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Deformations of Trianguline B-Pairs and Zariski Density of Two Dimensional Crystalline Representations

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Abstract. The aims of this article are to study the deformation theory of trianguline B-pairs and to construct a p-adic family of two dimensional trianguline representations for any p-adic field. The deformation theory is the generalization of Bellaïche-Chenevier's and Chenevier's works in the $\mathbb{Q}_p$-case, where they used (φ, Γ)-modules over the Robba ring instead of using B-pairs. Generalizing and modifying Kisin's theory of X_{fs} for any p-adic field, we construct a p-adic family of two dimensional trianguline representations. As an application of these theories, we prove a theorem concerning the Zariski density of two dimensional crystalline representations for any p-adic field, which is a generalization of Colmez's and Kisin's theorem for the $\mathbb{Q}_p$-case.

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1. Introduction

1.1. Background

Let p be a prime number and let K be a p-adic field, i.e. finite extension of $\mathbb{Q}_p$. The theory of trianguline representations which form a class of p-adic representations of $G_K := \mathrm{Gal}(\overline{K}/K)$, in particular the theory of their p-adic families turns out to be very important for the study of the families of p-adic Galois representations over the eigenvarieties which parametrize some p-adic automorphic representations. Inspired by Kisin's p-adic Hodge theoretic study of Coleman-Mazur eigencurve [Ki03], Colmez [Co08] defined the notion of trianguline representations by using Fontaine's and Kedlaya's theory of (φ, Γ)-modules over the Robba ring in the study of p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$. Based on their works, Bellaïche-Chenevier [Bel-Ch09] and Chenevier [Ch09b] studied deformation theory of trianguline representations and p-adic families of trianguline representations. These theories are the fundamental tools for their applications of the eigenvarieties to some number theoretic problems, e.g. a construction of non trivial elements in some Selmer groups. Because all their studies are limited to the case $K = \mathbb{Q}_p$, we didn't have any results concerning the p-adic Hodge theoretic properties of eigenvarieties over a number field F except when F is $\mathbb{Q}$ or more generally is a number field in which p splits completely.

On the other hands, in [Na09], the author of this article generalized many results of [Co08] for any p-adic field K. The author proved some fundamental properties of trianguline representations and then classified two dimensional trianguline representations for any p-adic field, where we studied trianguline representations by using B-pairs, which were defined by Berger in [Be09], instead of using (φ, Γ)-modules over the Robba ring.

The aim of this article is to generalize Kisin's, Bellaïche-Chenevier's and Chenevier's works for any p-adic field K, more precisely, to develop deformation theory of trianguline representations and to construct a p-adic family of two dimensional trianguline representations for any p-adic field K. The author expects that these generalizations are also fundamental for applications to p-adic Hodge theoretic study of eigenvarieties for more general number fields.

As an application of these theories, we prove some theorems (see Theorem 1.6 and Theorem 1.7 in Introduction) concerning the Zariski density of two dimensional crystalline representations for any p-adic field. These results are the generalizations of a theorem of Colmez and Kisin for $K = \mathbb{Q}_p$, which played some crucial roles in the proof of p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ ([Co10], [Ki10], [Pa10]).

In the next article [Na11] which is based on the results of this article, we construct a p-adic family of d-dimensional trianguline representations for any $d \in \mathbb{Z}_{\geqq 1}$ and for any K and prove some theorems concerning the Zariski density of d-dimensional crystalline representations for any d and K.

1.2. Overview

Here, we explain the contents of each section of this article.

In § 2, we study the deformation theory of trianguline B-pairs, which is the generalization of the studies of [Bel-Ch09], [Ch09b] for any p-adic field.

In § 2.1, we recall the definition of B-pairs and some fundamental properties of trianguline B-pairs proved in [Na09] and then we extend these notions to those over Artin local rings. Let E be a suitable finite extension of $\mathbb{Q}_p$ which is sufficiently large as in Notation below. We recall the definition of E-B-pairs of G_K, which is the E-coefficient version of B-pairs. Set $\mathbf{B}_e := \mathbf{B}_{\mathrm{cris}}^{\varphi=1}$. An E-B-pair is a pair $W = (W_e, W_{\mathrm{dR}}^+)$ where W_e is a finite free $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E$-module with a continuous semi-linear G_K-action such that $W_{\mathrm{dR}}^+ \subseteq W_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$ is a G_K-stable $\mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E$-lattice of W_{dR}. The category of E-representations of G_K is embedded in the category of E-B-pairs by $V \mapsto W(V) := (\mathbf{B}_e \otimes_{\mathbb{Q}_p} V, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} V)$. We say that an E-B-pair is split trianguline if W is a successive extension of rank one E-B-pairs, i.e. W has a filtration $0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$ such that W_i is a saturated sub E-B-pair of W and W_i/W_{i-1} is a rank one E-B-pair for any $1 \leqq i \leqq n$. We say that W is trianguline if $W \otimes_E E'$ is a E'-split trianguline E'-B-pair for a finite extension E' of E. We say that an E-representation V is split trianguline (resp. trianguline) if $W(V)$ is split trianguline (resp. trianguline). By these definitions, to study trianguline E-B-pairs, we first need to classify rank one E-B-pairs and then we need to calculate extensions between them, which were studied in [Co08] for $K=\mathbb{Q}_p$ and in [Na09] for general K. In § 2.1, we recall these results which we need to study the deformation theory of trianguline E-B-pairs. We define the Artin local ring coefficient version of B-pairs. Let $\mathcal{C}_E$ be the category of Artin local E-algebras with the residue fields isomorphic to E. For $A\in\mathcal{C}_E$, we say that $W:=(W_e,W^+_{\mathrm{dR}})$ is an A-B-pair if W_e is a finite free $\mathbf{B}_e\otimes_{\mathbb{Q}_p}A$-module with a continuous semi-linear G_K-action and $W^+_{\mathrm{dR}}\subseteq W_{\mathrm{dR}}:=\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbf{B}_e}W_e$ is a G_K-stable $\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}A$-lattice. We generalize some results of [Na09] for A-B-pairs.

In § 2, we study two types of deformations of split trianguline E-B-pairs. In § 2.2, first we study the usual deformation for all E-B-pairs, which is the generalization of Mazur's deformation theory of p-adic Galois representations. Let W be an E-B-pair and $A\in\mathcal{C}_E$. We say that (W_A,ι) is a deformation of W over A if W_A is an A-B-pair and $\iota: W_A\otimes_A E\xrightarrow{\sim}W$ is an isomorphism. We define the deformation functor of W, $D_W:\mathcal{C}_E\to(\mathrm{Sets})$ by $D_W(A):=\{$equivalent classes of deformations (W_A,ι) of W over A $\}$. We prove the following proposition concerning the pro-representability and the formal smoothness and the dimension formula of D_W.

Proposition 1.1 (Corollary 2.31). *Let W be an E-B-pair of rank n. If W satisfies the following conditions,*

(1) $\mathrm{End}_{G_K}(W)=E$,

(2) $\mathrm{H}^2(G_K,\mathrm{ad}(W))=0$,

then the functor D_W is pro-representable by a pro-object R_W of $\mathcal{C}_E$ such that

$$R_W\xrightarrow{\sim}E[[T_1,\cdots,T_d]]\quad \textit{for } d:=[K:\mathbb{Q}_p]n^2+1.$$

In § 2.3, we study the other more important type of deformations, i.e. the trianguline deformations. Let W be a split trianguline E-B-pair of rank n and $\mathcal{T}:0\subseteq W_1\subseteq W_2\subseteq\cdots\subseteq W_{n-1}\subseteq W_n=W$ be a fixed triangulation of W. For $A\in\mathcal{C}_E$, we say that $(W_A,\iota,\mathcal{T}_A)$ is a trianguline deformation of $(W,\mathcal{T})$ over A if (W_A,ι) is a deformation of W over A and $\mathcal{T}_A:0\subseteq W_{1,A}\subseteq\cdots\subseteq W_{n-1,A}\subseteq W_{n,A}=W_A$ is an A-triangulation of W_A (i.e. $W_{i,A}$ is a saturated sub A-B-pair of W_A such that $W_{i,A}/W_{i-1,A}$ is a rank one A-B-pair for any i) such that $\iota(W_{i,A}\otimes_A E)=W_i$ for any i. We define the trianguline deformation functor $D_{W,\mathcal{T}}:\mathcal{C}_E\to$ (Sets) of $(W,\mathcal{T})$ by $D_{W,\mathcal{T}}(A):=\{$equivalent classes of trianguline deformations $(W_A,\iota,\mathcal{T}_A)$ of $(W,\mathcal{T})$ over $A\}$. We prove the following proposition concerning the pro-representability and the formal smoothness and the dimension formula of this functor, which is the generalization of Proposition 3.6 of [Ch09b] for any p-adic field. For the notations, see the main body of the article.

PROPOSITION 1.2 (Proposition 2.41). *Let W be a split trianguline E-B-pair of rank n with a triangulation $\mathcal{T}:0\subseteq W_1\subseteq\cdots\subseteq W_{n-1}\subseteq W_n=W$. We assume that $(W,\mathcal{T})$ satisfies the following conditions,*

(0) $\mathrm{End}_{G_K}(W)=E$,

(1) *For any $1\leqq i<j\leqq n$, $\delta_j/\delta_i\neq\prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma\in\mathcal{P}}\in\prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\leqq 0}$*

(2) *For any $1\leqq i<j\leqq n$, $\delta_i/\delta_j\neq|\mathrm{N}_{K/\mathbb{Q}_p}|\prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma\in\mathcal{P}}\in\prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\geqq 1}$,*

then $D_{W,\mathcal{T}}$ is pro-representable by a quotient ring $R_{W,\mathcal{T}}$ of R_W such that

$$R_{W,\mathcal{T}}\xrightarrow{\sim}E[[T_1,\cdots,T_{d_n}]]\ \textit{for}\ d_n:=\frac{n(n+1)}{2}[K:\mathbb{Q}_p]+1.$$

In § 2.4, we define the notion of benign E-B-pairs, which forms a special good class of split trianguline and potentially crystalline E-B-pairs, and prove a theorem concerning the tangent spaces of the deformation rings of this class. In § 2.4.1, we define the notion of benign E-B-pairs, in [Ch09b] this class is called generic, in this article we follow the terminology of [Ki10]. Let W be a potentially crystalline E-B-pair of rank n such that $W|_{G_L}$ is crystalline for a finite totally ramified abelian extension L of K, which we call a crystabelline representation. We assume that $\mathbf{D}^L_{\mathrm{cris}}(W):=(\mathbf{B}_{\mathrm{cris}}\otimes_{\mathbf{B}_e}W_e)^{G_L}=K_0\otimes_{\mathbb{Q}_p}Ee_1\oplus\cdots\oplus K_0\otimes_{\mathbb{Q}_p}Ee_n$ such that $K_0\otimes_{\mathbb{Q}_p}Ee_i$ are preserved by $(\varphi,\mathrm{Gal}(L/K))$-action and that $\varphi^f(e_i)=\alpha_ie_i$ for some $\alpha_i\in E^\times$, here $f:=[K_0:\mathbb{Q}_p]$ and K_0 is the maximal unramified extension of $\mathbb{Q}_p$ in K. We denote by $\{k_{1,\sigma}.k_{2,\sigma},\cdots,k_{n,\sigma}\}_{\sigma:K\hookrightarrow\overline{K}}$ the Hodge-Tate weights of W such that $k_{1,\sigma}\geqq k_{2,\sigma}\geqq\cdots\geqq k_{n,\sigma}$ for any $\sigma:K\hookrightarrow\overline{K}$. In this paper, we define the Hodge-Tate weight of $\mathbb{Q}_p(1)$ by 1. Let $\mathfrak{S}_n$ be the n-th permutation group. For any $\tau \in \mathfrak{S}_n$, we can define a filtration on $\mathbf{D}^L_{\mathrm{cris}}(W)$ by $0 \subseteq K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(1)} \subseteq K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(1)} \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(2)} \subseteq \cdots \subseteq K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(1)} \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(n-1)} \subseteq \mathbf{D}^L_{\mathrm{cris}}(W)$ by sub E-filtered (φ, G_K)-modules, where the filtration of $K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(1)} \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(i)}$ is the one induced from that of $\mathbf{D}^L_{\mathrm{dR}}(W) = L \otimes_{K_0} \mathbf{D}^L_{\mathrm{cris}}(W)$. By the equivalence between the category of potentially crystalline B-pairs and the category of filtered (φ, G_K)-modules, for any $\tau \in \mathfrak{S}_n$, we obtain the triangulation $\mathcal{T}_\tau : 0 \subseteq W_{\tau,1} \subseteq W_{\tau,2} \subseteq \cdots \subseteq W_{\tau,n} = W$ such that $W_{\tau,i}$ is potentially crystalline and $\mathbf{D}^L_{\mathrm{cris}}(W_{\tau,i}) \xrightarrow{\sim} K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(1)} \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ee_{\tau(i)}$ for any $1 \leqq i \leqq n$.

Under this situation, we define the notion of benign E-B-pairs as follows.

DEFINITION 1.3. Let W be a potentially crystalline E-B-pair of rank n as above. Then we say that W is benign if W satisfies the following;

(1) For any $i \neq j$, we have $\alpha_i/\alpha_j \neq 1, p^f, p^{-f}$,

(2) For any $\sigma : K \hookrightarrow \overline{K}$, we have $k_{1,\sigma} > k_{2,\sigma} > \cdots > k_{n-1,\sigma} > k_{n,\sigma}$,

(3) For any $\tau \in \mathfrak{S}_n$ and $1 \leqq i \leqq n$, the Hodge-Tate weights of $W_{\tau,i}$ are $\{k_{1,\sigma}, k_{2,\sigma}, \cdots, k_{i,\sigma}\}_{\sigma:K\hookrightarrow\overline{K}}$.

In § 2.4.2, we prove the main theorem of § 2. Let W be a benign E-B-pair of rank n as above. For any $\tau \in \mathfrak{S}_n$, we can define the trianguline deformation functor $D_{W,\mathcal{T}_\tau}$ as above. Let R_W be the universal deformation ring of D_W, and let $R_{W,\mathcal{T}_\tau}$ be the universal deformation ring of $D_{W,\mathcal{T}_\tau}$ for each $\tau \in \mathfrak{S}_n$. Let denote by $t(R_W)$ and $t(R_{W,\mathcal{T}_\tau})$ the tangent spaces of R_W and $R_{W,\mathcal{T}_\tau}$ respectively. For each $\tau \in \mathfrak{S}_n$, $t(R_{W,\mathcal{T}_\tau})$ is a sub E-vector space of $t(R_W)$. The main theorem of § 2 is the following, which is the generalization of Theorem 3.19 of [Ch09b] for any p-adic field.

THEOREM 1.4 (Theorem 2.62). *Let W be a benign E-B-pair of rank n, then we have an equality*

$$\sum_{\tau\in\mathfrak{S}_n} t(R_{W,\mathcal{T}_\tau}) = t(R_W).$$

This theorem is a crucial local result for the applications to some Zariski density theorems of local or global p-adic Galois representations. In fact, using this theorem for $K = \mathbb{Q}_p$, Chenevier [Ch09b] proved a theorem concerning the Zariski density of the unitary automorphic p-adic Galois representations in the universal deformation spaces of three dimensional conjugate-selfdual p-adic representations of G_F for any CM field F in which p splits completely. Moreover, this theorem is also a crucial result for the proof of Zariski density of crystalline representations in the universal deformation spaces of p-adic Galois representations of p-adic fields. In the rest of this paper § 3 and §4, we apply this theorem only for the two dimensional case. Using this theorem for $K = \mathbb{Q}_p$, Chenevier [Ch10] recently proved the Zariski density of crystalline representations for higher dimensional and $K = \mathbb{Q}_p$ case. In the next paper [Na11], the author uses this theorem for proving the Zariski density of crystalline representations for higher dimensional and any p-adic field case.

In § 3, we construct some p-adic families of two dimensional trianguline representations for any p-adic field. To construct these, we generalize Kisin's theory of finite slope subspace X_{fs} in [Ki03] for any p-adic field. As in the case of $\mathbb{Q}_p$ ([Ki03], [Ki10]), this family is essential for the proof of the Zariski density of two dimensional crystalline representations in §4 and for the study of the p-adic Hodge theoretic properties of Hilbert modular eigenvarieties.

In § 3.1, we prove some propositions concerning Banach G_K-modules which we need for the construction of p-adic families of trianguline representations. In particular, we show that these Banach G_K-modules can be obtained naturally from some almost $\mathbb{C}_p$-representations [Fo03]. For us, one of the important properties of these Banach G_K-modules is orthonormalizability as Banach modules over some Banach algebras. All these properties follow from some general facts of almost $\mathbb{C}_p$-representations.

In § 3.2, for any separated rigid analytic space X over E and for any finite free $\mathcal{O}_X$-module M with a continuous G_K-action and for any invertible function Y on X, we construct a Zariski closed subspace X_{fs} of X, which is "roughly" defined as the subspace of X consisting of the points $x \in X$ such that $\mathbf{D}^+_{\mathrm{cris}}(M(x))^{\varphi^f = Y(x)} \neq 0$, where $M(x)$ is the fiber of M at x. For the precise characterization of X_{fs}, see Theorem 3.9. This construction is the generalization of Kisin's X_{fs} in §5 of [Ki03] for any p-adic field. After obtaining the results in § 3.1, the construction and the proof is almost all the same as that of [Ki03], but a difference is that we need to consider all the embeddings $\tau : K \hookrightarrow \overline{K}$, which makes the situation more complicated. For convenience of the readers or the author, we choose to re-prove this construction in full detail.

In § 3.3, we apply this construction to the rigid analytic space associated to the universal deformation ring of a two dimensional mod p-representation of G_K. Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a two dimensional mod p representation of G_K, where $\mathbb{F}$ is the residue field of E. For simplicity, in this paper, we assume that $\mathrm{End}_{\mathbb{F}[G_K]}(\bar{\rho}) = \mathbb{F}$, then there exists the universal deformation ring $R_{\bar{\rho}}$ of $\bar{\rho}$, which is a complete noetherian local $\mathcal{O}$-algebra, where $\mathcal{O}$ is the integer ring of E. Let $\mathfrak{X}(\bar{\rho})$ be the rigid analytic space over E associated to $R_{\bar{\rho}}$. The universal deformation V^{univ} of $\bar{\rho}$ over $R_{\bar{\rho}}$ defines a rank two free $\mathcal{O}_{\mathfrak{X}(\bar{\rho})}$-module $\widetilde{V}^{\mathrm{univ}}$ with a continuous $\mathcal{O}_{\mathfrak{X}(\bar{\rho})}$-linear G_K-action. The space $\mathfrak{X}(\bar{\rho})$ parametrizes p-adic representations V of G_K whose reductions are isomorphic to $\bar{\rho}$ for some G_K-stable lattices of V. Let $\mathcal{W}$ be the rigid analytic space over E which represents the functor $D_{\mathcal{W}} : \{$ rigid analytic spaces over $E\} \to$ (Sets) defined by $D_{\mathcal{W}}(X) := \{\delta : \mathcal{O}_K^{\times} \to \Gamma(X, \mathcal{O}_X^{\times})$: continuous group homomorphisms $\}$ for each rigid analytic space X over E. Let $\delta^{\mathrm{univ}} : \mathcal{O}_K^{\times} \to \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}}^{\times})$ be the universal homomorphism. If we fix a uniformizer $\pi_K \in K$, there exists a unique character $\widetilde{\delta}^{\mathrm{univ}} : G_K^{\mathrm{ab}} \to \Gamma(\mathcal{W}, \mathcal{O}_{\mathcal{W}}^{\times})$ such that $\widetilde{\delta}^{\mathrm{univ}} \circ \mathrm{rec}_K|_{\mathcal{O}_K^{\times}} = \delta^{\mathrm{univ}}$ and $\widetilde{\delta}^{\mathrm{univ}} \circ \mathrm{rec}_K(\pi_K) = 1$, where $\mathrm{rec}_K : K \hookrightarrow G_K^{\mathrm{ab}}$ is the reciprocity map of the local class field theory. We denote by $X(\bar{\rho}) := \mathfrak{X}(\bar{\rho}) \times_E \mathcal{W} \times_E \mathbb{G}_{m,E}^{\mathrm{an}}$ and denote by $p_1 : X(\bar{\rho}) \to \mathfrak{X}(\bar{\rho})$, $p_2 : X(\bar{\rho}) \to \mathcal{W}$ and $p_3 : X(\bar{\rho}) \to \mathbb{G}_{m,E}^{\mathrm{an}}$ the canonical projections. For $x \in X(\bar{\rho})$, we denote by $E(x)$ the residue field at x which is a finite extension of E. Let $x = [V] \in \mathfrak{X}(\bar{\rho})$ be a point which corresponds to a two dimensional trianguline representation V with a triangulation $\mathcal{T} : 0 \subseteq W(\delta_1) \subseteq W(V \otimes_{E(x)} E')$ for some E', then we define a point $x_{(V,\mathcal{T})} := ([V], \delta_1|_{\mathcal{O}_K}^{\times}, \delta_1(\pi_K)) \in X(\bar{\rho})$. We define $M := p_1^*(\widetilde{V}^{\mathrm{univ}})((p_2^*\widetilde{\delta}^{\mathrm{univ}})^{-1})$ a rank two $\mathcal{O}_{X(\bar{\rho})}$-module with a continuous $\mathcal{O}_{X(\bar{\rho})}$-linear G_K-action. Let $Y := p_3^*(T) \in \mathcal{O}_{X(\bar{\rho})}^{\times}$ be the pullback of the canonical coordinate T of $\mathbb{G}_{m,E}^{\mathrm{an}}$. If we apply the construction of X_{fs} to the triple $(X(\bar{\rho}), M, Y)$, we obtain a Zariski closed subspace $X(\bar{\rho})_{fs}$ of $X(\bar{\rho})$, which we denote by $\mathcal{E}(\bar{\rho}) := X(\bar{\rho})_{fs}$. The main result of § 3 is the following theorem (see Theorem 3.17 and Theorem 3.22 for more precise statements), which is a generalization of Proposition 10.4 and 10.6 of [Ki03].

