6.1. *For any $\varepsilon > 0$ and $k \in \{1,2\}$, we have that*

$$\lim_{m\to 0} \kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]} |J_{kl}(t)| > \varepsilon\Big) = 0, \quad l \in \{2,3,4\}.$$

Proof of Lemma 6.1 with $l = 2$. By Lemma 4.2, we have that

$$\begin{aligned}
&\kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]} |J_{k2}(t)| > \varepsilon\Big)\\
&\le \kappa_m\Big(\int_0^T ds 1_{[4m^{1/2}\tau,\infty)}(s)\\
&\qquad \times \int_{\mathbf{R}\times E} m 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau, 2m^{1/2}\tau]}(s-r)\lambda(dr,dx,dv) > \frac{\varepsilon}{2C}\Big)\\
&\quad + \sum_{j=1}^{2} \kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\\
&\qquad\qquad \times \int_{\mathbf{R}\times E} m^{1/2} 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\\
&\qquad\qquad \times |V_j(s) - V_j(\widetilde{r})|\lambda(dr,dx,dv) > \frac{\varepsilon}{4C}\Big).
\end{aligned} \tag{6.1}$$

We prove in the following that all of the terms on the right hand side of (6.1) converges to 0 as $m \to 0$.

For the first term, we have that

$$\begin{aligned}
&\kappa_m\Big(\int_0^T ds 1_{[4m^{1/2}\tau,\infty)}(s)\\
&\qquad \times \int_{\mathbf{R}\times E} m 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\lambda(dr,dx,dv) > \frac{\varepsilon}{2C}\Big)\\
&\le \frac{2C}{\varepsilon} m \cdot E^{\kappa_m}\Big[\int_0^T ds 1_{[4m^{1/2}\tau,\infty)}(s)\\
&\qquad\qquad \times \int_{\mathbf{R}\times E} 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\lambda(dr,dx,dv)\Big]\\
&\le \frac{2C}{\varepsilon} m \cdot (2R_0)^{d-1} T 4m^{1/2}\tau \int_{\mathbf{R}^d} \widetilde{\rho}_{\|U_1\|_\infty + \|U_2\|_\infty}(\frac{1}{2}|v|^2)|v| dv m^{-1}\\
&=: C_1 m^{1/2},
\end{aligned}$$

which converges to 0 as $m \to 0$.

We next deal with the second term on the right hand side of (6.1). Fix any $j \in \{1,2\}$. Let $A_{Jj1}(u_1,u_2) = |V_j(u_1) - \overline{V}_j(u_1)| + |V_j(u_2) - \overline{V}_j(u_2)| + |A_{\overline{V}j1}(u_1) - A_{\overline{V}j1}(u_2)| + |\eta_{\overline{V}j}(u_1)| + |\eta_{\overline{V}j}(u_2)|$. Then we have by (5.24) that

$$|V_j(s) - V_j(\widetilde{r})| \le A_{Jj1}(s,\widetilde{r}) + \Big| \int_{\widetilde{r}}^{s} m^{-1/2} g_{j1}(u)du \Big| + \Big| \int_{\widetilde{r}}^{s} m^{-1/2} g_{j2}(u)du \Big|. \tag{6.2}$$

Therefore, it suffices to prove that the followings hold:

$$\lim_{m\to 0} \kappa_m \Big( \int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s) \times \int_{\mathbf{R}\times E} m^{1/2} 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau, 2m^{1/2}\tau]}(s-r) \times A_{Jj1}(s,\widetilde{r}) \lambda(dr,dx,dv) > \frac{\varepsilon}{12C} \Big) = 0, \tag{6.3}$$

$$\lim_{m\to 0} \kappa_m \Big( \int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s) \times \int_{\mathbf{R}\times E} m^{1/2} 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau, 2m^{1/2}\tau]}(s-r) \times \Big| \int_{\widetilde{r}}^{s} m^{-1/2} g_{j1}(u)du \Big| \lambda(dr,dx,dv) > \frac{\varepsilon}{12C} \Big) = 0, \tag{6.4}$$

$$\lim_{m\to 0} \kappa_m \Big( \int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s) \times \int_{\mathbf{R}\times E} m^{1/2} 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau, 2m^{1/2}\tau]}(s-r) \times \Big| \int_{\widetilde{r}}^{s} m^{-1/2} g_{j1}(u)du \Big| \lambda(dr,dx,dv) > \frac{\varepsilon}{12C} \Big) = 0. \tag{6.5}$$

Let us first prove (6.3). By Lemma 5.10 (3) and Lemma 3.13 (4), we have that there exists a constant $C_2 > 0$ such that

$$\sup_{u_1,u_2\in[0,T], |u_1-u_2|\le 4m^{1/2}\tau} E^{\kappa_m}[A_{Jj1}(u_1,u_2)] \le C_2 \Big( m^{1/4} + y(m)^{1/2} \Big), \qquad \forall m \in (0,1]. \tag{6.6}$$

Therefore, with some proper constants $C_3, C_4 > 0$, we have that

$$\begin{aligned}
&\kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E} m^{1/2} 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r) \\
&\qquad\qquad \times A_{Jj1}(s,\widetilde{r})\lambda(dr,dx,dv) > \frac{\varepsilon}{12C}\Big) \\
&\le m^{1/2}\frac{12C}{\varepsilon}E^{\kappa_m}\Big[\int_0^T ds 1_{[4m^{1/2}\tau,\infty)}(s) \\
&\qquad\qquad \times \int_{\mathbf{R}\times E} 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r) \\
&\qquad\qquad \times A_{Jj1}(s,\widetilde{r})\lambda(dr,dx,dv)\Big] \\
&\le C_3 \sup_{u_1,u_2\in[0,T],|u_1-u_2|\le 4m^{1/2}\tau} E^{\kappa_m}[A_{Jj1}(u_1,u_2)] \\
&\le C_4\Big(m^{1/4}+y(m)^{1/2}\Big),
\end{aligned}$$

which converges to 0 as $m \to 0$. So (6.3) holds.

Let us next prove (6.4). Notice that in general for any $g$, we have that

$$\begin{aligned}
&\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}} 1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)dr\int_{\widetilde{r}}^s |g(u)|du \\
&\le \int_0^{T\wedge\sigma} du\int_u^{(T\wedge\sigma)\wedge(u+4m^{1/2}\tau)} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{s-2m^{1/2}\tau}^{s+2m^{1/2}\tau} dr|g(u)| \\
&\le (4m^{1/2}\tau)^2\int_0^{T\wedge\sigma}|g(u)|du, \qquad (6.7)
\end{aligned}$$

so with some proper constant $C_5, C_6 > 0$, we have that

$$\begin{aligned}
&\kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E} m^{1/2} 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r) \\
&\qquad\qquad \times \Big|\int_{\widetilde{r}}^s m^{-1/2}g_{j1}(u)du\Big|\lambda(dr,dx,dv) > \frac{\varepsilon}{12C}\Big) \\
&\le \frac{12C}{\varepsilon}E^{\kappa_m}\Big[\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s) \\
&\qquad\qquad \times \int_{\mathbf{R}\times E} 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r) \\
&\qquad\qquad \times \int_{\widetilde{r}}^s |g_{j1}(u)|du\lambda(dr,dx,dv)\Big]
\end{aligned}$$

$$\leq C_5 E^{\kappa_m}\Big[\int_0^{T\wedge\sigma}|g_{j1}(u)|du\Big]$$
$$\leq C_6 m^{1/2},$$

where we used Lemma 5.12 when passing to the last line. So (6.4) holds.

Finally, let us prove (6.5). We have that the following holds for any $K>0$.

$$\kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E} m^{1/2}1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)$$
$$\times\Big|\int_{\tilde{r}}^{s} m^{-1/2}g_{j2}(u)du\Big|\lambda(dr,dx,dv)>\frac{\varepsilon}{12C}\Big)$$
$$\leq \kappa_m\Big(\sup_{w\in[0,T\wedge\sigma]}|\overline{R_j}(w)|\cdot\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)$$
$$\times\int_{\mathbf{R}\times E} m^{1/2}1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)$$
$$\times\int_{\tilde{r}}^{s}|g_{j2}(u)|du\lambda(dr,dx,dv)>\frac{\varepsilon}{12C}\Big)$$
$$\leq \kappa_m\Big(\sup_{w\in[0,T\wedge\sigma]}|\overline{R_j}(w)|>K\Big)$$
$$+\kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)$$
$$\times\int_{\mathbf{R}\times E} m^{1/2}1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)$$
$$\times\int_{\tilde{r}}^{s}|g_{j2}(u)|du\lambda(dr,dx,dv)>\frac{\varepsilon}{12CK}\Big).$$

We have by Lemma 5.7 that

$$\kappa_m\Big(\sup_{w\in[0,T\wedge\sigma]}|\overline{R_j}(w)|>K\Big)\leq\frac{1}{K}E^{\kappa_m}\Big[\sup_{w\in[0,T\wedge\sigma]}|\overline{R_j}(w)|\Big]\leq\frac{C}{K},$$

which converges to 0 as $K\to\infty$. Therefore, in order to prove (6.5), it suffices to prove that

$$\kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E} m^{1/2}1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)$$

$$\times \int_{\widetilde{r}}^{s} |g_{j2}(u)| du \lambda(dr, dx, dv) > \delta \Big) \to 0, \quad m \to 0$$

for any $\delta > 0$.

On the other hand, by (6.7), we have that

$$\begin{aligned}
&\kappa_m \Big( \int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s) \int_{\mathbf{R}\times E} m^{1/2} 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau, 2m^{1/2}\tau]}(s-r) \\
&\qquad \times \int_{\widetilde{r}}^{s} |g_{j2}(u)| du \lambda(dr, dx, dv) > \delta \Big) \\
&\le \frac{1}{\delta} E^{\kappa_m} \Big[ \int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s) \\
&\qquad \times \int_{\mathbf{R}\times E} m^{1/2} 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau, 2m^{1/2}\tau]}(s-r) \\
&\qquad \times \int_{\widetilde{r}}^{s} |g_{j2}(u)| du \lambda(dr, dx, dv) \Big] \\
&\le \frac{1}{\delta} C (4m^{1/2}\tau)^2 E^{\kappa_m} \Big[ \int_0^{T\wedge\sigma} \frac{|\nabla_j \widetilde{U}(Q_1(u), Q_2(u))|}{1 + M_j^{-2} c^{-2} |\overline{P_j}(u-)|^2} du \Big] \le C m^{1/2},
\end{aligned}$$

where we used Lemma 5.20 in the last inequality.

This implies (6.5), and completes the proof of our assertion with $l = 2$. □

PROOF OF LEMMA 6.1 WITH $l = 3$. The idea is similar to that for $l = 2$.

