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Local Zeta Functions for Non-degenerate Laurent Polynomials Over p-adic Fields

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Abstract. In this article, we study local zeta functions attached to Laurent polynomials over p-adic fields, which are non-degenerate with respect to their Newton polytopes at infinity. As an application we obtain asymptotic expansions for p-adic oscillatory integrals attached to Laurent polynomials. We show the existence of two different asymptotic expansions for p-adic oscillatory integrals, one when the absolute value of the parameter approaches infinity, the other when the absolute value of the parameter approaches zero. These two asymptotic expansions are controlled by the poles of twisted local zeta functions of Igusa type.

1. Introduction

The local zeta functions for non-degenerate polynomials (or more generally for non-degenerate analytic functions) have been studied quite extensively. Initially these functions were studied by Varchenko in the Archimedean case, later Denef studied them in the non-Archimedean case, see e.g. [2], [4], [5], [8], [16], [19], [20], [23], [24], among others. In this article, we study local zeta functions attached to Laurent polynomials over p-adic fields, which are weakly non-degenerate with respect to their Newton polytopes at infinity, see Definition 2.4. This notion of non-degeneracy is weaker than the standard non-degeneracy condition of Khovanskii, see Definition 2.5 and Example 2.6. By using a variation of toric resolution of singularities, we show the existence of a meromorphic continuation for these zeta functions as rational functions of q^{-s}, see Theorem 3.3. We also extend Igusa's stationary phase method for oscillatory integrals (and certain

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exponential sums) depending on a p-adic parameter to the case of Laurent polynomials, see Theorem 4.2. Here, a new and interesting phenomenon occurs: there are two different asymptotic expansions for p-adic oscillatory integrals, one when the absolute value of the parameter approaches infinity, the other when the absolute value of the parameter approaches zero. These two asymptotic expansions are controlled by the poles of twisted local zeta functions.

The classical local zeta functions are connected with polynomial congruences mod p^m. In the case of Laurent polynomials the corresponding local zeta functions control the asymptotic behavior of the volumes of 'tubular neighborhoods' attached to the polynomials, see Theorem 3.8.

There are several important differences between the classical local zeta functions for non-degenerate polynomials and the local zeta functions studied here. First, the classical local zeta functions have only poles with negative real parts while the local zeta functions for Laurent polynomials have poles with positive and negative real parts. This fact makes more difficult the determination of the actual poles of these new local zeta functions. Second, the convergence of the integral defining the local zeta function (see Definition 3.2) is not a straightforward matter due to the presence of 'denominators.'

Finally we want to comment that our initial motivation was to find p-adic counterparts of certain estimates for exponential sums attached to non-degenerate Laurent polynomials over finite fields due to Adolphson and Sperber [1] and Denef and Loeser [6], see Corollary 4.3.

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2. Newton Polytopes, Non-degeneracy Conditions and Toric Manifolds

In this section, we review some basic results on toric manifolds, and non-degeneracy conditions for Laurent polynomials over a local field of characteristic zero. The results needed here are variations of the ones given in [14]-[15], [17], in the Archimedean setting. The material needed to adapt these results to the p-adic setting can be found in [12], [18].

2.1. Newton polytopes

We set $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geqslant 0\}$. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product of $\mathbb{R}^n$, and identify the dual space of $\mathbb{R}^n$ with $\mathbb{R}^n$ itself by means of it.

Let K be a local field of characteristic zero. Let

$$f(x) = \sum_{m \in \mathbb{Z}^n} a_m x^m \in K\left[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\right]$$

be a non-constant Laurent polynomial. Set $supp(f) := \{m \in \mathbb{Z}^n; a_m \neq 0\}$. We define *the Newton polytope* $\Gamma_\infty(f) := \Gamma_\infty$ *of* f *at infinity* as the convex hull of $supp(f)$ in $\mathbb{R}^n$. Note that, if $supp(f) = \{m_1, \ldots, m_l\}$, then

$$\Gamma_\infty = conv\,(m_1, \ldots, m_l) = \left\{ \sum_{i=1}^{l} \lambda_i m_i; \lambda_1, \ldots, \lambda_l \in \mathbb{R}_+, \sum_{i=1}^{l} \lambda_i = 1 \right\}.$$

In combinatorics a set like Γ_∞ is typically called a *rational (or lattice) polytope (i.e. a compact polyhedron).* From now on, we will use just *polytope* to mean *rational polytope* and *assume that* $\dim \Gamma_\infty = n$.

2.1.1 Faces

Let H be the hyperplane $\{x \in \mathbb{R}^n; \langle a, x \rangle = b\}$. Then H determines two closed half-spaces:

$$H^+ := \{x \in \mathbb{R}^n; \langle a, x \rangle \geq b\}$$

and

$$H^- := \{x \in \mathbb{R}^n; \langle a, x \rangle \leq b\}.$$

We say that H is *a supporting hyperplane* of Γ_∞, if $\Gamma_\infty \cap H \neq \emptyset$ and Γ_∞ is contained in one of the closed half-spaces determined by H.

The *dimension of a face* τ of Γ_∞ is the dimension of its affine span, and its *codimension* is $cod\,(\tau) = n - \dim(\tau)$. A face of codimension 1 is a *facet.* Faces of dimension 0 and 1 are called *vertices* and *edges* respectively. We denote by $vert(\Gamma_\infty)$ the set of vertices of Γ_∞. A face of Γ_∞ different from Γ_∞ is called *proper.*

Given $a \in \mathbb{R}^n$, we define

$$d(a, \Gamma_\infty) := d(a) = \inf\{\langle a, x \rangle; x \in \Gamma_\infty\}.$$

Note, that since a convex polytope is the convex hull of its vertices, we can take the infimum as v varies in $vert(\Gamma_\infty)$, which is a finite set, hence

$$d(a) = \min\left\{\langle a, x\rangle \,; x \in vert(\Gamma_\infty)\right\},$$

and $d(a) = \langle a, x_0\rangle$ for some $x_0 \in vert(\Gamma_\infty)$.

2.1.2 Primitive vectors and facets

Given a supporting hyperplane H containing a facet of Γ_∞, there exists a unique vector $a \in \mathbb{Z}^n \smallsetminus \{0\}$ perpendicular to H and directed into the polytope. This vector is called the *inward* normal to H. A vector $a = (a_1, \dots, a_n) \in \mathbb{Z}^n$ is called primitive if $g.c.d.(a_1, \dots, a_n) = 1$. Every facet has a unique primitive inward vector. We denote the set of all these vectors as $\mathfrak{D}(\Gamma_\infty)$.

2.2. Cones and fans

We now review the construction of conical subdivisions of $\mathbb{R}^n$ and $\mathbb{R}^n_+$ *subordinated to* Γ_∞. Such constructions are simple variations of some well-known ones, see e.g. [10], [15, The main example, Section 1.2], [22, Chapter 7], we also use [5], [19], for this reason we do not give proofs.

We recall that the *cone strictly spanned* by the vectors $a_1, \dots, a_r \in \mathbb{R}^n \setminus \{0\}$ is the set $\Delta^\circ = \{\lambda_1 a_1 + ... + \lambda_r a_r; \lambda_i \in \mathbb{R}_+, \lambda_i > 0\}$. Notice that the topological closure of Δ° is

$$\Delta = \left\{\lambda_1 a_1 + ... + \lambda_r a_r; \lambda_i \in \mathbb{R}_+, \lambda_i \geq 0\right\}, \tag{2.1}$$

c.f. Lemma 2.1. This set is typically called a *convex polyhedral cone.* If $a_1, \dots, a_r$ are linearly independent over $\mathbb{R}$, Δ° and Δ are called *simplicial cones.* If $a_1, \dots, a_r \in \mathbb{Z}^n$, we say Δ° and Δ are *rational cones.* If $\{a_1, \dots, a_r\}$ is a subset of a basis of the $\mathbb{Z}$-module $\mathbb{Z}^n$, we call Δ° and Δ *simple cones.* The justification for this 'unusual' approach is the following. The computation and the obtention of explicit formulas of local zeta functions require 'open cones', see Remark 3.5 (ii) and [5], [20], [24], while calculations using toroidal resolution of singularities require 'closed cones'.

We define *the first meet locus* of $a \in \mathbb{R}^n$ as

$$F(a, \Gamma_\infty) := F(a) = \left\{x \in \Gamma_\infty; \langle a, x\rangle = d\,(a)\right\}.$$

Note that $F(a)$ is a face of Γ_∞, and that $F(0) = \Gamma_\infty$.