Theorem 1.5. *$\mathcal{E}(\bar{\rho})$ satisfies the following properties.*

(1) *For any point $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$, V_x is a trianguline representation.*

(2) *Conversely, if $x = [V_x] \in \mathfrak{X}(\bar{\rho})$ is a point such that V_x is a split trianguline $E(x)$-representation with a triangulation $\mathcal{T} : 0 \subseteq W(\delta_1) \subseteq W(V_x)$ satisfying all the conditions in Proposition 1.2, then the point $x_{(V_x,\mathcal{T})} \in X(\bar{\rho})$ defined as above is contained in $\mathcal{E}(\bar{\rho})$.*

(3) *For each point $x_{(V_x,\mathcal{T})}$ as in (2), there exists an isomorphism $\hat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x_{(V_x,\mathcal{T})}} \xrightarrow{\sim} R_{V_x,\mathcal{T}}$, in particular $\mathcal{E}(\bar{\rho})$ is smooth of its dimension $3[K : \mathbb{Q}_p] + 1$ at such points.*

In the next paper [Na11], the author will generalize all these results for higher dimensional case.

In the final section § 4, as an application of § 2 (in the two dimensional case) and of § 3, we prove the following theorems concerning the Zariski density of two dimensional crystalline representations. We denote by $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}} := \{x = [V_x] \in \mathfrak{X}(\bar{\rho}) | V_x$ is crystalline with the Hodge-Tate weights $\{k_{1,\sigma}, k_{2,\sigma}\}_{\sigma:K\hookrightarrow\overline{K}}$ such that $k_{1,\sigma} \neq k_{2,\sigma}$ for any $\sigma\}$, $\mathfrak{X}(\bar{\rho})_{\mathrm{b}} := \{x \in \mathfrak{X}(\bar{\rho}) | V_x$ is crystalline and benign $\}$. We denote by $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}}$ the Zariski closure of $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ in $\mathfrak{X}(\bar{\rho})$.

Theorem 1.6 (Theorem 4.13). *If $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is non empty, then $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is also non empty and the closure $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}}$ is a union of irreducible components of $\mathfrak{X}(\bar{\rho})$.*

Moreover, under the following assumptions, we prove the following stronger results concerning the Zariski density.

Theorem 1.7 (Theorem 4.15, Theorem 4.16). *Assume the following conditions,*

(0) $\mathrm{End}_{G_K}(\bar{\rho}) = \mathbb{F}$,

(1) $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ *is not empty.*

Moreover, assume one of the following two conditions (2), (3),

(2) $\zeta_p \notin K$ *(ζ_p is a primitive root of unity) and $\bar{\rho}$ satisfies one of the following conditions (i), (ii),*

(i) *If $\bar{\rho}$ is absolutely reducible, then* $\bar{\rho} \otimes_{\mathbb{F}} \overline{\mathbb{F}} \not\sim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \otimes \chi$ *for any* $\chi : G_K \to \overline{\mathbb{F}}^{\times}$, *where ω is the mod p cyclotomic character,*

(ii) *If $\bar{\rho}$ is absolutely irreducible, then* $[K(\zeta_p) : K] \neq 2$ *or* $\bar{\rho}|_{I_K} \otimes_{\mathbb{F}} \overline{\mathbb{F}} \not\sim \begin{pmatrix} \chi_2^i & 0 \\ 0 & \chi_2^{ip^f} \end{pmatrix}$ *such that* $\chi_2^{i(p^f-1)} = \omega|_{I_K}$, *where* $\chi_2 : I_K \to \overline{\mathbb{F}}^{\times}$ *is a fundamental character of the second kind,*

(3) $\zeta_p \in K$ *and* $p \neq 2$,

then we have an equality $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}} = \mathfrak{X}(\bar{\rho})$.

This theorem generalizes the results of Colmez [Co08] and Kisin [Ki10] for $K = \mathbb{Q}_p$ to the case of any p-adic field. As is stated in the above paragraph, Chenevier recently proved similar results for higher dimensional and for $K = \mathbb{Q}_p$, and the author will prove these theorems in full generality (i.e. for higher dimensional and for any p-adic field) in the next paper.

Notation. Let p be a prime number. Let K be a finite extension of $\mathbb{Q}_p$, $\overline{K}$ be a fixed algebraic closure of K, K_0 be the maximal unramified extension of $\mathbb{Q}_p$ in K, K^{nor} be the Galois closure of K in $\overline{K}$. Let $G_K := \mathrm{Gal}(\overline{K}/K)$ be the absolute Galois group of K. Let $\mathcal{O}_K$ be the ring of integers of K, $\pi_K \in \mathcal{O}_K$ be a uniformizer of K, $k := \mathcal{O}_K/\pi_K\mathcal{O}_K$ be the residue field of K, $q = p^f := \sharp k$ be the order of k. Denote by $\chi_p : G_K \to \mathbb{Z}_p^{\times}$ the p-adic cyclotomic character (i.e. $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ for any p^n-th roots of unity and for any $g \in G_K$). Let $\mathbb{C}_p := \hat{\overline{K}}$ be the p-adic completion of $\overline{K}$, which is an algebraically closed p-adically completed field, and $\mathcal{O}_{\mathbb{C}_p}$ be its ring of integers. We denote by v_p the normalized valuation on $\mathbb{C}_p^{\times}$ such that $v_p(p) = 1$. We denote by $|-|_p : \mathbb{C}_p \to \mathbb{R}_{\geqq 0}$ the absolute value such that $|p|_p = \frac{1}{p}$. Let E be a finite extension of $\mathbb{Q}_p$ in $\overline{K}$ such that $K^{\mathrm{nor}} \subseteq E$. In this paper, we use the notation E as a coefficient field of representations. We denote by $\mathcal{P} := \{\sigma : K \hookrightarrow \overline{K}\} = \{\sigma : K \hookrightarrow E\}$ the set of $\mathbb{Q}_p$-algebra homomorphisms from K to $\overline{K}$ (or E). Let $\chi_{\mathrm{LT}} : G_K \to \mathcal{O}_K^{\times}$ be the Lubin-Tate character associated with the fixed uniformizer π_K. Let $\mathrm{rec}_K : K^{\times} \to G_K^{\mathrm{ab}}$ be the reciprocity map of local class field theory normalized such that $\mathrm{rec}_K(\pi_K)$ is a lifting of the inverse of q-th power Frobenius on k. We remark that $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K : K^\times \to \mathcal{O}_K^\times$ satisfies $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K(\pi_K) = 1$ and $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K|_{\mathcal{O}_K^\times} = \mathrm{id}_{\mathcal{O}_K^\times}$.

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2. Deformation Theory of Trianguline B-Pairs

2.1. Review of B-pairs

2.1.1 E-B-pairs

We start by recalling the definition of E-B-pairs ([Be09], [Na09]) and then recall some fundamental properties of them established in [Na09]. First, we recall some rings of p-adic periods [Fo94] which we need for defining B-pairs. Let $\widetilde{\mathbf{E}}^+ := \varprojlim_n \mathcal{O}_{\mathbb{C}_p} \xrightarrow{\sim} \varprojlim_n \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, where the limits are taken with respect to the p-th power map. It is known that $\widetilde{\mathbf{E}}^+$ is a complete valuation ring of characteristic p whose valuation is defined by $v(x) := v_p(x^{(0)})$ for $x = (x^{(n)})_{n \geqq 0} \in \varprojlim_n \mathcal{O}_{\mathbb{C}_p}$. We fix a system of p^n-th roots of unity $\{\varepsilon^{(n)}\}_{n \geqq 0}$ such that $\varepsilon^{(0)} = 1$, $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$ for any n, $\varepsilon^{(1)} \neq 1$. Then $\varepsilon := (\varepsilon^{(n)})_{n \geqq 0}$ is an element of $\widetilde{\mathbf{E}}^+$ satisfying $v(\varepsilon - 1) = p/(p-1)$. The topological group G_K acts on this ring continuously in natural way. We put $\widetilde{\mathbf{A}}^+ := W(\widetilde{\mathbf{E}}^+)$, where we denote by $W(R)$ the ring of Witt vectors in R for any perfect ring R. We put $\widetilde{\mathbf{B}}^+ := \widetilde{\mathbf{A}}^+[\frac{1}{p}]$. These rings are equipped with the weak topology and also have a natural continuous G_K-action and have a Frobenius action φ. We have a G_K-equivariant surjection

$\theta : \widetilde{\mathbf{A}}^+ \to \mathcal{O}_{\mathbb{C}_p} : \sum_{k=0}^{\infty} p^k[x_k] \mapsto \sum_{k=0}^{\infty} p^k x_k^{(0)}$, where $[\ ] : \widetilde{\mathbf{E}}^+ \to \widetilde{\mathbf{A}}^+$ is the Teichmüller lift. Inverting p, we obtain a surjection $\widetilde{\mathbf{B}}^+ \to \mathbb{C}_p$. We put $\mathbf{B}^+_{\mathrm{dR}} := \varprojlim_n \widetilde{\mathbf{B}}^+/(\mathrm{Ker}(\theta))^n$, which is a complete discrete valuation ring with the residue field $\mathbb{C}_p$ and is equipped with the projective limit topology of the $\mathbb{Q}_p$-Banach spaces $\widetilde{\mathbf{B}}^+/(\mathrm{Ker}(\theta))^n$ $(n \geqq 1)$ whose $\mathbb{Z}_p$-lattice is the image of the natural map $\widetilde{\mathbf{A}}^+ \to \widetilde{\mathbf{B}}^+/(\mathrm{Ker}(\theta))^n$. Let $\mathbf{A}_{\max}$ be the p-adic completion of $\widetilde{\mathbf{A}}^+[\frac{[\widetilde{p}]}{p}]$, where $\widetilde{p} := (p^{(n)})$ is an element in $\widetilde{\mathbf{E}}^+$ such that $p^{(0)} = p, (p^{(n+1)})^p = p^{(n)}$ for any n. We put $\mathbf{B}^+_{\max} := \mathbf{A}_{\max}[\frac{1}{p}]$. $\mathbf{A}_{\max}$ and $\mathbf{B}^+_{\max}$ have a continuous G_K-action and a Frobenius actions φ. We have a natural G_K-equivariant embedding $K \otimes_{K_0} \mathbf{B}^+_{\max} \hookrightarrow \mathbf{B}^+_{\mathrm{dR}}$. If we put $t := \log([\varepsilon]) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n}$, then we can see that $t \in \mathbf{A}_{\max}$, $\varphi(t) = pt, g(t) = \chi_p(g)t$ for any $g \in G_K$ and $\mathrm{Ker}(\theta) = t\mathbf{B}^+_{\mathrm{dR}} \subset \mathbf{B}^+_{\mathrm{dR}}$ is the maximal ideal. If we put $\mathbf{B}_{\max} := \mathbf{B}^+_{\max}[\frac{1}{t}], \mathbf{B}_{\mathrm{dR}} := \mathbf{B}^+_{\mathrm{dR}}[\frac{1}{t}]$, we have a natural embedding $K \otimes_{K_0} \mathbf{B}_{\max} \hookrightarrow \mathbf{B}_{\mathrm{dR}}$. We put $\mathbf{B}_e := \mathbf{B}^{\varphi=1}_{\max}$ which is equipped with the locally convex inductive limit topology of $\mathbf{B}_e = \cup_n (\frac{1}{t^n}\mathbf{B}^+_{\max})^{\varphi=1}$, where the topology on each $(\frac{1}{t^n}\mathbf{B}^+_{\max})^{\varphi=1} = \frac{1}{t^n}\mathbf{B}^{+,\varphi=p^n}_{\max}$ is induced that of $\mathbf{B}^+_{\max}$. We put $\mathrm{Fil}^i\mathbf{B}_{\mathrm{dR}} := t^i\mathbf{B}^+_{\mathrm{dR}}$ for any $i \in \mathbb{Z}$. On $\mathbf{B}_{\mathrm{dR}}$, we also equipped with the locally convex inductive limit topology of $\mathbf{B}_{\mathrm{dR}} = \varinjlim_n \frac{1}{t^n}\mathbf{B}^+_{\mathrm{dR}}$.

In this paper, we fix a coefficient field of p-adic representations or B-pairs. Hence we start by recalling the definition of E-coefficient versions of p-adic representations and B-pairs.

DEFINITION 2.1. An E-representation of G_K is a finite dimensional E-vector space V with a continuous E-linear action of G_K. We call E-representation for simplicity when there is no risk of confusion about K.

DEFINITION 2.2. A pair $W := (W_e, W^+_{\mathrm{dR}})$ is an E-B-pair if

(1) W_e is a finite $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E$-module which is free over $\mathbf{B}_e$ with a continuous semi-linear G_K-action.

(2) W^+_{dR} is a G_K-stable finite $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-submodule of $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$ which generates $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$ as a $\mathbf{B}_{\mathrm{dR}}$-module.

We have an exact fully faithful functor $W(-)$ from the category of E-representations to the category of E-B-pairs defined by

$$W(V) := (\mathbf{B}_e \otimes_{\mathbb{Q}_p} V, \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$$

for any E-representation V, where the fully faithfulness follows from the Bloch-Kato's fundamental short exact sequence,

$$0 \to \mathbb{Q}_p \to \mathbf{B}_e \oplus \mathbf{B}^+_{\mathrm{dR}} \to \mathbf{B}_{\mathrm{dR}} \to 0.$$

We remark that W_e is a free $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E$-module and W^+_{dR} is a free $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-module by Lemma 1.7, 1.8 of [Na09]. We define the rank of W by

$$\mathrm{rank}(W) := \mathrm{rank}_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} E}(W_e).$$

For E-B-pairs $W_1 := (W_{1,e}, W^+_{1,\mathrm{dR}})$ and $W_2 := (W_{2,e}, W^+_{2,\mathrm{dR}})$, we define the tensor product of W_1 and W_2 by

$$W_1 \otimes W_2 := (W_{1,e} \otimes_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} E} W_{2,e}, W^+_{1,\mathrm{dR}} \otimes_{\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E} W^+_{2,\mathrm{dR}})$$

and define the dual of W_1 by

$$W_1^{\vee} := (\mathrm{Hom}_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} E}(W_{1,e}, B_e \otimes_{\mathbb{Q}_p} E), W^{+,\vee}_{1,\mathrm{dR}})$$

where we define

$$W^{+,\vee}_{1,\mathrm{dR}} := \{f \in \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_{1,e}, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E) | f(W^+_{1,\mathrm{dR}}) \subseteq \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E\})$$

(remark: there is a mistake in the definition 1.9 of [Na09]). The category of E-B-pairs is not an abelian category. In particular, an inclusion $W_1 \hookrightarrow W_2$ does not have a quotient in the category of E-B-pairs in general. However we can always take the saturation

$$W_1^{\mathrm{sat}} := (W^{\mathrm{sat}}_{1.e}, W^{+,\mathrm{sat}}_{1,\mathrm{dR}})$$

such that W_1^{sat} sits in $W_1 \hookrightarrow W_1^{\mathrm{sat}} \hookrightarrow W_2$ and $W_{1,e} = W^{\mathrm{sat}}_{1,e}$ and W_2/W_1^{sat} is an E-B-pair (see Lemma 1.14 of [Na09]). We say that an inclusion $W_1 \hookrightarrow W_2$ is saturated if W_2/W_1 is an E-B-pair, i.e. $W_1 = W_1^{\mathrm{sat}}$.

Next, we recall the p-adic Hodge theory for B-pairs. Let $W = (W_e, W^+_{\mathrm{dR}})$ be an E-B-pair. We define

$$\mathbf{D}_{\mathrm{cris}}(W) := (\mathbf{B}_{\max} \otimes_{\mathbf{B}_e} W_e)^{G_K},\ \mathbf{D}^L_{\mathrm{cris}}(W) := (\mathbf{B}_{\max} \otimes_{\mathbf{B}_e} W_e)^{G_L}$$

for any finite extension L of K and define

$$\mathbf{D}_{\mathrm{dR}}(W) := (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e)^{G_K},\ \mathbf{D}_{\mathrm{HT}}(W) := (\mathbf{B}_{\mathrm{HT}} \otimes_{\mathbb{C}_p} (W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}))^{G_K},$$

here $\mathbf{B}_{\mathrm{HT}} := \mathbb{C}_p[T, T^{-1}]$ on which G_K acts by $g(aT^i) := \chi_p(g)^i g(a)T^i$ for any $g \in G_K, a \in \mathbb{C}_p, i \in \mathbb{Z}$.

DEFINITION 2.3. We say that W is crystalline (resp. de Rham, resp. Hodge-Tate) if $\dim_{K_*}\mathbf{D}_*(W) = [E : \mathbb{Q}_p]\mathrm{rank}(W)$ for $* =$ cris (resp. $* =$ dR, resp. $* =$ HT), where $K_* = K_0$ when $* =$ cris and $K_* = K$ when $* = \mathrm{dR}, \mathrm{HT}$. We say that W is potentially crystalline if $\dim_{L_0}(\mathbf{D}^L_{\mathrm{cris}}(W)) = [E : \mathbb{Q}_p]\mathrm{rank}(W)$ for a finite extension L of K, where L_0 is the maximal unramified extension of $\mathbb{Q}_p$ in L.

DEFINITION 2.4. Let L be a finite Galois extension of K. Set $G_{L/K} := \mathrm{Gal}(L/K)$. We say that D is an E-filtered $(\varphi, G_{L/K})$-module over K if

(1) D is a finite free $L_0 \otimes_{\mathbb{Q}_p} E$-module with a φ-semi-linear action $\varphi_D : D \xrightarrow{\sim} D$ and a semi-linear action of $G_{L/K}$ such that φ_D and the action of $G_{L/K}$ commute, where (φ-)semi-linear means that $\varphi_D(a \otimes b \cdot x) = \varphi(a) \otimes b \cdot \varphi_D(x)$, $g(a \otimes b \cdot x) = g(a) \otimes b \cdot g(x)$ for any $a \in L_0, b \in E, x \in D, g \in G_{L/K}$,

(2) $D_L := L \otimes_{L_0} D$ has a separated and exhausted decreasing $G_{L/K}$-stable filtration $\{\mathrm{Fil}^i D_L\}_{i\in\mathbb{Z}}$ by $L \otimes_{\mathbb{Q}_p} E$-submodules.

Let W be a potentially crystalline E-B-pair such that $W|_{G_L}$ is crystalline for a finite Galois extension L of K, then we define an E-filtered $(\varphi, G_{L/K})$-module's structure on $\mathbf{D}^L_{\mathrm{cris}}(W)$ as follows. First, $\mathbf{D}^L_{\mathrm{cris}}(W)$ has a Frobenius action induced from that on $\mathbf{B}_{\max}$ and has a $G_{L/K}$-action induced from those on $\mathbf{B}_{\max}$ and W_e. We define a filtration on $L \otimes_{L_0} \mathbf{D}^L_{\mathrm{cris}}(W) = L \otimes_K \mathbf{D}_{\mathrm{dR}}(W)$ by

$$\mathrm{Fil}^i(L \otimes_{L_0} \mathbf{D}^L_{\mathrm{cris}}(W)) := (L \otimes_K \mathbf{D}_{\mathrm{dR}}(W)) \cap t^i W^+_{\mathrm{dR}}$$

for any $i \in \mathbb{Z}$.

Let $D := L_0 e$ be a rank one $\mathbb{Q}_p$-filtered $(\varphi, G_{L/K})$-module with a base e, then we define $t_N(D) := v_p(\alpha)$ where $\varphi_D(e) = \alpha \cdot e$ and define $t_H(D) := i$ such that $\mathrm{Fil}^i D_L/\mathrm{Fil}^{i+1} D_L \neq 0$. For general D of rank d, we define $t_N(D) := t_N(\wedge^d D)$, $t_H(D) := t_H(\wedge^d D)$. We say that D is weakly admissible if $t_N(D) = t_H(D)$ and $t_N(D') \geqq t_H(D')$ for any sub $\mathbb{Q}_p$-filtered $(\varphi, G_{L/K})$-module D' of D.

Theorem 2.5. *Let L be a finite Galois extension of K, then we have the following results.*

(1) *The functor $W \mapsto \mathbf{D}^L_{\mathrm{cris}}(W)$ gives an equivalence of categories between the category of potentially crystalline E-B-pairs which are crystalline if restricted to G_L and the category of E-filtered $(\varphi, G_{L/K})$-modules over K.*

(2) *Restricting the above functor to E-representations, the functor $V \mapsto \mathbf{D}^L_{\mathrm{cris}}(V)$ gives an equivalence of categories between the category of potentially crystalline E-representations which are crystalline if restricted to G_L and the category of weakly admissible E-filtered $(\varphi, G_{L/K})$-modules over K.*

Proof. See Proposition 2.3.4 and Theorem 2.3.5 of [Be09] or Theorem 1.18 of [Na09]. □

Next, we recall the definition of trianguline E-B-pairs, whose deformation theory we study in detail in this chapter.

