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Transparent Boundary Conditions for a Diffusion Problem Modified by Hilfer Derivative

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Abstract. We consider a homogeneous fractional diffusion problem in an infinite reservoir sometimes called a “modified” diffusion equation. The equation involves a (nonlocal in time) memory term in the form of a time fractional derivative (of the Laplacian). For the sake of reducing the computational domain to a bounded one we establish appropriate “artificial” boundary conditions. This is to avoid the effect of reflected waves in case of a “solid” standard boundary. Then, an equivalent problem is studied in this bounded domain. To this end we use the Laplace-Fourier transform, the two-parameter Mittag-Leffler function and some properties of fractional derivatives.

1. Introduction

Of concern is a problem which arises in petroleum engineering. Namely, it consists of a diffusion equation with a fractional time derivative together with an initial data and boundary condition. In fact, as the domain is the whole real axis we assume that the solution vanishes at infinity. More precisely, we consider the problem

$$(1)\qquad \begin{cases} \frac{\partial p}{\partial t} - KD^{\alpha,\beta}\frac{\partial^2 p}{\partial x^2} = f\left(x,t\right), \ -\infty < x < \infty, \ t > 0, \\ p\left(x,0\right) = g\left(x\right), \ -\infty < x < \infty, \\ p\left(x,t\right) \to 0, \ \left|x\right| \to \infty. \end{cases}$$

The function $p = p(x,t)$ is the pressure of the fluid. The forcing function f may be due to wells in the reservoir (or some additional flow) and g is the initial pressure. These functions f and g are assumed to be compactly supported say in $\left(-L, L\right)$, $L > 0$. The coefficient K is the diffusion

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coefficient and involves several constants determined by the nature of the problem. The equation in this problem is the standard heat (or diffusion) equation modified by a fractional derivative introduced by Hilfer [16]. It has been derived by Caputo in [4,30] by modifying Darcy's law [5] for a problem which arises in porous media in case of a fractional derivative in the sense of Caputo which corresponds here to $\beta = 1$. In the present work our fractional derivative is more general and covers as special cases the Caputo fractional derivative (used in [4,30]) and the well-known Riemann-Liouville fractional derivative (see definitions in the next section). For more on fractional derivatives we refer the reader to [20,33,35,37,39].

Fractional differential equations are nowadays extensively investigated by many engineers and scientists because of their ability to model complex phenomena with a certain high accuracy. In fact, fractional calculus has proved to be an excellent tool in the study of many processes. It is widely used in physics, chemistry, control theory, electromagnetic, electrodynamic, aerodynamic, porous media, viscoelasticity, optics, biomedicine, signal processing, heat conduction, neuroscience, economy, ... (see [6,10,12,19,20,26-28,30,33,36,38,40,42-44]).

It has been observed and proved that the use of fractional calculus leads to a good fit of the experimental data and therefore describes better the behavior of many materials (see [3,9,16-18,21,29,34]). Hilfer [16-18] showed that time fractional derivatives are equivalent to infinitesimal generators of generalized time fractional evolutions arising in the transition from microscopic to macroscopic time scales. In [16] Hilfer showed that this transition from ordinary time derivative to fractional time derivative indeed arises in physical problems. The Hilfer's idea on time fractional evolution is presented in detail in Chapter 9 of the book [21], and in [reference 25 there]. Bagley and Torvik [3] used the fractional calculus to generalize the Kelvin-Voigt theory and showed that it has several attractive features.

It is worth noting that, for instance, the connection of the results obtained by solving fractional diffusion and fractional Fokker-Planck equations with those obtained from the continuous time random walk theory, see for example [21]. Other closely related works on the (fractional) wave equation and the (fractional) Langevin equation may be found, for instance, in [2,7,8,12,21,22,29,41].

Unlike the usual integer order derivatives (which are local in time), the fractional derivatives are non-local and this presents an advantage. A fractional derivative is defined as a convolution which involves all the prehistory of the solution. This feature facilitates the description of many hereditary and retarded phenomena and processes. However, its mathematical analysis is often challenging as, in addition to the fact that the memory term is non-local, the kernel in this convolution is neither "regular" nor "integrable". Moreover, the existence of a semi-group is not clear. Even in numerical treatments, small steps and large integration domains lead to huge computations.

When Darcy derived his equation, he assumed that the fluid mass flow rate (velocity) is proportional to the pore pressure (gradient). However, in anomalous media (and non-Newtonian fluids) this diffusion equation is not appropriate [7,9,28,31,34]. The reaction of the system is not instantaneous because of a delay between the flux and the gradient. Anomalous diffusion is characterized by the second moment (mean square displacement) $<x^2> \propto t^\sigma$ ($0 < \sigma < 1$, $\sigma = 1$, $1 < \sigma < 2$, $\sigma = 2$ correspond to sub-diffusion, normal diffusion, super-diffusion and ballistic diffusion, respectively). One is lead then to consider pseudo-differential operators, namely fractional derivatives.

In our case, Caputo [4] noticed that in many situations the diffusivity "constant" varies with time. The permeability of the medium changes with time as a consequence of the diminishing of the size of the pores due to the minerals precipitation in some geothermal areas (as opposed to enlargement of the pore size in case of chemical reactions). Under this observation of delay of the effect of fluid pressure, the permeability may depend on the previous pressure gradients. This is modelled by the insertion of a fractional derivative (there are also other approaches). As a consequence, the diffusivity constant is modified to a "pseudo-diffusivity" constant with dimension $L^2/T^{1-\alpha}$. It depends, in general, on the permeability, viscosity, porosity and compressibility of the fluid. Caputo introduced a fractional derivative (of order between 0 and 1) he introduced earlier and which nowadays bears his name (the Caputo fractional derivative, see definition below). The limiting case when the order of this derivative is equal to zero then we recover the conventional diffusion equation (Darcy's law).

Fractional diffusion equations (when the first time derivative is replaced by a fractional one) and modified fractional diffusion models appear naturally in many applications (see for instance [4,20,32,37,40] and references therein).