We define an equivalence relation on $\mathbb{R}^n$ by taking

$$a \sim a' \Longleftrightarrow F(a) = F(a').$$

If τ is a face of Γ_∞, we define *the cone associated to* τ as

$$\Delta_\tau^\circ = \left\{a \in \mathbb{R}_+^n; F(a) = \tau\right\}.$$

LEMMA 2.1. *Let τ be a proper face of Γ_∞. Then*
(1) Δ_τ° is a relatively open in the vector subspace of $\mathbb{R}^n$ spanned by Δ_τ°.
(2) The topological closure Δ_τ of Δ_τ° is a rational convex polyhedral cone with vertex at the origin, and

$$\Delta_\tau = \{a \in \mathbb{R}^n; F(a) \supset \tau\}.$$

(3) $\dim \Delta_\tau^\circ = \dim \Delta_\tau = n - \dim \tau$.
(4) The function $d(\cdot)$ is linear on Δ_τ.

We recall that a *rational strongly convex polyhedral cone* Δ is cone of form (2.1) with vertex at the origin of $\mathbb{R}^n$, and with $a_1, \ldots, a_r \in \mathbb{Z}^n$. It is also useful to recall that Δ is the solution set of a system of inequalities of the form $Ax \leq 0$, where A is a matrix with integer entries and $x \in \mathbb{R}^n$.

We recall that a *fan* $\mathcal{L}$ is a finite collection of *rational strongly convex polyhedral cones* $\{\Delta_i; i \in I\}$ in $\mathbb{R}^n$ such that: (i) if $\Delta_i \in \mathcal{L}$ and Δ is a face of Δ_i, then $\Delta \in \mathcal{L}$; (ii) if Δ_1, $\Delta_2 \in \mathcal{L}$, then $\Delta_1 \cap \Delta_2$ is a face of Δ_1 and Δ_2. The *support* of $\mathcal{L}$ is $|\mathcal{L}| := \cup_{i \in I} \Delta_i$. A fan $\mathcal{L}$ is called *simplicial* (resp. *simple*) if all its cones are simplicial (resp. simple). A fan $\mathcal{L}$ is called *subordinated to* Γ_∞, if every cone in $\mathcal{L}$ is contained in an equivalence class of $\sim$. We denote by *edges*$(\mathcal{L})$, the set of all edges (generators) of the cones in $\mathcal{L}$.

LEMMA 2.2. *The closures Δ_τ of the cones associated to the faces of Γ_∞ form a simplicial fan $\mathcal{F}$ subordinated to Γ_∞. Moreover, we have the following:*
(i) Let τ be a proper face of Γ_∞. Then the map

$$\begin{array}{ccc}\{\textit{faces of}\,\Gamma_\infty \textit{ that contain } \tau\} & \to & \{\textit{non-empty faces of } \Delta_\tau\} \\ \\ \sigma & \to & \Delta_\sigma\end{array}$$

is one-to-one and onto.
(ii) Let τ_1*,* τ_2 *be faces of* Γ_∞*. Suppose that* τ_1 *is a facet of* τ_2*, i.e.* τ_1 *has codimension one in* τ_2*, then* Δ_{τ_2} *is a facet of* Δ_{τ_1}*.*

Lemma 2.3. *(i) Let* τ *be a proper face of* Γ_∞*. Let* $\gamma_1, \ldots, \gamma_r$ *be the facets of* Γ_∞ *containing* τ*. Let* $a_1, \ldots, a_r \in \mathbb{Z}^n \setminus \{0\}$ *be the unique primitive inward vectors to* $\gamma_1, \ldots, \gamma_r$ *respectively. Then*

$$\Delta_\tau = \left\{ \sum_{i=1}^{r} \lambda_i a_i; \lambda_i \in \mathbb{R},\ \lambda_i \geq 0 \right\} \text{ and } \Delta_\tau^\circ = \left\{ \sum_{i=1}^{r} \lambda_i a_i; \lambda_i \in \mathbb{R},\ \lambda_i > 0 \right\}.$$

(ii) $\dim \Delta_\tau^\circ = \dim \Delta_\tau = n - \dim \tau$.

From the above discussion, we conclude that $\{\Delta_\tau\}$ is a fan *subordinated* to Γ_∞ with support $\mathbb{R}^n$. We now note that if $\Delta_\tau \cap \mathbb{R}^n_+ \neq \emptyset$, then $\Delta_\tau \cap \mathbb{R}^n_+$ is a strongly convex polyhedral cone. We denote by *Faces*$(\Delta_\tau \cap \mathbb{R}^n_+)$ the set of all the faces of cone $\Delta_\tau \cap \mathbb{R}^n_+$. Then $\cup_{\Delta_\tau \cap \mathbb{R}^n_+ \neq \emptyset}$*Faces* $(\Delta_\tau \cap \mathbb{R}^n_+)$ is a fan *subordinated* to Γ_∞ with support $\mathbb{R}^n_+$. Set Δ_τ^+ to be a face of $\Delta_\tau \cap \mathbb{R}^n_+ \neq \emptyset$, which is also a cone, then each cone Δ_τ^+ can be partitioned into a finite number of simplicial cones $\Delta_{\tau,i}^+$. By adding new rays, each simplicial cone can be partitioned further into a finite number of simple cones, see e.g. [13]. In this way we construct a *simple fan* $\mathcal{F}$ *subordinated* to Γ_∞. From now on, we fix a simple fan $\mathcal{F}$ subordinated to Γ_∞ with support $\mathbb{R}^n_+$.

Set $\mathcal{F}_0$ to be the cone $\mathbb{R}^n_+$ and its faces. We will say that $\mathcal{F}$ is *trivial* if $\mathcal{F} = \mathcal{F}_0$.

Given a fan subordinated to Γ_∞ with support $\mathbb{R}^n_+$, it is possible to obtain a *conical partition of* $\mathbb{R}^n_+ \setminus \{0\}$ (*subordinated* to Γ_∞) into open cones. This type of partitions play a central role in explicit calculations of local zeta functions.

2.3. Khovanskii non-degeneracy condition

Given $a \in \mathbb{R}^n_+$, we define *the face function of* $f(x) = \sum_m a_m x^m$ *with respect to* a as

$$f_a(x) = \sum_{m \in F(a, \Gamma_\infty)} a_m x^m.$$

We set $T^n(K) := \{x \in K^n; x_1 \ldots x_n \neq 0\}$, for the n-dimensional torus considered as a K-analytic manifold.

Definition 2.4. Let $f(x) = \sum_m a_m x^m \in K\left[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\right]$ be a non-constant Laurent polynomial, and let Γ_∞ be its Newton polytope. *We say that f is non-degenerate with respect to* $a \in \mathbb{R}_+^n$, if the system of equations

$$\{f_a(x) = 0, \nabla f_a(x) = 0\}$$

has no solutions in $T^n(K)$. We say that f is weakly non-degenerate *with respect to* Γ_∞, if f_a *is non-degenerate with respect to any* $a \in \mathbb{R}_+^n$.

We recall the standard non-degeneracy condition of Khovanskii.

Definition 2.5. Given a face τ of Γ_∞, *the face function of f with respect to τ* is $f_\tau(x) := \sum_{l \in \tau} c_l x^l$. We say that f *is non-degenerate with respect to* Γ_∞, if for every face τ of Γ_∞, including Γ_∞ itself, the system of equations

$$\{f_\tau(x) = 0, \nabla f_\tau(x) = 0\}$$

has no solutions in $T^n(K)$.

Example 2.6. Take $f(x, y) = \left(x^{-1} - y\right)^2 + x^2$. Then Γ_∞ is a triangle with vertices at $(-2, 0), (0, 2), (2, 0)$. The facet τ_1 containing the points $(-2, 0)$, $(0, 2)$ has $(1, -1)$ as inward vector, the facet τ_2 containing the points $(0, 2)$, $(2, 0)$ has $(-1, -1)$ as inward vector, and the facet τ_3 containing the points $(-2, 0)$, $(2, 0)$ has $(0, 1)$ as inward vector. Note that the fan $\mathcal{F}$ is trivial. In addition, f is degenerate with respect to Γ_∞, but f is weakly non-degenerate with respect to Γ_∞.

2.4. Toric manifolds

Let $A = \{a_{i,j}\} \in GL(n, \mathbb{Z})$ with $\det A = \pm 1$. We associate to A a birational morphism

$$\begin{array}{cccc} \Psi_A : & (K^\times)^n & \to & (K^\times)^n \\ & (z_1, \ldots, z_n) & \to & \left(z_1^{a_{1,1}} \cdots z_n^{a_{1,n}}, \ldots, z_1^{a_{n,1}} \cdots z_n^{a_{n,n}}\right). \end{array}$$

Note that Ψ_A is a group homomorphism of the algebraic group $(K^\times)^n$. It is clear that $\Psi_A \circ \Psi_B = \Psi_{AB}$ and $\Psi_A^{-1} = \Psi_{A^{-1}}$. In addition, if there exists a subset $J \subset \{1, \ldots, n\}$ such that $a_{i,j} \geq 0$ for any $i \in \{1, \ldots, n\}$ and $j \in J$, then Ψ_A extends to

$$\{z \in K^n; i \notin J \Rightarrow z_i \neq 0\} \to K^n.$$

In particular, if $a_{i,j} \geq 0$ for any $i, j \in \{1, \ldots, n\}$, then Ψ_A extends to a birational morphism $\Psi_A : K^n \to K^n$.