Definition 2.6. Let W be an E-B-pair of rank n, then we say that W is split trianguline if there exists a filtration

$$\mathcal{T} : 0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_n = W$$

by sub E-B-pairs such that W_i is saturated in W_{i+1} and W_{i+1}/W_i is a rank one E-B-pair for any $0 \leqq i \leqq n-1$. We say that W is trianguline if $W \otimes_E E'$, the base change of W to E', is a split trianguline E'-B-pair for a finite extension E' of E.

By this definition, to study split trianguline E-B-pairs, it is important to classify rank one E-B-pairs and calculate extension classes of rank one E-B pairs, which were studied in [Na09]. We recall some results concerning these.

THEOREM 2.7. *There exists a canonical one to one correspondence $\delta \mapsto W(\delta)$ between the set of continuous homomorphisms $\delta : K^{\times} \to E^{\times}$ and the set of isomorphism classes of rank one E-B-pairs.*

PROOF. See Proposition 3.1 of [Co08] for $K = \mathbb{Q}_p$ and Theorem 1.45 of [Na09] for general K. For the construction of $W(\delta)$, see §1.4 of [Na09]. □

This correspondence is compatible with the local class field theory, i.e. for any unitary homomorphism $\delta : K^{\times} \to \mathcal{O}_E^{\times}$, if we take the character $\widetilde{\delta} : G_K^{\mathrm{ab}} \to \mathcal{O}_E^{\times}$ satisfying $\widetilde{\delta} \circ \mathrm{rec}_K = \delta$, then we have a canonical isomorphism

$$W(\delta) \xrightarrow{\sim} W(E(\widetilde{\delta})).$$

This correspondence is also compatible with tensor products and with duals, i.e for continuous homomorphisms $\delta_1, \delta_2 : K^{\times} \to E^{\times}$, we have canonical isomorphisms

$$W(\delta_1) \otimes W(\delta_2) \xrightarrow{\sim} W(\delta_1\delta_2) \text{ and } W(\delta_1)^{\vee} \xrightarrow{\sim} W(\delta_1^{-1}).$$

There are some important examples of rank one E-B-pairs which we recall now. For any $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}$, we define a homomorphism

$$\prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma} : K^{\times} \to E^{\times} : y \mapsto \prod_{\sigma\in\mathcal{P}} \sigma(y)^{k_\sigma},$$

then we have an isomorphism

$$W(\prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}) \xrightarrow{\sim} (\mathbf{B}_e \otimes_{\mathbb{Q}_p} E, \oplus_{\sigma\in\mathcal{P}} t^{k_\sigma} \mathbf{B}_{\mathrm{dR}}^{+} \otimes_{K,\sigma} E)$$

(see Lemma 2.12 of [Na09]). Let $N_{K/\mathbb{Q}_p} : K^{\times} \to \mathbb{Q}_p^{\times}$ be the norm and $|-| : \mathbb{Q}_p^{\times} \to \mathbb{Q}^{\times} \hookrightarrow E^{\times}$ be the p-adic absolute value such that $|p| = \frac{1}{p}$, and we define $|N_{K/\mathbb{Q}_p}| : K^{\times} \to E^{\times}$ the composite of $N_{K/\mathbb{Q}_p}$ and $|-|$, then we have an isomorphism

$$W(|N_{K/\mathbb{Q}_p}| \prod_{\sigma\in\mathcal{P}} \sigma) \xrightarrow{\sim} W(E(\chi_p)),$$

which is the E-B-pair associated to the p-adic cyclotomic character χ_p. Next, we recall the definition and some properties of Galois cohomology of

E-B-pairs. For an E-B-pair $W := (W_e, W_{\mathrm{dR}}^+)$, we put $W_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$. We have natural inclusions $W_e \hookrightarrow W_{\mathrm{dR}}$ and $W_{\mathrm{dR}}^+ \hookrightarrow W_{\mathrm{dR}}$. We define the Galois cohomology $\mathrm{H}^i(G_K, W)$ of W as the cohomology of the continuous cochains of G_K with values in the complex

$$W_e \oplus W_{\mathrm{dR}}^+ \to W_{\mathrm{dR}} : (x, y) \mapsto x - y,$$

see the appendix of this article or §2.1 of [Na09] for the precise definition. As in the case of usual p-adic representations, we have the following isomorphisms of E-vector spaces

$$\mathrm{H}^0(G_K, W) \xrightarrow{\sim} \mathrm{Hom}_{G_K}(\mathbf{B}_E, W),\ \mathrm{H}^1(G_K, W) \xrightarrow{\sim} \mathrm{Ext}^1(\mathbf{B}_E, W),$$

where $\mathbf{B}_E := (\mathbf{B}_e \otimes_{\mathbb{Q}_p} E, \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E)$ is the trivial E-B-pair and $\mathrm{Hom}_{G_K}(-, -)$ is the group of homomorphisms of E-B-pairs and $\mathrm{Ext}^1(-,-)$ is the extension class group in the category of E-B-pairs. If V is an E-representation of G_K, we have a canonical isomorphism

$$\mathrm{H}^i(G_K, V) \xrightarrow{\sim} \mathrm{H}^i(G_K, W(V)),$$

which follows from the Bloch-Kato's fundamental short exact sequence. Moreover, we have the following theorem, the Euler-Poincaré characteristic formula and the Tate duality for B-pairs.

THEOREM 2.8. *Let W be an E-B-pair.*

(1) *For $i = 0, 1, 2$, $\mathrm{H}^i(G_K, W)$ is finite dimensional over E and $\mathrm{H}^i(G_K, W) = 0$ for $i \neq 0, 1, 2$.*

(2) $\sum_{i=0}^{2} (-1)^{i-1} \dim_E \mathrm{H}^i(G_K, W) = [K : \mathbb{Q}_p] \mathrm{rank}(W)$.

(3) *For any $i = 0, 1, 2$, there is a natural perfect pairing defined by cup product,*

$$\mathrm{H}^i(G_K, W) \times \mathrm{H}^{2-i}(G_K, W^\vee(\chi_p)) \to \mathrm{H}^2(G_K, W \otimes W^\vee(\chi_p)) \\ \to \mathrm{H}^2(G_K, W(E(\chi_p)) \xrightarrow{\sim} E,$$

where the last isomorphism is the Tate's trace map.

PROOF. See Theorem 5.9 and Theorem 5.10 in the appendix. □

REMARK 2.9. In [Li08], Liu proved all these results for the cohomology of (φ,Γ)-modules over the Robba ring. In the appendix of this article, we first prove the finiteness and the Euler-Poincaré formula (Theorem 5.9) for the Galois cohomology of B-pairs using the theory of almost $\mathbb{C}_p$-representations. Then, we prove the Tate duality (Theorem 5.10) for the Galois cohomology of B-pairs using the Liu's argument. After establishing these properties, we prove the comparison results (Theorem 5.11) between the cohomology of (φ,Γ)-modules with that of the corresponding B-pairs.

Using these formulae, we obtain the following dimension formulae of Galois cohomologies of rank one E-B-pairs.

PROPOSITION 2.10. *Let $\delta : K^{\times} \to E^{\times}$ be a continuous homomorphism, then we have*:

(1) $\mathrm{H}^0(G_K, W(\delta)) \xrightarrow{\sim} E$ *if* $\delta = \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *such that* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$, *and* $\mathrm{H}^0(G_K, W(\delta)) = 0$ *otherwise.*

(2) $\mathrm{H}^2(G_K, W(\delta)) \xrightarrow{\sim} E$ *if* $\delta = |N_{K/\mathbb{Q}_p}| \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *such that* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\geqq 1}$, *and* $\mathrm{H}^2(G_K, W(\delta)) = 0$ *otherwise.*

(3) $\dim_E \mathrm{H}^1(G_K, W(\delta)) = [K : \mathbb{Q}_p] + 1$ *if* $\delta = \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *such that* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$ *or* $\delta = |N_{K/\mathbb{Q}_p}| \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *such that* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\geqq 1}$, *and* $\dim_E \mathrm{H}^1(G_K, W(\delta)) = [K : \mathbb{Q}_p]$ *otherwise.*

PROOF. See Theorem 2.9 and Theorem 2.22 of [Co08] for $K = \mathbb{Q}_p$. For general K, the results can be proved by using Proposition 2.14 and Proposition 2.15 of [Na09] and Tate duality for B-pairs. □

2.1.2 B-pairs over Artin local rings

Here, we define B-pairs over Artin local rings, which we need to define the notion of deformations of E-B-pairs. Let $\mathcal{C}_E$ be the category of Artin local E-algebra A with the residue field E. The morphisms in $\mathcal{C}_E$ are given by local E-algebra homomorphisms. For $A \in \mathcal{C}_E$, we denote by $\mathfrak{m}_A$ the maximal ideal of A. We define the A-coefficient version of B-pairs as follows.

DEFINITION 2.11. We call a pair $W := (W_e, W^+_{\mathrm{dR}})$ an A-B-pair of G_K if

(1) W_e is a finite $\mathbf{B}_e \otimes_{\mathbb{Q}_p} A$-module which is flat over A and is free over $\mathbf{B}_e$, with a continuous semi-linear G_K-action.

(2) W^+_{dR} is a finite generated $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-submodule of $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$ which is stable by the G_K-action and which generates $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$ as a $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module such that $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ is flat over A.

For an A-B-pair $W := (W_e, W^+_{\mathrm{dR}})$, we put $W_{\mathrm{dR}} := \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e$.

We simply call an A-B-pair if there is no risk of confusing about K.

Lemma 2.12. *Let $W := (W_e, W^+_{\mathrm{dR}})$ be an A-B-pair. Then W_e is a finite free $\mathbf{B}_e \otimes_{\mathbb{Q}_p} A$-module, W^+_{dR} is a finite free $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module and $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ is a finite free $\mathbb{C}_p \otimes_{\mathbb{Q}_p} A$-module.*

Proof. First, we prove the assertion for W_e. Because the submodule $\mathfrak{m}_A W_e \subseteq W_e$ is a G_K-stable finite generated torsion free $\mathbf{B}_e$-module and because $\mathbf{B}_e$ is a Bézout domain by Proposition 1.1.9 of [Be08], $\mathfrak{m}_A W_e$ is a finite free $\mathbf{B}_e$-module by Lemma 2.4 of [Ke04]. By Lemma 2.1.4 of [Be08], the cokernel $W_e \otimes_A E$ is also a finite free $\mathbf{B}_e$-module (with an E-action). By Lemma 1.7 of [Na09], $W_e \otimes_A E$ is a finite free $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E$-module of some rank n. We take a $\mathbf{B}_e \otimes_{\mathbb{Q}_p} A$-linear morphism $f : (\mathbf{B}_e \otimes_{\mathbb{Q}_p} A)^n \to W_e$ which is a lift of a $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E$-linear isomorphism $(\mathbf{B}_e \otimes_{\mathbb{Q}_p} E)^n \xrightarrow{\sim} W_e \otimes_A E$. Because A is Artinian, f is surjective. Because W_e is A-flat, we have $\mathrm{Ker}(f) \otimes_A E = 0$, hence $\mathrm{Ker}(f) = 0$. Hence W_e is a free $\mathbf{B}_e \otimes_{\mathbb{Q}_p} A$-module.

Next, we prove that W^+_{dR} is a free $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module. Because W_e is a free $\mathbf{B}_e \otimes_{\mathbb{Q}_p} A$-module, W_{dR} is a free $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module, in particular this is flat over A. Because $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ is a flat A-module, $W_{\mathrm{dR}}/W^+_{\mathrm{dR}}$ is also a flat A-module. Hence W^+_{dR} is also flat over A. By the A-flatness of $W_{\mathrm{dR}}/W^+_{\mathrm{dR}}$, we have an inclusion $W^+_{\mathrm{dR}} \otimes_A E \hookrightarrow W_{\mathrm{dR}} \otimes_A E$, hence $W^+_{\mathrm{dR}} \otimes_A E$ is a finite generated torsion free $\mathbf{B}^+_{\mathrm{dR}}$-module, hence is a free $\mathbf{B}^+_{\mathrm{dR}}$-module. By Lemma 1.8 of [Na09], we can show that W^+_{dR} is a free $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module in the same way as in the case of W_e. The freeness over $\mathbb{C}_p \otimes_{\mathbb{Q}_p} A$ of $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ follows from the $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-freeness of W^+_{dR}. □

Definition 2.13. Let $W = (W_e, W^+_{\mathrm{dR}})$ be an A-B-pair. We define the rank of W by $\mathrm{rank}(W) := \mathrm{rank}_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} A}(W_e)$.

DEFINITION 2.14. Let $f : A \to A'$ be a morphism in $\mathcal{C}_E$, and let $W = (W_e, W_{\mathrm{dR}})$ be an A-B-pair. We define the base change of W to A' by

$$W \otimes_A A' := (W_e \otimes_A A', W^+_{\mathrm{dR}} \otimes_A A').$$

By Lemma 2.12, we can easily see that this is an A'-B-pair.

DEFINITION 2.15. Let $W_1 = (W_{e,1}, W^+_{\mathrm{dR},1})$, $W_2 = (W_{e,2}, W^+_{\mathrm{dR},2})$ be A-B-pairs. We define the tensor product of W_1 and W_2 by $W_1 \otimes W_2 := (W_{e,1} \otimes_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} A} W_{e,2}, W^+_{\mathrm{dR},1} \otimes_{\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A} W^+_{\mathrm{dR},2})$, and define the dual of W_1 by $W_1^{\vee} := (\mathrm{Hom}_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} A}(W_{e,1}, \mathbf{B}_e \otimes_{\mathbb{Q}_p} A), W^{+,\vee}_{\mathrm{dR},1})$. Here, $W^{+,\vee}_{\mathrm{dR},1} := \{f \in \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A}(W_{\mathrm{dR},1}, \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A) | f(W^+_{\mathrm{dR},1}) \subseteq \mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A\}$. By Lemma 2.12, we can easily see that these are A-B-pairs.

Next, we classify rank one A-B-pairs. Let $\delta : K^{\times} \to A^{\times}$ be a continuous homomorphism, then we define a rank one A-B-pair $W(\delta)$ as follows. Let $\bar{u} \in E^{\times}$ be the image of $u := \delta(\pi_K) \in A^{\times}$ by the canonical projection $A \to A/\mathfrak{m}_A = E$. We define a homomorphism $\delta_0 : K^{\times} \to A^{\times}$ such that $\delta_0|_{\mathcal{O}_K^{\times}} = \delta|_{\mathcal{O}_K^{\times}}$, $\delta_0(\pi_K) = u/\bar{u}$. Because $u/\bar{u} \in 1 + \mathfrak{m}_A$, $(u/\bar{u})^{p^n}$ $(n \to \infty)$ converges to $1 \in A^{\times}$. If we fix an isomorphism $K^{\times} \xrightarrow{\sim} \mathcal{O}_K^{\times} \times \mathbb{Z} : v\pi_K^n \mapsto (v, n)$ $(v \in \mathcal{O}_K^{\times})$, then δ_0 uniquely extends to a continuous homomorphism $\delta'_0 : \mathcal{O}_K^{\times} \times \hat{\mathbb{Z}} \to \mathcal{O}_K^{\times} \times \mathbb{Z}_p \to A^{\times}$, where the first map is induced by the natural projection $\hat{\mathbb{Z}} \to \mathbb{Z}_p$. By the local class field theory, then there exists a unique character $\widetilde{\delta}_0 : G_K^{\mathrm{ab}} \to A^{\times}$ such that $\delta_0 = \widetilde{\delta}_0 \circ \mathrm{rec}_K$, where $\mathrm{rec}_K : K^{\times} \to G_K^{\mathrm{ab}}$ is the reciprocity map which is normalized as in Notation. Using $\widetilde{\delta}_0$, we define an étale rank one A-B-pair $W(A(\widetilde{\delta}_0))$, which is the A-B-pair associated to the rank one A-representation $A(\widetilde{\delta}_0)$. Next, we define a non-étale rank one A-B-pair by using $\bar{u} \in E^{\times}$. For this, we first define a rank one E-filtered φ-module $D_{\bar{u}} := K_0 \otimes_{\mathbb{Q}_p} Ee_{\bar{u}}$ such that $\varphi^f(e_{\bar{u}}) := \bar{u}e_{\bar{u}}$ and $\mathrm{Fil}^0(K \otimes_{K_0} D_{\bar{u}}) := K \otimes_{K_0} D_{\bar{u}}$, $\mathrm{Fil}^1(K \otimes_{K_0} D_{\bar{u}}) := 0$. From this, we obtain the rank one crystalline E-B-pair $W(D_{\bar{u}})$ such that $\mathbf{D}_{\mathrm{cris}}(W(D_{\bar{u}})) \xrightarrow{\sim} D_{\bar{u}}$ which is pure of slope $\frac{v_p(\bar{u})}{f}$. By tensoring these, we define a rank one A-B-pair $W(\delta)$ by $W(\delta) := (W(D_{\bar{u}}) \otimes_E A) \otimes W(A(\widetilde{\delta}_0))$, which is pure of slope $\frac{v_p(\bar{u})}{f}$.

The following proposition is the A-coefficient version of Theorem 1.45 of [Na09].

PROPOSITION 2.16. *This construction $\delta \mapsto W(\delta)$ does not depend on the choice of uniformizer π_K, and gives a bijection between the set of continuous homomorphisms $\delta : K^\times \to A^\times$ and the set of isomorphism classes of rank one A-B-pairs.*

PROOF. The independence of the choice of uniformizer and the injection can be proved in the same way as in the proof of Theorem 1.45 of [Na09]. We prove the surjection. Let W be a rank one A-B-pair. As an E-B-pair, W is a successive extension of the rank one E-B-pair $W \otimes_A E$. By Lemma 1.42 of [Na09], $W \otimes_A E$ is pure of slope $\frac{n}{fe_E}$ for some $n \in \mathbb{Z}$. Then, W is also pure of slope $\frac{n}{fe_E}$ by Theorem 1.6.6 of [Ke08]. We define a rank one E-filtered φ-module $D_{\pi_E^n} := K_0 \otimes_{\mathbb{Q}_p} Ee_{\pi_E^n}$ in the same way as for $D_{\bar{u}}$, where π_E is a uniformizer of E. Then, $W \otimes (W(D_{\pi_E^n}) \otimes_E A)^\vee$ is pure of slope zero by Lemma 1.34 of [Na09]. Hence, there exists $\widetilde{\delta'} : G_K^{\mathrm{ab}} \to A^\times$ such that $W \otimes (W(D_{\pi_E^n}) \otimes_E A)^\vee \xrightarrow{\sim} W(A(\widetilde{\delta'}))$. We put $\delta' := \widetilde{\delta'} \circ \mathrm{rec}_K : K^\times \to A^\times$ and define $\delta : K^\times \to A^\times$ such that $\delta|_{\mathcal{O}_K^\times} := \delta'|_{\mathcal{O}_K^\times}$ and $\delta(\pi_K) := \delta'(\pi_K)\pi_E^n$. Then, we have an isomorphism $W \xrightarrow{\sim} W(\delta)$, which can be easily seen from the construction of $W(\delta)$. □

By the local class field theory, we have a canonical bijection $\delta \mapsto A(\widetilde{\delta})$ from the set of unitary continuous homomorphisms from $K^\times$ to $A^\times$ (here, "unitary" means that the image of the composition of δ with the projection $A^\times \to E^\times$ is contained in $\mathcal{O}_E^\times$) to the set of isomorphism class of rank one A-representations of G_K, where $\widetilde{\delta} : G_K^{ab} \to A^\times$ is the continuous homomorphism such that $\delta = \widetilde{\delta} \circ \mathrm{rec}_K$. By the definition of $W(\delta)$ and by the above proof, it is easy to see that there exists an isomorphism $W(\delta) \xrightarrow{\sim} W(A(\widetilde{\delta}))$ for any unitary homomorphism $\delta : K^\times \to A^\times$. Moreover, it is easy to see that, for any continuous homomorphisms $\delta_1, \delta_2 : K^\times \to A^\times$, we have isomorphisms $W(\delta_1) \otimes W(\delta_2) \xrightarrow{\sim} W(\delta_1\delta_2)$ and $W(\delta_1)^\vee \xrightarrow{\sim} W(\delta_1^{-1})$.

Next, we generalize the functor $\mathbf{D}_{\mathrm{cris}}$ to potentially crystalline A-B-pairs. First, we define the A-coefficient version of filtered (φ, G_K)-modules. Let L be a finite Galois extension of K, we denote by $G_{L/K} := \mathrm{Gal}(L/K)$.

DEFINITION 2.17. Let A be an object of $\mathcal{C}_E$. We say that D is an A-filtered $(\varphi, G_{L/K})$-module of K if D satisfies the following conditions.

(1) D is a finite $L_0 \otimes_{\mathbb{Q}_p} A$-module which is free as an A-module with a φ-semi-linear action $\varphi : D \xrightarrow{\sim} D$.

(2) $D_L := L \otimes_{L_0} D$ has a decreasing filtration $\mathrm{Fil}^i D_L$ by $L \otimes_{\mathbb{Q}_p} A$-submodules such that $\mathrm{Fil}^k D_L = 0$ and $\mathrm{Fil}^{-k} D_L = D_L$ for sufficiently large k and that $\mathrm{Fil}^k D_L / \mathrm{Fil}^{k+1} D_L$ are free A-modules for any k.

(3) $G_{L/K}$ acts on D by $L_0 \otimes_{\mathbb{Q}_p} A$-semi-linear automorphism which commutes with the action of φ and preserves the filtration.