We want to reduce the computational domain from $\mathbf{R}$ to $(-L,\ L)$ by considering an "equivalent" problem whose solution is exactly equal to the restriction of the solution of the original problem on $(-L, L)$. It has been observed that in case of the usual Dirichlet or Neumann boundary conditions waves are reflected inside the domain and affect greatly the computations there. To overcome this problem an appropriate "artificial" boundary condition is suggested here. This boundary condition is often called "transparent boundary condition". It is, in general, nonlocal in time (fractional derivative of order half) and space even when the equation does not contain a nonlocal term (fractional derivative in the present case). Nowadays there are numerical methods which deal with such memory terms. Transparent boundary conditions have been first devised for elliptic systems and then for the heat and the Schrödinger equation. One can find many papers in the literature in this regard. Our objective here is to implement transparent boundary conditions for the case of the modified fractional diffusion problem (1) with a generalized fractional derivative, namely the Hilfer fractional derivative introduced in [16,17] (see also [10]). This will extend the works in [13-15]. It is then in line with the work in [11].

Finally we mention that the fractional differential equation

$$^{C}D^{\alpha}p - K\frac{\partial^2 p}{\partial x^2} = f\left(x, t\right)$$

where $^{C}D^{\alpha}$ denotes the Caputo fractional derivative, has been investigated in [8,11,19,28,40] to cite but a few. For works closely related to ours we refer the reader to [38,23-26].

The plan of the paper is as follows: in the next section we present some definitions and results which will be useful in our proofs in the subsequent sections. In Section 3 we determine the reduced problem in a bounded domain. Section 4 contains the proof of the uniqueness of solutions and finally in the last section we find the explicit solution of our reduced problem.

2. Preliminaries

We gather here some material needed later to prove our results. They can be found in [20,33,35,37,39].

Definition 1. The Riemann-Liouville fractional integral of order α of f is defined by

$$\left(I_{0+}^{\alpha} f\ \right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad t > 0, \quad \alpha > 0$$

when the right-hand side exists.

Definition 2. The Riemann-Liouville fractional derivative of order α of f is defined by

$$({}^{RL}D_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha}\ f(s)\, ds, \quad t > 0, \quad 0 < \alpha < 1$$

when the right-hand side exists. Note that

$$({}^{RL}D_{0+}^{\alpha} f)(t) = \frac{d}{dt} \left(I^{1-\alpha} f\right)(t)\,.$$

Definition 3. The Caputo fractional derivative of order α of f is defined by

$$({}^{C}D_{0+}^{\alpha} f)(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s) ds, \quad t > 0, \quad 0 < \alpha < 1$$

(the prime here is for the derivative) when the right-hand side exists. Note that

$$({}^{C}D_{0+}^{\alpha} f)(t) = \left(I^{1-\alpha}\ \frac{d}{dt} f\right)(t)\,.$$

The relationship between these two types of derivatives is given by the following theorem.

Theorem 1. *We have*

$$({}^{RL}D_{0+}^{\alpha} f)(t) = ({}^{C}D_{0+}^{\alpha} f)(t) + \frac{t^{-\alpha}}{\Gamma\ (1-\alpha)} f\left(0^{+}\right), \quad t > 0, \quad 0 < \alpha < 1.$$

Definition 4. The Hilfer fractional derivative of f of order α and type β is defined by

$$(D_{0^+}^{\alpha,\beta} f)(t) = (I^{\beta(1-\alpha)} \frac{d}{dt} I^{(1-\alpha)(1-\beta)} f)(t), \quad 0 < \alpha < 1, \ 0 \leq \beta \leq 1$$

whenever the right hand side exists.

Note that when $\beta = 0$

$$(D_{0^+}^{\alpha,0} \ f)\,(t) = \frac{d}{dt}\left(I^{1-\alpha} \ f\right)(t)$$

which is the Riemann-Liouville fractional derivative (see Definition 2) and when $\beta = 1$

$$(D_{0^+}^{\alpha,1} f)\,(t) = \left(I^{1-\alpha} \ \frac{d}{dt} f\right)(t)$$

which is the Caputo fractional derivative (see Definition 3).

For $0 < \alpha < 1$, the Laplace transforms of these derivatives are given by

$$\mathcal{L}\left[({}^{RL}D_{0^+}^{\alpha} f)(t)\right](s) = s^{\alpha}\mathcal{L}\left[f(t)\right](s) - \left(I_{0^+}^{1-\alpha} f \ \right)\left(0^+\right), \tag{2}$$

$$\mathcal{L}\left[({}^{C}D_{0^+}^{\alpha} f) \ (t)\right](s) = s^{\alpha}\mathcal{L}\left[f(t)\right](s) - s^{\alpha-1} f\left(0^+\right), \tag{3}$$

$$\begin{aligned} \mathcal{L}\left[(D_{0^+}^{\alpha,\beta} f)(t)\right](s) &= s^{\alpha}\mathcal{L}\left[f(t)\right](s) \\ &-s^{\beta(\alpha-1)}\left(I_{0^+}^{(1-\alpha)(1-\beta)} f\right)\left(0^+\right), \ 0 \leq \beta \leq 1, \end{aligned} \tag{4}$$

respectively. It is clear that the difference in these Laplace transforms is in the "initial" data $f\left(0^+\right)$, $\left(I_{0^+}^{1-\alpha} \ f\right)\left(0^+\right)$ and $\left(I_{0^+}^{(1-\alpha)(1-\beta)} f \ \right)\left(0^+\right)$ (this last one is a natural initial data for the Hilfer derivative).

We refer the readers to Hilfer's papers for many applications of this (Hilfer) derivative (like [16-18] and others).