Let $\mathcal{F}$ be the fixed simple fan subordinated to Γ_∞ with support $\mathbb{R}_+^n$, and let $\mathcal{F}_0$ be the trivial fan as before. Let Δ be an $n-$dimensional simple cone in $\mathcal{F}$ and let τ be the vertex of Γ_∞ such that $F(a) = \tau$ for any $a \in \Delta$. Assume that Δ is spanned by a basis $a_j = (a_{i,j})_{1 \leq i \leq n}$. We set $A := \{a_{i,j}\}$ and identify A with Δ, in particular $\Psi_A := \Psi_\Delta$.

We attach to Δ a copy K_Δ^n of K^n with coordinates $y_\Delta := y = (y_1, \ldots, y_n)$ and define the projection morphisms

$$\begin{array}{cccc} \sigma_\Delta : & K_\Delta^n & \to & K^n \\ & & & \\ & (y_1, \ldots, y_n) & \to & (x_1, \ldots, x_n), \end{array}$$

with $x_i = \prod_j y_j^{a_{i,j}}$ for $i \in \{1, \ldots, n\}$. Thus $\sigma_\Delta(y) = \Psi_\Delta(y)$. We now take $\bigsqcup_{\substack{\Delta \in \mathcal{F} \\ \dim(\Delta)=n}} K_\Delta^n$ and define an equivalence relation in this disjoint union. Take $z_\Delta \in K_\Delta^n$ and $z_{\Delta'} \in K_{\Delta'}^n$. We define $z_\Delta \sim z_{\Delta'}$ if the birational map $\Psi_{\Delta'^{-1}\Delta} : K_\Delta^n \to K_{\Delta'}^n$ is well defined on $z_\Delta \in K_\Delta^n$ and $z_{\Delta'} = \Psi_{\Delta'^{-1}\Delta}(z_\Delta)$. Then $\sim$ is an equivalence relation, c.f. [17, p. 72]. Let $X(\mathcal{F})$ be the quotient space $\bigsqcup_{\substack{\Delta \in \mathcal{F} \\ \dim(\Delta)=n}} K_\Delta^n / \sim$. As the gluing maps are K-bianalytic maps, $X(\mathcal{F})$ is a K-analytic manifold (in the sense of Serre) with coordinate charts $(K_\Delta^n, \sigma_\Delta)$. We also have a canonical projection map $\sigma : X(\mathcal{F}) \to K^n$ defined by $\sigma|_{K_\Delta^n}([y]) = \sigma_\Delta(y)$ where $[y]$ is the equivalence class of $y \in K_\Delta^n$. This map is proper. In [17, p. 75-79] this fact is proved, in the complex setting, using sequences, this proof can be adapted to the case of p-adic fields.

We have a canonical embedding morphism $i_\Delta : (K^\times)^n \to (K_\Delta^\times)^n$ defined by $i_\Delta(z) = \Psi_{\Delta^{-1}}(z)$. This is compatible with $\sim$ and thus we have an embedding morphism $i : (K^\times)^n \to X(\mathcal{F})$. The image is an open dense subset of $X(\mathcal{F})$, this image is an n-dimensional K-analytic torus.

Let $T^n(K) = \{x \in K^n;\ x_1 \ldots x_n \neq 0\}$ be the n-dimensional K-analytic torus. Then, the mapping $\sigma :\ \sigma^{-1}(T^n(K)) \to T^n(K)$ is a K-analytic isomorphism.

It is well-known that one can define the toric manifold $X(\mathcal{F})$ associated to a simple fan $\mathcal{F}$ as an algebraic variety over K, and that the morphism induced by a subdivision is a proper morphism of algebraic varieties, c.f. [13, Chapter I, Theorems 6,7,8]. Thus, since $X(\mathcal{F}_0) = K^n$, $\sigma : X(\mathcal{F}) \to K^n$ is a proper morphism of algebraic varieties. By considering the K-analytic manifolds associated we obtain a morphism $\sigma : X(\mathcal{F}) \to K^n$ of K-analytic manifolds.

2.4.1 Resolution of singularities

For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \smallsetminus \{0\}$, we set $\|a\| := a_1 + \ldots + a_n$. Take $f(x) = \sum_{m \in \mathbb{Z}} c_m x^m \in K\left[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\right]$ a non constant Laurent polynomial and define

$$f : T^n(K) \to K.$$

The pair $(X(\mathcal{F}), \sigma)$ works as an embedded resolution of singularities for f. In this section we give explicit formulas for $f \circ \sigma$ and $\sigma^*(dx_1 \wedge \ldots \wedge dx_n)$ around a point of $X(\mathcal{F})$.

Let Δ be an $n-$dimensional cone in $\mathcal{F}$ spanned by $a_1, \ldots, a_n$ and let τ be the vertex of Γ_∞ such that $F(a) = \tau$ for any $a \in \Delta$. By the explicit description of σ_Δ above, we have

$$f_\Delta(y) := (f \circ \sigma_\Delta)(y) = \sum_{m \in \operatorname{supp}(f)} c_m \prod_{j=1}^{n} y_j^{\langle a_j, m\rangle}. \tag{2.2}$$

We have $\langle a_j, m\rangle \geq d(a_j)$ by the definition of $d(a_j)$. The equalities for all j hold if and only if the set $\{m\}$ coincides with the vertex τ. This implies that $f_\Delta(y)$ is written in the form

$$(f \circ \sigma_\Delta)(y) = \varepsilon(y) \left(\prod_{j=1}^{n} y_j^{d(a_j)}\right), \ \varepsilon(y) \in K[y_1, \ldots, y_n], \ \varepsilon(0) \neq 0. \tag{2.3}$$

In particular, there exists a neighborhood $V_0 \subset K_\Delta^n$ of the origin such that $|\varepsilon(y)|_K = |\varepsilon(0)|_K \neq 0$ for any $y \in V_0$. The above description of σ_Δ also implies

$$\sigma_\Delta^*(dx_1 \wedge \ldots \wedge dx_n) = (\pm 1)\left(\prod_{j=1}^{n} y_j^{\|a_j\|-1}\right) dy_1 \wedge \ldots \wedge dy_n, \tag{2.4}$$

for any $y \in V_0$.

Let $b \neq 0$ be a point of $K_\Delta^n \smallsetminus \left(K_\Delta^\times\right)^n$. By renaming the coordinates, we assume $b = (0, \dots, 0, b_{r+1}, \dots, b_n)$ with $b_i \in K^\times$ for $r+1 \leq i \leq n$. Let Δ' be the face of Δ spanned by $a_1, \dots, a_r$ and let τ' be the face of Γ_∞ such that $F(a) = \tau'$ for all $a \in \Delta'$. Then, for $m \in$supp(f), $\langle a_j, m\rangle = d\left(a_j\right)$ holds for all $j \in \{1, \dots, r\}$ if and only if $m \in \tau'$. Hence we may write (2.2) as

$$\left(f \circ \sigma_\Delta\right)(y) = \left(\prod_{j=1}^{r} y_j^{d(a_j)}\right)\left(f_{\Delta,\tau'}\left(y\right) + h_{\Delta,\tau'}(y)\right),$$

with

$$f_{\Delta,\tau'}\left(y\right) = \sum_{m \notin \tau' \cap \mathrm{supp}(f)} c_m \prod_{j=r+1}^{n} y_j^{\langle a_j, m\rangle} \in K\left[y_{r+1}, y_{r+1}^{-1}, \dots, y_n, y_n^{-1}\right],$$

$$h_{\Delta,\tau'}(y) \in \sum_{j=1}^{r} y_j K\left[y_1, \dots, y_r, y_{r+1}, y_{r+1}^{-1}, \dots, y_n, y_n^{-1}\right].$$

Note that $h_{\Delta,\tau'}\left(b\right) = 0$. Two cases happen: (i) $f_{\Delta,\tau'}\left(b\right) \neq 0$, (ii) $f_{\Delta,\tau'}\left(b\right) = 0$. In the first case,

$$\left(f \circ \sigma_\Delta\right)(y) = \varepsilon\left(y\right)\left(\prod_{j=1}^{r} y_j^{d(a_j)}\right), \tag{2.5}$$

$$\varepsilon\left(y\right) \in K\left[y_1, \dots, y_r, y_{r+1}, y_{r+1}^{-1}, \dots, y_n, y_n^{-1}\right],$$

with $\varepsilon\left(b\right) \neq 0$, and

$$\sigma_\Delta^*\left(dx_1 \wedge \ldots \wedge dx_n\right) = \eta\left(y\right)\left(\prod_{j=1}^{r} y_j^{\|a_j\|-1}\right) dy_1 \wedge \ldots \wedge dy_n, \tag{2.6}$$

$$\eta\left(y\right) \in K\left[y_1, \dots, y_r, y_{r+1}, y_{r+1}^{-1}, \dots, y_n, y_n^{-1}\right],\ \eta\left(b\right) \neq 0.$$

In particular, there exists an open neighborhood $V_b \subset K_\Delta^n$ of b such that $\left|\varepsilon\left(y\right)\right|_K = \left|\varepsilon\left(b\right)\right|_K$ and $\left|\eta\left(y\right)\right|_K = \left|\eta\left(b\right)\right|_K$ for $y \in V_b$.