First notice that $\nabla^2 U_i \Big( Q_i(\widetilde{r}) - \psi(m^{-1/2}(s-r), x, v; \vec{Q}(\widetilde{r})) \Big) \ne 0$ implies $|x| \le R_0$ and $s - r \in [-m^{1/2}\tau, 2m^{1/2}\tau]$. So by the continuity of $\psi^0$ with respect to $Q$ (Lemma 3.5), we have that there exists a constant $C > 0$ such that

$$\begin{aligned}
|J_{k3}(t)| \le C \int_0^t ds 1_{[4m^{1/2}\tau,\infty)}(s) &\int_{\mathbf{R}\times E} (\mu_\omega + \lambda)(dr, dx, dv) \\
&\times 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau, 2m^{1/2}\tau]}(s-r) \\
&\times \sum_{j=1}^{2} \Big| Q_j(s) - Q_j(\widetilde{r}) - (s - \widetilde{r}) V_j(\widetilde{r}) \Big|.
\end{aligned}$$

Therefore, it suffices to prove that

$$\lim_{m\to 0} \kappa_m \Big( \int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s) \int_{\mathbf{R}\times E} (\mu_\omega + \lambda)(dr, dx, dv)$$

$$\times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)$$
$$\times \left|Q_j(s)-Q_j(\widetilde{r})-(s-\widetilde{r})V_j(\widetilde{r})\right|>\varepsilon\Big)=0$$

for $j\in\{1,2\}$.

We have by (6.2) that

$$\left|Q_j(s)-Q_j(\widetilde{r})-(s-\widetilde{r})V_j(\widetilde{r})\right|\le\int_{\widetilde{r}}^{s}\left|V_j(u)-V_j(\widetilde{r})\right|du$$
$$\le\int_{\widetilde{r}}^{s}A_{Jj1}(u,\widetilde{r})du+\int_{\widetilde{r}}^{s}du\int_{\widetilde{r}}^{u}m^{-1/2}|g_{j1}(w)|dw$$
$$+\int_{\widetilde{r}}^{s}du\int_{\widetilde{r}}^{u}m^{-1/2}|g_{j2}(w)|dw.$$

We prove in the following that the probabilities corresponding to each of the terms above converge to 0 as $m\to 0$.

First, by (6.6), we have that

$$\kappa_m\Big(\int_0^{T\wedge\sigma}ds1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E}(\mu_\omega+\lambda)(dr,dx,dv)$$
$$\times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\int_{\widetilde{r}}^{s}|A_{Jj1}(u,\widetilde{r})|du>\varepsilon\Big)$$
$$\le\frac{2}{\varepsilon}\int_0^{T}ds1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E}\lambda(dr,dx,dv)$$
$$\times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\int_{\widetilde{r}}^{s}duE^{\kappa_m}\Big[A_{Jj1}(u,\widetilde{r})\Big]$$
$$\le C_2\Big(m^{1/4}+y(m)^{1/2}\Big)\to 0,\qquad m\to 0.$$

Next, for the term with respect to $g_{j1}$, notice that

$$\int_0^{T\wedge\sigma}ds1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}}dr1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\int_{\widetilde{r}}^{s}du\int_{\widetilde{r}}^{u}dw|g_{j1}(w)|$$
$$\le(4m^{1/2}\tau)^3\int_0^{T\wedge\sigma}|g_{j1}(w)|dw,$$

so by Lemma 5.12, we get that there exist constants $C_7,C_8>0$ such that

$$\kappa_m\Big(\int_0^{T\wedge\sigma}ds1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E}(\mu_\omega+\lambda)(dr,dx,dv)$$

$$
\begin{aligned}
&\qquad \times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\\
&\qquad \times \int_{\tilde{r}}^{s} du \int_{\tilde{r}}^{u} m^{-1/2}|g_{j1}(w)|dw > \varepsilon\Big)\\
&\le \frac{2}{\varepsilon} m^{-1/2} E^{\kappa_m}\Big[\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E}\lambda(dr,dx,dv)\\
&\qquad \times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\int_{\tilde{r}}^{s} du \int_{\tilde{r}}^{u} |g_{j1}(w)|dw\Big]\\
&\le C_7 E^{\kappa_m}\Big[\int_0^{T\wedge\sigma}|g_{j1}(w)|dw\Big] \le C_8 m^{1/2}.
\end{aligned}
$$

Finally, for the term with respect to $g_{j2}$, we have that there exist constants $C_9, C_{10}, C_{11} > 0$ (that do not depend on $m$ and $K$) such that the following holds for any $K > 0$:

$$
\begin{aligned}
&\kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E}(\mu_\omega+\lambda)(dr,dx,dv)\\
&\qquad \times 1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\\
&\qquad \times \int_{\tilde{r}}^{s} du \int_{\tilde{r}}^{u} m^{-1/2}|g_{j2}(w)|dw > \varepsilon\Big)\\
&\le \kappa_m\Big(\sup_{w\in[0,T\wedge\sigma]}|\overline{R_j}(w)| > K\Big)\\
&\quad + \kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\\
&\qquad \times \int_{\mathbf{R}\times E}(\mu_\omega+\lambda)(dr,dx,dv)1_{[0,R_0)}(|x|)1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\\
&\qquad \times \int_{\tilde{r}}^{s} du \int_{\tilde{r}}^{u} dw \frac{M_j^{-2}c^{-2}(\nabla_j\widetilde{U}(Q_1(w),Q_2(w)),\overline{P_j}(w-))}{M_j(1+M_j^{-2}c^{-2}|\overline{P_j}(w-)|^2)} > \frac{\varepsilon}{K}\Big)\\
&\le \frac{1}{K}E^{\kappa_m}\Big[\sup_{w\in[0,T\wedge\sigma]}|\overline{R_j}(w)|\Big]\\
&\quad + \frac{2K}{\varepsilon}E^{\kappa_m}\Big[\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s)\int_{\mathbf{R}\times E}\lambda(dr,dx,dv)1_{[0,R_0)}(|x|)\\
&\qquad \times 1_{[-m^{1/2}\tau,2m^{1/2}\tau]}(s-r)\\
&\qquad \times \int_{\tilde{r}}^{s} du \int_{\tilde{r}}^{u} dw \frac{M_j^{-2}c^{-2}(\nabla_j\widetilde{U}(Q_1(w),Q_2(w)),\overline{P_j}(w-))}{M_j(1+M_j^{-2}c^{-2}|\overline{P_j}(w-)|^2)}\Big]
\end{aligned}
$$

$$\leq C_9 \frac{1}{K} + C_{10} m^{-1/2} (4m^{1/2}\tau)^3 m^{-1} \times$$
$$\times E^{\kappa_m}\Big[\int_0^{T\wedge\sigma} \frac{M_j^{-2} c^{-2} (\nabla_j \widetilde{U}(Q_1(w), Q_2(w)), \overline{P_j}(w-))}{M_j(1 + M_j^{-2} c^{-2} |\overline{P_j}(w-)|^2)} dw\Big] K$$
$$\leq C_9 \frac{1}{K} + C_{11} m^{1/2} K.$$

So

$$\lim_{m\to 0} \kappa_m\Big(\int_0^{T\wedge\sigma} ds 1_{[4m^{1/2}\tau,\infty)}(s) \int_{\mathbf{R}\times E} (\mu_\omega + \lambda)(dr, dx, dv)$$
$$\times 1_{[0,R_0)}(|x|) 1_{[-m^{1/2}\tau, 2m^{1/2}\tau]}(s - r)$$
$$\times \int_{\tilde{r}}^{s} du \int_{\tilde{r}}^{u} m^{-1/2} |g_{j2}(w)| dw > \varepsilon\Big) = 0.$$

This completes the proof of our assertion for $l = 3$. □

**Proof of Lemma 6.1 with $l = 4$.** The idea is similar to that for $l = 2, 3$, and we omit it here. □

## 7. Determine $(P_1(t), P_2(t))$ by $(R_1(t), R_2(t), Y(t), H(t))$

As mentioned in Sections 1 and 2, in the instant that the two massive particles arrive at the boundary of the two phases from the uniform motion phase, the value of $(R_1(t), R_2(t), Y(t), H(t))$ is sufficient to determine the value of $(P_1(t), P_2(t))$. We study this fact in this section.

Consider the equation (2.7), and let $b_0(a_1, a_2, d)$ be as in (2.8). Let us first prove the following.

Lemma 7.1. *For any given $a_1, a_2, b, d \in \mathbf{R}$, we have that*

(1) *If $b < b_0(a_1, a_2, d)$, then (2.7) has no real solution,*

(2) *if $b > b_0(a_1, a_2, d)$, then (2.7) has exactly one real solution $x = x(a_1, a_2, b, d)$ given by*

$$x(a_1, a_2, b, d) = \frac{d}{2} + \frac{c^2 d(a_1^2 - a_2^2 + c^2(M_1^2 - M_2^2))}{2(b^2 - c^2 d^2))} - \frac{b}{2c(b^2 - c^2 d^2)}$$

$$\times \sqrt{\left[b^2 + c^2(a_1^2 - a_2^2 - d^2) + c^4(M_1^2 - M_2^2)\right]^2 - 4c^2(M_1^2c^2 + a_1^2)(b^2 - c^2d^2)}. \tag{7.1}$$

PROOF. For any $a_1, a_2, d \in \mathbf{R}$, let

$$\begin{aligned} g_{a_1,a_2,d}(x) &= M_1c^2\sqrt{1 + M_1^{-2}c^{-2}(|a_1|^2 + x^2)} \\ &\quad + M_2c^2\sqrt{1 + M_2^{-2}c^{-2}(|a_2|^2 + (d-x)^2)}. \end{aligned}$$

Then $g_{a_1,a_2,d}$ is bounded from below, and $\lim_{x\to\pm\infty} g_{a_1,a_2,d}(x) = +\infty$. By definition, $b_0 := b_0(a_1, a_2, d) = \inf_{x\in\mathbf{R}} g_{a_1,a_2,d}(x)$. Also,

$$\begin{aligned} g'_{a_1,a_2,d}(x) &= \frac{x}{M_1\sqrt{1 + M_1^{-2}c^{-2}(|a_1|^2 + x^2)}} \\ &\quad + \frac{x-d}{M_2\sqrt{1 + M_2^{-2}c^{-2}(|a_2|^2 + (d-x)^2)}}, \\ g''_{a_1,a_2,d}(x) &= \frac{1 + M_1^{-2}c^{-2}|a_1|^2}{M_1(1 + M_1^{-2}c^{-2}(|a_1|^2 + x^2))^{3/2}} \\ &\quad + \frac{1 + M_2^{-2}c^{-2}|a_2|^2}{M_2(1 + M_2^{-2}c^{-2}(|a_2|^2 + (d-x)^2))^{3/2}}. \end{aligned}$$

So $g''_{a_1,a_2,d}(x) > 0$ for all $x \in \mathbf{R}$. Therefore, there exists a unique number $x_0 \in \mathbf{R}$ such that $g_{a_1,a_2,d}(x_0) = b_0$, and the last condition in (2.7) is equivalent to $g'_{a_1,a_2,d}(x) < 0$. Notice that $b_0 > c|d|$. Indeed, $b_0 = M_1c^2\sqrt{1 + M_1^{-2}c^{-2}(a_1^2 + x_0^2)} + M_2c^2\sqrt{1 + M_2^{-2}c^{-2}(a_2^2 + (d-x_0)^2)} > M_1c^2 \cdot M_1^{-1}c^{-1}|x_0| + M_2c^2 \cdot M_2^{-1}c^{-1}|d - x_0| = c(|x_0| + |d - x_0|) \geq c|d|$.