REMARK 2.18. Using the φ-structure on D, we can easily see that D is a free $L_0 \otimes_{\mathbb{Q}_p} A$-module.

Let $W_A := (W_{A,e}, W^+_{A,\mathrm{dR}})$ be an A-B-pair such that $W_A|_{G_L}$ is crystalline as an E-B-pair for a finite Galois extension L of K. As in the case of E-B-pairs, we define $\mathbf{D}^L_{\mathrm{cris}}(W_A) := (\mathbf{B}_{\max} \otimes_{\mathbf{B}_e} W_e)^{G_L}$ with a φ-action induced from that on $\mathbf{B}_{\max}$, then the natural map $L \otimes_{L_0} \mathbf{D}^L_{\mathrm{cris}}(W_A) \to \mathbf{D}^L_{\mathrm{dR}}(W_A) := (\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e)^{G_L}$ is isomorphism. We define $\mathrm{Fil}^k \mathbf{D}^L_{\mathrm{dR}}(W_A) := \mathbf{D}^L_{\mathrm{dR}}(W_A) \cap t^k W^+_{\mathrm{dR}}$ for any $k \in \mathbb{Z}$. These are naturally equipped with a $G_{L/K}$-action.

LEMMA 2.19. *In the above situation, $\mathbf{D}^L_{\mathrm{cris}}(W_A)$ is an A-filtered $(\varphi, G_{L/K})$-module of K.*

PROOF. It suffices only to show the A-freeness of $\mathbf{D}^L_{\mathrm{cris}}(W_A)$, $\mathrm{Fil}^k \mathbf{D}^L_{\mathrm{dR}}(W_A)$, $\mathrm{Fil}^k \mathbf{D}^L_{\mathrm{dR}}(W_A)/\mathrm{Fil}^{k+1} \mathbf{D}^L_{\mathrm{dR}}(W_A)$. Here, we only prove the A-freeness of $\mathbf{D}^L_{\mathrm{cris}}(W_A)$, other cases can be proved in a similar way. By the exactness of $\mathbf{D}^L_{\mathrm{cris}}$ for E-B-pairs which are crystalline when restricted to G_L, we have a natural isomorphism $\mathbf{D}^L_{\mathrm{cris}}(W_A) \otimes_A N \xrightarrow{\sim} \mathbf{D}^L_{\mathrm{cris}}(W_A \otimes_A N)$ for any finite A-module N. From this, for any A-linear injection $N_1 \hookrightarrow N_2$ of finite A-modules, we have an inclusion $\mathbf{D}^L_{\mathrm{cris}}(W_A) \otimes_A N_1 = \mathbf{D}^L_{\mathrm{cris}}(W_A \otimes_A N_1) \hookrightarrow \mathbf{D}^L_{\mathrm{cris}}(W_A \otimes_A N_2) = \mathbf{D}^L_{\mathrm{cris}}(W_A) \otimes_A N_2$ because $W_{A,e}$ is A-flat. Hence, $\mathbf{D}^L_{\mathrm{cris}}(W_A)$ is A-flat. □

Conversely, let D be an A-filtered $(\varphi, G_{L/K})$-module of K, then we define $W_e(D) := (\mathbf{B}_{\mathrm{cris}} \otimes_{L_0} D)^{\varphi=1}$. We have a natural isomorphism $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e(D) \xrightarrow{\sim} \mathbf{B}_{\mathrm{dR}} \otimes_L D_L$. We define $W^+_{\mathrm{dR}}(D) := \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_L D_L) \subseteq \mathbf{B}_{\mathrm{dR}} \otimes_{\mathbf{B}_e} W_e(D)$. We write $W(D) := (W_e(D), W^+_{\mathrm{dR}}(D))$ which is a potentially crystalline E-B-pair with an A-action.

LEMMA 2.20. *In the above situation, $W(D)$ is a potentially crystalline A-B-pair.*

PROOF. It suffices to show the A-flatness of $W_e(D)$ and $W^+_{\mathrm{dR}}(D)/tW^+_{\mathrm{dR}}(D)$. We can prove these in the same way as in Lemma 2.19 by using the exactness of the functor $W(D)$ and the A-flatness of D and $\mathrm{Fil}^k D_L/\mathrm{Fil}^{k+1}D_L$ for any k. □

COROLLARY 2.21. *For any $A \in \mathcal{C}_E$, the functor $\mathbf{D}^L_{\mathrm{cris}}$ gives an equivalence of categories between the category of potentially crystalline A-B-pairs which are crystalline if restricted to G_L and the category of A-filtered $(\varphi, G_{L/K})$-modules of K.*

PROOF. This follows from Lemma 2.19 and Lemma 2.20 and Theorem 2.5. □

Next, we prove some lemmas which will be used in later sections.

Let W_A be an A-B-pair which is not potentially crystalline in general and L be a finite Galois extension of K, then we can define $\mathbf{D}^L_{\mathrm{cris}}(W_A)$ in the same way as in the case where W_A is potentially crystalline. This is an E-filtered $(\varphi, G_{L/K})$-module, but this may not be an A-filtered $(\varphi, G_{L/K})$-module in general.

LEMMA 2.22. *Let W be an E-B-pair. Let $D \subseteq \mathbf{D}_{\mathrm{cris}}(W)$ be a rank one sub E-filtered φ-module whose filtration is induced from that of $\mathbf{D}_{\mathrm{cris}}(W)$, then there exists a natural saturated inclusion $W(D) \hookrightarrow W$.*

PROOF. Twisting W by a suitable crystalline character of the form $\prod_{\sigma\in\mathcal{P}} \sigma(\chi_{\mathrm{LT}})^{k_\sigma}$, we may assume that $\mathrm{Fil}^0(D_K) = D_K$ and $\mathrm{Fil}^1(D_K) = 0$, where we put $D_K := K \otimes_{K_0} D$. We have natural inclusions $W(D)_e = (\mathbf{B}_{\max} \otimes_{K_0} D)^{\varphi=1} \subseteq (\mathbf{B}_{\max} \otimes_{K_0} \mathbf{D}_{\mathrm{cris}}(W))^{\varphi=1} \subseteq (\mathbf{B}_{\max} \otimes_{\mathbf{B}_e} W_e)^{\varphi=1} = W_e$ and, under the above assumption, $W^+_{\mathrm{dR}}(D) = \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_K D_K) = \mathbf{B}^+_{\mathrm{dR}} \otimes_K D_K \subseteq \mathrm{Fil}^0(\mathbf{B}_{\mathrm{dR}} \otimes_K \mathbf{D}_{\mathrm{dR}}(W)) \subseteq W^+_{\mathrm{dR}}$, which define an inclusion $W(D) \hookrightarrow W$. Hence, it suffices to show that this inclusion is saturated, i.e. suffices to show that we have $\mathbf{B}^+_{\mathrm{dR}} \otimes_K D_K = (\mathbf{B}_{\mathrm{dR}} \otimes_K D_K) \cap W^+_{\mathrm{dR}}$. We can write $(\mathbf{B}_{\mathrm{dR}} \otimes_K D_K) \cap W^+_{\mathrm{dR}} = \oplus_{\sigma\in\mathcal{P}} \frac{1}{t^{k_\sigma}} \mathbf{B}^+_{\mathrm{dR}} \otimes_{K,\sigma} D_{K,\sigma}$ for some $k_\sigma \in \mathbb{Z}_{\geqq 0}$, where we decompose D_K by $D_K \xrightarrow{\sim} \oplus_{\sigma\in\mathcal{P}} D_K \otimes_{K\otimes_{\mathbb{Q}_p} E, \sigma\otimes id_E} E =: \oplus_{\sigma\in\mathcal{P}} D_{K,\sigma}$. If $k_\sigma \geqq 1$ for some $\sigma \in \mathcal{P}$, then $D_{K,\sigma} \subseteq t^{k_\sigma} W^+_{\mathrm{dR}}$. Because the filtration on D is induced from $\mathbf{D}_{\mathrm{cris}}(W)$, this implies that $\mathrm{Fil}^{k_\sigma} D_{K,\sigma} = D_{K,\sigma}$, this contradicts to $\mathrm{Fil}^1 D_{K,\sigma} = 0$. Hence $k_\sigma = 0$ for any $\sigma \in \mathcal{P}$, which implies that $\mathbf{B}^+_{\mathrm{dR}} \otimes_K D_K = (\mathbf{B}_{\mathrm{dR}} \otimes_K D_K) \cap W^+_{\mathrm{dR}}$. □

Lemma 2.23. *Let W_A be an A-B-pair. Let $D \subseteq \mathbf{D}_{\mathrm{cris}}(W_A)$ be a sub E-filtered φ-module which is an A-filtered φ-module of rank one, where the filtration on D is the one induced from that of $\mathbf{D}_{\mathrm{cris}}(W_A)$. We assume that the natural map $D\otimes_A E \to \mathbf{D}_{\mathrm{cris}}(W_A \otimes_A E)$ remains an injection. Then, we have a natural injection of A-B-pairs $W(D) \hookrightarrow W_A$ such that the cokernel $W_A/W(D)$ is also an A-B-pair.*

Proof. In the same way as in the above proof, we have a natural injection $W(D) \hookrightarrow W_A$. Because the natural map $D\otimes_A E \to \mathbf{D}_{\mathrm{cris}}(W_A\otimes_A E)$ is an injection, we obtain an injection $W(D)\otimes_A E \xrightarrow{\sim} W(D\otimes_A E) \hookrightarrow W_A\otimes_A E$ and this injection is saturated by the above lemma. Hence, it suffices to show that if $W_1 \hookrightarrow W_2$ is an inclusion of A-B-pairs such that $W_1 \otimes_A E \to W_2 \otimes_A E$ remains to be injective and saturated, then the cokernel W_2/W_1 exists and is an A-B-pair. We put $W_{3,e}$, $W^+_{3,\mathrm{dR}}$ the cokernels of $W_{1,e} \hookrightarrow W_{2,e}$, $W^+_{1,\mathrm{dR}} \hookrightarrow W^+_{2,\mathrm{dR}}$ respectively. By Lemma 2.2.3 (i) of [Bel-Ch09], these are A-flat. Hence, it suffices to show that these are free over $\mathbf{B}_e$, $\mathbf{B}^+_{\mathrm{dR}}$ respectively. We can prove this claim in the same way as in Lemma 2.2.3 (iii) of [Bel-Ch09]. $\square$

2.2. Deformations of B-pairs

In this subsection, we develop the deformation theory of B-pairs, which is a natural generalization of Mazur's deformation theory of p-adic Galois representations.

Definition 2.24. Let A be an object in $\mathcal{C}_E$, and let W be an E-B-pair. We say that a pair (W_A, ι) is a deformation of W over A if W_A is an A-B-pair and $\iota : W_A \otimes_A E \xrightarrow{\sim} W$ is an isomorphism of E-B-pairs. Let (W_A, ι), (W'_A, ι') be two deformations of W over A. Then, we say that (W_A, ι) and (W'_A, ι') are equivalent if there exists an isomorphism $f : W_A \xrightarrow{\sim} W'_A$ of A-B-pairs which satisfies $\iota = \iota' \circ \bar{f}$, where $\bar{f} : W_A \otimes_A E \xrightarrow{\sim} W'_A \otimes_A E$ is the isomorphism naturally induced by f.

Definition 2.25. Let W be an E-B-pair, then we define the deformation functor D_W from the category $\mathcal{C}_E$ to the category of sets by

$$D_W(A) := \{ \text{ equivalent classes } (W_A, \iota) \text{ of deformations of } W \text{ over } A\}$$

for $A \in \mathcal{C}_E$.

We simply denote by W_A if there is no risk of confusing about ι.

Next, we prove the pro-representability of the functor D_W under suitable conditions. For this, we recall Schlessinger's criterion for pro-representability of functors from $\mathcal{C}_E$ to the category of sets. We call a morphism $f : A' \to A$ in $\mathcal{C}_E$ a small extension if it is surjective and the kernel $\mathrm{Ker}(f) = (t)$ is generated by a nonzero single element $t \in A'$ and $\mathrm{Ker}(f) \cdot \mathfrak{m}_{A'} = 0$. $E[\varepsilon]$ is the ring defined by $E[\varepsilon] := E[X]/(X^2) : \varepsilon \mapsto \overline{X}$.

THEOREM 2.26. *Let F be a functor from $\mathcal{C}_E$ to the category of sets such that $F(E)$ consists of a single element. For morphisms $A' \to A$, $A" \to A$ in $\mathcal{C}_E$, consider the natural map*

(1) $F(A' \times_A A") \to F(A') \times_{F(A)} F(A")$,

then F is pro-representable if and only if F satisfies properties (H_1), (H_2), (H_3), (H_4) below:

(H_1) (1) *is surjective if $A" \to A$ is surjective.*

(H_2) (1) *is bijective when $A = E$ and $A" = E[\varepsilon]$.*

(H_3) $\dim_E(t_F) < \infty$ (*where $t_F := F(E[\varepsilon])$ and, under the condition (H_2), it is known that t_F naturally has a structure of an E-vector space*).

(H_4) (1) *is bijective if $A' = A"$ and $A' \to A$ is a small extension.*

PROOF. See [Schl68] or §18 of [Ma97]. □

Using this criterion, we prove the pro-representability of D_W.

PROPOSITION 2.27. *Let W be an E-B-pair. If $\mathrm{End}_{G_K}(W) = E$, then D_W is pro-representable by a complete noetherian local E-algebra R_W with the residue field E.*

To prove this proposition, we first prove some lemmas.

LEMMA 2.28. *Let $\mathrm{ad}(W) := \mathrm{Hom}(W, W)(\xrightarrow{\sim} W \otimes W^{\vee})$ be the internal endomorphism of W, then there exists an isomorphism of E-vector spaces*

$$D_W(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}(W)).$$

PROOF. Let $W_{E[\varepsilon]} := (W_{E[\varepsilon],e}, W^+_{E[\varepsilon],\mathrm{dR}})$ be a deformation of W over $E[\varepsilon]$. From this, we define an element in $\mathrm{H}^1(G_K, \mathrm{ad}(W))$ as follows. Because we have natural isomorphisms $\varepsilon W_{E[\varepsilon],e} \xrightarrow{\sim} W_e$ and $W_{E[\varepsilon],e}/\varepsilon W_{E[\varepsilon],e} \xrightarrow{\sim} W_e$ (here we put $W := (W_e, W^+_{\mathrm{dR}})$), we have a natural exact sequence of $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E[G_K]$-modules

$$0 \to W_e \to W_{E[\varepsilon],e} \to W_e \to 0.$$

We fix an isomorphism of $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E$-modules $W_{E[\varepsilon],e} \xrightarrow{\sim} W_e e_1 \oplus W_e e_2$ such that first factor $W_e e_1$is equal to $\varepsilon W_{E[\varepsilon]}$ as $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E[G_K]$-module and that the above natural projection maps the second factor $W_e e_2$ to W_e by $xe_2 \mapsto x$ for any $x \in W_e$. We define a continuous one cocycle by

$$c_e : G_K \to \mathrm{Hom}_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e) \text{ by } g(ye_2) := c_e(g)(gy)e_1 + gye_2$$

for any $g \in G_K$ and $y \in W_e$. For W^+_{dR}, we fix an isomorphism $W^+_{E[\varepsilon],\mathrm{dR}} \xrightarrow{\sim} W^+_{\mathrm{dR}} e_1 \oplus W^+_{\mathrm{dR}} e'_2$ as in the case of W_e, then we define a one cocycle by

$$c_{\mathrm{dR}} : G_K \to \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W^+_{\mathrm{dR}}, W^+_{\mathrm{dR}}) \text{ by } g(ye'_2) := c_{\mathrm{dR}}(g)(gy)e_1 + gye'_2$$

for any $g \in G_K$ and $y \in W^+_{\mathrm{dR}}$. Next, we define $c \in \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}, W_{\mathrm{dR}})$ as follows. Tensoring $W_{E[\varepsilon],e}$ and $W^+_{E[\varepsilon],\mathrm{dR}}$ with $\mathbf{B}_{\mathrm{dR}}$ over $\mathbf{B}_e$ or $\mathbf{B}^+_{\mathrm{dR}}$, we have an isomorphism $f : W_{\mathrm{dR}} e_1 \oplus W_{\mathrm{dR}} e_2 \xrightarrow{\sim} W_{E[\varepsilon],\mathrm{dR}} \xrightarrow{\sim} W_{\mathrm{dR}} e_1 \oplus W_{\mathrm{dR}} e'_2$ of $\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-modules. We define

$$c : W_{\mathrm{dR}} \to W_{\mathrm{dR}} \text{ by } f(ye_2) := c(y)e_1 + ye'_2$$

for any $y \in W_{\mathrm{dR}}$. By the definition, the triple $(c_e,\, c_{\mathrm{dR}},\, c)$ satisfies

$$c_e(g) - c_{\mathrm{dR}}(g) = gc - c \text{ in } \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}, W_{\mathrm{dR}})$$

for any $g \in G_K$, i.e. the triple $(c_e, c_{\mathrm{dR}}, c)$ defines an element of $\mathrm{H}^1(G_K, \mathrm{ad}(W))$ by the definition of Galois cohomology of B-pairs (§ 2.1 of [Na09]), then it is standard to check that this definition is independent of the choice of a fixed isomorphism $W_{E[\varepsilon],e} \xrightarrow{\sim} W_e e_1 \oplus W_e e_2$, etc, and it is easy to check that this map defines an isomorphism $D_W(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}(W))$. □

Lemma 2.29. *Let W_A be a deformation of W over A. If $\mathrm{End}_{G_K}(W) = E$, then $\mathrm{End}_{G_K}(W_A) = A$.*

Proof. We prove this lemma by induction on the length of A. When $A = E$, this is trivial. We assume that the lemma is proved for the rings of length n and assume that A is of length $n+1$. We take a small extension $f : A \to A'$. Because we have $\mathrm{End}_{G_K}(W) = \mathrm{H}^0(G_K, W^\vee \otimes W)$, we have the following short exact sequence

$$0 \to \mathrm{Ker}(f) \otimes_E \mathrm{End}_{G_K}(W) \to \mathrm{End}_{G_K}(W_A) \to \mathrm{End}_{G_K}(W_A \otimes_A A').$$

From this and the induction hypothesis, we have

$$\begin{aligned} \mathrm{length}(\mathrm{End}_{G_K}(W_A)) &\leqq \mathrm{length}(\mathrm{End}_{G_K}(W_A \otimes_A A')) \\ &\quad + \mathrm{length}(\mathrm{Ker}(f) \otimes_E \mathrm{End}_{G_K}(W)) \\ &= \mathrm{length}(A') + 1 = \mathrm{length}(A). \end{aligned}$$

On the other hand, we have a natural inclusion $A \subseteq \mathrm{End}_{G_K}(W_A)$. Comparing the length, we obtain an equality $A = \mathrm{End}_{G_K}(W_A)$. □

Proof of Proposition 2.27. Let W be an E-B-pair of rank n satisfying that $\mathrm{End}_{G_K}(W) = E$. For this W, we check the conditions (H_i) $(i = 1 \sim 4)$ of Schlessinger's criterion. First, by Lemma 2.28, we have

$$\dim_E(D_W(E[\varepsilon])) = \dim_E(\mathrm{H}^1(G_K, \mathrm{ad}(W))) < \infty,$$

hence (H_3) is satisfied. Next we check the condition (H_1). Let $f : A' \to A$, $g : A" \to A$ be morphisms in $\mathcal{C}_E$ such that g is a surjection. Let $([W_{A'}], [W_{A"}])$ be an element in $D_W(A') \times_{D_W(A)} D_W(A")$. We take deformations $W_{A'} := (W_{A',e}, W^+_{A',\mathrm{dR}})$, $W_{A"} := (W_{A",e}, W^+_{A",\mathrm{dR}})$ over A' and $A"$ which are representatives of equivalent classes $[W_{A'}]$ and $[W_{A"}]$ respectively, then we have an isomorphism $h : W_{A'} \otimes_{A'} A \xrightarrow{\sim} W_{A"} \otimes_{A"} A =: W_A := (W_{A,e}, W^+_{A,\mathrm{dR}})$ which defines an equivalent class in $D_W(A)$. We fix a basis $e_1, \cdots, e_n$ of $W_{A',e}$ as a $\mathbf{B}_e \otimes_{\mathbb{Q}_p} A'$ -module and denote by $\overline{e}_1, \cdots, \overline{e}_n$ the basis of $W_{A',e} \otimes_{A'} A$ induced from $e_1, \cdots, e_n$. By the surjectivity of $g : A" \to A$ and by the $A"$-flatness of $W_{A",e}$, we can take a basis $\widetilde{e}_1, \cdots, \widetilde{e}_n$ of $W_{A",e}$ such that the basis $\overline{\widetilde{e}}_1, \cdots, \overline{\widetilde{e}}_n$ of $W_{A",e} \otimes_{A"} A$ induced from $\widetilde{e}_1, \cdots, \widetilde{e}_n$ satisfies $h(\overline{e}_i) = \overline{\widetilde{e}}_i$ for any i. If we define $W^{'"}_e$ by

$$W^{'"}_e := W_{A',e} \times_{W_{A,e}} W_{A",e} := \{(x, y) \in W_{A',e} \times W_{A",e} | h(\bar{x}) = \bar{y}\},$$

then $W_e^{'"}$ is a free $\mathbf{B}_e \otimes_{\mathbb{Q}_p} (A' \times_A A")$-module with a basis $(e_1, \widetilde{e}_1), \cdots, (e_n, \widetilde{e}_n)$. In the same way, we define $W_{\mathrm{dR}}^{'"+}$ by $W_{\mathrm{dR}}^{'"+} := W_{A',\mathrm{dR}}^{+} \times_{W_{A,\mathrm{dR}}^{+}} W_{A",\mathrm{dR}}^{+}$, which is a free $\mathbf{B}_{\mathrm{dR}}^{+} \otimes_{\mathbb{Q}_p} (A' \times_A A")$-module. If we put $W_{A'} \times_{W_A} W_{A"} := (W_e^{'"}, W_{\mathrm{dR}}^{'"+})$, then this is a $(A' \times_A A")$-B-pair which is a deformation of W over $A' \times_A A"$ such that the equivalent class $[W_{A'} \times_{W_A} W_{A"}] \in D_W(A' \times_A A")$ maps $([W_{A'}], [W_{A"}]) \in D_W(A') \times_{D_W(A)} D_W(A")$. Hence, we have checked the condition (H_1).