Suppose that $f_{\Delta,\tau'}\left(b\right) = 0$. We claim that there exists $l \in \{r+1, \dots, n\}$ such that $\frac{\partial f_{\Delta,\tau'}}{\partial y_l}\left(b\right) \neq 0$. Choose $b_i \in K^\times$, for $1 \leq i \leq r$ and set

$$\widetilde{b} = (b_1, \dots, b_r, b_{r+1}, \dots, b_n) \in \left(K^\times\right)^n.$$

Put $f_{\tau'}(x) = \sum_{m \in \tau'} c_m x^m$. Then

$$f_{\tau'} \circ \sigma_\Delta(y) = f_{\Delta,\tau'}(y) \prod_{j=1}^{r} y_j^{d(a_j)}. \tag{2.7}$$

Hence $f_{\tau'} \circ \sigma_\Delta\left(\widetilde{b}\right) = 0$. Since $\sigma_\Delta : (K^\times)^n \to T^n(K)$ is an isomorphism of K-analytic manifolds, the non-degeneracy of f implies $\nabla\left(f_{\tau'} \circ \sigma_\Delta\right)\left(\widetilde{b}\right) \neq 0$. Since $f_{\Delta,\tau'}(y) \in K\left[y_{r+1}, y_{r+1}^{-1}, \ldots, y_n, y_n^{-1}\right]$, we have $f_{\Delta,\tau'}\left(\widetilde{b}\right) = f_{\Delta,\tau'}(b) = 0$ and $\frac{\partial f_{\Delta,\tau'}}{\partial y_l}\left(\widetilde{b}\right) = \frac{\partial f_{\Delta,\tau'}}{\partial y_l}(b)$ for $r+1 \leq l \leq n$. By (2.7), we obtain

$$\frac{\partial\left(f_{\tau'} \circ \sigma_\Delta\right)}{\partial y_l}\left(\widetilde{b}\right) = \begin{cases} 0 & \text{if} \quad 1 \leq l \leq r \\ \frac{\partial f_{\Delta,\tau'}}{\partial y_l}(b) \prod_{j=1}^{r} b_j^{d(a_j)} & \text{if} \quad r+1 \leq l \leq n. \end{cases}$$

This implies the desired claim. By renaming the coordinates if necessary, we assume $\frac{\partial f_{\Delta,\tau'}}{\partial y_{r+1}}(b) \neq 0$. Since $\frac{\partial h_{\Delta,\tau'}}{\partial y_{r+1}}(b) = 0$, letting $y'_{r+1} = f_{\Delta,\tau'}(y) + h_{\Delta,\tau'}(y)$ and $y'_j = y_j$ for $j \neq r+1$, we see that $y' = (y_1, \ldots, y_r, y'_{r+1}, y_{r+2}, \ldots, y_n)$ becomes a coordinate system in a neighborhood V_b of b and obtain

$$\left(f \circ \sigma_\Delta\right)\left(y'\right) = \left(\prod_{j=1}^{r} y_j'^{d(a_j)}\right) y'_{r+1}. \tag{2.8}$$

From (2.4), we also obtain

$$\begin{aligned} &\sigma_\Delta^*\left(dx_1 \wedge \ldots \wedge dx_n\right) \\ &= \eta\left(y'\right)\left(\prod_{j=1}^{r} y_j'^{\|a_j\|-1}\right) dy'_1 \wedge \ldots \wedge dy'_n, \end{aligned} \tag{2.9}$$

where $\eta\left(y'\right)$ is a K-analytic function on V_b such that $\left|\eta\left(b'\right)\right|_K \neq 0$, where b' denotes the coordinates of b with respect to the new coordinate system y'. There exists an open neighborhood $V'_b \subset V_b$ of b' such that $\left|\eta\left(y'\right)\right|_K = \left|\eta\left(b'\right)\right|_K \neq 0$ for any $y' \in V'_b$.

Finally, suppose that $b \in \left(K_\Delta^\times\right)^n$, which implies $\sigma_\Delta(b) \neq 0$. If $f\left(\sigma_\Delta(b)\right) = 0$, by using the weak non-degeneracy of f with respect to

Γ_∞ there exists $i \in \{1, 2, \ldots, n\}$ such that $\frac{\partial f}{\partial x_i}(\sigma_\Delta(b)) \neq 0$. Now, since $\sigma_\Delta|_{T^n(K)}$ is a $K-$analytic isomorphism, we may define a new coordinate system $y' = (y'_1, \ldots, y'_n)$ on a neighborhood V_b of b as follows:

$$(y'_1, \ldots, y'_n) = (f \circ \sigma_\Delta, x_1 \circ \sigma_\Delta, \ldots, x_{i-1} \circ \sigma_\Delta, x_{i+1} \circ \sigma_\Delta, \ldots, x_n \circ \sigma_\Delta,).$$

With this new coordinate system we have

$$\sigma^*_\Delta (dx_1 \wedge \ldots \wedge dx_n) = (-1)^{i-1} \left[\frac{\partial f}{\partial x_i}(\sigma_\Delta(b))\right]^{-1} dy'_1 \wedge \ldots \wedge dy'_n.$$

Therefore

$$(f \circ \sigma_\Delta)(y) = y'_1, \tag{2.10}$$

$$\sigma^*_\Delta (dx_1 \wedge \ldots \wedge dx_n) = \eta\left(y'\right) dy'_1 \wedge \ldots \wedge dy'_n, \tag{2.11}$$

with $\eta(y')$ a K-analytic function defined on V_b such that $|\eta(b)|_K \neq 0$ and $|\eta(y')|_K = |\eta(b')|_K$for any $y \in V_b$.

If $f(\sigma_\Delta(b)) \neq 0$, we define a new coordinate system $y' = (y'_1, \ldots, y'_n)$ by $y'_i = x_i \circ \sigma_\Delta$. Then there exists a neighborhood V_b of b such that $|(f \circ \sigma_\Delta)(y)|_K = |(f \circ \sigma_\Delta)(b)|_K$ and $\sigma^*_\Delta (dx_1 \wedge \ldots \wedge dx_n) = dy_1 \wedge \ldots \wedge dy_n$ for any $y \in V_b$.

2.5. A hypothesis on the critical locus of f

We consider f as a regular function on $T^n(K)$. The critical set of f is $C_f := C_f(K) = T^n(K) \cap \{\nabla f(x) = 0\}$. Later on we will use the following hypothesis:

$$C_f \subset f^{-1}(0). \tag{H1}$$

Let $b \in X(\mathcal{F})$ and $a = \sigma(b)$. If $f(a) \neq 0$, by hypothesis H1, there is a local coordinate system of the form $y' = \left(f(a)^{-1} f(x) - 1, y_2, \ldots, y_n\right)$ in a neighborhood V_b of b, then

$$\begin{aligned} (f \circ \sigma)(y) &= f(a)\left(1 + y'_1\right), \\ \sigma^*(dx_1 \wedge \ldots \wedge dx_n) &= \eta\left(y'\right) dy'_1 \wedge \ldots \wedge dy'_n, \end{aligned} \tag{2.12}$$

and $|\eta(y')|_K = |\eta(b')|_K$ for any $y \in V_b$.

3. Local Zeta Functions

In this section we attach to a Laurent polynomial in n variables a local zeta function and show that it has a meromorphic continuation to the whole complex plane. We also give some results about the poles of the meromorphic continuation.

3.1. Quasicharacters

Let K be a $p-$adic field, i.e. $[K:\mathbb{Q}_p]<\infty$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers. Let R_K be the valuation ring of K, P_K the maximal ideal of R_K, and $\overline{K}=R_K/P_K$ the residue field of K. The cardinality of the residue field of K is denoted by q, thus $\overline{K}=\mathbb{F}_q$. For $z\in K$, $ord\,(z)\in\mathbb{Z}\cup\{+\infty\}$ denotes the valuation of z, and $|z|_K=q^{-ord(z)}$, $ac\ z=z\mathfrak{p}^{-ord(z)}$, where $\mathfrak{p}$ is a fixed uniformizing parameter of R_K.