The equation $g_{a_1,a_2,d}(x) = b$ has no real solution when $b < b_0$, and has exactly two real solutions $x_1, x_2 \in \mathbf{R}$ (depending on $a_1, a_2, b, d$) when $b > b_0$, with $x_1 < x_0$ and $x_2 > x_0$, hence $g'_{a_1,a_2,d}(x_1) < 0$ and $g'_{a_1,a_2,d}(x_2) > 0$. So $x_1$ is the only solution $x(a_1, a_2, b, d)$ of (2.7).

Finally, let us find the explicit formular of $x_1$. If $x$ satisfies $g_{a_1,a_2,d}(x) = b$, then

$$b - M_1c^2\sqrt{1 + M_1^{-2}c^{-2}(|a_1|^2 + x^2)} = M_2c^2\sqrt{1 + M_2^{-2}c^{-2}(|a_2|^2 + (d-x)^2)}.$$

Taking the squares of the both sides, we get that $2c^2dx + \Big[b^2 + c^2(a_1^2 - a_2^2 - d^2) + c^4(M_1^2 - M_2^2)\Big] = 2bM_1c^2\sqrt{1 + M_1^{-2}c^{-2}(|a_1|^2 + x^2)}$. Taking the squares of the both sides once more, by a simple calculation, we get that

$$4c^2(b^2 - c^2d^2)x^2 - 4c^2d\Big[b^2 + c^2(a_1^2 - a_2^2 - d^2) + c^4(M_1^2 - M_2^2)\Big]x + \Big\{4b^2M_1^2c^4 + 4b^2c^2a_1^2 - \Big[b^2 + c^2(a_1^2 - a_2^2 - d^2) + c^4(M_1^2 - M_2^2)\Big]^2\Big\} = 0.$$

This is a quadratic equation, so it has at most two possible real solutions given by

$$x_1, x_2 = \frac{d}{2} + \frac{c^2d(a_1^2 - a_2^2 + c^2(M_1^2 - M_2^2))}{2(b^2 - c^2d^2)} \pm \frac{b}{2c(b^2 - c^2d^2)} \times \sqrt{\Big[b^2 + c^2(a_1^2 - a_2^2 - d^2) + c^4(M_1^2 - M_2^2)\Big]^2 - 4c^2(M_1^2c^2 + a_1^2)(b^2 - c^2d^2)}.$$

This combined with $x(a_1, a_2, b, d) = x_1 < x_2$ gives us (7.1), and completes the proof of our assertion. □

By the definition of $b_0(\,\cdot\,,\,\cdot\,,\,\cdot\,)$, we have that $b_0(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is a continuous function, and $(M_1 + M_2)c^2 \le b_0(a_1, a_2, d) \le \sum_{j=1}^{2} M_jc^2 \cdot \sqrt{1 + M_j^{-2}c^{-2}(a_j^2 + (d/2)^2)}$.

In the following, let us derive heuristically the expression of $(P_1(t), P_2(t))$ in the instant that the particles enter the diffusion phase from the uniform motion phase. Let

$$H_0(R_1, R_2, Y, Q_1, Q_2) = b_0(|R_1|, |R_2|, Y \cdot \frac{Q_2 - Q_1}{|Q_2 - Q_1|}).$$

So we have the following:

Lemma 7.2. *$H_0(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ is a continuous function, and there exists a constant $C > 0$ such that $(M_1 + M_2)c^2 \le H_0(R_1, R_2, Y, Q_1, Q_2) \le C(1 + |R_1| + |R_2| + |Y|)$.*

When $b > b_0(a_1, a_2, d)$, let $S(a_1, a_2, b, d) = (S^1(a_1, a_2, b, d), S^2(a_1, a_2, b, d)) := (x(a_1, a_2, b, d), d - x(a_1, a_2, b, d))$, where $x(a_1, a_2, b, d)$ is the unique solution of (2.7) given by (7.1). Also, for $k \in \{1,2\}$, $m \in (0,1]$ and $t \in [0,T]$, let $S_k(t) = S_k^{(m)}(t) \in \mathbf{R}$ be the coordinate of $P_k(t)$ in $Q_2(t)-Q_1(t)$-direction, *i.e.*, $P_k(t) = R_k(t) + S_k(t)\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}$. Then we have that as long as $|Q_2(t)-Q_1(t)| > R_1 + R_2$, $(S_1(t), S_2(t))$ satisfies

$$\begin{aligned}&S_1(t) + S_2(t) = Y(t) \cdot \frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|},\\ &M_1c^2\sqrt{1 + M_1^{-2}c^{-2}(|R_1(t)|^2 + |S_1(t)|^2)}\\ &\qquad + M_2c^2\sqrt{1 + M_2^{-2}c^{-2}(|R_2(t)|^2 + |S_2(t)|^2)} = H(t).\end{aligned}$$

This is true for any $m \leq 1$, hence keeps true after taking limit $m \to 0$. Also, in the limit process, if the particles cross the boundary and enter the diffusion phase from the uniform motion phase, we have that $|Q_2(t) - Q_1(t)|^2$ increases, hence $(Q_2(t) - Q_1(t)) \cdot (V_2(t) - V_1(t)) > 0$, equivalently, $\frac{S_1(t)}{M_1\sqrt{1+M_1^{-2}c^{-2}(|R_1(t)|^2+S_1(t)^2)}} - \frac{S_2(t)}{M_2\sqrt{1+M_2^{-2}c^{-2}(|R_2(t)|^2+|S_2(t)|^2)}} < 0$. Therefore, in this instant, we must have that

$$(S_1(t), S_2(t)) = S\Big(|R_1(t)|, |R_2(t)|, H(t), Y(t) \cdot \frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|})\Big).$$

This determines the value of $(P_1(t), P_2(t))$ of the instant that the particles cross the boundary and enter the diffusion phase. Precisely, $\vec{P}(t)$ is given by $\vec{P}(R_1(t), R_2(t), Y(t), H(t), Q_1(t), Q_2(t))$ with $\vec{P}(R_1, R_2, Y, H, Q_1, Q_2)$ defined by

$$\begin{aligned}&\vec{P}(R_1, R_2, Y, H, Q_1, Q_2)\\ &= \vec{R} + S\Big(|R_1|, |R_2|, H, Y \cdot \frac{Q_2 - Q_1}{|Q_2 - Q_1|})\Big)\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}.\end{aligned}$$

In conclusion, when the particles reach the boundary from the uniform motion phase, they enter the diffusion phase if and only if $H(t) > H_0(R_1(t), R_2(t), Y(t), Q_1(t), Q_2(t))$, and in this case, the new velocities of the massive particles in the instant of boundary-crossing is given by $\vec{P}(t) = \vec{P}(R_1(t), R_2(t), Y(t), H(t), Q_1(t), Q_2(t))$. The mathematically rigorous proof of this fact is given in Subsection 8.2.

## 8. Convergence

Use the notation $X = (Q_1, Q_2, V_1, V_2, R_1, R_2, Y, H)$, and for any $m \in (0,1]$, let $\mu_m$ denote the distribution of $\{X^{(m)}(t); t \in [0,\infty)\}$. $\mu_m$ is a probability on $C([0,\infty); \mathbf{R}^{2d}) \times D([0,\infty); \mathbf{R}^{2d}) \times C([0,\infty); \mathbf{R}^{3d+1})$. Write the canonical process on this space as $X = (Q_1, Q_2, V_1, V_2, R_1, R_2, Y, H)$, too.

By Section 5, we have that $\{\mu_m; m \in (0,1]\}$ is tight. we prove in this section that any of its cluster point(s) as $m \to 0$ must be equal to $\mu_0$, the unique probability measure that satisfies $(\mu 1) \sim (\mu 5)$ in Theorem 2.2. Choose any sequence $m_n \to 0$ $(n \to \infty)$ such that $\mu_{m_n}$ converges, and write the limit of it as $P_\infty$. We prove from now on that $P_\infty$ satisfies $(\mu 1) \sim (\mu 5)$.

The fact that $P_\infty$ satisfies $(\mu 1)$ is trivial, and the fact that $P_\infty$ satisfies $(\mu 2)$ is a direct consequence of Corollary 5.5.

For $(\mu 3)$, first notice that by Lemma 5.16, we have $P_\infty$-almost surely that $|V_1(t)| = |V_2(t)| = c$ if $|Q_1(t) - Q_2(t)| \in (r_1, r_2)$. Also, the last term in (5.26), the decomposition of $V_k(t)$, is equal to 0 as long as $|Q_1(t) - Q_2(t)| > r_2$. Now, the fact that $P_\infty$ satisfies $(\mu 3)$ can be derived by the same argument as we used in the proof of Theorem 2.1. More precisely, for any $f \in C_0^\infty(\mathbf{R}^{7d+1})$ with $\mathrm{supp}(f) \subset \Big((B_1 \cup B_2) \times \mathbf{R}^{5d+1}\Big)$, consider $f(\vec{Q}(t), \vec{V}(t), \vec{R}(t), Y(t), H(t))$. Apply Ito's formula to its approximation $f(\vec{Q}(t), \vec{V}(t) - \eta_{V\cdot}(t), \vec{R}(t) - \eta_{R\cdot}(t), Y(t) - \eta_1(t) - \eta_2(t), H(t) - \eta_H(t))$ stopped at $\sigma$, then by Lemmas 5.11, 5.8 (1), 5.1 (1), 5.21, with the help of Lemmas 3.16 (3) and 4.1, we have that when $m \to 0$, $\Big\{ f(\vec{Q}(t\wedge\sigma), \vec{V}(t\wedge\sigma), \vec{R}(t \wedge \sigma), Y(t\wedge\sigma), H(t\wedge\sigma)) - \int_0^{t\wedge\sigma} Lf(\vec{Q}(s), \vec{V}(s), \vec{R}(s), Y(s), H(s))ds; t \geq 0 \Big\}$ is a continuous martingale under $P_\infty$. Since $P_\infty(\sigma = \infty) = 1$ by Corollary 5.5, this completes the proof of the fact that $P_\infty$ satisfies $(\mu 3)$.