Lemma 2. *Assume that $f(t)$ is continuous on $[0, A]$ for some $A > 0$, then*

$$\lim{}_{t\to 0^+} \ I_{0^+}^{\alpha} f\,(t) = 0, \quad \alpha > 0.$$

PROOF. Let $0 \le t \le A$, then

$$\left| \int_0^t (t-s)^{\alpha-1} f(s)\, ds \right| \le \int_0^t (t-s)^{\alpha-1} |f(s)|\, ds$$
$$\le M \int_0^t (t-s)^{\alpha-1}\, ds \le M \left. \frac{-(t-s)^{\alpha}}{\alpha} \right|_0^t = \frac{M}{\alpha} t^{\alpha}$$

where M is a bound for $f(t)$ on $[0, A]$. $\square$

LEMMA 3. *For $f \in L^p(0,b)$, $1 \le p \le \infty$, $\alpha > 0$ and $\beta > 0$ we have the following power rule*

$$\left(I_{0^+}^{\alpha}\; I_{0^+}^{\beta} f\right)(t) = \left(I_{0^+}^{\alpha+\beta}\; f\right)(t)$$

for a.e. $t \in (0,b)$. This relation holds at any point of $(0,b)$ if $\alpha + \beta > 1$.

DEFINITION 5. A function $k \in L^1_{loc}[0, +\infty)$ is called positive definite if

$$\int_0^t w(s) \int_0^s k(s-z) w(z) dz ds \ge 0,\ t \ge 0$$

for every $w \in C[0, +\infty)$.

DEFINITION 6. The function $k(t)$ is said to be strongly positive definite if there exists a positive constant γ such that the mapping $t \to k(t) - \gamma e^{-t}$ is positive definite.

The function $k(t) = t^{-\alpha}$, $0 < \alpha < 1$ is an example of a strongly positive definite function.

3. The Unbounded Domain Problem

For simplicity we shall consider a semi-infinite axis $[-L, +\infty)$ as the original domain. We divide

(5) $$\Omega = \{(x,t)\,,\ -L\ \le x < +\infty,\ 0 < t \le T\}$$

into a bounded domain

(6) $$\Omega_b = \{(x,t)\,,\ -L\ \le x < 0,\ 0 < t \le T\}$$

and an unbounded one

$$\Omega_u = \{(x,t)\,,\ 0\ \leq x < +\infty,\ 0 < t \leq T\}\,. \tag{7}$$

In between we have the artificial boundary

$$\Gamma_0 = \{(x,t)\,,\ x = 0,\ 0 < t \leq T\} \tag{8}$$

on which we need to find an appropriate condition ensuring the continuity of the flux.

THEOREM 4. *The natural boundary condition on the artificial boundary* Γ_0 *is*

$$\frac{\partial}{\partial x} D^{\alpha,\beta} p\,(0,t) = \frac{2K}{\pi}\int_0^\infty w^2 \cos wx D^{\alpha,\beta}$$
$$\times \int_0^t h(t-\tau)\tau^{-\alpha} E_{1-\alpha,1-\alpha}(-Kw^2\tau^{1-\alpha})d\tau dw,\ 0 < t \leq T.$$

PROOF. We start by the unbounded domain and solve the problem

$$\begin{cases} \frac{\partial p}{\partial t} - KD^{\alpha,\beta}\frac{\partial^2 p}{\partial x^2} = 0,\ in\ \Omega_u \\ p\,(x,0) = 0,\ 0\ \leq x < +\infty \\ p\,(0,t) = h(t),\ 0 < t \leq T \\ p\,(x,t) \to 0,\ x \to \infty,\ 0 < t \leq T. \end{cases} \tag{9}$$

It will be solved by the Laplace-Fourier transform method. Applying the Laplace transform first to both sides of the equation in (9), we find (see (4) and Lemma 2)

$$s\bar{p}(x,s) = Ks^\alpha \frac{\partial^2 \bar{p}}{\partial x^2},\ 0\ \leq x < +\infty \tag{10}$$

where $\bar{p}(x,s) = \mathcal{L}[p\,(x,t)](s)$. Then, we apply the Fourier sine transform to both sides of the equation (10) to get

$$s^{1-\alpha}\widehat{\overline{p}}\,(w,s) = -Kw^2\widehat{\overline{p}}\,(w,s) + \sqrt{\frac{2}{\pi}}wK\overline{p}(0,s)$$

or

$$\left(s^{1-\alpha}+Kw^2\right)\widehat{\overline{p}}\,(w,s)=\sqrt{\frac{2}{\pi}}Kw\overline{p}\,(0,s)=\sqrt{\frac{2}{\pi}}Kw\overline{h}(s).$$

Therefore,

$$\widehat{\overline{p}}\,(w,s)=\sqrt{\frac{2}{\pi}}Kw\frac{\overline{h}(s)}{s^{1-\alpha}+Kw^2}. \tag{11}$$

We recall here the definitions

$$\mathcal{F}_s\{f(x)\}=F_s(w)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx dx,$$

$$\mathcal{F}_s^{-1}\{F_s(w)\}=f(x)=\sqrt{\frac{2}{\pi}}\int_0^\infty F_s(w)\sin wx dw.$$

Now taking the inverse Fourier sine transform of (11) we obtain

$$\begin{aligned}\overline{p}\,(x,s)&=\sqrt{\frac{2}{\pi}}\int_0^\infty \sin wx\sqrt{\frac{2}{\pi}}Kw\frac{\overline{h}(s)}{s^{1-\alpha}+Kw^2}dw\\ &=\frac{2}{\pi}\overline{h}\,(s)\,K\int_0^\infty\frac{w\sin wx}{s^{1-\alpha}+Kw^2}dw.\end{aligned} \tag{12}$$

Next, we take the inverse Laplace transform of (12) to find our solution $p(x,t)$. To this end notice that the right hand side of (12) is the product of two functions and

$$\frac{1}{s^{1-\alpha}+Kw^2}=\mathcal{L}\left[t^{-\alpha}E_{1-\alpha,1-\alpha}(-Kw^2t^{1-\alpha})\right]. \tag{13}$$

This follows immediately from the formula (see [37])

$$\mathcal{L}\left[t^{\beta-1}E_{\alpha,\beta}(\pm at^{\alpha})\right]=\frac{s^{\alpha-\beta}}{s^{\alpha}\mp a} \tag{14}$$

where

$$E_{\alpha,\beta}(z):=\sum\nolimits_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)},\ \alpha,\beta\in\mathbf{C};\ \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0$$

and Γ stands for the Gamma function. Therefore, from (12) and (13) we infer that

$$p(x,t) = \frac{2K}{\pi}\int_0^\infty w \sin wx \times \int_0^t h(t-\tau)\tau^{-\alpha} E_{1-\alpha,1-\alpha}(-Kw^2\tau^{1-\alpha}) d\tau dw. \tag{15}$$

The natural boundary-condition on Γ_0 would be

$$\frac{\partial}{\partial x} D^{\alpha,\beta} p(x,t) = \frac{2K}{\pi}\int_0^\infty w^2 \cos wx D^{\alpha,\beta} \times \int_0^t h(t-\tau)\tau^{-\alpha} E_{1-\alpha,1-\alpha}(-Kw^2\tau^{1-\alpha}) d\tau dw \tag{16}$$

taken at $x = 0$. The proof is complete. □

Note that the inner integral in (16) is a convolution. To compute its fractional derivative we need the following lemma.