Finally, we prove that if $g : A" \to A$ is a surjectition, then the natural map $D_W(A' \times_A A") \to D_W(A') \times_{D_W(A)} D_W(A")$ is bijective, which proves the conditions (H_2) and (H_4), hence proves the pro-representability of D_W. Let $W_1^{'"}$, $W_2^{'"}$ be deformations of W over $A' \times_A A"$ such that $[W_1^{'"} \otimes_{A' \times_A A"} A'] = [W_2^{'"} \otimes_{A' \times_A A"} A']$ in $D_W(A')$ and $[W_1^{'"} \otimes_{A' \times_A A"} A"] = [W_2^{'"} \otimes_{A' \times_A A"} A"]$ in $D_W(A")$. Under this situation, we want to show that $[W_1^{'"}] = [W_2^{'"}]$ in $D_W(A' \times_A A")$. We put $W_{1A'} := W_1^{'"} \otimes_{A' \times_A A"} A'$, $W_{1A"} := W_1^{'"} \otimes_{A' \times_A A"} A"$, $W_{1A} := W_1^{'"} \otimes_{A' \times_A A"} A$, and similarly for $W_{2A'}$, $W_{2A"}$, W_{2A}, then we have natural isomorphisms $W_1^{'"} \xrightarrow{\sim} W_{1A'} \times_{W_{1A}} W_{1A"}$ and $W_2^{'"} \xrightarrow{\sim} W_{2A'} \times_{W_{2A}} W_{2A"}$ defined as in the previous paragraph. Because we have $[W_{1A'}] = [W_{2A'}]$ and $[W_{1A"}] = [W_{2A"}]$, we have isomorphisms $h' : W_{1A'} \xrightarrow{\sim} W_{2A'}$ and $h" : W_{1A"} \xrightarrow{\sim} W_{2A"}$. By the base change to A, we obtain an automorphism $\bar{h}' \circ \bar{h}"^{-1} : W_{2A} \xrightarrow{\sim} W_{1A} \xrightarrow{\sim} W_{2A}$. By Lemma 2.29 and by the surjectivity of $g : A"^{\times} \to A^{\times}$, we can find an automorphism $\widetilde{h} : W_{2A"} \xrightarrow{\sim} W_{2A"}$ such that $\bar{\widetilde{h}} = \bar{h}' \circ \bar{h}"^{-1}$. If we define a morphism

$$h^{'"} : W_{1A'} \times_{W_{1A}} W_{1A"} \to W_{2A"} \times_{W_{2A}} W_{2A'} : (x, y) \mapsto (h_1(x), \widetilde{h} \circ h_2(y)),$$

then we can see that this is well-defined and is isomorphism. Hence, we finish to prove the proposition. □

PROPOSITION 2.30. *Let $W := (W_e, W_{\mathrm{dR}}^{+})$ be an E-B-pair of rank n. If $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$, then the functor D_W is formally smooth.*

PROOF. Let $A' \to A$ be a small extension in $\mathcal{C}_E$, we denote the kernel by $I \subseteq A'$. Let $W_A := (W_{e,A}, W_{\mathrm{dR},A}^{+})$ be a deformation of W over A, then it suffices to show that there exists an A'-B-pair $W_{A'}$ such that $W_{A'} \otimes_{A'} A \xrightarrow{\sim} W_A$. We fix a basis of $W_{e,A}$ as a $\mathbf{B}_e \otimes_{\mathbb{Q}_p} A$-module. Using this basis and the G_K-action on $W_{e,A}$, we obtain a continuous one cocycle

$$\rho_e : G_K \to \mathrm{GL}_n(\mathbf{B}_e \otimes_{\mathbb{Q}_p} A).$$

In the same way, if we fix a basis of $W^+_{\mathrm{dR},A}$ as a $\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A$-module, we obtain a continuous one cocycle

$$\rho_{\mathrm{dR}} : G_K \to \mathrm{GL}_n(\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A).$$

From the canonical isomorphism $W_{e,A} \otimes_{\mathbf{B}_e} \mathbf{B}_{\mathrm{dR}} \xrightarrow{\sim} W^+_{\mathrm{dR},A} \otimes_{\mathbf{B}^+_{\mathrm{dR}}} \mathbf{B}_{\mathrm{dR}}$, we obtain a matrix $P \in \mathrm{GL}_n(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A)$ which satisfies

$$P\rho_e(g)g(P)^{-1} = \rho_{\mathrm{dR}}(g) \text{ for any } g \in G_K.$$

We fix an E-linear section $s : A \to A'$ of $A' \to A$ and fix a lifting $\widetilde{P} \in \mathrm{GL}_n(\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A')$ of P. Using this section, we obtain continuous liftings

$$\widetilde{\rho}_e := s \circ \rho_e : G_K \to \mathrm{GL}_n(\mathbf{B}_e \otimes_{\mathbb{Q}_p} A')$$

of ρ_e and

$$\widetilde{\rho}_{\mathrm{dR}} := s \circ \rho_{\mathrm{dR}} : G_K \to \mathrm{GL}_n(\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} A')$$

of ρ_{dR}. Using these liftings, we define

$$c_e : G_K \times G_K \to I \otimes_E \mathrm{Hom}_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e)$$

by

$$\begin{aligned}\widetilde{\rho}_e(g_1g_2)g_1(\widetilde{\rho}_e(g_2))^{-1}\widetilde{\rho}_e(g_1)^{-1} &= I_n + c_e(g_1, g_2) \in I_n + I \otimes_{A'} \mathrm{M}_n(\mathbf{B}_e \otimes_{\mathbb{Q}_p} A') \\ &= I_n + I \otimes_E \mathrm{Hom}_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e)\end{aligned}$$

for any $g_1, g_2 \in G_K$, where I_n is the identity matrix. In the same way, we define

$$c_{\mathrm{dR}} : G_K \times G_K \to I \otimes_E \mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W^+_{\mathrm{dR}}, W^+_{\mathrm{dR}})$$

by

$$\widetilde{\rho}_{\mathrm{dR}}(g_1g_2)g_1(\widetilde{\rho}_{\mathrm{dR}}(g_2))^{-1}\widetilde{\rho}_{\mathrm{dR}}(g_1)^{-1} = I_n + c_{\mathrm{dR}}(g_1, g_2).$$

We define

$$c : G_K \to I \otimes_E \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}, W_{\mathrm{dR}})$$

by

$$\widetilde{P}\widetilde{\rho}_e(g)g(\widetilde{P})^{-1}\widetilde{\rho}_{\mathrm{dR}}(g)^{-1} = I_n + c(g) \in I_n + I \otimes_E \mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E}(W_{\mathrm{dR}}, W_{\mathrm{dR}}).$$

These c_e and c_{dR} are continuous two cocycles, i.e. these satisfy

$$g_1c_*(g_2,g_3)-c_*(g_1g_2,g_3)+c_*(g_1,g_2g_3)-c_*(g_1,g_2)=0$$

for any $g_1,g_2,g_3\in G_K$ $(*=e,\mathrm{dR})$. Moreover, we can check that c_e and c_{dR} and c satisfy

$$c_e(g_1,g_2)-c_{\mathrm{dR}}(g_1,g_2)=g_1(c(g_2))-c(g_1g_2)+c(g_1)$$

for any $g_1,g_2,g_3\in G_K$, here we note that the isomorphism $\mathrm{Hom}_{\mathbf{B}_e\otimes_{\mathbb{Q}_p}E}(W_e, W_e)\otimes_{\mathbf{B}_e}\mathbf{B}_{\mathrm{dR}}\xrightarrow{\sim}\mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E}(W^+_{\mathrm{dR}},W^+_{\mathrm{dR}})\otimes_{\mathbf{B}^+_{\mathrm{dR}}}\mathbf{B}_{\mathrm{dR}}$ is given by $c\mapsto\bar{P}^{-1}c\bar{P}$, where $\bar{P}\in\mathrm{GL}_n(\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E)$ is the reduction of $P\in\mathrm{GL}_n(\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}A)$. By the definition of Galois cohomology of B-pairs, these mean that the triple (c_e,c_{dR},c) defines an element $[(c_e,c_{\mathrm{dR}},c)]$ in $I\otimes_E\mathrm{H}^2(G_K,\mathrm{ad}(W))$. We can show that $[(c_e,c_{\mathrm{dR}},c)]$ doesn't depend on the choice of s or $\widetilde{P}$, i.e. it depends only on W_A. Under the assumption that $\mathrm{H}^2(G_K,\mathrm{ad}(W))=0$, there exists a triple (f_e,f_{dR},f), where $f_e:G_K\to I\otimes_E\mathrm{Hom}_{\mathbf{B}_e\otimes_{\mathbb{Q}_p}E}(W_e,W_e)$ and $f_{\mathrm{dR}}:G_K\to I\otimes_E\mathrm{Hom}_{\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E}(W^+_{\mathrm{dR}},W^+_{\mathrm{dR}})$ are continuous maps and $f\in I\otimes_E\mathrm{Hom}_{\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E}(W_{\mathrm{dR}},W_{\mathrm{dR}})$, satisfying that

$$c_e(g_1,g_2)=g_1f_e(g_2)-f_e(g_1g_2)+f_e(g_1)$$

and

$$c_{\mathrm{dR}}(g_1,g_2)=g_1f_{\mathrm{dR}}(g_2)-f_{\mathrm{dR}}(g_1g_2)+f_{\mathrm{dR}}(g_1)$$

and

$$c(g_1)=f_{\mathrm{dR}}(g_1)-\bar{P}^{-1}f_e(g_1)\bar{P}+(g_1f-f)$$

for any $g_1,g_2\in G_K$. Using these, we define new liftings $\rho'_e:G_K\to\mathrm{GL}_n(\mathbf{B}_e\otimes_{\mathbb{Q}_p}A')$ by

$$\rho'_e(g):=(1+f_e(g))\widetilde{\rho}_e(g),$$

and $\rho'_{\mathrm{dR}}(g):G_K\to\mathrm{GL}_n(\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}A')$ by

$$\rho'_{\mathrm{dR}}(g):=(1+f_{\mathrm{dR}}(g))\widetilde{\rho}_{\mathrm{dR}}(g),$$

and define a matrix

$$P':=(1+f)\widetilde{P}\in\mathrm{GL}_n(\mathbf{B}_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}A').$$

Then, we can check that these satisfy the equalities

$$\rho'_e(g_1g_2) = \rho'_e(g_1)g_1(\rho'_e(g_2)) \text{ and } \rho'_{\mathrm{dR}}(g_1g_2) = \rho'_{\mathrm{dR}}(g_1)g_1(\rho'_{\mathrm{dR}}(g_2))$$

and

$$P'\rho'_e(g_1)g_1(P')^{-1} = \rho'_{\mathrm{dR}}(g_1)$$

for any $g_1, g_2 \in G_K$. By the definition of A'-B-pair, these equalities mean that the triple $(\rho'_e, \rho'_{\mathrm{dR}}, P')$ defines an A'-B-pair which is a lift of W_A, which proves the proposition. □

Corollary 2.31. *Let W be an E-B-pair of rank n. If $\mathrm{End}_{G_K}(W) = E$ and $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$, then the functor D_W is pro-representable by R_W such that*

$$R_W \xrightarrow{\sim} E[[T_1, \cdots, T_d]] \quad \textit{for } d := [K : \mathbb{Q}_p]n^2 + 1.$$

Proof. The existence and the formal smoothness of R_W follow from Proposition 2.27 and Proposition 2.30. For its dimension, we have

$$\begin{aligned}\dim_E D_W(E[\varepsilon]) &= \dim_E \mathrm{H}^1(G_K, \mathrm{ad}(W)) \\ &= [K : \mathbb{Q}_p]n^2 + \dim_E \mathrm{H}^0(G_K, \mathrm{ad}(W)) \\ &\quad + \dim_E \mathrm{H}^2(G_K, \mathrm{ad}(W)) = [K : \mathbb{Q}_p]n^2 + 1\end{aligned}$$

by Theorem 2.8 and Lemma 2.28. □

2.3. Trianguline deformations of trianguline B-pairs

In this subsection, we define the trianguline deformation functor for split trianguline E-B-pairs and prove the pro-representability and the formal smoothness under some conditions, and calculate the dimension of the universal deformation ring of this functor. These are the generalizations of Bellaïche-Chenevier's works in the $\mathbb{Q}_p$-case. In § 2 of [Bel-Ch09], Bellaïche-Chenevier proved all these results in the $\mathbb{Q}_p$-case by using (φ, Γ)-modules over the Robba ring and Colmez's theory of trianguline representations [Co08]. We generalize their results by using B-pairs and the theory of trianguline representations for any p-adic field ([Na09] or § 2.1).

We first define the notion of split trianguline A-B-pairs as follows.

DEFINITION 2.32. Let W be an A-B-pairs of rank n. We say that W is a split trianguline A-B-pair if there exists a sequence of sub A-B-pairs

$$\mathcal{T} : 0 = W_0 \subseteq W_1 \subseteq W_2 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$$

such that W_i is saturated in W_{i+1} and the quotient W_{i+1}/W_i is a rank one A-B-pair for any $0 \leqq i \leqq n-1$.

By Proposition 2.16, there exists a continuous homomorphisms $\delta_i : K^\times \to A^\times$ such that $W_i/W_{i-1} \xrightarrow{\sim} W(\delta_i)$ for each $1 \leqq i \leqq n$. We say that the ordered set $\{\delta_i\}_{i=1}^n$ is the parameter of the triangulation $\mathcal{T}$.

Next, we define the trianguline deformation functor. Let W be a split trianguline E-B-pair of rank n. We fix a triangulation

$$\mathcal{T} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$$

of W. Under this situation, we define the trianguline deformation as follows.

DEFINITION 2.33. Let A be an object in $\mathcal{C}_E$. We say that $(W_A, \iota, \mathcal{T}_A)$ is a trianguline deformation of $(W, \mathcal{T})$ over A if (W_A, ι) is a deformation of W over A and W_A is a split trianguline A-B-pair with a triangulation

$$\mathcal{T}_A : 0 \subseteq W_{1,A} \subseteq \cdots \subseteq W_{n,A} = W_A$$

such that $\iota(W_{i,A} \otimes_A E) = W_i$ for any $1 \leqq i \leqq n$. Let $(W_A, \iota, \mathcal{T}_A)$ and $(W'_A, \iota', \mathcal{T}'_A)$ be two trianguline deformations of $(W, \mathcal{T})$ over A. We say that $(W_A, \iota, \mathcal{T}_A)$ and $(W'_A, \iota', \mathcal{T}'_A)$ are equivalent if there exists an isomorphism of A-B-pairs $f : W_A \xrightarrow{\sim} W'_A$ satisfying that $\iota = \iota' \circ \bar{f}$ and $f(W_{i,A}) = W'_{i,A}$ for any $1 \leqq i \leqq n$.

DEFINITION 2.34. Let W be a split trianguline E-B-pair with a triangulation $\mathcal{T}$. We define the trianguline deformation functor $D_{W,\mathcal{T}}$ from the category $\mathcal{C}_E$ to the category of sets by

$$D_{W,\mathcal{T}}(A) := \{\text{equivalent classes } (W_A, \iota, \mathcal{T}_A) \text{ of trianguline deformations of } (W, \mathcal{T}) \text{ over } A\}.$$

for $A \in \mathcal{C}_E$.

By definition, we have a natural map of functors from $D_{W,\mathcal{T}}$ to D_W by forgetting the triangulations, i.e. defined by

$$D_{W,\mathcal{T}}(A) \to D_W(A) : [(W_A, \iota, \mathcal{T}_A)] \mapsto [(W_A, \iota)].$$

In general, $D_{W,\mathcal{T}}$ is not a subfunctor of D_W by this map, i.e. a deformation W_A can have many liftings of the triangulation $\mathcal{T}$. Here, we give a sufficient condition for $D_{W,\mathcal{T}}$ to be a subfunctor of D_W. Let $\{\delta_i\}_{i=1}^n$ be the parameter of triangulation $\mathcal{T}$.

Lemma 2.35. *Assume that* $\delta_j/\delta_i \neq \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ *for any* $1 \leqq i < j \leqq n$ *and* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$, *then the functor* $D_{W,\mathcal{T}}$ *is a subfunctor of* D_W.

Proof. Let W_A be a deformation of W over A, let $0 \subseteq W_{A,1} \subseteq \cdots \subseteq W_{A,n-1} \subseteq W_A$ and $0 \subseteq W'_{A,1} \subseteq \cdots \subseteq W'_{A,n-1} \subseteq W_A$ be two triangulations which are lifts of $\mathcal{T}$. It suffices to show the equalities $W_{A,i} = W'_{A,i}$ for all i. By induction, it suffices to show the equality $W_{A,1} = W'_{A,1}$. To prove this, we first consider $\mathrm{Hom}_{G_K}(W_{1,A}, W_A)$. This is equal to $\mathrm{H}^0(G_K, W_{1,A}^\vee \otimes W_A)$. Because $\mathrm{H}^0(G_K, -)$ is left exact and because $\mathrm{H}^0(G_K, W(\delta)) = 0$ for any $\delta : K^\times \to E^\times$ such that $\delta \neq \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ for any $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$ by Proposition 2.10, we have

$$\mathrm{H}^0(G_K, W_{1,A}^\vee \otimes (W_{i+1,A}/W_{i,A})) = \mathrm{H}^0(G_K, W_{1,A}^\vee \otimes (W'_{i+1,A}/W'_{i,A})) = 0$$

for any $i \geqq 1$. Hence, we obtain equalities

$$\mathrm{Hom}_{G_K}(W_{1,A}, W_{1,A}) = \mathrm{Hom}_{G_K}(W_{1,A}, W_A) = \mathrm{Hom}_{G_K}(W_{1,A}, W'_{1,A}).$$

This means that the given inclusion $W_{1,A} \hookrightarrow W_A$ factors through $W'_{1,A} \hookrightarrow W_A$. By the same reason, the inclusion $W'_{1,A} \hookrightarrow W_A$ also factors through $W_{1,A} \hookrightarrow W_A$. Hence, we obtain an equality $W_{1,A} = W'_{1,A'}$. □

Next, we prove relative representability of $D_{W,\mathcal{T}}$. Before doing this, we need to define the following functor which is the B-pair version of Lemma 2.3.8 of [Bel-Ch09]. Let $W = (W_e, W^+_{\mathrm{dR}})$ be an E-B-pair. Then we define a functor by

$$F(W) := \{x \in W_e \cap W^+_{\mathrm{dR}} | \exists n \in \mathbb{Z}_{\geqq 1}, (\sigma_1 - 1)(\sigma_2 - 1)\cdots(\sigma_n - 1)x = 0, \\ \forall \sigma_1, \cdots, \sigma_n \in G_K\}$$

which is an E-vector space with a G_K-action, hence F is a left exact functor form the category of E-B-pairs to that of $E[G_K]$-modules. By this definition, we obtain the following lemma, which is the B-pair version of Lemma 2.3.8 of [Bel-Ch09].

LEMMA 2.36. *Let $\delta : K^{\times} \to E^{\times}$ be a continuous homomorphism, then $F(W(\delta)) \neq 0$ if and only if $\mathrm{H}^0(G_K, W(\delta)) \neq 0$.*

PROOF. The proof is essentially the same as that of Lemma 2.3.8 of [Bel-Ch09], so we omit it. □

Using this lemma, we prove the relative representability of $D_{W,\mathcal{T}}$.

PROPOSITION 2.37. *Let W be a trianguline representations with a triangulation $\mathcal{T}$ such that the parameter $\{\delta_i\}_{i=1}^n$ satisfies $\delta_j/\delta_i \neq \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ for any $1 \leqq i < j \leqq n$ and $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$, then the natural map of functors $D_{W,\mathcal{T}} \to D_W$ is relatively representable.*

PROOF. By §23 of [Ma97], it suffices to check that the map $D_{W,\mathcal{T}} \to D_W$ satisfies the fallowing three conditions (1), (2), (3).

(1) For any map $A \to A'$ in $\mathcal{C}_E$ and $W_A \in D_{W,\mathcal{T}}(A)$, we have $W_A \otimes_A A' \in D_{W,\mathcal{T}}(A')$.

(2) For any maps $A' \to A$, $A" \to A$ in $\mathcal{C}_E$ and $W^{'"} \in D_W(A' \times_A A")$, if $W^{'"} \otimes_{A'\times_A A"} A' \in D_{W,\mathcal{T}}(A')$ and $W^{'"} \otimes_{A'\times_A A"} A" \in D_{W,\mathcal{T}}(A")$, then we also have $W^{'"} \in D_{W,\mathcal{T}}(A' \times_A A")$.