Finally, for $(\mu 4)$ and $(\mu 5)$, first notice that by Lemma 5.16, we have $P_\infty$-almost surely that $|V_1(t)| = |V_2(t)| = c$ if $|Q_1(t) - Q_2(t)| \in (r_1, r_2)$. So in order to prove that $P_\infty$ satisfies $(\mu 4)$, it suffices to prove that the followings hold $P_\infty$-almost surely: (1) $V_k$ is continuous in the domain $|Q_1 - Q_2| \in (r_1, r_2)$, and (2) $|V_k(t)| = c$ when $|Q_1(t) - Q_1(t)| = r_1$. We prove the assertion (1) in §8.1, and in §8.2, we prove that it has the required behavior at boundaries $|Q_1(t) - Q_2(t)| = r_1$ and $|Q_1(t) - Q_2(t)| = r_2$.

**8.1.** $V_k$ **is continuous when** $\vec{Q} \in B_2$

By (5.28), it suffices to prove the following.

Lemma 8.1. *For any* $\varepsilon_1, \varepsilon_2 > 0$ *and* $k \in \{1, 2\}$, *we have that*

$$\lim_{m \to 0} \kappa_m \Big( \int_{0+}^{T \wedge \sigma} m^{-1/2} (|g_{k1}(t)| + |g_{k2}(t)|) \times 1_{\{|Q_1(t) - Q_2(t)| \in (r_1 + \varepsilon_2, r_2 - \varepsilon_2)\}} dt > \varepsilon_1 \Big) = 0.$$

Let us first prepare the following.

Lemma 8.2. *For any* $\varepsilon_2 > 0$, *there exists a constant* $C > 0$ *such that*

$$E^{\kappa_m} \Big[ \sup_{t \in [0, T \wedge \sigma]} \frac{1_{\{|Q_1(t) - Q_2(t)| \in (r_1 + \varepsilon_2, r_2 - \varepsilon_2)\}}}{(1 + M_k^{-2} c^{-2} |\overline{P_k}(t)|^2)^{1/2}} \Big] \le C m^{1/2}, \quad m \in (0, 1].$$

Proof. As in the proof of Lemma 5.17, there exists a $\delta > 0$ such that if $|Q_2(s) - Q_1(s)| \in (r_1 + \varepsilon, r_2 - \varepsilon)$, then $\widetilde{U}(Q_1(s), Q_2(s)) < -\delta$. If $\overline{H}(s) > -\frac{\delta}{2} m^{-1/2}$ and $|\overline{Y}(s)| < \frac{\delta}{8} m^{-1/2} c^{-1}$ in addition, then for any $m \le 1 \wedge \left( \frac{\delta}{4} c^{-2} (M_1 + M_2)^{-1} \right)^2$, we have that

$$|\overline{P_k}(s)| \ge \frac{\delta}{16} m^{-1/2} c^{-1}.$$

Therefore,

$$\begin{aligned}
&E^{\kappa_m} \Big[ \sup_{t \in [0, T \wedge \sigma]} \frac{1_{\{|Q_1(t) - Q_2(t)| \in (r_1 + \varepsilon_2, r_2 - \varepsilon_2)\}}}{(1 + M_k^{-2} c^{-2} |\overline{P_k}(u)|^2)^{1/2}} \Big] \\
&\le \Big( 1 + M_k^{-2} c^{-2} \Big( \frac{\delta}{16} m^{-1/2} c^{-1} \Big)^2 \Big)^{-1/2} + \kappa_m \Big( \sup_{t \in [0, T \wedge \sigma]} |\overline{H}(t)| > \frac{\delta}{2} m^{-1/2} \Big) \\
&\qquad + \kappa_m \Big( \sup_{t \in [0, T \wedge \sigma]} |\overline{Y}(t)| > \frac{\delta}{8} m^{-1/2} c^{-1} \Big) \\
&\le \Big( 1 + M_k^{-2} c^{-2} \Big( \frac{\delta}{16} m^{-1/2} c^{-1} \Big)^2 \Big)^{-1/2} \\
&\qquad + \Big( \frac{\delta}{2} m^{-1/2} \Big)^{-2} E^{\kappa_m} \Big[ \sup_{t \in [0, T \wedge \sigma]} |\overline{H}(t)|^2 \Big]
\end{aligned}$$

$$+ \Big(\frac{\delta}{8} m^{-1/2} c^{-1}\Big)^{-2} E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]} |\overline{Y}(t)|^2\Big].$$

This combined with Lemmas 5.4 and 5.7 implies our assertion. □

PROOF OF LEMMA 8.1. For any $\varepsilon_1, \varepsilon_2 > 0$ and $k \in \{1,2\}$, we have that the following holds for any $K_1, K_2 > 0$.

$$\begin{aligned}
&\kappa_m\Big(\int_{0+}^{T\wedge\sigma} m^{-1/2}|g_{k1}(t)| 1_{\{|Q_1(t)-Q_2(t)|\in(r_1+\varepsilon_2, r_2-\varepsilon_2)\}} dt > \varepsilon_1\Big) \\
&\le \kappa_m\Big(\int_{0+}^{T\wedge\sigma} m^{-1/2} \frac{1_{\{|Q_1(t)-Q_2(t)|\in(r_1+\varepsilon_2, r_2-\varepsilon_2)\}}}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(t-)|^2)^{1/2}} dt > K_1\Big) \\
&\quad + \kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]} \Big(1+M_k^{-2}c^{-2}|\overline{R_k}(t)|^2\Big) > K_2\Big) \\
&\quad + \kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]} \frac{1_{\{|Q_1(t)-Q_2(t)|\in(r_1+\varepsilon_2, r_2-\varepsilon_2)\}}}{1+M_k^{-2}c^{-2}|\overline{P_k}(t-)|^2} > \frac{\varepsilon_1}{\|\nabla_k \widetilde{U}\|_\infty K_1 K_2}\Big) \\
&\le \frac{1}{K_1} E^{\kappa_m}\Big[\int_{0+}^{T\wedge\sigma} m^{-1/2} \frac{1_{\{|Q_1(t)-Q_2(t)|\in(r_1+\varepsilon_2, r_2-\varepsilon_2)\}}}{M_k(1+M_k^{-2}c^{-2}|\overline{P_k}(t-)|^2)^{1/2}} dt\Big] \\
&\quad + \frac{1}{K_2}\Big(1+M_k^{-2}c^{-2} E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]} |\overline{R_k}(t)|^2\Big]\Big) \\
&\quad + \frac{\|\nabla_k \widetilde{U}\|_\infty K_1 K_2}{\varepsilon_1} E^{\kappa_m}\Big[\sup_{t\in[0,T\wedge\sigma]} \frac{1_{\{|Q_1(t)-Q_2(t)|\in(r_1+\varepsilon_2, r_2-\varepsilon_2)\}}}{1+M_k^{-2}c^{-2}|\overline{P_k}(t-)|^2}\Big].
\end{aligned}$$

By Lemmas 5.17 and 5.7, the first two expectations on the right hand side above are bounded for $m \in (0,1]$. Also, the last expectation on the right hand side above converges to 0 as $m \to 0$ by Lemma 8.2. Therefore, by taking first $K_1, K_2 > 0$ large enough and then $m \to 0$, we get that

$$\lim_{m\to 0} \kappa_m\Big(\int_{0+}^{T\wedge\sigma} m^{-1/2}|g_{k1}(t)| 1_{\{|Q_1(t)-Q_2(t)|\in(r_1+\varepsilon_2, r_2-\varepsilon_2)\}} dt > \varepsilon_1\Big) = 0$$

for any $\varepsilon_1 > 0$.

Similarly, we have for any $K_1, K_2 > 0$ that

$$\kappa_m\Big(\int_{0+}^{T\wedge\sigma} m^{-1/2}|g_{k2}(t)| 1_{\{|Q_1(t)-Q_2(t)|\in(r_1+\varepsilon_2, r_2-\varepsilon_2)\}} dt > \varepsilon_1\Big)$$

$$\leq \kappa_m\Big(\int_{0+}^{T\wedge\sigma} m^{-1/2}\frac{1_{\{|Q_1(t)-Q_2(t)|\in(r_1+\varepsilon_2,r_2-\varepsilon_2)\}}}{1+M_k^{-2}c^{-2}|\overline{P_k}(t-)|^2}dt > K_1\Big)$$
$$+\kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]}|\overline{R_k}(t)|^2 > K_2\Big)$$
$$+\kappa_m\Big(\sup_{t\in[0,T\wedge\sigma]}\frac{1}{(1+M_k^{-2}c^{-2}|\overline{P_k}(t-)|^2)^{1/2}} > \frac{\varepsilon_1}{K_k^{-3}c^{-2}\|\nabla_k\widetilde{U}\|_\infty K_1K_2}\Big).$$

So by exactly the same way as above, we get our assertion for $g_{k2}$. □

### 8.2. Behavior at boundaries

First, we prove that $|V_1(t)| = |V_2(t)| = c$ when $|Q_1(t) - Q_2(t)| = r_1$. It suffices to prove the following.