We equip K^n with the norm $\|(x_1,\ldots,x_n)\|_K:=\max\left(|x_1|_K,\ldots,|x_n|_K\right)$. Then $(K^n,\|\cdot\|_K)$ is a complete metric space and the metric topology is equal to the product topology.

Let ω be a quasicharacter of $K^\times$, i.e. a continuous homomorphism from $K^\times$ into $\mathbb{C}^\times$. The set of quasicharacters form an Abelian group denoted as $\Omega\left(K^\times\right)$. We define an element ω_s of $\Omega\left(K^\times\right)$ for every $s\in\mathbb{C}$ as $\omega_s\left(x\right)=|x|_K^s=q^{-sord(x)}$. If, for every ω in $\Omega\left(K^\times\right)$, we choose $s\in\mathbb{C}$ satisfying $\omega\left(\mathfrak{p}\right)=q^{-s}$, then $\omega\left(x\right)=\omega_s\left(x\right)\chi\left(ac\ x\right)$ in which $\chi:=\omega\mid_{R_K^\times}$. We denote the conductor of χ as $c\left(\chi\right)$. Hence $\Omega\left(K^\times\right)$ is isomorphic to $\mathbb{C}/\left(2\pi\sqrt{-1}/\ln q\right)\times\left(R_K^\times\right)^*$, where $\left(R_K^\times\right)^*$ is the group of characters of $R_K^\times$, and $\Omega\left(K^\times\right)$ is a one dimensional complex manifold. We note that $\sigma\left(\omega\right):=\mathrm{Re}(s)$ depends only on ω, and $|\omega\left(x\right)|=\omega_{\sigma(\omega)}\left(x\right)$. Given an interval (a,b), we define an open subset of $\Omega\left(K^\times\right)$ by

$$\Omega_{(a,b)}\left(K^\times\right)=\left\{\omega\in\Omega\left(K^\times\right);\sigma\left(\omega\right)\in(a,b)\right\}.$$

For further details we refer the reader to [12].

3.2. Meromorphic continuation of local zeta functions

The following result will be used later frequently.

Lemma 3.1. *Take $a \in K$, $\omega \in \Omega(K^{\times})$ and $N \in \mathbb{Z} \setminus \{0\}$. Take also $n, e \in \mathbb{N}$, with $n > 0$, and put $\chi = \omega \mid_{R_K^{\times}}$. Then*

$$\int\limits_{a+\mathfrak{p}^e R_K \setminus \{0\}} \omega(z)^N |z|_K^{n-1} |dz| = \begin{cases} \left(1-q^{-1}\right) \frac{q^{-en-eNs}}{1-q^{-n-Ns}} & \text{if} \quad \begin{array}{c} a \in \mathfrak{p}^e R_K \\ \chi^N = 1 \end{array} \\ \\ q^{-e} \omega(a)^N |a|_K^{n-1} & \text{if} \quad \begin{array}{c} a \notin \mathfrak{p}^e R_K \\ \chi^N \mid_{U'} = 1 \end{array} \\ \\ 0 & \text{all other cases,} \end{cases}$$

in which $U' = 1 + \mathfrak{p}^e a^{-1} R_K$. In addition, the integral converges on $\mathrm{Re}(s) > \frac{-n}{N}$, *if $N > 0$, and on* $\mathrm{Re}(s) < \frac{n}{|N|}$, *if $N < 0$. In addition, if $N = 0$ the above integral converges to a non-zero value.*

Proof. The proof of the lemma is an easy variation of the one given for Lemma 8.2.1 in [12]. □

We recall that a locally constant function on K^n with compact support is called a *Bruhat-Schwartz function*, these functions form a $\mathbb{C}$-vector space denoted as $S(K^n)$.

For $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \setminus \{0\}$, set $\|a\| = a_1 + \ldots + a_n$ as before, and

$$\mathcal{P}(a) := \begin{cases} \left\{ -\frac{\|a\|}{d(a)} + \frac{2\pi\sqrt{-1}\mathbb{Z}}{d(a)\ln q} \right\} & \text{if} \quad d(a) \neq 0 \\ \\ \varnothing & \text{if} \quad d(a) = 0. \end{cases}$$

Let $\mathcal{F}$ be the fixed simple fan subordinated to Γ_∞ as before. Denote by *edges*$(\mathcal{F})$, the set of all edges of the cones in $\mathcal{F}$ as before. Set

$$A(\mathcal{F}) := \bigcup_{\substack{a \in edges(\mathcal{F}) \\ d(a) \neq 0}} \left\{ \frac{\|a\|}{-d(a)};\ d(a) < 0 \right\},$$

$$B(\mathcal{F}) := \bigcup_{\substack{a \in edges(\mathcal{F}) \\ d(a) \neq 0}} \left\{ \frac{\|a\|}{-d(a)};\ d(a) > 0 \right\},$$

$$\alpha := \alpha\left(\mathcal{F}\right) = \begin{cases} \min_{\gamma \in A(\mathcal{F})} \gamma, & \text{if} \quad A\left(\mathcal{F}\right) \neq \emptyset \\ \\ +\infty, & \text{if} \quad A\left(\mathcal{F}\right) = \emptyset, \end{cases}$$

and

$$\beta := \beta\left(\mathcal{F}\right) = \max_{\gamma \in B(\mathcal{F}) \cup \{-1\}} \gamma.$$

Definition 3.2. Given f a Laurent polynomial, Φ a Bruhat-Schwartz function, and $\omega \in \Omega\left(K^{\times}\right)$, we attach to these data the following local zeta function:

$$Z_{\Phi}\left(\omega, f\right) = Z_{\Phi}\left(s, \chi, f\right) = \int\limits_{T^{n}(K)} \Phi\left(x\right) \omega\left(f\left(x\right)\right) \left|dx\right|,$$

where $|dx|$ is the normalized Haar measure of K^n, which is the measure induced by an n-degree differential form dx.

Theorem 3.3. *Let f be weakly non- degenerate Laurent polynomial with respect to Γ_{∞}, and let $\mathcal{F}$ be a fixed simple, non trivial, fan subordinated to Γ_{∞}. Then the following assertions hold:*
(i) $Z_{\Phi}\left(\omega, f\right)$ converges for $\omega \in \Omega_{(\beta,\alpha)}\left(K^{\times}\right)$.
(ii) $Z_{\Phi}\left(\omega, f\right)$ has a meromorphic continuation to $\Omega\left(K^{\times}\right)$ as a rational function of $\omega\left(q\right)$, and the poles belong to

$$\bigcup_{a \in edges(\mathcal{F})} \mathcal{P}(a) \cup \left\{-1 + \frac{2\pi\sqrt{-1}\mathbb{Z}}{\ln q}\right\}.$$

In addition, the multiplicity of any pole is $\leq n$.

Proof. We pick a pair $\left(X\left(\mathcal{F}\right), \sigma\right)$ as in Section 2.4 and use all the notation introduced there. By using the fact that $\sigma : \sigma^{-1}\left(T^{n}\left(K\right)\right) \rightarrow T^{n}\left(K\right)$ is a K-analytic isomorphism, we have

$$\begin{aligned} Z_{\Phi}(\omega, f) &= \int\limits_{T^{n}(K)} \Phi\left(x\right) \omega\left(f\left(x\right)\right) \left|dx\right| \\ &= \int\limits_{\sigma^{-1}(T^{n}(K))} \Phi \circ \sigma\left(y\right) \omega\left(f \circ \sigma\left(y\right)\right) \left|\sigma^{*}\left(dx\right)\right|. \end{aligned}$$

Since σ is a proper map and S =supp(Φ) is compact open, we see that $\sigma^{-1}(S)$ is a compact subset of $X(\mathcal{F})$. For every point $b \in \sigma^{-1}(S)$ there exists a neighborhood V_b such that (2.2)-(2.11) hold, and by the compactness of $\sigma^{-1}(S)$, there is a finite covering of $\sigma^{-1}(S)$, say U_i, $i = 1, 2, \ldots, M$, where all these formulas hold. Now by taking $U_1, U_2 \smallsetminus U_1, \ldots, U_k \smallsetminus \cup_{i=1}^{k-1} U_i$, etc., we may assume that the U_i are already disjoint and non-empty. After embedding each of these subsets in K^n and decomposing them into cosets modulo P_K^e, where e is a fixed natural number, we get a disjoint open covering V_i, $i = 1, 2, \ldots, M'$ of $S \cap T^n(K)$ such that each $V_i = c_i + (P_K^e)^n$, $c_i \in K^n$ for $i = 1, 2, \ldots, M'$. In addition, we choose the open sets V_i's in such way that $\omega(\varepsilon(y))$ is constant on V_i.