Lemma 5. *Assume that* $\left(I^{(1-\alpha)(1-\beta)}f\right)(t) \in C^1([0,T])$ *and* $g(t) \in C(0,T] \cap L^1(0,T)$, $0 < \alpha < 1$, $0 \le \beta \le 1$. *Then*

$$D^{\alpha,\beta}\left(\int_0^t f(t-s)g(s)ds\right) = \int_0^t \left(D^{\alpha,\beta}f\right)(t-s)g(s)ds + \lim_{t\to 0^+}\left(I^{(1-\alpha)(1-\beta)}f\right)(t)\left(I^{\beta(1-\alpha)}g\right)(t),\ t>0.$$

That is,

$$D^{\alpha,\beta}(f*g)(t) = \left(D^{\alpha,\beta}f*g\right)(t) + \lim_{t\to 0^+}\left(I^{(1-\alpha)(1-\beta)}f\right)(t)\left(I^{\beta(1-\alpha)}g\right)(t).$$

Proof. By Definition 4 of the Hilfer fractional derivative

$$D^{\alpha,\beta}(f*g)(t) = \left(I^{\beta(1-\alpha)}DI^{(1-\alpha)(1-\beta)}\right)(f*g)(t),\ t>0.$$

Then, in view of the definition of the fractional integral (Definition 1) we may write

$$D^{\alpha,\beta}(f*g)(t) = \frac{1}{\Gamma[(1-\alpha)(1-\beta)]} I^{\beta(1-\alpha)} D \times \int_0^t h(t-\tau)^{-\beta-\alpha(1-\beta)}\left(\int_0^\tau f(t-s)g(s)ds\right)d\tau.$$

By our hypotheses we can apply Fubini's theorem (see [1]) to derive

$$D^{\alpha,\beta}(f*g)(t)$$
$$= \frac{1}{\Gamma[(1-\alpha)(1-\beta)]} I^{\beta(1-\alpha)} D \int_0^t \left(\int_s^t \frac{f(\tau-s)g(s)}{(t-\tau)^{\beta+\alpha(1-\beta)}} d\tau \right) ds.$$

Now Leibniz formula for the differentiation gives

$$\Gamma[(1-\alpha)(1-\beta)]\, D^{\alpha,\beta}(f*g)(t)$$
$$= I^{\beta(1-\alpha)} \left\{ \int_0^t \frac{\partial}{\partial t} \left(\int_s^t \frac{f(\tau-s)\, g(s)\, d\tau}{(t-\tau)^{\beta+\alpha(1-\beta)}} \right) ds \right.$$
$$\left. + \lim_{s\to t^-} \int_s^t \frac{f(\tau-s)\, g(s)\, d\tau}{(t-\tau)^{\beta+\alpha(1-\beta)}} \right\}$$
$$= I^{\beta(1-\alpha)} \left\{ \int_0^t \frac{\partial}{\partial t} \left(\int_0^{t-s} \frac{f(\sigma)\, d\sigma}{(t-s-\sigma)^{\beta+\alpha(1-\beta)}} \right) g(s)\, ds \right.$$
$$\left. + \lim_{s\to t^-} \int_s^t \frac{f(\tau-s)\, g(s)\, d\tau}{(t-\tau)^{\beta+\alpha(1-\beta)}} \right\}.$$

Clearly (see Lemma 3)

$$D^{\alpha,\beta}(f*g)(t) = \frac{1}{\Gamma[(1-\alpha)(1-\beta)]}$$
$$\times \left\{ \int_0^t I^{\beta(1-\alpha)} \frac{\partial}{\partial t} \left(\int_0^{t-s} \frac{f(\sigma)\, d\sigma}{(t-s-\sigma)^{\beta+\alpha(1-\beta)}} \right) g(s)\, ds \right.$$
$$\left. + I^{\beta(1-\alpha)} \left(\lim_{s\to t^-} \int_s^t \frac{f(\tau-s)\, g(s)\, d\tau}{(t-\tau)^{\beta+\alpha(1-\beta)}} \right) \right\}. \tag{17}$$

The last term in (17) is equal to

$$I^{\beta(1-\alpha)} \left(\lim_{s\to t^-} \int_s^t \frac{f(\tau-s)\, d\tau}{(t-\tau)^{\beta+\alpha(1-\beta)}}\; g(s) \right)$$
$$= \left(\lim_{\sigma\to 0} \int_0^\sigma \frac{f(u)\, du}{(\sigma-u)^{\beta+\alpha(1-\beta)}} \right) I^{\beta(1-\alpha)} g(t).$$

In case f is continuous at zero, it is bounded nearby zero and the limit is equal to zero (see Lemma 2). Hence,

$$
\begin{aligned}
&D^{\alpha,\beta}(f*g)(t)\\
&=\frac{1}{\Gamma\left[(1-\alpha)(1-\beta)\right]}\int_0^t I^{\beta(1-\alpha)}\frac{\partial}{\partial t}\left(\int_0^{t-s}\frac{f(\sigma)\,d\sigma}{(t-s-\sigma)^{\beta+\alpha(1-\beta)}}\right)g(s)\,ds\\
&\quad+\lim{}_{t\to0^+}\left(I^{(1-\alpha)(1-\beta)}f\right)(t)\,.\left(I^{\beta(1-\alpha)}g\right)(t)\,,\ t>0.
\end{aligned}
$$