Lemma 8.3.

$$P_\infty\Big(\int_0^{T\wedge\sigma}(c^2-|V_k(t)|^2)1_{\{|Q_1(t)-Q_2(t)|\in(r_1-\varepsilon,r_1+\varepsilon)\}}dt = 0\Big) = 1.$$

Proof. By assumption (T1), we have that $|\nabla_k\widetilde{U}(q_1,q_2)| = |U'(|q_1 - q_2|)| > 0$ if $|q_1 - q_2| = r_1$. So there exists a function $g \in C_b^1(\mathbf{R}^d)$ and a constant $\delta > 0$ such that $\delta 1_{\{|q_1-q_2|\in(r_1-\varepsilon,r_1+\varepsilon)\}} \leq g(q_1 - q_2) \leq |\nabla_k\widetilde{U}(q_1,q_2)|$. Therefore, for any $a > 0$, we have that

$$P_\infty\Big(\int_0^{T\wedge\sigma}(c^2-|V_k(t)|^2)1_{\{|Q_1(t)-Q_2(t)|\in(r_1-\varepsilon,r_1+\varepsilon)\}}dt > a\Big)$$
$$\leq \delta^{-1}P_\infty\Big(\int_0^{T\wedge\sigma}(c^2-|V_k(t)|^2)g(Q_1(t)-Q_2(t))dt > a\Big)$$
$$\leq \delta^{-1}\lim_{n\to\infty}\kappa_{m_n}\Big(\int_0^{T\wedge\sigma}(c^2-|V_k(t)|^2)g(Q_1(t)-Q_2(t))dt > a\Big)$$
$$\leq \delta^{-1}\lim_{n\to\infty}a^{-1}m_n^{1/2}E^{\kappa_{m_n}}\Big[m_n^{-1/2}\int_0^{T\wedge\sigma}(c^2-|V_k(t)|^2)$$
$$\times|\nabla_k\widetilde{U}(Q_1(t),Q_2(t))|dt\Big].$$

The expectation on the right hand side above is bounded for $n \in \mathbf{N}$ by Lemma 5.20, so

$$P_\infty\Big(\int_0^{T\wedge\sigma}(c^2-|V_k(t)|^2)1_{\{|Q_1(t)-Q_2(t)|\in(r_1-\varepsilon,r_1+\varepsilon)\}}dt > a\Big) = 0, \qquad a > 0.$$

Therefore,

$$P_\infty\Big(\int_0^{T\wedge\sigma}(c^2-|V_k(t)|^2)1_{\{|Q_1(t)-Q_2(t)|\in(r_1-\varepsilon,r_1+\varepsilon)\}}dt=0\Big)=1.\ \square$$

Finally, we prove that $P_\infty$ satisfies ($\mu5$) by proving Lemmas 8.4 $\sim$ 8.6 given below. Indeed, it is easy to be seen that Lemma 8.6 implies (1) of ($\mu5$) and Lemma 8.5 implies (3); also, Lemma 8.4 implies (2): under $P_\infty$, if $|Q_1(t)-Q_2(t)|=r_2$ and $H(t)<H_0(t)$ for some $t$, then there exists a $\delta>0$ such that $H(s)<H_0(s)$ and $|Q_1(s)-Q_2(s)|\in(r_2-\varepsilon,r_2+\varepsilon)$ for any $s\in(t-\delta,t+\delta)$, therefore, by Lemma 8.4, $|V_k(s)|=c$ for any $s\in(t-\delta,t+\delta)$, hence $|Q_1(s)-Q_2(s)|\in(r_1,r_2)$. This combined with ($\mu4$) implies that $V_k(t)=\pm c\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}$. Since $|Q_1(t)-Q_2(t)|=r_2$ and $|Q_1(s)-Q_2(s)|<r_2$ for $s\in(t,t+\delta)$, this combined with the right-continuity of $V_k$ implies that $V_1(t)=-V_2(t)=c\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}$.

Let $H_0(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ be the one defined in Lemma 7.1. We prove the following three lemmas.

Lemma 8.4. *For any $k\in\{1,2\}$, we have that*

$$P_\infty\Big(\int_0^{T\wedge\sigma}1_{\{H(t)<H_0(R_1(t),R_2(t),Y(t),Q_1(t),Q_2(t))\}}\times 1_{\{|Q_2(t)-Q_1(t)|\in(r_2-\varepsilon,r_2+\varepsilon)\}}(c^2-|V_k(t)|^2)dt=0\Big)=1.$$

Lemma 8.5. *We have $P_\infty$-almost surely that if $|Q_1(t)-Q_2(t)|=r_2$, $(Q_1(t)-Q_2(t))\cdot(V_1(t-)-V_2(t-))>0$ and $H(t)>H_0(R_1(t),R_2(t),Y(t),Q_1(t),Q_2(t))$, then $V_k(t)=\frac{P_k(t)}{M_k\sqrt{1+M_k^{-2}c^{-2}|P_k(t)|^2}}$, $k=1,2$ with $\vec{P}(t)=(P_1(t),P_2(t))$ given by $\vec{P}(t)=\vec{P}(R_1(t),R_2(t),Y(t),H(t),Q_1(t),Q_2(t))$.*

Lemma 8.6. *We have $P_\infty$-almost surely that if $|Q_1(t)-Q_2(t)|=r_2$ and $(Q_1(t)-Q_2(t))\cdot(V_1(t-)-V_2(t-))<0$, then $V_1(t)=-V_2(t)=c\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}$.*

We prove Lemmas 8.4 $\sim$ 8.6 in the following. For the sake of simplicity, we write $H_0(\vec{R}(t),Y(t),\vec{Q}(t))$ as $H_0(t)$ in our proofs.

PROOF OF LEMMA 8.4. Let $\varepsilon > 0$ be a constant such that $U'(x) < 0$ as long as $x \in (r_2 - 2\varepsilon, r_2)$. So $|\nabla \widetilde{U}(q)| > 0$ as long as $|q| \in (r_2 - 2\varepsilon, r_2)$. It suffices to prove that

$$P_\infty\Big( \int_0^{T\wedge\sigma} 1_{\{H(t)<H_0(t)-2\varepsilon_1\}} 1_{\{|Q_2(t)-Q_1(t)|\in(r_2-\varepsilon,r_2+\varepsilon)\}} \times (c^2 - |V_k(t)|^2)dt > \varepsilon_3\Big) = 0$$

for any $\varepsilon_1, \varepsilon_3 > 0$.

Choose $f_1 \in C_b^\infty(\mathbf{R})$ and $f_2 \in C_0^\infty(\mathbf{R}^d)$ such that

$$1_{\{x>2\varepsilon_1\}} \le f_1(x) \le 1_{\{x>\varepsilon_1\}},$$
$$1_{\{|x|\in(r_2-\varepsilon,r_2+\varepsilon)\}} \le f_2(x) \le 1_{\{|x|\in(r_2-2\varepsilon,r_2+2\varepsilon)\}}.$$

Since $\int_0^{T\wedge\sigma} f_1(H_0(t) - H(t)) f_2(Q_1(t) - Q_2(t))(c^2 - |V_k(t)|^2)dt$ is continuous with respect to $(R_1, R_2, H, Y, Q_1, Q_2, V_k)$, we have that

$$\begin{aligned}
&P_\infty\Big( \int_0^{T\wedge\sigma} 1_{\{H(t)<H_0(t)-2\varepsilon_1\}} 1_{\{|Q_1(t)-Q_2(t)|\in(r_2-\varepsilon,r_2+\varepsilon)\}} \times (c^2 - |V_k(t)|^2)dt > \varepsilon_3\Big)\\
&\le P_\infty\Big( \int_0^{T\wedge\sigma} f_1(H_0(t) - H(t)) f_2(Q_1(t) - Q_2(t))(c^2 - |V_k(t)|^2)dt > \varepsilon_3\Big)\\
&\le \lim_{n\to\infty} \kappa_{m_n}\Big( \int_0^{T\wedge\sigma} f_1(H_0(t) - H(t)) f_2(Q_1(t) - Q_2(t)) \times (c^2 - |V_k(t)|^2)dt > \varepsilon_3\Big).
\end{aligned}$$

Therefore, in order to get our assertion, it suffices to prove that

$$\lim_{m\to 0} P\Big( \int_0^{T\wedge\sigma} f_1(H_0^{(m)}(t) - H^{(m)}(t)) f_2(Q_1^{(m)}(t) - Q_2^{(m)}(t)) \times (c^2 - |V_k^{(m)}(t)|^2)dt > \varepsilon_3\Big) = 0. \tag{8.1}$$

We prove this in the following.

Notice that for any $m \in (0, 1]$, we have that $H^{(m)}(t) = \sum_{j=1}^2 \sqrt{1 + |P_j^{(m)}(t)|^2} + m^{-1/2}\widetilde{U}(\vec{Q}^{(m)}(t)) \ge H_0(\vec{R^{(m)}}(t), Y^{(m)}(t), \vec{Q^{(m)}}(t)) +$

$m^{-1/2}\widetilde{U}(\vec{Q^{(m)}}(t))$, therefore, if $H^{(m)}(t) < H_0(\vec{R^{(m)}}(t), Y^{(m)}(t), \vec{Q^{(m)}}(t))$, then $\widetilde{U}(\vec{Q^{(m)}}(t)) < 0$, this combined with $t \le \sigma$ implies that $|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_1, r_2)$. so if $|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - 2\varepsilon, r_2 + 2\varepsilon)$ in addition, we get that $|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - 2\varepsilon, r_2)$. So for any $\delta \in (0, 2\varepsilon)$ and $m \in (0,1]$, we have that

$$
\begin{aligned}
&P\Big(\int_0^{T\wedge\sigma} f_1(H_0^{(m)}(t) - H^{(m)}(t)) f_2(Q_1^{(m)}(t) - Q_2^{(m)}(t)) \\
&\qquad\qquad \times (c^2 - |V_k^{(m)}(t)|^2)dt > \varepsilon_3\Big) \\
&\le P\Big(\int_0^{T\wedge\sigma} 1_{\{|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - 2\varepsilon, r_2 - \delta]\}} (c^2 - |V_k^{(m)}(t)|^2)dt > \varepsilon_3/2\Big) \\
&\quad + P\Big(\int_0^{T\wedge\sigma} 1_{\{H^{(m)}(t) < H_0^{(m)}(t) - \varepsilon_1\}} 1_{\{|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - \delta, r_2)\}} \\
&\qquad\qquad \times (c^2 - |V_k^{(m)}(t)|^2)dt > \varepsilon_3/2\Big).
\end{aligned}
\tag{8.2}
$$

For the first term on the right hand side of (8.2), we have that

$$
\begin{aligned}
&P\Big(\int_0^{T\wedge\sigma} 1_{\{|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - 2\varepsilon, r_2 - \delta]\}} (c^2 - |V_k^{(m)}(t)|^2)dt > \varepsilon_3/2\Big) \\
&\le (\varepsilon_3/2)^{-1} E\Big[\int_0^{T\wedge\sigma} 1_{\{|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - 2\varepsilon, r_2 - \delta]\}} (c^2 - |V_k^{(m)}(t)|^2)dt\Big],
\end{aligned}
$$

which, by Lemma 5.17, converges to 0 as $m \to 0$ for any $\delta > 0$.