(3) For any inclusion $A \hookrightarrow A'$ in $\mathcal{C}_E$ and $W_A \in D_W(A)$, if $W_A \otimes_A A' \in D_{W,\mathcal{T}}(A')$, then we have $W_A \in D_{W,\mathcal{T}}(A)$.

The condition (1) is trivial. For (2), let $W^{'"} \in D_W(A' \times_A A")$ be a deformation such that $W_{A'} := W^{'"} \otimes_{A'\times_A A"} A' \in D_{W,\mathcal{T}}(A')$ and $W_{A"} := W^{'"} \otimes_{A'\times_A A"} A" \in D_{W,\mathcal{T}}(A")$. We put $W_A := W^{'"} \otimes_{A'\times_A A"} A$. In the same way as in the proof of Proposition 2.27, we have an isomorphism $W^{'"} \xrightarrow{\sim} W_{A'} \times_{W_A} W_{A"}$. By Lemma 2.35, the triangulations of W_A induced from $W_{A'}$ and $W_{A"}$ coincide, hence these triangulations induce a triangulation of $W^{'"} \xrightarrow{\sim} W_{A'} \times_{W_A} W_{A"}$, i.e. $W^{'"} \in D_{W,\mathcal{T}}(A' \times_A A")$.

Finally, we check the condition (3). The proof is essentially the same as that of Proposition 2.3.9 of [Bel-Ch09], but here we give the proof for convenience of readers. Let $W \in D_W(A)$ and $A \hookrightarrow A'$ be an inclusion such that $W_A \otimes_A A' \in D_{W,\mathcal{T}}(A')$. Let $0 \subseteq W_{1,A'} \subseteq \cdots \subseteq W_{n-1,A'} \subseteq W_A \otimes_A A'$ be a triangulation which is a lifting of $\mathcal{T}$. By induction on the rank of W, it suffices to show that there exists a rank one sub A-B-pair $W_{1,A} \subseteq W_A$ such that $W_{1,A} \otimes_A A' = W_{1,A'}$ and that $W_A/W_{1,A}$ is an A-B-pair. By Proposition 2.16, there exists a continuous homomorphism $\delta_{1,A'} : K^\times \to A'^\times$ such that $W_{1,A'} \xrightarrow{\sim} W(\delta_{1,A'})$. Twisting W by δ_1^{-1}, we may assume that $\delta_{1,A'} \equiv 1 \pmod{\mathfrak{m}_{A'}}$. Under this assumption, we apply the functor $F(-)$. Because $\delta_{1,A'}$ is unitary, there exists a continuous character $\widetilde{\delta}_{1,A'} : G_K^{\mathrm{ab}} \to A'^\times$ such that $W(\delta_{1,A'}) \xrightarrow{\sim} W(A'(\widetilde{\delta}_{1,A'})) = (\mathbf{B}_e \otimes_{\mathbb{Q}_p} A'(\widetilde{\delta}_{1,A'}), \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} A'(\widetilde{\delta}_{1,A'}))$, hence we have $W_{1,A',e} \cap W_{1,A',\mathrm{dR}}^+ = A'(\widetilde{\delta}_{1,A'})$. Moreover, because the image of $\delta_{1,A'}$ is in $1 + \mathfrak{m}_{A'}$, we also have $F(W_{1,A'}) = A'(\widetilde{\delta}_{1,A'})$. Next, because $(W_A \otimes_A A')/W_{1,A'}$ is a successive extension of $W(\delta_i\delta_1^{-1})$ $(i \geqq 2)$ as an E-B-pair, the left exactness of F implies that $F((W_A \otimes_A A')/W_{1,A'}) = 0$ by Lemma 2.36. Applying F to the short exact sequence $0 \to W_{1,A'} \to W_A \otimes_A A' \to (W_A \otimes_A A')/W_{1,A'} \to 0$, we obtain $A'(\widetilde{\delta}_{1,A'}) \xrightarrow{\sim} F(W_{1,A'}) = F(W_A \otimes_A A')$. In the same way, we obtain $E = F(W_1) = F(W)$. Then, by the left exactness and by considering the length, we can show that $F(W_A)$ is a free A-module of rank one and that the natural map $F(W_A) \to F(W)$ induced by the natural quotient map $W_A \to W$ is a surjection and that the natural map $F(W_A) \otimes_A A' \to F(W_A \otimes_A A')$ is isomorphism. If we define the character $\widetilde{\delta}_{1,A} : G_K^{\mathrm{ab}} \to A^\times$ such that $F(W_A) \xrightarrow{\sim} A(\widetilde{\delta}_{1,A})$ and define $W_{1,A}$ as the image of the natural map $(\mathbf{B}_e \otimes_{\mathbb{Q}_p} F(W_A), \mathbf{B}_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} F(W_A)) \to W_A$ induced form $F(W) \hookrightarrow W_{A,e}, F(W) \hookrightarrow W_{A,\mathrm{dR}}^+$, then we can check that $W_{1,A}$ is a rank one A-B-pair and that the quotient $W_A/W_{1,A}$ is also an A-B-pair and that $W_{1,A} \otimes_A A' \xrightarrow{\sim} W_{1,A'}$, which proves the condition (3), hence we finish to prove the proposition. □

Corollary 2.38. *Let W be a trianguline E-B-pair with a triangulation $\mathcal{T}$. Assume that $\mathrm{End}_{G_K}(W) = E$ and that the parameter $\{\delta_i\}_{i=1}^n$ of $\mathcal{T}$ satisfies $\delta_j/\delta_i \neq \prod_{\sigma\in\mathcal{P}} \sigma^{k_\sigma}$ for any $1 \leqq i < j \leqq n$ and $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$, then the functor $D_{W,\mathcal{T}}$ is pro-representable by a quotient $R_{W,\mathcal{T}}$ of R_W.*

Proof. This follows from Proposition 2.27 and Proposition 2.37. □

Next, we prove the formal smoothness of the functor $D_{W,\mathcal{T}}$.

PROPOSITION 2.39. *Let W be a trianguline E-B-pair of rank n with a triangulation $\mathcal{T}$. Assume that the parameter $\{\delta_i\}_{i=1}^n$ of $\mathcal{T}$ satisfies $\delta_i/\delta_j \neq |\mathrm{N}_{K/\mathbb{Q}_p}|\prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$ for any $1 \leqq i < j \leqq n$ and $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\geqq 1}$, then the functor $D_{W,\mathcal{T}}$ is formally smooth.*

PROOF. We prove this proposition by induction of the rank of W. When W is of rank one, then we have $D_{W,\mathcal{T}} = D_W$ and $\mathrm{ad}(W)$ is the trivial E-B-pair. Hence $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$ by Proposition 2.10, and $D_{W,\mathcal{T}}$ is formally smooth by Proposition 2.30. Let's assume that the proposition is proved for all trianguline E-B-pairs of rank less or equal $n-1$. Let W be an E-B-pair of rank n with a triangulation $\mathcal{T} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$ whose parameter $\{\delta_i\}_{i=1}^n$ satisfies the condition in the proposition. Let $A' \to A$ be a small extension in $\mathcal{C}_E$, and let W_A be a trianguline deformation of $(W, \mathcal{T})$ with a triangulation $\mathcal{T}_A : 0 \subseteq W_{1,A} \subseteq \cdots \subseteq W_{n-1,A} \subseteq W_{n,A} = W_A$ which is a lift of $\mathcal{T}$. Then, it suffices to show that there exists a split trianguline A'-B-pair $W_{A'}$ with a triangulation $0 \subseteq W_{1,A'} \subseteq \cdots \subseteq W_{n-1.A'} \subseteq W_{n,A'} = W_{A'}$ which is a lift of W_A and $\mathcal{T}_A$. We take such a lift as follows. Applying the induction hypothesis to W_{n-1}, there exists a trianguline A'-B-pair $W_{n-1,A'}$ with a triangulation $0 \subseteq W_{1,A'} \subseteq \cdots \subseteq W_{n-2,A'} \subseteq W_{n-1,A'}$ which is a lift of $W_{n-1,A}$ and $0 \subseteq W_{1,A} \subseteq \cdots \subseteq W_{n-1,A}$. We put $\mathrm{gr}_n W_A := W_A/W_{n-1,A}$. By the rank one case and by Proposition 2.16, there exists a continuous homomorphism $\delta_{n,A'} : K^\times \to A'^\times$ such that the rank one A'-B-pair $W(\delta_{n,A'})$ satisfies $W(\delta_{n,A'}) \otimes_{A'} A = W(\delta_{n,A}) \xrightarrow{\sim} \mathrm{gr}_n W_A$, where $\delta_{n,A} : K^\times \to A^\times$ is the composition of $\delta_{n,A'}$ with $A' \to A$. We see the isomorphism class $[W_A]$ as an element in $\mathrm{Ext}^1(W(\delta_{n,A}), W_{n-1,A}) \xrightarrow{\sim} \mathrm{H}^1(G_K, W_{n-1,A}(\delta_{n,A}^{-1}))$. If we take the long exact sequence associated to

$$0 \to I \otimes_E W_{n-1}(\delta_n^{-1}) \to W_{n-1,A'}(\delta_{n,A'}^{-1}) \to W_{n-1,A}(\delta_{n,A}^{-1}) \to 0,$$

where $I \subseteq A'$ is the kernel of $A' \to A$, then we obtain a long exact sequence

$$\cdots \to \mathrm{H}^1(G_K, W_{n-1,A'}(\delta_{n,A'}^{-1})) \to \mathrm{H}^1(G_K, W_{n-1,A}(\delta_{n,A}^{-1})) \\ \to I \otimes_E \mathrm{H}^2(G_K, W_{n-1}(\delta_n^{-1})) \to \cdots$$

By the assumption on $\{\delta_i\}_{i=1}^n$ and by Proposition 2.10, we have $\mathrm{H}^2(G_K, W_{n-1}(\delta_n^{-1})) = 0$. Hence, we can take a lift $[W_{A'}] \in \mathrm{Ext}^1(W(\delta_{n,A'}),$

$W_{n-1,A'}) \xrightarrow{\sim} \mathrm{H}^1(G_K, W_{n-1,A'}(\delta_{n,A'}^{-1}))$ of $[W_A]$, which proves the proposition. $\square$

Next, we calculate the dimension of $D_{W,\mathcal{T}}$. For this, we interpret the tangent space $D_{W,\mathcal{T}}(E[\varepsilon])$ in terms of Galois cohomology of B-pair as in Lemma 2.28. Let W be a trianguline E-B-pair with a triangulation $\mathcal{T}: 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$, then we define an E-B-pair $\mathrm{ad}_{\mathcal{T}}(W)$ by

$$\mathrm{ad}_{\mathcal{T}}(W) := \{f \in \mathrm{ad}(W) | f(W_i) \subseteq W_i \text{ for any } 1 \leqq i \leqq n\}.$$

LEMMA 2.40. *Let W be a trianguline E-B-pair, then there exists a canonical bijection of sets*

$$D_{W,\mathcal{T}}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}}(W)).$$

In particular, if $D_{W,\mathcal{T}}$ has a canonical structure of E-vector space (*see the condition* (2) *in Schlessinger's criterion* 2.26), *then this bijection is an E-linear isomorphism.*

PROOF. The construction of the map $D_{W,\mathcal{T}}(E[\varepsilon]) \to \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}}(W))$ is the same as in the proof of Lemma 2.28. We put $\mathrm{ad}_{\mathcal{T}}(W) := (\mathrm{ad}_{\mathcal{T}}(W_e), \mathrm{ad}_{\mathcal{T}}(W_{\mathrm{dR}}^+))$. Let $W_{E[\varepsilon]} := (W_{e,E[\varepsilon]}, W_{\mathrm{dR},E[\varepsilon]}^+)$ be a trianguline deformation of $(W, \mathcal{T})$ over $E[\varepsilon]$ with a lifted triangulation $\mathcal{T}_{E[\epsilon]}: 0 \subseteq W_{1,E[\varepsilon]}, \subseteq \cdots \subseteq W_{n-1,E[\varepsilon]} \subseteq W_{n,E[\varepsilon]} = W_{E[\varepsilon]}$. Then, we can take a splitting $W_{e,E[\varepsilon]} = W_e e_1 \oplus W_e e_2$ as a filtered $\mathbf{B}_e \otimes_{\mathbb{Q}_p} E$-module such that $W_e e_1 = \varepsilon W_{e,E[\varepsilon]}$ and that the natural map $W_e e_2 \hookrightarrow W_{e,E[\varepsilon]} \to W_{e,E[\varepsilon]}/\varepsilon W_{e,E[\varepsilon]} \xrightarrow{\sim} W_e$ sends ye_2 to y for any $y \in W_e$. If we define $c_e: G_K \to \mathrm{Hom}_{\mathbf{B}_e \otimes_{\mathbb{Q}_p} E}(W_e, W_e)$ in the same way as in the proof of Lemma 2.28, we can check that the image of c_e is contained in $\mathrm{ad}_{\mathcal{T}}(W_e)$. In the same way, we can define $c_{\mathrm{dR}}: G_K \to \mathrm{ad}_{\mathcal{T}}(W_{\mathrm{dR}}^+)$ from a filtered splitting $W_{\mathrm{dR},E[\varepsilon]}^+ = W_{\mathrm{dR}}^+ e_1 \oplus W_{\mathrm{dR}}^+ e_2'$. Moreover, we can define $c \in \mathrm{ad}_{\mathcal{T}}(W_{\mathrm{dR}})$ by $ye_2 = c(y)e_1 + ye_2'$ for $y \in W_{\mathrm{dR}}$. Then, the map

$$D_{W,\mathcal{T}}(E[\varepsilon]) \to \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}}(W)) : [(W_{E[\varepsilon]}, \mathcal{T}_{E[\varepsilon]})] \mapsto [(c_e, c_{\mathrm{dR}}, c)]$$

defines a bijection, and this is an E-linear isomorphism when $D_{W,\mathcal{T}}(E[\varepsilon])$ has a canonical structure of an E-vector space. $\square$

We calculate the dimension of $R_{W,\mathcal{T}}$.

PROPOSITION 2.41. *Let W be a split trianguline E-B-pair of rank n with a triangulation $\mathcal{T}: 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-1} \subseteq W_n = W$. We assume that $(W, \mathcal{T})$ satisfies the following conditions,*

(0) $\mathrm{End}_{G_K}(W) = E$,

(1) $\delta_j/\delta_i \neq \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ *for any* $1 \leqq i < j \leqq n$ *and* $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\leqq 0}$,

(2) $\delta_i/\delta_j \neq |\mathrm{N}_{K/\mathbb{Q}_p}| \prod_{\sigma \in \mathcal{P}} \sigma^{k_\sigma}$ *for any* $1 \leqq i < j \leqq n$ *and* $\{k_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} \mathbb{Z}_{\geqq 1}$,

then the universal trianguline deformation ring $R_{W,\mathcal{T}}$ is a quotient ring of R_W such that

$$R_{W,\mathcal{T}} \xrightarrow{\sim} E[[T_1, \cdots, T_{d_n}]] \text{ for } d_n := \frac{n(n+1)}{2}[K : \mathbb{Q}_p] + 1.$$

PROOF. By Proposition 2.37 and Proposition 2.39 and Lemma 2.40, it suffices to show that $\dim_E \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = d_n$. We prove this by induction on the rank n of W. When $n = 1$, then $\mathrm{ad}_{\mathcal{T}}(W) = \mathrm{ad}(W)$ is the trivial E-B-pair, hence the proposition follows from Proposition 2.10. Let $(W, \mathcal{T})$ be a split trianguline E-B-pair of rank n satisfying all the conditions in the propositionas. Put $\mathcal{T}_{n-1} : 0 \subseteq W_1 \subseteq \cdots \subseteq W_{n-2} \subseteq W_{n-1}$ the triangulation of $W_{n-1} (\subseteq W)$ which is induced from $\mathcal{T}$. Then, for any $f \in \mathrm{ad}_{\mathcal{T}}(W)$, the restriction of f to W_{n-1} is an element of $\mathrm{ad}_{\mathcal{T}_{n-1}}(W_{n-1})$ and this defines a short exact sequence of E-B-pair

$$0 \to \mathrm{Hom}(W(\delta_n), W) \to \mathrm{ad}_{\mathcal{T}}(W) \to \mathrm{ad}_{\mathcal{T}_{n-1}}(W_{n-1}) \to 0.$$

From this, we obtain

$$\mathrm{rank}(\mathrm{ad}_{\mathcal{T}}(W)) = \mathrm{rank}(\mathrm{ad}_{\mathcal{T}_{n-1}}(W_{n-1})) + n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

by induction. By Theorem 2.8, it suffices to show that $\mathrm{H}^0(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = E$ and $\mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = 0$. For H^0, this follows from the following natural inclusions

$$E \subseteq \mathrm{H}^0(G_K, \mathrm{ad}_{\mathcal{T}}(W)) \subseteq \mathrm{H}^0(G_K, \mathrm{ad}(W)) = E.$$

We prove $\mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = 0$ by induction of the rank of W. When $n = 1$, this follows from Proposition 2.10. When W is of rank n, from the above short exact sequence, we obtain the following long exact sequence

$$\cdots \to \mathrm{H}^2(G_K, \mathrm{Hom}(W(\delta_n), W)) \to \mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}}(W)) \\ \to \mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}_{n-1}}(W_{n-1})) \to 0.$$

Because we have $\mathrm{H}^2(G_K, \mathrm{Hom}(W(\delta_n), W)) = 0$ by Proposition 2.10 and by the assumption on $\{\delta_i\}_{i=1}^n$, we obtain the equality $\mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}}(W)) = 0$ by induction, which proves the proposition. □

2.4. Deformations of benign B-pairs

In this final subsection, we study benign representations or more generally benign B-pairs which is a class of potentially crystalline and trianguline B-pairs and have some very good properties for trianguline deformations and play a crucial role in the problem of the Zariski density of modular Galois (or crystalline) representations in some deformation spaces of global (or local) p-adic Galois representations. This class was defined by Kisin in the case where $K = \mathbb{Q}_p$ and the rank is 2 in [Ki03] and [Ki10]. He studied some deformation theoretic properties of this class in [Ki03] and used these in a crucial way in his proof of Zariski density of two dimensional crystalline representations of $G_{\mathbb{Q}_p}$. For higher dimensional and the $\mathbb{Q}_p$-case, Bellaïche-Chenevier [Bel-Ch09] and Chenevier [Ch09b] were the first ones who noticed the importance of benign representations in the study of p-adic families of trianguline representations. In particular, Chenevier [Ch09b] (where he calls "generic" instead of benign) discovered and proved a crucial theorem concerning the tangent spaces of the universal deformation rings of benign representations. In fact, by using this theorem, Chenevier [Ch09b] proved some theorems concerning the Zariski density of modular Galois representations in some deformation spaces of global p-adic representations.

The aim of this subsection is to generalize the definition of benign representations and the Chenevier's theorem for any K.

2.4.1 Benign B-pairs

Let $P(X) \in \mathcal{O}_K[X]$ be a polynomial such that $P(X) \equiv \pi_K X \pmod{X^2}$ and $P(X) \equiv X^q \pmod{\pi_K}$, where $q := p^f$ and $f := [K_0 : \mathbb{Q}_p]$. We take the Lubin-Tate formal group law $\mathcal{F}$ over $\mathcal{O}_K$ such that $[\pi_K] = P(X)$, where $[-] : \mathcal{O}_K \xrightarrow{\sim} \mathrm{End}(\mathcal{F})$. We denote by K_n the abelian extension of K generated by

$[\pi_K^n]$-torsion points of $\mathcal{F}(\overline{K})$ for any n, then we have a canonical isomorphism $\chi_{\mathrm{LT},n} : \mathrm{Gal}(K_n/K) \xrightarrow{\sim} (\mathcal{O}_K^\times/\pi^n\mathcal{O}_K)^\times$. We put $K_{\mathrm{LT}} := \cup_{n=1}^\infty K_n$ and $G_n := \mathrm{Gal}(K_n/K)$.

In [Ki10], [Bel-Ch09] or [Ch09b] etc, benign representation is defined as a special class of crystalline representations. But, as we show in the sequel, we can easily generalize the main theorem to some potentially crystalline representations. Hence, before defining benign representations, we first define the following class of potentially crystalline representations.

DEFINITION 2.42. Let W be an E-Bpair. We say that W is crystabelline if $W|_{G_L}$ is a crystalline E-B-pair of G_L for a finite abelian extension L of K.

REMARK 2.43. Because a finite abelian extension L of K is contained in K_mL' for some $m \geqq 0$ and for a finite unramified extension L' of K, by using Hilbert 90, we can easily show that W is crystabelline if and only if $W|_{G_{K_m}}$ is crystalline for some $m \geqq 1$.