The proof of the lemma is complete. □

Our artificial boundary condition is therefore

$$
\begin{aligned}
\frac{\partial}{\partial x}D^{\alpha,\beta}p(0,t)&=\frac{2K}{\pi}\int_0^\infty w^2\\
&\times\left\{\int_0^t D^{\alpha,\beta}h(t-\tau)\,\tau^{-\alpha}E_{1-\alpha,1-\alpha}\left(-Kw^2\tau^{1-\alpha}\right)d\tau\right.\\
&\quad+\lim_{t\to0^+}\left(I^{(1-\alpha)(1-\beta)}h\right)(t)\\
&\quad\left.\cdot\, I^{\beta(1-\alpha)}\left(t^{-\alpha}E_{1-\alpha,1-\alpha}(-Kw^2t^{1-\alpha})\right)\right\}dw.
\end{aligned}
\tag{18}
$$

We are now in position to set the equivalent reduced problem on Ω_b

$$
\begin{cases}
\frac{\partial p}{\partial t}-KD^{\alpha,\beta}\frac{\partial^2 p}{\partial x^2}=f(x,t)\ \ in\ \Omega_b\\
p(x,0)=g(x)\,,\ -L\le x\le 0\\
p(-L,t)=\varphi(t)\,,\ 0<t\le T\\
\frac{\partial}{\partial x}D^{\alpha,\beta}\,p(0,t)=\psi(t),\ 0<t\le T
\end{cases}
\tag{19}
$$

where $\psi(t)$ is equal to the right hand side of (18). It is important to keep in mind that $\psi(t)$ depends on p.

4. The Equivalent Reduced Problem

In this section we prove uniqueness of solutions to Problem (19).

THEOREM 6. *The solution of problem (19) is unique.*

PROOF. Assume that there exist two solutions p_1 and p_2 to the problem (19). If $p^* = p_1 - p_2$, it follows that p^* is solution of the problem

$$\begin{cases} \frac{\partial p^*}{\partial t} = KD^{\alpha,\beta}\frac{\partial^2 p^*}{\partial x^2}, \ -L \le x \le 0, \ 0 < t \le T \\ p^*(x,0) = 0, \ -L \le x \le 0 \\ p^*(-L,t) = 0, \ 0 < t \le T \\ \frac{\partial}{\partial x}D^{\alpha,\beta}p^*(0,t) = \psi^*(t), \ 0 < t \le T \end{cases} \tag{20}$$

where

$$\psi^*(t) = \frac{2K}{\pi}\int_0^\infty w^2$$
$$\times \left(\int_0^t D^{\alpha,\beta}p^*(0,t-\tau)\,\tau^{-\alpha}E_{1-\alpha,1-\alpha}\left(-Kw^2\tau^{1-\alpha}\right)d\tau \right) dw.$$

Multiplying the equation in (20) by $\frac{\partial p^*}{\partial t}$ and integrating over $(0,T)\times(-L,0)$ we obtain

$$\int_0^T \int_{-L}^0 \left(\frac{\partial p^*}{\partial t}\right)^2 dxdt - K\int_0^T \int_{-L}^0 \frac{\partial p^*}{\partial t} D^{\alpha,\beta}\frac{\partial^2 p^*}{\partial x^2} dxdt = 0$$

or

$$\int_0^T \int_{-L}^0 \left(\frac{\partial p^*}{\partial t}\right)^2 dxdt - K\int_0^T \frac{\partial p^*}{\partial t}\ D^{\alpha,\beta}\frac{\partial p^*}{\partial x}\Big|_{-L}^0 dt$$
$$+K\int_0^T \int_{-L}^0 \frac{\partial^2 p^*}{\partial x \partial t} D^{\alpha,\beta}\frac{\partial p^*}{\partial x} dxdt = 0. \tag{21}$$

From the relationship between the Riemann-Liouville derivative and the Caputo derivative (see Theorem 1)

$$D^\gamma f(t) = \frac{f(0)t^{-\gamma}}{\Gamma(1-\gamma)} + \frac{1}{\Gamma(1-\gamma)}\int_0^t (t-s)^{-\gamma} f'(s)\,ds$$

we deduce that

$$D^{\alpha,\beta}\frac{\partial p^*}{\partial x} = I^{\beta(1-\alpha)}\left\{ \frac{t^{-\beta-\alpha(1-\beta)}}{\Gamma\left[(1-\alpha)(1-\beta)\right]}\frac{\partial p^*}{\partial x}(x,0) + I^{(1-\alpha)(1-\beta)}D\frac{\partial p^*}{\partial x} \right\}.$$

As $p^*(x,0) = 0$ we have $\frac{\partial p^*}{\partial x}(x,0) = 0$. Hence,

$$D^{\alpha,\beta}\frac{\partial p^*}{\partial x} = I^{1-\alpha}D\frac{\partial p^*}{\partial x} = I^{1-\alpha}\frac{\partial^2 p^*}{\partial x \partial t}\left(= {}^C D^\alpha \frac{\partial p^*}{\partial x}\right), \tag{22}$$

by Definition 3.

In view of (21) an (22) we may write

$$\int_0^T \int_{-L}^0 \left(\frac{\partial p^*}{\partial t}\right)^2 dxdt - K \int_0^T \frac{\partial p^*}{\partial t}(0,t)\, D^{\alpha,\beta} \frac{\partial p^*}{\partial x}(0,t)\, dt$$

$$+K \int_0^T \int_{-L}^0 \frac{\partial^2 p^*}{\partial x \partial t} I^{1-\alpha} \frac{\partial^2 p^*}{\partial x \partial t} dxdt = 0. \tag{23}$$

As the kernel $t^{-\alpha}$, $0 < \alpha < 1$ is a positive definite function (in fact a strongly positive definite function) the last term in the left hand side of (23) is nonnegative. We infer that the second term must by nonpositive i.e.