So in order to get our assertion, it suffices to prove that

$$
\begin{aligned}
&\lim_{\delta\to 0} \sup_{m\in(0,1]} P\Big(\int_0^{T\wedge\sigma} 1_{\{H^{(m)}(t) < H_0^{(m)}(t) - \varepsilon_1\}} 1_{\{|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - \delta, r_2)\}} \\
&\qquad\qquad \times (c^2 - |V_k^{(m)}(t)|^2)dt > \varepsilon_3/2\Big) = 0.
\end{aligned}
\tag{8.3}
$$

Let us first prepare the following fact:

$$
\frac{m^{-1/2}|\widetilde{U}(\vec{Q^{(m)}}(t))|}{H^{(m)}(t) + m^{-1/2}|\widetilde{U}(\vec{Q^{(m)}}(t))|} \ge \frac{\varepsilon_1}{H_0^{(m)}(t)}
$$

if $|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - \delta, r_2)$ and $H^{(m)}(t) < H_0^{(m)}(t) - \varepsilon_1$. Indeed, since $|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - \delta, r_2)$ and $\delta < \varepsilon$, we have that

$$
H^{(m)}(t) + m^{-1/2}|\widetilde{U}(\vec{Q^{(m)}}(t))| = H^{(m)}(t) - m^{-1/2}\widetilde{U}(\vec{Q^{(m)}}(t))
$$

$$
\begin{aligned}
&= \sum_{j=1}^{2} M_j c^2 \sqrt{1 + M_j^{-2} c^{-2} |P_j^{(m)}(t)|^2} \\
&\geq H_0^{(m)}(t) > 0,
\end{aligned}
$$

combining this with the assumption $H^{(m)}(t) < H_0^{(m)}(t) - \varepsilon_1$, we get that

$$
\frac{H^{(m)}(t)}{H^{(m)}(t) + m^{-1/2}|\widetilde{U}(\vec{Q^{(m)}}(t))|} \leq \frac{H_0^{(m)}(t) - \varepsilon_1}{H_0^{(m)}(t)}.
$$

Subtracting the both sides of the above from 1, we get our assertion.

Let $a(\delta) := \inf_{x \in (r_2 - \delta, r_2)} \frac{|U'(x)|}{|U(x)|}$. Then by our assumption (T1), we have that $a(\delta) \to \infty$ as $\delta \to 0$. Also,

$$
\begin{aligned}
H^{(m)}(t) + m^{-1/2}|\widetilde{U}(\vec{Q}^{(m)}(t))| &= \sum_{j=1}^{2} M_j c^2 \sqrt{1 + M_j^{-2} c^{-2} |P_j^{(m)}(t)|^2} \\
\geq M_k c^2 \sqrt{1 + M_k^{-2} c^{-2} |P_k^{(m)}(t)|^2} &= M_k c^3 (c^2 - |V_j^{(m)}(t)|^2)^{-1/2}.
\end{aligned}
$$

Therefore, since $|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - \delta, r_2)$, we have that

$$
\begin{aligned}
& m^{-1/2} |\nabla_k \widetilde{U}(\vec{Q^{(m)}}(t))| (c^2 - |V_k^{(m)}(t)|^2)^{1/2} \\
&\geq \frac{|\nabla_k \widetilde{U}(\vec{Q^{(m)}}(t))|}{|\widetilde{U}(\vec{Q^{(m)}}(t))|} \cdot \frac{m^{-1/2}|\widetilde{U}(\vec{Q^{(m)}}(t))|}{H^{(m)}(t) + m^{-1/2}|\widetilde{U}(\vec{Q}^{(m)}(t))|} \cdot M_k c^3 \\
&\geq a(\delta) \frac{\varepsilon_1 M_k c^3}{H_0^{(m)}(t)} 1_{\{H^{(m)}(t) < H_0^{(m)}(t) - \varepsilon_1\}} 1_{\{|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - \delta, r_2)\}},
\end{aligned}
$$

hence

$$
\begin{aligned}
& 1_{\{H^{(m)}(t) < H_0^{(m)}(t) - \varepsilon_1\}} 1_{\{|Q_1^{(m)}(t) - Q_2^{(m)}(t)| \in (r_2 - \delta, r_2)\}} (c^2 - |V_k^{(m)}(t)|^2) \\
&\leq \frac{1}{a(\delta)} \frac{H_0^{(m)}(t)}{\varepsilon_1 M_k c^3} m^{-1/2} |\nabla_k \widetilde{U}(\vec{Q^{(m)}}(t))| (c^2 - |V_k^{(m)}(t)|^2)^{3/2} \\
&= \frac{1}{a(\delta)} \frac{H_0^{(m)}(t)}{\varepsilon_1 M_k c^3} m^{-1/2} |\nabla_k \widetilde{U}(\vec{Q^{(m)}}(t))| \frac{c^3}{(1 + M_k^{-2} c^{-2} |P_k^{(m)}(t)|^2)^{3/2}}.
\end{aligned}
$$

Therefore, for any constant $A > 0$, we have that

$$
\begin{aligned}
&\sup_{m\in(0,1]} P\Big(\int_0^{T\wedge\sigma} 1_{\{H^{(m)}(t)<H_0^{(m)}(t)-\varepsilon_1\}} 1_{\{|Q_1^{(m)}(t)-Q_2^{(m)}(t)|\in(r_2-\delta,r_2)\}} \\
&\qquad\qquad \times (c^2-|V_k^{(m)}(t)|^2)dt > \varepsilon_3/2\Big) \\
&\le \sup_{m\in(0,1]} P\Big(\int_0^{T\wedge\sigma} H_0^{(m)}(t) m^{-1/2} |\nabla_k \widetilde{U}(\vec{Q^{(m)}}(t))| \\
&\qquad\qquad \times \frac{dt}{(1+M_k^{-2}c^{-2}|P_k^{(m)}(t)|^2)^{3/2}} > \frac{\varepsilon_1\varepsilon_3 M_k a(\delta)}{2}\Big) \\
&\le \frac{1}{A} \sup_{m\in(0,1]} E\Big[\sup_{t\in[0,T]} H_0^{(m)}(t)\Big] \\
&\qquad + \frac{2A}{\varepsilon_1\varepsilon_3 M_k a(\delta)} \sup_{m\in(0,1]} E\Big[\int_0^{T\wedge\sigma} \frac{m^{-1/2}|\nabla_k \widetilde{U}(\vec{Q^{(m)}}(t))|}{(1+M_k^{-2}c^{-2}|P_k^{(m)}(t)|^2)^{3/2}} dt\Big].
\end{aligned}
$$

The last expectation on the right hand side above is bounded for $m \in (0,1]$ by Lemma 5.20, and $a(\delta) \to \infty$ as $\delta \to 0$, so the second term on the right side above converges to 0 as $\delta \to 0$ for any $A$ fixed. Also, by Lemmas 7.2, 5.7 and 5.1, we have that

$$
\sup_{m\in(0,1]} E\Big[\sup_{t\in[0,T]} H_0^{(m)}(t)\Big] < \infty,
$$

hence the first term on the right hand side above converges to 0 as $A \to \infty$. Combining the above, we get (8.3). This completes the proof of our assertion. □

Before proving Lemmas 8.5 and 8.6, let us make some preparation. First, to simplify notations, for any $a, b \in \mathbf{R}^d$, let $x_a b := b \cdot \frac{a}{|a|}$, the coordinate of $b$ in $a$-direction. For any $a_1, a_2, d \in \mathbf{R}$, let $g_{a_1,a_2,d}$ be as defined in the proof of Lemma 7.1. Let $r_3 \in (r_1, r_2)$ be any constant such that $U'(r) > 0$ as long as $|r| \in (r_3, r_2)$. Here $U$ is the function such that $\widetilde{U}(q_1, q_2) = U(|q_1 - q_2|)$, as defined before. We have that $\inf\Big\{|g'_{a_1,a_2,d}(x)|; g_{a_1,a_2,d}(x) \ge b_0(a_1,a_2,d) + \varepsilon_1\Big\}$ is strictly positive and continuous with respect to $(a_1, a_2, d)$. Therefore, for any $A > 0$ and $\varepsilon > 0$, we have that

$$
\delta(A,\varepsilon) := \inf\Big\{\inf\Big\{|g'_{a_1,a_2,d}(x)|; g_{a_1,a_2,d}(x) \ge b_0(a_1,a_2,d) + \varepsilon_1\Big\};
$$

$$\times\, |a_1| \le A, |a_2| \le A, |d| \le A\Big\} \wedge \frac{r_2}{r_1} > 0.$$

For any $t_1 < t_2 < t_3$ and $\varepsilon, \delta, A > 0$, we define several conditions as follows.

($\mathrm{H1}_{t_1,t_3,\varepsilon}$) The following holds for any $s \in [t_1, t_3]$:

$$H(s) > H_0(s) + \varepsilon, |Q_2(s) - Q_1(s)| > r_3, \tag{8.4}$$

($\mathrm{H2}_{t_1,t_2,\delta}$) $|Q_2(t_2) - Q_1(t_2)|^2 - |Q_2(t_1) - Q_1(t_1)|^2 > 2r_1\delta(t_2 - t_1)$,

($\mathrm{H3}_{t_2,t_3,\delta}$) $|Q_2(t_3) - Q_1(t_3)|^2 - |Q_2(t_2) - Q_1(t_2)|^2 < 2r_1\delta(t_3 - t_2)$,

($\mathrm{H4}_{t_1,t_2,\delta}$) $|Q_2(t_2) - Q_1(t_2)|^2 - |Q_2(t_1) - Q_1(t_1)|^2 < -2r_1\delta(t_2 - t_1)$,

($\mathrm{H5}_{t_2,t_3,\delta}$) $|Q_2(t_3) - Q_1(t_3)|^2 - |Q_2(t_2) - Q_1(t_2)|^2 > -2r_1\delta(t_3 - t_2)$,

($\mathrm{H6}_A$) The following holds for any $s \in [0, T]$:

$$|R_1(s)| < A, |R_2(s)| < A, |x_{Q_2(s)-Q_1(s)}Y(s)| < A. \tag{8.5}$$

Also, define

$$\begin{aligned}
G_1(\varepsilon, \delta) := \Big\{\omega :\ & \text{there exist } t_1, t_2, t_3 \in [0, T \wedge \sigma], \text{ such that} \\
& t_1 < t_2 < t_3 \text{ and } (\mathrm{H1}_{t_1,t_3,\varepsilon}), (\mathrm{H2}_{t_1,t_2,\delta}) \\
& \text{and } (\mathrm{H3}_{t_2,t_3,\delta}) \text{ are satisfied}\Big\}, \\
G_2(\varepsilon, \delta) := \Big\{\omega :\ & \text{there exist } t_1, t_2, t_3 \in [0, T \wedge \sigma], \text{ such that} \\
& t_1 < t_2 < t_3 \text{ and } (\mathrm{H1}_{t_1,t_3,\varepsilon}), (\mathrm{H4}_{t_1,t_2,\delta}) \\
& \text{and } (\mathrm{H5}_{t_2,t_3,\delta}) \text{ are satisfied}\Big\}, \\
G_A := \Big\{\omega :\ & (\mathrm{H6}_A) \text{ is satisfied}\Big\}.
\end{aligned}$$

Then $G_1(\varepsilon, \delta)$, $G_2(\varepsilon, \delta)$ and $G_A$ are open sets, and the following holds.