Therefore $Z_\Phi(\omega, f)$ becomes a finite sum of integrals of the following types: First, if (2.3)-(2.4) or (2.5)-(2.6) hold, then

$$J_0(\omega) = q^{-e(n-r)} \Phi(\sigma(b))\, \omega(\varepsilon(b))\, |\eta(b)|_K \times \prod_{j=1}^{r} \int_{c_j + \mathfrak{p}^e R_K \smallsetminus \{0\}} \omega(y_j)^{d(a_j)} |y_j|_K^{\|a_j\|-1} |dy_j|, \tag{3.1}$$

where b is point in $X(\mathcal{F})$, $c = (c_1, \ldots, c_n) \in K^n$, $e \in \mathbb{N}$, and $1 \le r \le n$. We include (2.3)-(2.4) and (2.5)-(2.6) in the same case by allowing $r = n$; Second, if (2.8)-(2.9) hold, then

$$\begin{aligned} J_1(\omega) = q^{-e(n-r-1)} \Phi(\sigma(b'))\, |\eta(b')|_K & \times \left(\prod_{j=1}^{r} \int_{c_j + \mathfrak{p}^e R_K \smallsetminus \{0\}} \omega(y'_j)^{d(a_j)} |y'_j|_K^{\|a_j\|-1} |dy'_j| \right) \\ & \times \left(\int_{c_{r+1} + \mathfrak{p}^e R_K \smallsetminus \{0\}} \omega(y'_{r+1})\, |dy'_{r+1}| \right), \end{aligned} \tag{3.2}$$

where $1 \le r \le n-1$; Third, if (2.10)-(2.11) hold, then

$$J_2(\omega) = q^{-(n-1)e} \Phi(\sigma(b'))\, |\eta(b')|_K \int_{c_1 + \mathfrak{p}^e R_K} \omega(y'_1)\, |dy'_1|. \tag{3.3}$$

Finally, we note if $f \circ \sigma(b) \neq 0$, then by the discussion at the last paragraph of Section 2.4.1, the corresponding integral is a holomorphic function of s.

The parts (i)-(ii) follow by applying Lemma 3.1 to integrals (3.1)-(3.3). □

REMARK 3.4. Let $f(x) = \sum c_m x^m$ be a weakly non-degenerate Laurent polynomial with coefficients in $R_K^\times$. Assume that $\mathcal{F} = \mathcal{F}_0$, and that Φ is the characteristic function of R_K^n, and $\omega = \omega_s$. Then the first meet locus of any integer vector in $\mathbb{R}_+^n$ is a point, say $m_0 = (m_{0,1}, \ldots, m_{0,n})$. In addition,

$$\begin{aligned} Z_\Phi(\omega_s, f) &= \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} \int_{\mathfrak{p}^{a_1} R_K^\times \times\cdots\times \mathfrak{p}^{a_n} R_K^\times} |f(x)|_K^s \, |dx| \\ &= \left(1-q^{-1}\right)^n \sum_{(a_1,\ldots,a_n)\in\mathbb{N}^n} q^{-\|a\|-\langle a,m_0\rangle s} = \prod_{i=1}^n \left(\frac{1-q^{-1}}{1-q^{-1-m_{0,i}s}}\right). \end{aligned}$$

It is not difficult to show that in general case, we have

$$Z_\Phi(\omega, f) = \frac{L(q^{-s})}{\prod_{i=1}^n \left(1-q^{-1-m_{0,i}s}\right)},$$

where $L(q^{-s})$ is a polynomial in q^{-s} with rational coefficients.

REMARK 3.5. (i) Take $f(x,y) = x^{-2} - 2x^{-1}y + y^2 + x^2$, as in Example 2.6, then $\mathcal{F}$ is the trivial fan. Take Φ the characteristic function of $(\mathfrak{p}R_K)^2$, and $\omega = \omega_s$. Then

$$\begin{aligned} Z_\Phi(\omega_s, f) &= \int_{(\mathfrak{p}R_K \smallsetminus \{0\})^2} |f(x,y)|_K^s \, |dxdy| \\ &= \sum_{a=1}^\infty \sum_{b=1}^\infty \int_{\mathfrak{p}^a R_K^\times \times \mathfrak{p}^b R_K^\times} |f(x,y)|_K^s \, |dxdy| = \frac{\left(1-q^{-1}\right) q^{-2+2s}}{1-q^{-1+2s}}. \end{aligned}$$

Note that the integral converges for $\operatorname{Re}(s) < \frac{1}{2}$. Thus local zeta functions $Z_\Phi(\omega, f)$ may have poles with positive real parts.

(ii) In dimension 2, an explicit formula for $Z_\Phi(\omega_s, f)$, when Φ is the characteristic function of $(R_K \smallsetminus \{0\})^2$, similar to the one given in [5] holds. Let $\mathcal{L}$ be a simple conical partition of $\mathbb{R}^2 \smallsetminus \{0\}$ subordinated to Γ_∞. Then, the intersection of $\mathcal{L}$ with the first quadrant of $\mathbb{R}^2$ gives a simple conical partition of the first quadrant, in addition, the corresponding skeleton is the union the vectors in *edges* $(\mathcal{L})$ contained in the first quadrant and the vectors of a canonical basis of $\mathbb{R}^2$. This simple construction does not work in dimensions greater than two.

From now on, we will assume that $\mathcal{F}$ is not trivial.

3.3. Some additional remarks on poles of local zeta functions

As a consequence of Theorem 3.3, the mapping $\Phi \to Z_\Phi(\omega, f)$ defines a meromorphic distribution on $S(K^n)$. Denote this functional by $Z_\bullet(\omega)$. The set of poles of $Z_\bullet(\omega)$ is the set of poles of all the meromorphic functions $Z_\Phi(\omega, f)$ when Φ runs through $S(K^n)$.

Lemma 3.6. *Assume that $A(\mathcal{F}) \neq \emptyset$. Given $l \in \mathbb{N}$, with $1 \leq l \leq n$, define*

$$\mathcal{L}_l(\alpha) = \left\{ \begin{array}{c} \Delta \in \mathcal{F}; \Delta \textit{ has exactly } l \textit{ edges, } a_k, \\ \textit{satisfying } \frac{\|a_k\|}{-d(a_k)} = \alpha \textit{ for } k = 1, \dots, l. \end{array} \right\}$$

If $\max_l \{\mathcal{L}_l(\alpha) \neq \emptyset\} = n$, *then* $Z_\bullet(\omega)$ *has a pole* s *with multiplicity* n *satisfying* $\operatorname{Re}(s) = \alpha$.

Proof. We use all the notation introduced in Paragraph 2.4.1. Pick $\Phi > 0$ (later we will impose more restrictions on Φ) and $\omega = \omega_s$. To prove the result, it is sufficient to show that

$$\lim_{s \to \alpha} \left(1 - q^{s-\alpha}\right)^n Z_\Phi(\omega_s, f) > 0. \tag{3.4}$$

Since $Z_\Phi(\omega, f)$ is a finite sum of integrals of types $J_i(\omega_s)$, $i = 0, 1, 2$, see (3.1)-(3.3), it is sufficient to show the following:

$$\lim_{s \to \alpha} \left(1 - q^{s-\alpha}\right)^n J_i(\omega_s) \geq 0, \ i = 0, 1, 2 \tag{3.5}$$

and

$$\lim_{s \to \alpha} \left(1 - q^{s-\alpha}\right)^n J_0(\omega_s) > 0. \tag{3.6}$$

Let Δ be a cone in $\mathcal{L}_n(\alpha)$ spanned by a_i, $i = 1, \ldots, n$, with $\frac{\|a_i\|}{-d(a_i)} = \alpha$ for $i = 1, \ldots, n$. Take b in $X(\mathcal{F})$ to be the origin of the chart $(K_\Delta^n, \sigma_\Delta)$ corresponding to Δ and use formulas (2.3)-(2.4). Furthermore, we pick Φ in such a way that the neighborhood V_0 of the origin where (2.5)-(2.9) are valid be equal to $(\mathfrak{p}^e R_K)^n$. Then by using Lemma 3.1, $J_0(\omega_s)$ equals

$$\text{(3.7)} \qquad \frac{\left(1-q^{-1}\right)^n \Phi(\sigma(b)) \, |\varepsilon(b)|_K^s \, |\eta(b)|_K \, q^{-e\left\{\sum_{j=1}^n d(a_j)\right\}(s-\alpha)}}{(1-q^{s-\alpha})^n} \times \left(\prod_{j=1}^n \frac{1}{\prod\limits_{\varsigma_j \neq 1, \varsigma_j^{d(a_j)}=1} (1-\varsigma_j q^{s-\alpha})} \right).$$

Then

$$\text{(3.8)} \qquad \lim_{s\to\alpha} \left(1-q^{s-\alpha}\right)^n J_0(\omega_s) = \frac{\left(1-q^{-1}\right)^n \Phi(\sigma(b)) \, |\varepsilon(b)|_K^\alpha \, |\eta(b)|_K}{\prod\limits_{j=1}^n |d(a_j)|} > 0.$$