$$K \int_0^T \frac{\partial p^*}{\partial t}(0,t)\, D^{\alpha,\beta} \frac{\partial p^*}{\partial x}(0,t)\, dt \geq \int_0^T \int_{-L}^0 \left(\frac{\partial p^*}{\partial t}\right)^2 dxdt \geq 0.$$

Next, we consider the problem in the unbounded part of the domain

$$\begin{cases} \frac{\partial q}{\partial t} - K D^{\alpha,\beta} \frac{\partial^2 q}{\partial x^2} = 0 \ in \ \Omega_u \\ q(x,0) = 0, \ 0 < x < +\infty \\ q(0,t) = p^*(0,t), \ 0 \leq t \leq T \\ q(x,t) \to 0, \ when \ x \to +\infty. \end{cases} \tag{24}$$

Clearly, this problem has a unique solution. Moreover, multiplying the equation in (24) by $\frac{\partial q}{\partial t}$ and integrating over Ω_u we find

$$\int_0^T \int_0^\infty \left(\frac{\partial q}{\partial t}\right)^2 dxdt - K \int_0^T \int_0^\infty \frac{\partial q}{\partial t} D^{\alpha,\beta} \frac{\partial^2 q}{\partial x^2} dxdt = 0$$

and as above we find

$$\int_0^T \int_0^\infty \left(\frac{\partial q}{\partial t}\right)^2 dxdt - K \int_0^T \frac{\partial q}{\partial t}\, D^{\alpha,\beta} \frac{\partial q}{\partial x}\bigg|_0^\infty dt$$

$$+K \int_0^T \int_0^\infty \frac{\partial^2 q}{\partial x \partial t} I^{1-\alpha} \frac{\partial^2 q}{\partial x \partial t} dxdt = 0. \tag{25}$$

It is not difficult to see that

$$\int_0^T \frac{\partial q}{\partial t}\, D^{\alpha,\beta} \frac{\partial q}{\partial x}\bigg|_0^\infty dt = -\int_0^T \frac{\partial q}{\partial t}(0,t)\, D^{\alpha,\beta} \frac{\partial q}{\partial x}(0,t) dt$$

$$= -\int_0^T \frac{\partial p^*}{\partial t}(0,t)\, D^{\alpha,\beta} \frac{\partial p^*}{\partial x}(0,t)\, dt \tag{26}$$

because $\frac{\partial q}{\partial t}(0,t) = \frac{\partial p^*}{\partial t}(0,t)$ and $\left.\frac{\partial q}{\partial x}\right|_{x=0} = \left.\frac{\partial p^*}{\partial x}\right|_{x=0}$.

The relations (25) and (26) imply that

$$\begin{aligned} &\int_0^T \int_0^\infty \left(\frac{\partial q}{\partial t}\right)^2 dxdt + K \int_0^T \frac{\partial p^*}{\partial t}(0,t)\, D^{\alpha,\beta} \frac{\partial p^*}{\partial x}(0,t)\, dt \\ &+K \int_0^T \int_0^\infty \frac{\partial^2 q}{\partial x \partial t} I^{1-\alpha} \frac{\partial^2 q}{\partial x \partial t} dxdt = 0 \end{aligned} \tag{27}$$

which in turn implies that the second term in the left hand side of (24) must be nonpositive. This observation, together with the previous one, shows that this (second term) is identically equal to zero. It results (from (23)) that $\frac{\partial p^*}{\partial t}$ is also identically equal to zero over Ω_b. In view of the condition $p^*(-L,t) = 0$ we deduce that $p^* \equiv 0$. Consequently, the solution of problem (19) is unique. The proof is complete. □

The solution of (19) will be divided into

$$p(x,t) = U(x,t) + V(x)\, H(t)$$

where $U(x,t)$ is solution of problem

$$\begin{cases} \frac{\partial U}{\partial t} - K D^{\alpha,\beta} \frac{\partial^2 U}{\partial x^2} = f(x,t) \ \ in\ \Omega_b \\ U(x,0) = g(x),\ -L \le x \le 0 \\ U(-L,t) = 0,\ 0 < t \le T \\ \frac{\partial}{\partial x} D^{\alpha,\beta} U(0,t) = 0,\ 0 < t \le T \end{cases} \tag{28}$$

and $V(x)\, H(t)$ is a function which must satisfy $V(-L)\, H(t) = \varphi(t)$ and $\frac{\partial V}{\partial x}(0)\, D^{\alpha,\beta} H(t) = \psi(t)$.

This requires the compatibility condition

$$\frac{\partial V}{\partial x}(0) D^{\alpha,\beta} \frac{\varphi(t)}{V(-L)} = \psi(t) \ or\ D^{\alpha,\beta} \varphi(t) = \frac{V(-L)}{V'(0)} \psi(t). \tag{29}$$

5. Solution of Problem (28)

For simplicity, we shall assume that the source in (28) is of the form $f(t)l(x)$.

THEOREM 7. *The solution of Problem (28) is given by*

$$U(x,t) = \sum\nolimits_n \cos\frac{(2n+1)\pi x}{2L}$$
$$\times\left[T_n(0)\,E_{1-\alpha,1}\left(-K\lambda_n^2 t^{1-\alpha}\right) + C_n\int_0^t f(t-s)\,E_{1-\alpha,1}\left(-K\lambda_n^2 s^{1-\alpha}\right)ds\right]$$

where

$$T_n(0) = \frac{\int_{-L}^0 g(x)\cos\frac{(2n+1)\pi x}{2L}dx}{\int_{-L}^0 \cos^2\frac{(2n+1)\pi x}{2L}dx}.$$

PROOF. Let us look to a solution for the homogeneous equation in (28) in the form

$$U(x,t) = X(x)\,T(t).$$

We have

$$X(x)\,T'(t) = KD^{\alpha,\beta}\left(X''(x)\,T(t)\right) = KD^{\alpha,\beta}X''(x)\,T(t),$$
$$-L \le x \le 0,\ 0 < t \le T.$$

Therefore,

$$\frac{X''}{X} = \frac{T'}{KD^{\alpha,\beta}T} = -\lambda^2,\ \lambda \in \mathbb{R}$$

or

$$\left\{\begin{array}{l} X'' + \lambda^2 X = 0, \\ T' + \lambda^2 KD^{\alpha,\beta}T = 0. \end{array}\right. \tag{30}$$