Let W be a crystabelline E-B-pair of rank n such that $W|_{G_{K_m}}$ is crystalline for some m. Because K_m is totally ramified over K, $\mathbf{D}_{\mathrm{cris}}^{K_m}(W) := (\mathbf{B}_{\max}\otimes_{\mathbb{Q}_p} W)^{G_{K_m}}$ is a free $K_0\otimes_{\mathbb{Q}_p} E$-module of rank n. We take an embedding $\sigma : K_0 \hookrightarrow \overline{E}$. This defines a map $\sigma : K_0 \otimes_{\mathbb{Q}_p} E \to \overline{E} : x\otimes y \to \sigma(x)y$. Using this map, we define the σ-component

$$\mathbf{D}_{\mathrm{cris}}^{K_m}(W)_\sigma := \mathbf{D}_{\mathrm{cris}}^{K_m}(W)\otimes_{K_0\otimes_{\mathbb{Q}_p}E,\sigma}\overline{E},$$

this has a $\overline{E}$-linear φ^f-action and a $\overline{E}$-linear G_m-action. Let $\{\alpha_1,\cdots,\alpha_n\}$ be the solution in $\overline{E}$ (with multiplicities) of the characteristic polynomial $\det_{\overline{E}}(T\cdot \mathrm{id}-\varphi^f|_{\mathbf{D}_{\mathrm{cris}}^{K_m}(W)_\sigma}) \in \overline{E}[T]$. Because the actions of φ^f and G_m commute, each generalized φ^f-eigenvector subspace of $\mathbf{D}_{\mathrm{cris}}^{K_m}(W)_\sigma$ is preserved by the action of G_m. Hence we can take a $\overline{E}$-basis $\{e_{1,\sigma},\cdots,e_{n,\sigma}\}$ of $\mathbf{D}_{\mathrm{cris}}^{K_m}(W)_\sigma$ such that $e_{i,\sigma}$ is a generalized eigenvector of φ^f for the eigenvalue $\alpha_i\in\overline{E}^\times$ and G_m acts on $e_{i,\sigma}$ by a character $\tilde{\delta}_i : G_m\to\overline{E}^\times$ for any i. We change the numbering of $\{\alpha_1,\cdots,\alpha_n\}$ so that the basis $e_{1,\sigma},e_{2,\sigma},\cdots,e_{n,\sigma}$ gives a φ^f-Jordan decomposition of $\mathbf{D}_{\mathrm{cris}}^{K_m}(W)_\sigma$ by this order. Because we have $\{\sigma,\varphi^{-1}\sigma,\cdots,\varphi^{-(f-1)}\sigma\} = \mathrm{Hom}_{\mathbb{Q}_p}(K_0,\overline{E})$ and

$$\varphi^i : \mathbf{D}_{\mathrm{cris}}^{K_m}(W)_\sigma \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris}}^{K_m}(W)_{\varphi^{-i}\sigma} : x\otimes y\mapsto \varphi^i(x)\otimes y$$

($x \in \mathbf{D}_{\mathrm{cris}}^{K_m}(W)$ and $y \in \overline{E}$) is a $\overline{E}[\varphi^f, G_m]$-isomorphism, the set $\{\alpha_1, \cdots, \alpha_n\}$ doesn't depend on the choice of $\sigma : K_0 \hookrightarrow \overline{E}$. If we put

$$e_i := e_{i,\sigma} + \varphi(e_{i,\sigma}) + \cdots + \varphi^{f-1}(e_{i,\sigma}) \in \mathbf{D}_{\mathrm{cris}}^{K_m}(W) \otimes_E \overline{E},$$

we have

$$\mathbf{D}_{\mathrm{cris}}^{K_m}(W) \otimes_E \overline{E} = K_0 \otimes_{\mathbb{Q}_p} \overline{E} e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} \overline{E} e_n$$

such that the subspace $K_0 \otimes_{\mathbb{Q}_p} \overline{E} e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} \overline{E} e_i$ is preserved by φ and the action of G_m for any i. Moreover, if we take a sufficiently large finite extension E' of E, then we have $e_i \in \mathbf{D}_{\mathrm{cris}}^{K_m}(W) \otimes_E E'$ and

$$\mathbf{D}_{\mathrm{cris}}^{K_m}(W) \otimes_E E' = K_0 \otimes_{\mathbb{Q}_p} E' e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} E' e_n$$

and $\alpha_i \in E'$ and $\widetilde{\delta}_i : G_m \to E'^{\times}$ for any i.

Using these arguments, we first study a relation between crystabelline E-B-pairs and trianguline E-B-pairs.

Lemma 2.44. *Let W be an E-B-pair of rank n. The following conditions are equivalent,*

1. *W is crystabelline,*
2. *W is trianguline (i.e. $W \otimes_E E'$ is a split trianguline E'-B-pair for a finite extension E' of E) and potentially crystalline.*

Proof. First we assume that W is crystabelline. By the above argument, for a sufficiently large finite extension E' of E, we have $\mathbf{D}_{\mathrm{cris}}^{K_m}(W) \otimes_E E' = K_0 \otimes_{\mathbb{Q}_p} E' e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} E' e_n$ as above and $K_0 \otimes_{\mathbb{Q}_p} E' e_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} E' e_i$ is a sub E'-filtered (φ, G_m)-module of $\mathbf{D}_{\mathrm{cris}}^{K_m}(W \otimes_E E')$ for any i. Hence $W \otimes_E E'$ is split trianguline and potentially crystalline by Theorem 2.5.

Next we assume that W is trianguline and potentially crystalline. By extending the coefficient, we may assume that W is split trianguline. We take a triangulation $0 \subseteq W_1 \subseteq \cdots \subseteq W_n = W$ of W. Because the sub or quotient B-pairs of crystalline B-pairs are again crystalline, W_i and W_i/W_{i-1} are all potentially crystalline. Because W_i/W_{i-1} is of rank one, $W_i/W_{i-1}|_{G_m}$ is crystalline for any i and for any sufficiently large m. We claim that $W|_{G_m}$ is also crystalline. We prove this claim by induction on the rank n of W. When $n=1$, this is trivial. We assume that the claim is proved for the rank $n-1$ case, hence $W_{n-1}|_{G_m}$ is crystalline. If we put $W/W_{n-1}\xrightarrow{\sim} W(\delta_n)$, we have $[W]\in \mathrm{H}^1(G_K, W_{n-1}(\delta_n^{-1}))$. By the assumption, there exists a finite Galois extension L of K_m such that $[W]$ is contained in $\mathrm{Ker}(\mathrm{H}^1(G_K, W_{n-1}(\delta_n^{-1}))\rightarrow \mathrm{H}^1(G_L, \mathbf{B}_{\max}\otimes_{\mathbf{B}_e}(W_{n-1}(\delta_n^{-1}))_e))$. Hence, it suffices to show that the natural map $\mathrm{H}^1(G_{K_m}, \mathbf{B}_{\max}\otimes_{\mathbf{B}_e}(W_{n-1}(\delta_n^{-1}))_e)\rightarrow \mathrm{H}^1(G_L, \mathbf{B}_{\max}\otimes_{\mathbf{B}_e}(W_{n-1}(\delta_n^{-1}))_e)$ is an injection. By the inflation restriction sequence, the kernel of this map is $\mathrm{H}^1(\mathrm{Gal}(L/K_m), \mathbf{D}^L_{\mathrm{cris}}(W_{n-1}(\delta_n^{-1})))=0$. Hence $W|_{G_m}$ is crystalline, i.e. W is crystabelline. □

From now on, we consider a crystabelline E-B-pair W of rank n satisfying that $\mathbf{D}^{K_m}_{\mathrm{cris}}(W)=K_0\otimes_{\mathbb{Q}_p}Ee_1\oplus\cdots\oplus K_0\otimes_{\mathbb{Q}_p}Ee_n$ such that $K_0\otimes_{\mathbb{Q}_p}Ee_i$ is preserved by (φ, G_m) and $\varphi^f(e_i)=\alpha_i e_i$ for some $\alpha_i\in E^{\times}$ such that $\alpha_i\not=\alpha_j$ for any $i\not= j$. Let $\mathfrak{S}_n$ be the n-th permutation group. For any $\tau\in\mathfrak{S}_n$, we define a filtration on $\mathbf{D}^{K_m}_{\mathrm{cris}}(W)$ by E-filtered (φ, G_m)-modules as follows,

$$\mathcal{F}_{\tau}: 0\subseteq F_{\tau,1}\subseteq\cdots\subseteq F_{\tau,n-1}\subseteq F_{\tau,n}=\mathbf{D}^{K_m}_{\mathrm{cris}}(W)$$

such that

$$F_{\tau,i}:=K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(1)}\oplus\cdots\oplus K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(i)}$$

for any $1\leqq i\leqq n$, where the filtration on $F_{\tau,i}$ is induced from that on $\mathbf{D}^{K_m}_{\mathrm{cris}}(W)$. We put $\mathrm{gr}_{\tau,i}\mathbf{D}^{K_m}_{\mathrm{cris}}(W):=F_{\tau,i}/F_{\tau,i-1}$ for any $1\leqq i\leqq n$. By Theorem 2.5, there exists a filtration

$$\mathcal{T}_{\tau}: 0\subseteq W_{\tau,1}\subseteq\cdots\subseteq W_{\tau,n-1}\subseteq W_{\tau,n}=W$$

such that $W_{\tau,i}|_{G_m}$ is crystalline and

$$\mathbf{D}^{K_m}_{\mathrm{cris}}(W_{\tau,i})=F_{\tau,i}.$$

For any i, $W_{\tau,i}/W_{\tau,i-1}$ is a rank one crystabelline E-B-pair such that $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_{\tau,i}/W_{\tau,i-1})\xrightarrow{\sim}\mathrm{gr}_{\tau,i}\mathbf{D}^{K_m}_{\mathrm{cris}}(W)$. By Lemma 4.1 of [Na09] and by its proof, there exists a set of integers $\{k_{(\tau,i),\sigma}\}_{\sigma\in\mathcal{P}}$ and a homomorphism $\widetilde{\delta}_i: K^{\times}\rightarrow E^{\times}$ satisfying $\widetilde{\delta}_i|_{1+\pi_K^m\mathcal{O}_K}=1$ and $\widetilde{\delta}_i(\pi_K)=1$, such that

$$W_{\tau,i}/W_{\tau,i-1}\xrightarrow{\sim} W(\delta_{\alpha_{\tau(i)}}\widetilde{\delta}_{\tau(i)}\prod_{\sigma\in\mathcal{P}}\sigma^{k_{(\tau,i),\sigma}})$$

for any $1 \leqq i \leqq n$, where $\delta_{\alpha_i} : K^{\times} \to E^{\times}$ is the homomorphism such that $\delta_{\alpha_i}|_{\mathcal{O}_K^{\times}} = 1$ and $\delta_{\alpha_i}(\pi_K) = \alpha_i$. For any $\sigma \in \mathcal{P}$, the set $\{k_{(\tau,1),\sigma}, k_{(\tau,2),\sigma}, \cdots, k_{(\tau,n),\sigma}\}$ is independent of $\tau \in \mathfrak{S}_n$ because these numbers are the σ-components of the Hodge-Tate weights of W. We denote this set (with multiplicities) by $\{k_{1,\sigma}, k_{2,\sigma}, \cdots, k_{n,\sigma}\}$ such that $k_{1,\sigma} \geqq k_{2,\sigma} \geqq \cdots \geqq k_{n,\sigma}$ for any $\sigma \in \mathcal{P}$. Under these notations, we define the notion of benign E-B-pair as follows.

Definition 2.45. Let W be a rank n crystabelline E-B-pair as above. We say that W is a benign E-B-pair if the following conditions hold:

(1) For any $i \neq j$, we have $\alpha_i/\alpha_j \neq 1, p^f, p^{-f}$.

(2) For any $\sigma \in \mathcal{P}$, we have $k_{1,\sigma} > k_{2,\sigma} > \cdots > k_{n-1,\sigma} > k_{n,\sigma}$.

(3) For any $\tau \in \mathfrak{S}_n$ and $\sigma \in \mathcal{P}$, we have $k_{(\tau,i),\sigma} = k_{i,\sigma}$ for any $1 \leqq i \leqq n$.

Remark 2.46. By definition, if W is a benign, then we have

$$W_{\tau,i}/W_{\tau,i-1} \xrightarrow{\sim} W(\delta_{\alpha_{\tau(i)}} \widetilde{\delta}_{\tau(i)} \prod_{\sigma \in \mathcal{P}} \sigma^{k_{i,\sigma}})$$

for any $\tau \in \mathfrak{S}_n$ and $1 \leqq i \leqq n$.

Lemma 2.47. *Let W be a benign E-B-pair. If W_1 is a saturated sub E-B-pair of W, then W_1 and W/W_1 are also benign E-B-pairs.*

Proof. This follows from the definition and the fact that all the sub or quotient E-B-pairs of crystabelline E-B-pairs are crystabelline E-B-pairs. □

2.4.2 Deformations of benign B-pairs

Lemma 2.48. *Let W be a potentially crystalline E-B-pair satisfying the condition* (1) *of Definition* 2.45, *then we have* $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$ *and* $(W, \mathcal{T}_\tau)$ *satisfies the conditions in Proposition* 2.41 *except the condition* $\mathrm{End}_{G_K}(W) = E$ *for any* $\tau \in \mathfrak{S}_n$.

Proof. That $\mathrm{H}^2(G_K, \mathrm{ad}(W)) = 0$ follows from the condition (1) of Definition 2.45 and from (2) of Proposition 2.10 because $\mathrm{ad}(W)$ is split trianguline whose graded components are of the forms $(W_{\tau,i}/W_{\tau,i-1}) \otimes (W_{\tau,j}/W_{\tau,j-1})^{\vee}$ for any fixed $\tau \in \mathfrak{S}_n$. Other statements follow from the condition (1) of Definition 2.45. □

LEMMA 2.49. *Let W be a benign E-B-pair of rank n, then $\mathrm{End}_{G_K}(W) = E$.*

PROOF. We prove this by induction on n, the rank of W. If $n = 1$, $\mathrm{End}_{G_K}(W) = \mathrm{H}^0(G_K, (\mathbf{B}_e, \mathbf{B}^+_{\mathrm{dR}})) = E$. We assume that the lemma is proved for $n-1$. Let W be a benign E-B-pair of rank n. We take an element $\tau \in \mathfrak{S}_n$ and consider the filtration $\mathcal{T}_\tau : 0 \subseteq W_{\tau,1} \subseteq \cdots \subseteq W_{\tau,n-1} \subseteq W_{\tau,n} = W$. By Lemma 2.47, $W_{\tau,n-1}$ is benign of rank $n-1$, hence we have $\mathrm{End}_{G_K}(W_{\tau,n-1}) = E$ by induction. Let $f : W \to W$ be a non-zero morphism of E-B-pairs. By (1) of Definition 2.45 and by Proposition 2.10, we have $\mathrm{Hom}_{G_K}(W_{\tau,n-1}, W/W_{\tau,n-1}) = 0$. Hence we have $f(W_{\tau,n-1}) \subseteq W_{\tau,n-1}$. Because we have $\mathrm{End}_{G_K}(W_{\tau,n-1}) = E$, then we have $f|_{W_{\tau,n-1}} = a \cdot \mathrm{id}_{W_{\tau,n-1}}$ for some $a \in E$. If $a = 0$, then $f : W \to W$ factors through a non-zero morphism $f' : W/W_{\tau,n-1} \to W$. Because $\mathrm{Hom}_{G_K}(W/W_{\tau,n-1}, W_{\tau,n-1}) = 0$ by (1) of Definition 2.45 and by Proposition 2.10, the natural map $\mathrm{Hom}_{G_K}(W/W_{\tau,n-1}, W) \hookrightarrow \mathrm{Hom}_{G_K}(W/W_{\tau,n-1}, W/W_{\tau,n-1}) = E$ is injective, hence the composition of f' with the natural projection $W \to W/W_{\tau,n-1}$ induces an isomorphism $W/W_{\tau,n-1} \xrightarrow{\sim} W/W_{\tau,n-1}$. This implies that the short exact sequence $0 \to W_{\tau,n-1} \to W \to W/W_{\tau,n-1} \to 0$ splits. If we take a section $s : W/W_{\tau,n-1} \hookrightarrow W$, then we can choose $\tau' \in \mathfrak{S}_n$ such that $W_{\tau',1} = s(W/W_{\tau,n-1})$, then this τ' doesn't satisfy the condition (3) in the definition of benign B-pairs. It's contradiction. Hence the above a must not be zero. If $a \neq 0$, consider the map $f - a \cdot \mathrm{id}_W \in \mathrm{End}_{G_K}(W)$, then the same argument as above implies that $f = a \cdot \mathrm{id}_W$. Hence we obtain the equality $\mathrm{End}_{G_K}(W) = E$. □

COROLLARY 2.50. *Let W be a benign E-B-pair of rank n. The functor D_W is pro-representable by R_W which is formally smooth of its dimension $n^2[K : \mathbb{Q}_p] + 1$. For any $\tau \in \mathfrak{S}_n$, the functor $D_{W,\mathcal{T}_\tau}$ is pro-representable by a quotient $R_{W,\mathcal{T}_\tau}$ of R_W which is formally smooth of its dimension $\frac{n(n+1)}{2}[K : \mathbb{Q}_p] + 1$.*

PROOF. This follows from Proposition 2.41. □

Next, we want to consider the relation between R_W and $R_{W,\mathcal{T}_\tau}$ for all $\tau \in \mathfrak{S}_n$. In particular, we want to compare the tangent space of R_W and the sum of tangent spaces of $R_{W,\mathcal{T}_\tau}$ for all $\tau \in \mathfrak{S}_n$. For this, we first need to recall the potentially crystalline deformation functor.

Definition 2.51. Let W be a potentially crystalline E-B-pair. We define the potentially crystalline deformation functor D_W^{cris} which is a subfunctor of D_W defined by

$$D_W^{\mathrm{cris}}(A) := \{[W_A] \in D_W(A) | W_A \text{ is potentially crystalline } \}$$

for $A \in \mathcal{C}_E$.

Lemma 2.52. *Let W be a potentially crystalline E-B-pair. If $\mathrm{End}_{G_K}(W) = E$, then D_W^{cris} is pro-representable by a quotient R_W^{cris} of R_W which is formally smooth of its dimension equal to*

$$\dim_E(\mathbf{D}_{\mathrm{dR}}(\mathrm{ad}(W))/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(\mathrm{ad}(W))) + \dim_E(\mathrm{H}^0(G_K, \mathrm{ad}(W))).$$

Proof. For the pro-representability, by Proposition 2.30, it suffices to relatively representability of $D_W^{\mathrm{cris}} \hookrightarrow D_W$ as in the proof of Proposition 2.37. In this case, the conditions (1) and (2) are trivial and (3) follows from the fact that any sub E-B-pair of a potentially crystalline E-B-pair is again potentially crystalline. The formal smoothness follows from Proposition 3.1.2 and Lemma 3.2.1 of [Ki08] by using the deformations of filtered (φ, G_K)-modules. For the dimension, we take a finite Galois extension L of K such that $W|_{G_L}$ is crystalline. By the same argument as in the proof of Lemma 2.44, any $W_A \in D_W^{\mathrm{cris}}(A)$ is crystalline when restricted to G_L. It's easy to check that the map $D_W(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}(W))$ induces an isomorphism $D_W^{\mathrm{cris}}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{H}^1(G_K, \mathrm{ad}(W)) \rightarrow \mathrm{H}^1(G_L, \mathbf{B}_{\mathrm{max}} \otimes_{\mathbf{B}_e} \mathrm{ad}(W)_e))$. In the same way as in the proof of Lemma 2.44, the natural map $\mathrm{H}^1(G_K, \mathbf{B}_{\mathrm{max}} \otimes_{\mathbf{B}_e} \mathrm{ad}(W)_e) \rightarrow \mathrm{H}^1(G_L, \mathbf{B}_{\mathrm{max}} \otimes_{\mathbf{B}_e} \mathrm{ad}(W)_e)$ is an injection. Hence we obtain an isomorphism

$$D_W^{\mathrm{cris}}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{H}^1(G_K, \mathrm{ad}(W)) \rightarrow \mathrm{H}^1(G_K, \mathbf{B}_{\mathrm{max}} \otimes_{\mathbf{B}_e} \mathrm{ad}(W)_e)).$$

We can calculate the dimension of this group in the same way as in the proof of Proposition 2.7 [Na09]. □

COROLLARY 2.53. *Let W be a benign E-B-pair of rank n, then R_W^{cris} is formally smooth of its dimension $\frac{(n-1)n}{2}[K:\mathbb{Q}_p]+1$.*

PROOF. This follows from Lemma 2.49 and the equality

$$\dim_E(\mathbf{D}_{\mathrm{dR}}(\mathrm{ad}(W))/\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(\mathrm{ad}(W)))=\frac{(n-1)n}{2}[K:\mathbb{Q}_p],$$

which follows from the condition (2) in Definition 2.45. □

DEFINITION 2.54. Let W be a benign E-B-pair of rank n such that $W|_{G_{K_m}}$ is crystalline. For any $\tau\in\mathfrak{S}_n$, we define a rank one saturated crystabelline E-B-pair $W'_\tau\subseteq W$ such that $\mathbf{D}_{\mathrm{cris}}^{K_m}(W'_\tau)=K_0\otimes_{\mathbb{Q}_p}Ee_{\tau(n)}\subseteq\mathbf{D}_{\mathrm{cris}}^{K_m}(W)$ and define a subfunctor $D_{W,\tau}^{\mathrm{cris}}$ of D_W by

$$\begin{aligned}D_{W,\tau}^{\mathrm{cris}}(A):=\{[W_A]\in D_W(A)|\ &\text{there exists a rank one crystabelline}\\ &\text{saturated sub } A\text{-}B\text{-pair } W'_A\subseteq W_A \text{ such that } W'_A\otimes_A E=W'_\tau\}.\end{aligned}$$

LEMMA 2.55. *Under the above condition. The functor $D_{W,\tau}^{\mathrm{cris}}$ is pro-representable by a quotient $R_{W,\tau}^{\mathrm{cris}}$ of R_W which is formally smooth and of its dimension $n(n-1)[K:\mathbb{Q}_p]+1$.*