We note that the previous limit does not depend on the branch of the complex logarithm used to defined the complex power of q. By using a similar reasoning, one verifies that

$$\lim_{s\to\alpha} \left(1-q^{s-\alpha}\right)^n J_1(\omega_s) = \lim_{s\to\alpha} \left(1-q^{s-\alpha}\right)^n J_2(\omega_s) = 0.$$

On the other hand, if $\Delta \notin \mathcal{L}_n(\alpha)$, then $\lim_{s\to\alpha} (1-q^{s-\alpha})^n J_i(\omega_s) = 0$, for $i = 0, 1, 2$. □

Lemma 3.7. *Assume that $B(\mathcal{F}) \neq \emptyset$. Given $l \in \mathbb{N}$, with $1 \leq l \leq n$, define*

$$\mathcal{M}_l(\beta) = \left\{ \begin{array}{c} \Delta \in \mathcal{F}; \Delta \text{ has exactly } l \text{ edges, } a_k, \text{satisfying} \\ \frac{\|a_k\|}{-d(a_k)} = \beta, \text{ for } k = 1, \ldots, l. \end{array} \right\}$$

If $\max_l \{\mathcal{M}_l(\beta_f) \neq \emptyset\} = n$, *then* $Z_\bullet(\omega)$ *has a pole* s *of multiplicity* n *satisfying* $\mathrm{Re}(s) = \beta$.

Proof. It is similar to the proof of Lemma 3.6. □

3.4. Volumes of tubes

The classical local zeta functions attached to polynomials are connected with the number of solutions of polynomial congruences. The local zeta functions attached to Laurent polynomials are connected with the volumes of certain tubes determined by the Laurent polynomial.

Theorem 3.8. *Let f be a Laurent polynomial which is weakly non-degenerate with respect to Γ_∞. Set for $m \in \mathbb{N} \smallsetminus \{0\}$,*

$$V_{-m}(f,\Phi) := vol\left(\left\{x \in supp(\Phi) \cap T^n(K) ; |f(x)|_K = q^{-m}\right\}\right)$$

and

$$V_m(f,\Phi) := vol\left(\left\{x \in supp(\Phi) \cap T^n(K) ; |f(x)|_K = q^{m}\right\}\right).$$

Then the following assertions hold.
(i) Assume that $Z_\bullet(\omega)$ has at least one pole with negative real part. Then for m big enough, $V_{-m}(f,\Phi)$ has an asymptotic expansion of the form

$$V_{-m}(f,\Phi) = \sum_\gamma c_m(\gamma,f)\, m^{j_\gamma} q^{\gamma m}$$

where γ runs through all of the poles of $Z_\Phi(s,\chi_{triv},f)$ for which $\mathrm{Re}(\gamma) \in B(\mathcal{F})$, $j_\gamma \leq$ (the multiplicity of γ)-1, and the $c_m(\gamma,f)$ are complex constants. Furthermore

$$V_{-m}(f,\Phi) \leq Cm^{n-1}q^{m\beta} \text{ for } m \geq 0,$$

where C is a positive constant.
(ii) Assume that $Z_\bullet(\omega)$ has at least one pole with positive real part. If $|f|_K$ is not bounded on supp$(\Phi) \cap T^n(K)$, then for m big enough, $V_m(f,\Phi)$ has an asymptotic expansion of the form

$$V_m(f,\Phi) = \sum_\gamma c_m(\gamma,f)\, m^{j_\gamma} q^{-\gamma m},$$

where γ runs through all of the poles of $Z_\Phi(s,\chi_{triv},f)$ for which $\mathrm{Re}(\gamma) \in A(\mathcal{F})$, $j_\gamma \leq$ (the multiplicity of γ)-1, and the $c_m(\gamma,f)$ are complex constants. Furthermore

$$V_m(f,\Phi) \leq Cm^{n-1}q^{-m\alpha}, \text{ for } m \geq 0,$$

where C is a positive constant.

PROOF. We first note that

$$\begin{aligned}
&Z_{\Phi}(s,\chi_{triv},f)\\
&= \int\limits_{T^n(K)} \Phi\left(x\right)\left|f\left(x\right)\right|_K^s\left|dx\right|\\
&= \sum_{m\in\mathbb{Z}} vol\left(\left\{x\in\operatorname{supp}\left(\Phi\right);\left|f\left(x\right)\right|_K=q^{-m}\right\}t^m\right),\text{ with } t:=q^{-s},
\end{aligned}$$

for $\beta < \mathrm{Re}(s) < \alpha$. Now, the announced results follow from Theorem 3.3 by expanding $Z_{\Phi}(s,\chi_{triv},f)$ into partial fractions over the complex numbers. Since two variables t, t^{-1} are involved in the calculations and since we will need this technique later, we present here some details. For the sake of simplicity, we give the proof of the case $n=2$, the generalization to arbitrary n is straightforward.

For $m\in\mathbb{Z}\setminus\{0\}$, we write $m=|m|\,sgn\left(m\right)=|m|\left(\pm1\right)$. We also set $U_f:=\left\{\varsigma\in\mathbb{C}:\varsigma^f=1\right\}$ for $f\in\mathbb{N}\setminus\{0\}$. By using the identity

$$1-q^{-e}t^{\pm f}=\left(1-q^{\frac{-e}{f}}t^{\pm1}\right)\prod_{\varsigma\in U_f\setminus\{1\}}\left(1-q^{\frac{-e}{f}}\varsigma t^{\pm1}\right),\ e,f\in\mathbb{N}\setminus\{0\},$$

we have

$$\frac{1}{1-q^{-\|a_k\|}t^{d(a_k)}}=\frac{1}{1-q^{-\|a_k\|}t^{|d(a_k)|(\pm1)}}=\sum_{\varsigma\in U_{|d(a_k)|}}c_{\varsigma}\left(\sum_{l=0}^{+\infty}q^{\frac{-\|a_k\|}{|d(a_k)|}l}\varsigma^l t^{\pm l}\right)$$

for some constants $c_{\varsigma}\in\mathbb{C}$. Note that $\pm l=l\left\{sgn(d\left(a_k\right))\right\}$. If $\frac{-\|a_i\|}{|d(a_i)|}\neq\frac{-\|a_j\|}{|d(a_j)|}$, then

$$\begin{aligned}
\frac{1}{\left(1-q^{-\|a_i\|}t^{d(a_i)}\right)\left(1-q^{-\|a_j\|}t^{d(a_j)}\right)}&=\sum_{\varsigma\in U_{|d(a_i)|}}d_{\varsigma}\left(\sum_{l=0}^{+\infty}q^{\frac{-\|a_i\|}{|d(a_i)|}l}\varsigma^l t^{\pm l}\right)\\
&\quad+\sum_{\varsigma\in U_{\left|d\left(a_j\right)\right|}}h_{\varsigma}\left(\sum_{l=0}^{+\infty}q^{\frac{-\left\|a_j\right\|}{\left|d\left(a_j\right)\right|}l}\varsigma^l t^{\pm l}\right)
\end{aligned}$$

for some constants $d_\varsigma, h_\varsigma \in \mathbb{C}$. If $\frac{-\|a_i\|}{|d(a_i)|} = \frac{-\|a_j\|}{|d(a_j)|}$, then

$$\frac{1}{\left(1-q^{-\|a_i\|}t^{d(a_i)}\right)\left(1-q^{-\|a_j\|}t^{d(a_j)}\right)}$$

$$= \sum_{\varsigma \in U_{|d(a_i)|} \cap U_{|d(a_j)|}} \left\{ \frac{d_\varsigma}{\left(1-q^{\frac{-\|a_i\|}{|d(a_i)|}}\varsigma t^{\pm 1}\right)^2} + \frac{f_\varsigma}{1-q^{\frac{-\|a_i\|}{|d(a_i)|}}\varsigma t^{\pm 1}} \right\}$$

$$+ \sum_{\substack{\varsigma \in U_{|d(a_i)|} \\ \varsigma \notin U_{|d(a_i)|} \cap U_{|d(a_j)|}}} g_\varsigma \left(\sum_{l=0}^{+\infty} q^{\frac{-\|a_i\|}{|d(a_i)|}l} \varsigma^l t^{\pm l} \right)$$

$$+ \sum_{\substack{\varsigma \in U_{|d(aj)|} \\ \varsigma \notin U_{|d(a_i)|} \cap U_{|d(a_j)|}}} h_\varsigma \left(\sum_{l=0}^{+\infty} q^{\frac{-\|a_j\|}{|d(a_j)|}l} \varsigma^l t^{\pm l} \right)$$

for some constants $d_\varsigma, f_\varsigma, g_\varsigma, h_\varsigma \in \mathbb{C}$. Note that

$$\frac{1}{\left(1-q^{\frac{-\|a_i\|}{|d(a_i)|}}\varsigma t^{\pm 1}\right)^2} = \sum_{l=0}^{+\infty} (l+1)\, q^{\frac{-\|a_i\|}{|d(a_i)|}l} \varsigma^l t^{\pm l}.$$