Lemma 8.7. *We have for any $\delta \in (0, \delta(A, \varepsilon)]$ that*

(1) $P_\infty\Big(G_1(\varepsilon, \delta) \cap G_A\Big) = 0$,

(2) $P_\infty\Big(G_2(\varepsilon,\delta)\cap G_A\Big)=0.$

Proof. We give the proof of the first assertion. The second one can be gotten in exactly the same way.

Since $G_1(\varepsilon,\delta)\cap G_A$ is an open set, and $\mu_{m_n}$ converges to $P_\infty$ weakly, it suffices to prove the assertion with $P_\infty$ substituted by $\mu_m$ for any $m\in(0,1]$. We do it in the following. Fix any $m\in(0,1]$.

Notice that for any $s\geq 0$, if both (8.4) and (8.5) are satisfied, then we have that under $\kappa_m$,

$$\begin{aligned}
&g_{|R_1(s)|,|R_2(s)|,x_{Q_2(s)-Q_1(s)}Y(s)}(x_{Q_2(s)-Q_1(s)}P_1(s))\\
&=\sum_{j=1}^{2}M_jc^2\sqrt{1+M_j^{-2}c^{-2}|P_j(s)|^2}\\
&\geq H(s)>H_0(s)+\varepsilon=b_0\Big(|R_1(s)|,|R_2(s)|,x_{Q_2(s)-Q_1(s)}Y(s)\Big)+\varepsilon,
\end{aligned}$$

therefore,

$$\begin{aligned}
&\Big|\pi_{Q_2(s)-Q_1(s)}(V_2(s)-V_1(s))\Big|^2\\
&=\Big|\frac{\pi_{Q_2(s)-Q_1(s)}Y(s)-\pi_{Q_2(s)-Q_1(s)}P_1(s)}{\sqrt{1+|R_2(s)|^2+|\pi_{Q_2(s)-Q_1(s)}Y(s)-\pi_{Q_2(s)-Q_1(s)}P_1(s)|^2}}\\
&\qquad-\frac{\pi_{Q_2(s)-Q_1(s)}P_1(s)}{\sqrt{1+|R_2(s)|^2+|\pi_{Q_2(s)-Q_1(s)}P_1(s)|^2}}\Big|^2\\
&=\Big|g'_{|R_1(s)|,|R_2(s)|,x_{Q_2(s)-Q_1(s)}Y(s)}(x_{Q_2(s)-Q_1(s)}P_1(s))\Big|^2\\
&\geq\delta(A,\varepsilon)^2,
\end{aligned}$$

hence $\Big\{(Q_2(s)-Q_1(s))\cdot(V_2(s)-V_1(s))\Big\}^2=|Q_2(s)-Q_1(s)|^2\Big|x_{Q_2(s)-Q_1(s)}\cdot(V_2(s)-V_1(s))\Big|^2>r_1^2\delta(A,\varepsilon_1)^2$. In conclusion, we have that under $\mu_m$,

$$(8.4)+(8.5)\Rightarrow\Big\{(Q_2(s)-Q_1(s))\cdot(V_2(s)-V_1(s))\Big\}^2>r_1^2\delta(A,\varepsilon_1)^2.$$

Therefore, we have as a consequence of ($\mathrm{H1}_{t_1,t_3,\varepsilon}$) + ($\mathrm{H6}_A$) that

$$\Big\{(Q_2(s)-Q_1(s))\cdot(V_2(s)-V_1(s))\Big\}^2>r_1^2\delta(A,\varepsilon_1)^2,\qquad s\in[t_1,t_3]. \tag{8.6}$$

On the other hand, if ($\mathrm{H2}_{t_1,t_2,\delta}$) and ($\mathrm{H3}_{t_2,t_3,\delta}$) are satisfied, then there exists a $u \in [t_1, t_3]$ such that

$$\frac{d}{du}\Big(|Q_2(u) - Q_1(u)|^2 - 2r_1\delta u\Big) = 0. \tag{8.7}$$

In particular,

$$(Q_2(u) - Q_1(u)) \cdot (V_2(u) - V_1(u)) > 0. \tag{8.8}$$

Since $\vec{Q}$ and $\vec{V}$ are continuous under $\kappa_m$, this combined with (8.6) implies that

$$(Q_2(s) - Q_1(s)) \cdot (V_2(s) - V_1(s)) > r_1\delta(A, \varepsilon), \quad \text{for any } s \in [t_1, t_3],$$

hence

$$\frac{d}{ds}\Big(|Q_2(s) - Q_1(s)|^2 - 2r_1\delta(A, \varepsilon)s\Big) > 0, \quad \text{for any } s \in [t_1, t_3].$$

When $\delta \le \delta(A, \varepsilon)$, this contradicts (8.7). Therefore, $\kappa_m\Big(G_1(\varepsilon, \delta) \cap G_A\Big) = 0$ for any $\delta \in (0, \delta(A, \varepsilon)]$. This completes the proof of our assertion. □

PROOF OF LEMMA 8.5. Write $H_0(t) := H_0(R_1(t), R_2(t), Y(t), Q_1(t), Q_2(t))$. It suffices to prove the assertion with the condition $H(t) > H_0(t)$ substituted by $H(t) > H_0(t) + \varepsilon$ for any $\varepsilon > 0$. Fix any $\varepsilon > 0$ through this proof.

Assume that under $\kappa_m$, we have $|Q_1(t) - Q_2(t)| = r_2$, $H(t) > H_0(t) + \varepsilon$ and $(Q_1(t) - Q_2(t)) \cdot (V_1(t-) - V_2(t-)) > 0$ for some $t \in [0, T \wedge \sigma)$. Then there exists a $\delta > 0$ such that $H(s) > H_0(s) + \varepsilon$ and $|Q_1(s) - Q_2(s)| > r_3$ for any $s \in [t - \delta, t + \delta]$, *i.e.*, ($\mathrm{H1}_{t-\delta,t+\delta,\varepsilon}$) is satisfied.

Also, since $G_A$ is monotone increasing with respect to $A$ and $\lim_{A\to\infty} P_\infty(G_A) = 1$, there exists an $A > 0$ large enough such that our $\omega$ is in $G_A$.

Finally, re-choosing $\delta > 0$ if necessary, we have by assumption that $V_1(s) = -V_2(s) = -\frac{Q_2(t)-Q_1(t)}{|Q_2(t)-Q_1(t)|}$ for any $s \in [t - \delta, t)$. Without loss of generaliy, we assume that $\delta < r_2 - \frac{1}{2}r_1\delta(A, \varepsilon)$. So for any $s \in [t - \delta, t)$, we have that

$$Q_1(s) = Q_1(t) + (t - s)\frac{Q_2(t) - Q_1(t)}{|Q_2(t) - Q_1(t)|},$$

$$Q_2(s) = Q_2(t) - (t-s)\frac{Q_2(t) - Q_1(t)}{|Q_2(t) - Q_1(t)|},$$

hence

$$\begin{aligned}
&|Q_2(t) - Q_1(t)|^2 - |Q_2(s) - Q_1(s)|^2 \\
&= \Big(1 - \Big(1 - \frac{2(t-s)}{r_2}\Big)^2\Big) r_2^2 \\
&= 4(t-s)\Big(r_2 - (t-s)\Big) > 4(t-s)(r_2 - \delta) \\
&\geq 2r_1\delta(A,\varepsilon)(t-s),
\end{aligned}$$

where when passing to the last line, we used the fact that $\delta < r_2 - \frac{1}{2}r_1\delta(A,\varepsilon)$. Therefore, $(\mathrm{H2}_{s,t,\delta})$ is also satisfied for any $s \in [t-\delta, t)$.

On the other hand, for any $\delta \in (0, \delta(A,\varepsilon)]$, we have by Lemma 8.7 (1) that $P_\infty(G_1(\varepsilon,\delta) \cap G_A) = 0$, therefore, under our condition, for any $t_3 \in [t, t+\delta)$, we have $P_\infty$-almost surely that $|Q_2(t_3) - Q_1(t_3)|^2 - |Q_2(t) - Q_1(t)|^2 > 2r_1\delta(t_3 - t)$, in particular, $|Q_2(t_3) - Q_1(t_3)| > r_2$, hence $\vec{V}(t_3) = P(\vec{R}(t_3), Y(t_3), H(t_3), \vec{Q}(t_3))$. Since $\vec{V}$ is right-continuous, and $(\vec{R}, Y, H, \vec{Q})$ is continuous, taking $t_3 \to t+$, we get our assertion. □

Proof of Lemma 8.6. First, for any $\varepsilon > 0$, we have that $\inf\Big\{g_{a_1,a_2,d}(x) - b_0(a_1,a_2,d); |g'_{a_1,a_2,d}(x)| > \varepsilon\Big\}$ is continuous with respect to $(a_1,a_2,d)$, and is strictly positive for any given $(a_1,a_2,d)$, therefore,

$$\begin{aligned}
\overline{\delta}(A,\varepsilon) := \inf\Big\{\inf\Big\{g_{a_1,a_2,d}(x) - b_0(a_1,a_2,d); |g'_{a_1,a_2,d}(x)| > \varepsilon\Big\}; \\
|a_1| \leq A, |a_2| \leq A, |d| \leq A\Big\} > 0.
\end{aligned}$$

Also, by a similar calculation as in the proof of Lemma 8.5, it is easy to be seen that for any $s \geq 0$,

$$\begin{aligned}
&|x_{Q_2(s)-Q_1(s)}(V_2(s) - V_1(s))| > \varepsilon, |Q_2(s) - Q_1(s)| > r_2, \\
&\qquad |R_1(s)| \leq A, |R_2(s)| \leq A, |Y(s)| \leq A \\
&\Rightarrow H(s) \geq H_0(s) + \overline{\delta}(A,\varepsilon). \qquad (8.9)
\end{aligned}$$

Now, for any $\varepsilon > 0$, if $|Q_2(t) - Q_1(t)| = r_2$ and $(Q_2(t) - Q_1(t)) \cdot (V_2(t-) - V_1(t-)) < -\varepsilon$ for some $t \in [0, T \wedge \sigma)$, then there exists a $\delta > 0$ such that for any $s \in [t-\delta, t)$, we have

$$(8.10) \qquad \begin{aligned} &|Q_2(s) - Q_1(s)| \in (r_2, 2r_2), \\ &(Q_2(s) - Q_1(s)) \cdot (V_2(s) - V_1(s)) < -\varepsilon, \end{aligned}$$

hence

$$(8.11) \qquad |Q_2(s) - Q_1(s)| > r_2, \quad \Big|\pi_{Q_2(s)-Q_1(s)}(V_2(s) - V_1(s))\Big| > \frac{\varepsilon}{2r_2}.$$

Also, as before, since $\lim_{A\to\infty} P_\infty(G_A) = 1$, there exists an $A$ large enough such that our $\omega$ is in $G_A$. This combined with (8.11) and (8.9) implies that

$$H(s) > H_0(s) + \overline{\delta}(A, \frac{\varepsilon}{2r_2}).$$

This is true for any $s \in [t-\delta, t)$, so by continuity, by re-choosing $\delta > 0$ if necessary, we have that

$$(8.12) \qquad H(s) > H_0(s) + \frac{1}{2}\overline{\delta}(A, \frac{\varepsilon}{2r_2}), \qquad \text{for all } s \in [t-\delta, t+\delta].$$

*i.e.*, $(\mathrm{H1}_{t-\delta, t+\delta, \frac{1}{2}\overline{\delta}(\frac{\varepsilon}{2r_2})})$ is satisfied.