The first equation in (30) has

$$X(x) = C_1\cos\lambda x + C_2\sin\lambda x$$

with $C_1, C_2 \in \mathbb{R}$ as a general solution. Its derivative is

$$X'(x) = -\lambda C_1\sin\lambda x + \lambda C_2\cos\lambda x.$$

The boundary condition

$$U(-L,t) = 0 = X(-L)\,T(t)\,,\ \forall t$$

implies that $X(-L) = 0$. The second boundary condition

$$D^{\alpha,\beta}\frac{\partial U}{\partial x}(0,t) = D^{\alpha,\beta}\left(X'(0)\,T(t)\right) = X'(0)\,D^{\alpha,\beta}T(t)\,,\ \forall t$$

being equal to zero implies that $X'(0) = 0$. So we need to solve the system

$$\left\{\begin{array}{l} X'' + \lambda^2 X = 0 \\ X(-L) = 0,\ X'(0) = 0 \end{array}\right. \tag{31}$$

which has $X(x) = C_1 \cos \lambda x$ as a solution and we find $X_n(x) = \cos\frac{(2n+1)\pi x}{2L}$.

Back to the nonhomogeneous problem (28), we need

$$\frac{\partial}{\partial t}\left(\sum\nolimits_n X_n(x)T_n(t)\right) = KD^{\alpha,\beta}\left(\sum\nolimits_n X_n''(x)T_n(t)\right) + f(t)\,l(x)$$

or

$$\begin{aligned}\sum\nolimits_n X_n(x)T_n'(t) =& K\sum\nolimits_n \left(-\lambda_n^2 X_n(x)\right) D^{\alpha,\beta}T_n(t)\\ &+ f(t)\sum\nolimits_n C_n X_n(x).\end{aligned} \tag{32}$$

We infer that

$$T_n'(t) + K\lambda_n^2 D^{\alpha,\beta}T_n(t) = C_n f(t). \tag{33}$$

Taking the Laplace transform of (33) we obtain

$$s\overline{T}_n(s) + K\lambda_n^2[s^{\alpha}\overline{T}_n(s) - s^{\beta(\alpha-1)}\left(I^{(1-\alpha)(1-\beta)}T_n\right)(0)] = C_n\overline{f}(s) + T_n(0)\,.$$

As $T_n(t)$ are differentiable near zero, they are continuous at zero and therefore (see Lemma 2)

$$\left(I^{(1-\alpha)(1-\beta)}T_n\right)(0) = 0.$$

Thus,

$$\left(s + K\lambda_n^2 s^{\alpha}\right)\overline{T}_n(s) = T_n(0) + C_n\overline{f}(s)$$

or

$$\overline{T}_n(s) = \frac{T_n(0) + C_n \overline{f}(s)}{s + K\lambda_n^2 s^\alpha} = \frac{T_n(0) + C_n \overline{f}(s)}{s^{1-\alpha} + K\lambda_n^2} s^{-\alpha}. \tag{34}$$

Applying the inverse Laplace transform to (34) we obtain

$$T_n(t) = T_n(0)\mathcal{L}^{-1}\left[\frac{s^{-\alpha}}{s^{1-\alpha} + K\lambda_n^2}\right] + C_n \mathcal{L}^{-1}\left[\frac{\overline{f}(s)\, s^{-\alpha}}{s^{1-\alpha} + K\lambda_n^2}\right]$$
$$= T_n(0) E_{1-\alpha,1}\left(-K\lambda_n^2 t^{1-\alpha}\right) + C_n \int_0^t f(t-s) E_{1-\alpha,1}\left(-K\lambda_n^2 s^{1-\alpha}\right) ds.$$

Using the initial data

$$U(x,0) = \sum\nolimits_n X_n(x) T_n(0) = g(x)$$

we find

$$T_n(0) = \frac{2}{L}\int_{-L}^{0} g(x) \cos\frac{(2n+1)\pi x}{2L} dx$$

by using the formula

$$T_n(0) = \frac{\int_{-L}^0 g(x)\cos\frac{(2n+1)\pi x}{2L} dx}{\int_{-L}^0 \cos^2\frac{(2n+1)\pi x}{2L} dx}.$$

The proof is complete. □

Example 1. Let $l(x) = C$ where C is a constant. Then,

$$l(x) = \sum\nolimits_n \frac{2}{L}\frac{2LC(-1)^n}{(2n+1)\pi}\cos\frac{(2n+1)\pi x}{2L} = \sum\nolimits_n \frac{4C(-1)^n}{(2n+1)\pi}\cos\frac{(2n+1)\pi x}{2L}$$

and the solution is given by

$$U(x,t) = \sum\nolimits_n \Big[T_n(0) E_{1-\alpha,1}\left(-K\lambda_n^2 t^{1-\alpha}\right)$$
$$+ \frac{4C(-1)^n}{(2n+1)\pi}\int_0^t f(t-s) E_{1-\alpha,1}\left(-K\lambda_n^2 s^{1-\alpha}\right) ds\Big] \cos\frac{(2n+1)\pi x}{2L}.$$