Therefore for m big enough,

$$V_{-m}(f,\Phi) = \sum_{\gamma} c_m(\gamma, f)\, m^{j_\gamma} q^{\gamma m} \tag{3.9}$$

where γ runs through all of the poles of $Z_\Phi(s, \chi_{triv}, f)$ such that $\operatorname{Re}(\gamma) \in B(\mathcal{F})$, $j_\gamma \leq$ (the multiplicity of γ)-1, and the $c(m,\gamma)$ are complex constants. The first part follows from (3.9). The second part is established in a similar form. $\square$

3.5. Vanishing of local zeta functions

THEOREM 3.9. *Let f be a weakly non-degenerate Laurent polynomial satisfying $C_f \subset f^{-1}(0)$. There exists a constant $e(\Phi) \in \mathbb{N}$, such that $Z_\Phi(s, \chi, f) = 0$ unless $c(\chi) \leq e(\Phi)$.*

PROOF. The proof follows from formulas (2.3)-(2.12), Lemma 3.1, by using the same argument given by Igusa for Theorem 8.4.1 in [12]. □

4. Oscillatory Integrals

In this section we extend Igusa's stationary phase method for p-adic oscillatory integrals ([11], [12], [3]) to the case of non-degenerate Laurent polynomials.

4.1. Additive characters

Given

$$z = \sum_{n=n_0}^{\infty} z_n p^n \in \mathbb{Q}_p, \text{ with } z_n \in \{0, \dots, p-1\} \text{ and } z_{n_0} \neq 0,$$

we set

$$\{z\}_p := \begin{cases} 0 & \text{if} \quad n_0 \geq 0 \\ \\ \sum_{n=n_0}^{-1} z_n p^n & \text{if} \quad n_0 < 0, \end{cases}$$

the fractional part of z. Then $\exp(2\pi\sqrt{-1}\,\{z\}_p)$, $z \in \mathbb{Q}_p$, is an additive character on $\mathbb{Q}_p$ trivial on $\mathbb{Z}_p$ but not on $p^{-1}\mathbb{Z}_p$.

We recall that there exists an integer $d \geq 0$ such that $Tr_{K/\mathbb{Q}_p}(z) \in \mathbb{Z}_p$ for $|z|_K \leq q^d$ but $Tr_{K/\mathbb{Q}_p}(z_0) \notin \mathbb{Z}_p$ for some z_0 with $|z_0|_K = q^{d+1}$. The integer d is called *the exponent of the different* of $K/\mathbb{Q}_p$. It is known that $d \geq e-1$, where e is the ramification index of $K/\mathbb{Q}_p$, see e.g. [21, Chap. VIII, Corollary of Proposition 1]. The additive character

$$\varkappa(z) = \exp(2\pi\sqrt{-1}\left\{Tr_{K/\mathbb{Q}_p}(\mathfrak{p}^{-d}z)\right\}_p),\ z \in K,$$

is *a standard character* of K, i.e. $\varkappa$ is trivial on R_K but not on P_K^{-1}. For our purposes, it is more convenient to use

$$\Psi(z) = \exp(2\pi\sqrt{-1}\left\{Tr_{K/\mathbb{Q}_p}(z)\right\}_p),\ z \in K,$$

instead of $\varkappa(\cdot)$. This particular choice is due to the fact that we use Denef's approach for estimating oscillatory integrals, see [3, Proposition 1.4.4].

4.2. Asymptotic expansion of oscillatory integrals

Given $\Phi \in S(K^n)$ and f a Laurent polynomial as before, we define

$$E_\Phi(z,f) = E_\Phi(z) = \int\limits_{T^n(K)} \Phi(x)\,\Psi(zf(x))\,|dx|,$$

for $z = u\mathfrak{p}^{-m}$, with $u \in R_K^\times$, and $m \in \mathbb{Z}$. We call a such integral an *oscillatory integral.*

Let $\mathrm{Coeff}_{t^k} Z_\Phi(s,\chi,f)$ denote the coefficient c_k in the power expansion of $Z_\Phi(s,\chi,f)$ in the variable $t = q^{-s}$.

Proposition 4.1. *With the above notation,*

$$E_\Phi\left(u\mathfrak{p}^{-m}\right) = Z_\Phi(0,\chi_{triv}) + Coeff_{t^{m-1}} \frac{(t-q)\,Z_\Phi(s,\chi_{triv})}{(q-1)\,(1-t)} + \sum_{\chi\neq\chi_{triv}} g_{\chi^{-1}}\chi(u)\,Coeff_{t^{m-c(\chi)}} Z_\Phi(s,\chi),$$

where $c(\chi)$ denotes the conductor of χ, and g_χ denotes the Gaussian sum

$$g_\chi = (q-1)^{-1}\,q^{1-c(\chi)} \sum_{v\in\left(R_K/P_K^{c(\chi)}\right)^\times} \chi(v)\,\Psi\left(v/\mathfrak{p}^{c(\chi)}\right).$$

Proof. The proof is similar to the proof of Proposition 1.4.4 in [3]. □

Theorem 4.2. *Let f be a Laurent polynomial which is weakly non-degenerate with respect to Γ_∞. Assume that $C_f \subset f^{-1}(0)$. Let $\mathcal{F}$ be a nontrivial simple fan subordinated to Γ_∞ as before. Then the following assertions hold.*
(i) Assume that $Z_\bullet(\omega)$ has at least one pole with negative real part. Then for $|z|_K$ big enough $E_\Phi(z)$ is a finite $\mathbb{C}-$linear combination of functions of the form

$$\chi(ac\ z)\,|z|_K^\lambda \left(\log_q |z|_K\right)^{j_\lambda}$$

with coefficients independent of z, and $\lambda \in \mathbb{C}$ a pole with negative real part of $\left(1-q^{-s-1}\right) Z_\Phi(s,\chi_{triv})$ or of $Z_\Phi(s,\chi)$, $\chi \neq \chi_{triv}$, and with $j_\lambda \leq$(multiplicity

of pole λ) -1. Moreover all the poles λ, with negative real part, appear effectively in this linear combination.
(ii) Furthermore,

$$|E_\Phi(z)| \leq C\,(K)\,|z|_K^{\beta}\left(\log_q |z|_K\right)^{n-1}$$

for $|z|_K$ big enough, where $C\,(K)$ is a positive constant.
(iii) Assume that $Z_\bullet\,(\omega)$ has at least one pole with positive real part. Then for $|z|_K$ small enough $E_\Phi(z) - Z_\Phi(0, \chi_{triv})$ is a finite $\mathbb{C}-$linear combination of functions of the form

$$\chi\,(ac\ z)\,|z|_K^{\lambda}\left(\log_q |z|_K\right)^{j_\lambda}$$

with coefficients independent of z, and $\lambda \in \mathbb{C}$ a pole with positive real part of $Z_\Phi(s,\chi)$, and with $j_\lambda \leq$(multiplicity of pole λ)-1. Moreover all the poles λ, with positive real part, appear effectively in this linear combination.
(iv) Furthermore,

$$|E_\Phi(z) - Z_\Phi(0,\chi_{triv})| \leq C\,(K)\,|z|_K^{\alpha}\left(\log_q |z|_K\right)^{n-1}$$

for $|z|_K$ small enough, where $C\,(K)$ is a positive constant.

Proof. The results follow from Theorem 3.3, Proposition 4.1 and Theorem 3.9, by writing $Z_\Phi(s,\chi)$ in partial fractions, as in the proof of Theorem 3.8. □

In general $E_\Phi(z,f)$ cannot be expressed as a finite sum of exponential sums mod $\mathfrak{p}^m$. The following result shows that, under additional hypotheses, $E_\Phi(z,f)$ becomes an exponential sum mod $\mathfrak{p}^m$.

Corollary 4.3. *Let $f\,(x) = \frac{\widehat{f}(x)}{\prod_{i=1}^{r} x_i^{d_i}}$, $1 \leq r \leq n-1$, be a non-degenerate Laurent polynomial as before. Set*

$$S_m\,(f) := q^{-mn} \sum_{x \in \left(R_K^\times / P_K^m\right)^r \times \left(R_K / P_K^m\right)^{n-r}} \Psi\,(zf\,(x))\,,$$

where $z = u\mathfrak{p}^{-m}$, with $u \in R_K^\times$ and $m \geq 1$. Then, for m big enough,

$$|S_m\,(f)| \leq C m^{n-1} q^{m\beta}.$$

Proof. Take Φ to be the characteristic function of $\left(R_K^{\times}\right)^r \times R_K^{n-r}$, then $E_\Phi(z,f) = S_m(f)$. Now the result follows from Theorem 4.2 (i). $\square$

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