On the other hand, let $\widetilde{\delta}(A, \varepsilon) := \frac{\varepsilon}{r_1} \wedge \delta\Big(A, \frac{1}{2}\overline{\delta}(A, \frac{\varepsilon}{2r_2})\Big)$. Then $\widetilde{\delta}(A, \varepsilon) > 0$, and for any $s \in [t-\delta, t)$, we have by (8.10) that

$$\begin{aligned} & |Q_2(t) - Q_1(t)|^2 - |Q_2(s) - Q_1(s)|^2 \\ = & \int_s^t 2(Q_2(u) - Q_1(u)) \cdot (V_2(u) - V_1(u)) du \\ < & -2\varepsilon(t-s) \\ \leq & -2r_1\widetilde{\delta}(A, \varepsilon)(t-s). \end{aligned}$$

*i.e.*, $(\mathrm{H4}_{t-\delta, t, \widetilde{\delta}(A,\varepsilon)})$ is also satisfied.

Therefore, for any $\delta' \in (0, \widetilde{\delta}(A, \varepsilon)]$, since $P_\infty\Big(G_2(\frac{1}{2}\overline{\delta}(A, \frac{\varepsilon}{2r_2}), \delta')\Big) = 0$ by Lemma 8.7 (2), we have $P_\infty$-almost surely that for any $t_3 \in (t, t+\delta]$, the following holds: $|Q_2(t_3) - Q_1(t_3)|^2 - |Q_2(t) - Q_1(t)|^2 < -2r_1\delta'(t_3 - t)$, in particular, $|Q_2(t_3) - Q_1(t_3)| \in (r_1, r_2)$. By $(\mu 4)$, this implies that $V_1(t_3) = -V_2(t_3) = \frac{Q_2(t_3) - Q_1(t_3)}{|Q_2(t_3) - Q_1(t_3)|}$. Since $\vec{Q}$ is continuous and $\vec{V}$ is right-continuous, taking $t_3 \to t+$ on the both sides, we get our assertion. □

This completes the proof of the fact that $P_\infty$ satisfies $(\mu 1) \sim (\mu 5)$. This combined with the uniqueness of the probability measure that satisfies these conditions completes the proof of Theorem 2.2.

## 9. Examples

In this section, we prove that the class of examples given by Example 1 in Section 2 satisfy the condition (T1) if $\lambda > 0$ is small enough.

Use the same notations as in Example 1. Write $F(u) = \int_0^\infty dt\rho(t)(t-u)^{\frac{d}{2}-2}$. Then by a simple calculation, we have that our function $U$ (recall that $U$ is the function such that $\widetilde{U}(Q) = U(|Q|))$ is given by

$$\begin{aligned} U(q) &= \widetilde{U_\lambda}(q) \\ &= -\int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr \int_0^{g_\lambda((q-x)^2+r^2)} ds \int_0^{g_\lambda(x^2+r^2)} dvF(s+v). \end{aligned}$$

We prove in the following that $\widetilde{U_\lambda}$ satisfies (T1) when $\lambda > 0$ is small enough.

First, we have that

$$q \in [a_1 + a_2, 2a_2) \Longrightarrow \widetilde{U_\lambda}(q) < 0.$$

Indeed, if $g_\lambda((q-x)^2+r^2) \neq 0$, then $|q-x| < a_2$; similarly, if $g_\lambda(x^2+r^2) \neq 0$, then $|x| < a_2$. So if $q \in [a_1 + a_2, 2a_2)$ in addition, then $|q-x| + |x| \geq q \geq a_1 + a_2$, hence $|q-x| > a_1$ and $|x| > a_1$. So $g_\lambda((q-x)^2 + r^2) = \lambda g_1((q-x)^2 + r^2) < 0$ and $g_\lambda(x^2 + r^2) = \lambda g_1(x^2 + r^2) < 0$. Therefore, $\widetilde{U_\lambda}(q) < 0$.

Next, since $2a_1 = a_0 + a_2$, we have that $\widetilde{U_0}(a_0 + a_2) = 0$, hence

$$\begin{aligned} &\lim_{\lambda \to 0+} \frac{1}{\lambda} \widetilde{U_\lambda}(a_0 + a_2) \\ = &- \int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr \int_0^{g_0((q-x)^2+r^2)} ds g_1(x^2 + r^2) F(s) \\ &\qquad - \int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr g_1((q-x)^2 + r^2) \int_0^{g_0(x^2+r^2)} dvF(v) \\ &> 0. \end{aligned}$$

So there exists a $\lambda_0 > 0$ small enough such that $\widetilde{U_\lambda}(a_0 + a_2) > 0$ for any $\lambda \in (0, \lambda_0]$.

Therefore, for any $\lambda \in (0, \lambda_0]$, there exists at least one point $q \in (a_0 + a_2, a_1 + a_2)$ such that $\widetilde{U_\lambda}(q) = 0$. Let $q_\lambda$ be the biggest one, *i.e.*, let

$$q_\lambda := \sup\{q \in (a_0 + a_2, a_1 + a_2); \widetilde{U_\lambda}(q) = 0\} \in (a_0 + a_2, a_1 + a_2).$$

We prove in the following that $\widetilde{U_\lambda}'(q_\lambda) < 0$ for $\lambda > 0$ small enough.

First notice that by definition of $g_\lambda$, we have that

$$\int_0^{g_\lambda((q-x)^2+r^2)} = \int_0^{g_0((q-x)^2+r^2)} + \int_0^{\lambda g_1((q-x)^2+r^2)}.$$

Also, as claimed before, for any $q \in (a_0 + a_2, a_1 + a_2)$, we have that $g_0((q - x)^2 + r^2)$ and $g_0(x^2 + r^2)$ could not be non-zero at the same time, so by a simple calcaulation, we have that

$$\begin{aligned} \widetilde{U_\lambda}(q) = & -2\int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr \int_0^{\lambda g_1((q-x)^2+r^2)} ds \int_0^{g_0(x^2+r^2)} dv F(s+v) \\ & - \int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr \int_0^{\lambda g_1((q-x)^2+r^2)} ds \int_0^{\lambda g_1(x^2+r^2)} dv F(s+v). \end{aligned} \tag{9.1}$$

Since $F(\,\cdot\,)$ is bounded, positive and bounded away from 0, the above implies that there exist constants $C_{q1}, C_{q2} > 0$ (that do not depend on $\lambda$) such that $\widetilde{U_\lambda}(q) \geq C_{q1}\lambda - C_{q2}\lambda^2$ for any $\lambda > 0$. So for any $q \in (a_0 + a_2, a_1 + a_2)$, if $\lambda > 0$ is small enough such that $C_{q1}\lambda - C_{q2}\lambda^2 > 0$, then $\widetilde{U_\lambda}(q) > 0$, hence $q_\lambda \in (q, a_1 + a_2)$. Therefore, we have that

$$\lim_{\lambda \to 0} q_\lambda = a_1 + a_2.$$

Combining this with (9.1) and the fact that $\widetilde{U_\lambda}(q_\lambda) = 0$, (re-choose $\lambda_0 > 0$ if necessary), we get that there exists a constant $C > 0$ that does not depend on $\lambda$ such that

$$\begin{aligned} & -\int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr g_1((q_\lambda - x)^2 + r^2) g_0(x^2 + r^2) \\ & \geq C\lambda, \quad \forall \lambda \in (0, \lambda_0]. \end{aligned} \tag{9.2}$$

Also, by (9.1), we have that

$$\begin{aligned} & \widetilde{U_\lambda}'(q_\lambda) \\ = & -\int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr \int_0^{g_0(x^2+r^2)} dv \lambda \\ & \quad \times g_1'((q_\lambda - x)^2 + r^2) F(v + g_\lambda((q_\lambda - x)^2 + r^2)) 2(q_\lambda - x) \end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr \int_0^{\lambda g_1(x^2+r^2)} dv \\
&\qquad \times g_0'((q_\lambda - x)^2 + r^2) F(v + g_\lambda((q_\lambda - x)^2 + r^2)) 2(q_\lambda - x) \\
&- \int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr \int_0^{\lambda g_1(x^2+r^2)} dv \lambda \\
&\qquad \times g_1'((q_\lambda - x)^2 + r^2) F(v + g_\lambda((q_\lambda - x)^2 + r^2)) 2(q_\lambda - x),
\end{aligned}
$$

so there exist constants $C_1, C_2 > 0$ such that

$$
\widetilde{U_\lambda}'(q_\lambda) \le -C_1 \int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr g_1'((q_\lambda - x)^2 + r^2) g_0(x^2 + r^2)\lambda + C_2 \lambda^2.
$$

So in order to prove that $\widetilde{U_\lambda}'(q_\lambda) < 0$ for $\lambda > 0$ small enough, it suffices to prove that $\int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr g_1'((q_\lambda - x)^2 + r^2) g_0(x^2 + r^2) >> \lambda$ as $\lambda \to 0$. By (9.2), it in turn suffices to prove that

$$
\frac{\int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr g_1'((q_\lambda - x)^2 + r^2) g_0(x^2 + r^2)}{\int_{\mathbf{R}} dx \int_{[0,\infty)} r^{d-2} dr g_1((q_\lambda - x)^2 + r^2) g_0(x^2 + r^2)} \to -\infty, \quad \lambda \to 0.
$$

Since $q_\lambda < a_1 + a_2$ and $q_\lambda \to a_1 + a_2$, this is an easy consequence of the assumption that $g_0 \le 0$, $g_1 \ge 0$ and $\lim_{x \to R_2^2 -} \frac{g_1'(x)}{g_1(x)} = -\infty$.

In a similar way, we get that the last condition in (T1) is also satisfied.

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