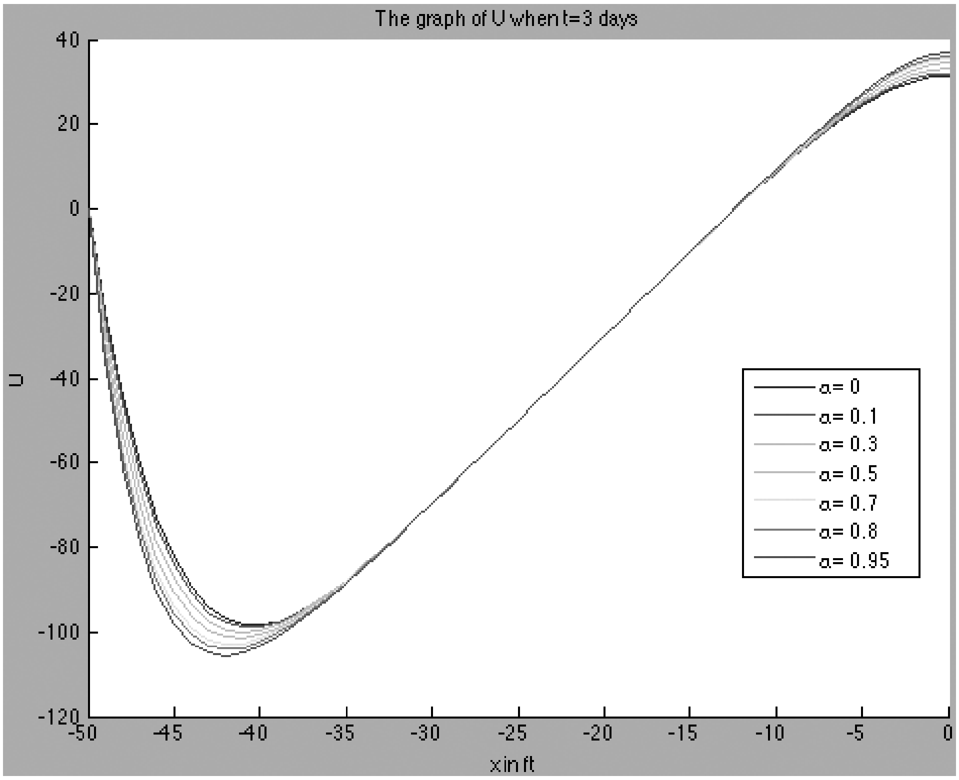
Example 2. Let $l(x) = \sin x$, then

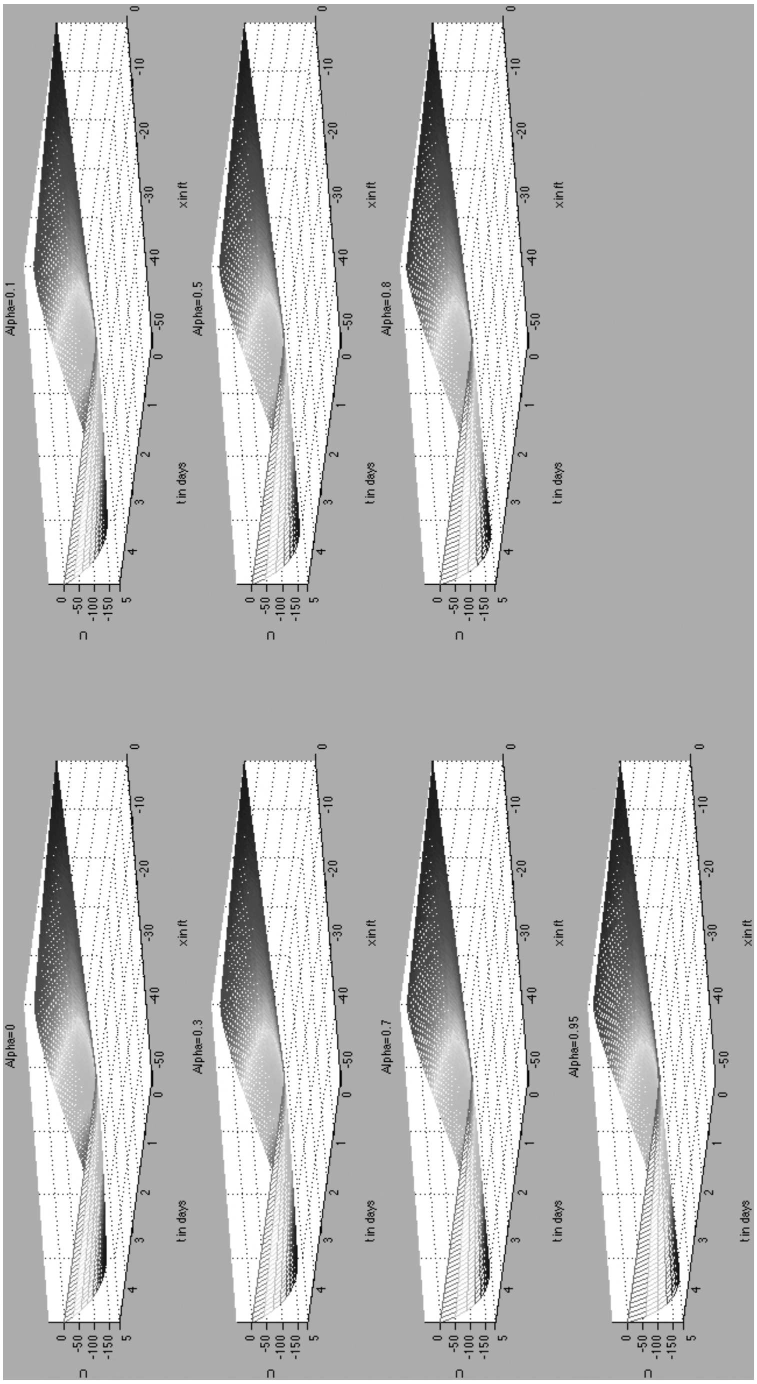
$$C_n = \frac{4L}{(2n+1)^2\pi^2 - 4L^2}$$
$$+2\left\{\frac{\cos\left[\frac{(2n+1)\pi}{2} - L\right]}{2L-(2n+1)\pi} + \frac{\cos\left[\frac{(2n+1)\pi}{2} + L\right]}{2L+(2n+1)\pi}\right\}.$$

Example 3. Let $l(x) = x$, then

$$C_n = \frac{2}{L}\left[\frac{2L^2(-1)^{n+1}}{(2n+1)\pi} + \frac{4L^2}{(2n+1)^2\pi^2}\right].$$

The graphs below are for different values of the order α and correspond to $f(t) = 1$, $l(x) = x$, $g(x) = x + L$, $L = 50$ and $k = 10$.




Alpha=0
Alpha=0.1
Alpha=0.3
Alpha=0.5
Alpha=0.7
Alpha=0.8
Alpha=0.95
t in days
x in ft
u

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