PROOF. The relatively representability of $D_{W,\tau}^{\mathrm{cris}}\hookrightarrow D_W$ and the formal smoothness easily follows from the combination of proofs of Proposition 2.37 and Proposition 2.39 and Lemma 2.52. Here, we only prove the dimension formula. Let $\mathrm{ad}_\tau(W):=\{f\in\mathrm{ad}(W)|f(W'_\tau)\subseteq W'_\tau\}$, then we have the following short exact sequence

$$0\to\mathrm{Hom}(W/W'_\tau,W)\to\mathrm{ad}_\tau(W)\to\mathrm{ad}(W'_\tau)\to 0.$$

Taking the long exact sequence, we obtain the following short exact sequence

$$\begin{aligned}0\to\mathrm{H}^1(G_K,\mathrm{Hom}(W/W'_\tau,W))&\to\mathrm{H}^1(G_K,\mathrm{ad}_\tau(W))\\ &\to\mathrm{H}^1(G_K,\mathrm{ad}(W'_\tau))\to 0\end{aligned}$$

by Proposition 2.10. We define a subspace $\mathrm{H}^1_{f,\tau}(G_K, \mathrm{ad}_\tau(W))$ of $\mathrm{H}^1(G_K, \mathrm{ad}_\tau(W))$ as the inverse image of $\mathrm{H}^1_f(G_K, \mathrm{ad}(W'_\tau))(\subseteq \mathrm{H}^1(G_K, \mathrm{ad}(W'_\tau)))$. Hence we obtain a short exact sequence

$$0 \to \mathrm{H}^1(G_K, \mathrm{Hom}(W/W'_\tau, W)) \to \mathrm{H}^1_{f,\tau}(G_K, \mathrm{ad}_\tau(W)) \to \mathrm{H}^1_f(G_K, \mathrm{ad}(W'_\tau)) \to 0.$$

In the same way as in Lemma 2.52, we can show that the natural isomorphism $D_W(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}(W))$ induces an isomorphism

$$D^{\mathrm{cris}}_{W,\tau}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1_{f,\tau}(G_K, \mathrm{ad}_\tau(W)).$$

By Theorem 2.8 and Proposition 2.10, we obtain the equality

$$\dim_E \mathrm{H}^1(G_K, \mathrm{Hom}(W/W'_\tau, W)) = n(n-1)[K : \mathbb{Q}_p].$$

Because $\mathrm{ad}(W'_\tau)$ is the trivial E-B-pair, we have $\dim_E \mathrm{H}^1_f(G_K, \mathrm{ad}(W'_\tau)) = 1$ by Proposition 2.7 of [Na09]. Hence we obtain the equality

$$\dim_E \mathrm{H}^1_{f,\tau}(G_K, \mathrm{ad}_\tau(W)) = n(n-1)[K : \mathbb{Q}_p] + 1.$$

This proves that the dimension of $R^{\mathrm{cris}}_{W,\tau}$ is $n(n-1)[K : \mathbb{Q}_p] + 1$. □

LEMMA 2.56. *Let W be a benign E-B-pair of rank n. Let W_A be a deformation of W over A which is potentially crystalline, then $[W_A] \in D_{W,\mathcal{T}_\tau}(A)$ and $[W_A] \in D^{\mathrm{cris}}_{W,\tau}(A)$ for any $\tau \in \mathfrak{S}_n$.*

PROOF. Let W_A be as above. If $W|_{G_{K_m}}$ is crystalline, then $W_A|_{G_{K_m}}$ is crystalline by the proof of Lemma 2.52. Hence it suffices to show that $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A)$ is of the form $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A) = K_0 \otimes_{\mathbb{Q}_p} Ae_1 \oplus K_0 \otimes_{\mathbb{Q}_p} Ae_2 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ae_n$ such that $K_0 \otimes_{\mathbb{Q}_p} Ae_i$ is preserved by (φ, G_m) and $\varphi^f(e_i) = \widetilde{\alpha}_i e_i$ for a lift $\widetilde{\alpha}_i \in A^\times$ of $\alpha_i \in E^\times$ for any $1 \leqq i \leqq n$. To prove this claim, we first note that $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A)$ is a free $K_0 \otimes_{\mathbb{Q}_p} A$-module of rank n and $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A) \otimes_A E \xrightarrow{\sim} \mathbf{D}^{K_m}_{\mathrm{cris}}(W)$ by Proposition 1.3.4 and Proposition 1.3.5 [Ki09]. Then, for any $\sigma : K_0 \hookrightarrow E$, $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A)_\sigma$ is of the form $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A)_\sigma = Ae_{1,\sigma} \oplus \cdots \oplus Ae_{n,\sigma}$ such that $\varphi^f(e_{i,\sigma}) \equiv \alpha_i e_{i,\sigma} \pmod{\mathfrak{m}_A}$ for any $1 \leqq i \leqq n$. By an easy linear algebra, we can take an A-basis $e'_{1,\sigma}, e'_{2,\sigma}, \cdots, e'_{n,\sigma}$ of $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A)_\sigma$ such that $\varphi^f(e'_{i,\sigma}) = \widetilde{\alpha}_i e'_{i,\sigma}$ for a lift $\widetilde{\alpha}_i \in A^\times$ of α_i for any i. Because the actions of

φ and G_m commute and we have $\alpha_i \neq \alpha_j$, $Ae'_{i,\sigma}$ is stable by G_m. If we take $e_i := e'_{i,\sigma} + \varphi(e'_{i,\sigma}) + \cdots \varphi^{f-1}(e'_{i,\sigma}) \in \mathbf{D}^{K_m}_{\mathrm{cris}}(W_A)$, then $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A)$ can be written by $\mathbf{D}^{K_m}_{\mathrm{cris}}(W_A) = K_0 \otimes_{\mathbb{Q}_p} Ae_1 \oplus \cdots \oplus K_0 \otimes_{\mathbb{Q}_p} Ae_n$ satisfying the property of the claim. □

Lemma 2.57. *Let W be a benign E-B-pair of rank n such that $W|_{G_{K_m}}$ is crystalline, and let $\tau \in \mathfrak{S}_n$. Let $[W_A] \in D_{W,\mathcal{T}_\tau}(A)$ be a trianguline deformation over A with a lifting $0 \subseteq W_{1,A} \subseteq W_{2,A} \subseteq \cdots W_{n,A} = W_A$ of the triangulation $\mathcal{T}_\tau$. If $W_{i,A}/W_{i-1,A}$ is Hodge-Tate for any $1 \leqq i \leqq n$, then $W_A|_{G_{K_m}}$ is crystalline.*

Proof. First, we prove that $(W_{i,A}/W_{i-1,A})|_{G_{K_m}}$ is crystalline with the Hodge-Tate weights $\{k_{i,\sigma}\}_{\sigma\in\mathcal{P}}$. Because $W_{i,A}/W_{i-1,A}$ is written as a successive extension of $W_{\tau,i}/W_{\tau,i-1}$, $W_{i,A}/W_{i-1,A}$ has the Hodge-Tate weights $\{k_{i,\sigma}\}_{\sigma\in\mathcal{P}}$ with multiplicity. Twisting W_A by the crystalline character $\delta^{-1}_{\alpha_{\tau(i)}} \prod_{\sigma\in\mathcal{P}} \sigma^{-k_{i,\sigma}} : K^\times \to A^\times$, we may assume that $W_{i,A}/W_{i-1,A}$ is an étale Hodge-Tate A-B-pair of rank one with the Hodge-Tate weight zero and is a deformation of an étale potentially unramified E-B-pair $W(\widetilde{\delta}_{\tau(i)})$. By Sen's theorem ([Se73] or Proposition 5.24 of [Be02]), $W_{i,A}/W_{i-1,A}$ is potentially unramified, hence there exists a unitary homomorphism $\delta : K^\times \to A^\times$ such that $\delta|_{\mathcal{O}_K^\times}$ has a finite image and $W_{i,A}/W_{i-1,A} \xrightarrow{\sim} W(\delta)$ and δ is a lift of $\widetilde{\delta}_{\tau(i)}$. Because $(1+\mathfrak{m}_A) \cap A^\times_{\mathrm{torsion}} = \{1\}$, then we have $\delta|_{\mathcal{O}_K^\times} = \widetilde{\delta}_{\tau(i)}|_{\mathcal{O}_K^\times} : \mathcal{O}_K^\times \to A^\times$, hence $W_{i,A}/W_{i-1,A}|_{G_{K_m}}$ is crystalline. Next, we prove that $W_A|_{G_{K_m}}$ is crystalline by induction on the rank of W. When $n = 1$, we just have proved this. Assume that the lemma is proved for $n-1$, then $W_{n-1,A}|_{G_{K_m}}$ is crystalline. If we put $W_A/W_{n-1,A} \xrightarrow{\sim} W(\delta_{A,n})$, then we have $[W_A] \in \mathrm{H}^1(G_K, W_{n-1,A}(\delta^{-1}_{A,n}))$. By considering the Hodge-Tate weights of W_A and the condition (3) of Definition 2.45, we have $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(W_{n-1,A}(\delta^{-1}_{A,n})) = 0$. Comparing the dimensions, we obtain the equality $\mathrm{H}^1_f(G_K, W_{A,n-1}(\delta^{-1}_{A,n})) = \mathrm{H}^1(G_K, W_{A,n-1}(\delta^{-1}_{A,n}))$ by Proposition 2.7 of [Na09]. In particular, $W_A|_{G_{K_m}}$ is crystalline. □

Lemma 2.58. *Let W be a benign E-B-pair of rank n, and let W_1 be a rank one crystabelline sub E-B-pair of W. Then, the saturation $W_1^{sat} := (W^{sat}_{1,e}, W^{+,sat}_{1,\mathrm{dR}})$ of W_1 in W is crystabelline and the natural map $\mathrm{Hom}_{G_K}(W_1^{sat}, W) \to \mathrm{Hom}_{G_K}(W_1, W)$ induced by the natural inclusion $W_1 \hookrightarrow W_1^{sat}$ is isomorphism between one dimensional E-vector spaces.*

PROOF. Because we have $W_{1,e} = W_{1,e}^{sat}$ by Lemma 1.14 of [Na09], so W_1^{sat} is crystabelline. By the definition of benign E-B-pairs, the Hodge-Tate weights of W_1^{sat} are $\{k_{1,\sigma}\}_{\sigma\in\mathcal{P}}$. Consider the following short exact sequence of complexes of G_K-modules defined in p.890 of [Na09]

$$0 \to \mathrm{C}^\bullet(W \otimes (W_1^{sat})^\vee) \to \mathrm{C}^\bullet(W \otimes W_1^\vee) \to ((W \otimes W_1^\vee)^+_{\mathrm{dR}}/(W \otimes (W_1^{sat})^\vee)^+_{\mathrm{dR}})[0] \to 0,$$

where, for an E-B-pair W, we denote by $C^\bullet(W)$ the complex

$$\mathrm{C}^0(W) := W_e \oplus W_{\mathrm{dR}}^+ \xrightarrow{(x,y)\mapsto x-y} W_{\mathrm{dR}} =: \mathrm{C}^1(W).$$

From this, we obtain an exact sequence

$$0 \to \mathrm{H}^0(G_K, W \otimes (W_1^{sat})^\vee) \to \mathrm{H}^0(G_K, W \otimes W_1^\vee) \to \mathrm{H}^0(G_K, (W \otimes W_1^\vee)^+_{\mathrm{dR}}/(W \otimes (W_1^{sat})^\vee)^+_{\mathrm{dR}}) \to \cdots$$

By the condition (3) in Definition 2.45, we have

$$\dim_E \mathrm{H}^0(G_K, (W \otimes (W_1^{sat})^\vee)^+_{\mathrm{dR}}) = \dim_E \mathrm{H}^0(G_K, (W \otimes W_1^\vee)^+_{\mathrm{dR}}) = n[K : \mathbb{Q}_p].$$

Hence, by a standard argument of the theory of $\mathbf{B}_{\mathrm{dR}}^+$-representations, we obtain the equality $\mathrm{H}^0(G_K, (W \otimes W_1^\vee)^+_{\mathrm{dR}}/(W \otimes (W_1^{sat})^\vee)^+_{\mathrm{dR}}) = 0$. Hence the map $\mathrm{H}^0(G_K, W \otimes (W_1^{sat})^\vee)) \xrightarrow{\sim} \mathrm{H}^0(G_K, W \otimes (W_1)^\vee)$ is isomorphism. Finally, for the dimension, we have $\dim_E \mathrm{H}^0(G_K, W \otimes (W_1^{sat})^\vee) = 1$ by the condition (1) in Definition 2.45 and by Proposition 2.10 of [Na09]. □

LEMMA 2.59. *Let W be a benign E-B-pair, and let W_A be a deformation of W over A. If there exists a crystabelline sub A-B-pair $W_{1,A} \subseteq W_A$ of rank one such that the base change $W_1 := W_{1,A} \otimes_A E \hookrightarrow W_A \otimes_A E$ remains to be injective, then the saturation $W_{1,A}^{sat}$ of $W_{1,A}$ in W_A as an E-B-pair is a crystabelline A-B-pair and $W_A/W_{1,A}^{sat}$ is an A-B-pair and $W_{1,A}^{sat} \otimes_A E \xrightarrow{\sim} W_1^{sat} (\subseteq W)$.*

PROOF. First, by Proposition 2.14 of [Na09], there exists $\{l_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\leqq 0}$ such that $W_1^{sat} \xrightarrow{\sim} W_1 \otimes W(\prod_{\sigma\in\mathcal{P}} \sigma^{l_\sigma})$. We claim that the inclusion

$$\mathrm{Hom}_{G_K}(W_{1,A} \otimes W_A(\prod_{\sigma\in\mathcal{P}} \sigma^{l_\sigma}), W_A) \to \mathrm{Hom}_{G_K}(W_{1,A}, W_A)$$

induced by the natural inclusion $W_{1,A} \hookrightarrow W_{1,A} \otimes W_A(\prod_{\sigma\in\mathcal{P}} \sigma^{l_\sigma})$ is isomorphism and that these groups are rank one free A-modules. By the same argument as in Lemma 2.58, the cokernel of $\mathrm{Hom}_{G_K}(W_{1,A} \otimes W_A(\prod_{\sigma\in\mathcal{P}} \sigma^{l_\sigma}), W_A) \to \mathrm{Hom}_{G_K}(W_{1,A}, W_A)$ is contained in $\mathrm{H}^0(G_K, (W_A \otimes W_{1,A}^{\vee})^+_{\mathrm{dR}}/(W_A \otimes W_{1,A}^{\vee} \otimes W_A(\prod_{\sigma\in\mathcal{P}} \sigma^{-l_\sigma}))^+_{\mathrm{dR}})$, which is zero by the proof of Lemma 2.58. Hence the natural inclusion

$$\mathrm{Hom}_{G_K}(W_{1,A} \otimes W_A(\prod_{\sigma\in\mathcal{P}} \sigma^{l_\sigma}), W_A) \xrightarrow{\sim} \mathrm{Hom}_{G_K}(W_{1,A}, W_A)$$

is isomorphism. Next, we prove that $\mathrm{Hom}_{G_K}(W_{1,A}, W_A)$ is a free A-module of rank one by induction on the length of A. When $A = E$, this claim is proved in Lemma 2.58. Assume that A is of length n and assume that the claim is proved for $W_{A'} := W \otimes_A A'$ for a small extension $A \to A'$. We denote by I the kernel of $A \to A'$, $W_{1,A'} := W_{1,A} \otimes_A A'$. From the exact sequence

$$0 \to I \otimes_E \mathrm{Hom}_{G_K}(W_1, W) \to \mathrm{Hom}_{G_K}(W_{1,A}, W_A) \to \mathrm{Hom}_{G_K}(W_{1,A'}, W_{A'})$$

and the induction hypothesis, we have $\mathrm{length}\mathrm{Hom}_{G_K}(W_{1,A}, W_A) \leqq \mathrm{length}A$. On the other hand, the fact that the given inclusion $\iota : W_{1,A} \hookrightarrow W_A$ remains to be injective after tensoring E and the fact that $\dim_E \mathrm{Hom}_{G_K}(W_1, W) = 1$ and the induction hypothesis imply that the map $A \to \mathrm{Hom}_{G_K}(W_{1,A}, W_A) : a \mapsto a \cdot \iota$ is an injection. Hence we obtain the equality $\mathrm{length}(A) = \mathrm{length}\mathrm{Hom}_{G_K}(W_{1,A}, W_A)$. These facts prove the claim for A. From this claim, the given inclusion $\iota : W_{1,A} \hookrightarrow W_A$ factors through a map

$$\widetilde{\iota} : W'_{1,A} := W_{1,A} \otimes W_A(\prod_{\sigma} \sigma^{l_\sigma}) \to W_A.$$

Because the injectiveness of morphisms of B-pairs is determined only by the W_e-part of B-pairs and we have $W_{1,A,e} = (W_{1,A} \otimes W_A(\prod_{\sigma\in\mathcal{P}} \sigma^{l_\sigma}))_e$, the map $\widetilde{\iota}$ is also an injection. Under this situation, we claim that the map $\widetilde{\iota}$ gives an isomorphism $W_{1,A}^{sat} \xrightarrow{\sim} W'_{1,A}$ and satisfies all the properties in this lemma. By induction on the length of A, we may assume that this claim is proved for A'. First, we prove that $W_A/W'_{1,A}$ is an E-B-pair. To prove this, by Lemma 2.1.4 of [Be08], it suffices to show that $W^+_{A,\mathrm{dR}}/W'^{+}_{1,A,\mathrm{dR}}$ is a free $\mathbf{B}^+_{\mathrm{dR}}$-module. By the snake lemma, we have the following short exact sequence

$$0 \to I \otimes_E W^+_{\mathrm{dR}}/W^{sat+}_{1,\mathrm{dR}} \to W^+_{A,\mathrm{dR}}/W'^+_{1,A,\mathrm{dR}} \to W^+_{A',\mathrm{dR}}/(W'^+_{1,A,\mathrm{dR}} \otimes_A A') \to 0.$$

From this and the induction hypothesis, $W^+_{A,\mathrm{dR}}/W'^+_{1,A,\mathrm{dR}}$ is a free $\mathbf{B}^+_{\mathrm{dR}}$-module. Finally, we prove the A-flatness of $W_A/W'_{1,A}$. This follows from the fact that the map $\widetilde{\iota} \otimes \mathrm{id}_E : W'_{1,A} \otimes_A E \hookrightarrow W_A \otimes_A E$ is saturated, which can be seen from the proof of the above first claim. Hence $W_A/W'_{1,A}$ is an A-B-pair. We finish the proof of the lemma. □

Lemma 2.60. *Let W be a benign E-B-pair of rank n. For any $\tau \in \mathfrak{S}_n$, we have*

$$D_{W,\mathcal{T}_\tau}(E[\epsilon]) + D^{\mathrm{cris}}_{W,\tau}(E[\varepsilon]) = D_W(E[\varepsilon]).$$

Proof. By Corollary 2.50 and Lemma 2.52 and Lemma 2.55, we obtain an equality

$$\dim_E D_W(E[\varepsilon]) + \dim_E D^{\mathrm{cris}}_W(E[\varepsilon]) = \dim_E D_{W,\mathcal{T}_\tau}(E[\varepsilon]) + \dim_E D^{\mathrm{cris}}_{W,\tau}(E[\varepsilon]).$$

Hence, it suffices to show that we have an equality

$$D_{W,\mathcal{T}_\tau}(A) \cap D^{\mathrm{cris}}_{W,\tau}(A) = D^{\mathrm{cris}}_W(A)$$

for any $A \in \mathcal{C}_E$. We first note that the inclusion $D^{\mathrm{cris}}_W(A) \subseteq D_{W,\mathcal{T}_\tau}(A) \cap D_{W,\tau}(A)$ follows from Lemma 2.56.

We prove the opposite inclusion $D_{W,\mathcal{T}_\tau}(A) \cap D^{\mathrm{cris}}_{W,\tau}(A) \subseteq D^{\mathrm{cris}}_W(A)$ by induction on the rank n of W. When $n = 1$, this is trivial. Let W be of rank n and assume that the lemma is proved for $n-1$. Take any $[W_A] \in D_{W,\mathcal{T}_\tau}(A) \cap D^{\mathrm{cris}}_{W,\tau}(A)$. By the definitions of $D_{W,\mathcal{T}_\tau}$ and $D^{\mathrm{cris}}_{W,\tau}$, then there exist an A-triangulation $0 \subseteq W_{1,A} \subseteq W_{2,A} \subseteq \cdots \subseteq W_{n-1,A} \subseteq W_{n,A} = W_A$ such that $W_{i,A} \otimes_A E \xrightarrow{\sim} W_{\tau,i}$ for any i and a saturated crystabelline rank one A-B-pair $W'_{1,A} \subseteq W_A$ such that $W'_{1,A} \otimes_A E \xrightarrow{\sim} W'_\tau$. We first claim that the composition of $W'_{1,A} \hookrightarrow W_A$ with $W_A \to W_A/W_{1,A}$ is an injection. Because $\mathrm{Ker}(W'_{1,A} \to W_A/W_{1,A})$ is a sub E-B-pair of $W_{1,A}$ and we have $\mathrm{Hom}_{G_K}(\mathrm{Ker}(W'_{1,A} \to W_A/W_{1,A}), W_{1,A}) = 0$ by the condition (1) of Definition 2.45 and Proposition 2.14 of [Na09], hence the composition map $W'_{1,A} \to W_A/W_{1,A}$ is an injection. Hence, the saturation $(W'_{1,A})^{sat}$ of

$W'_{1,A}$ in $W_A/W_{1,A}$ is a rank one crystabelline A-B-pair satisfying the similar conditions as those of $W'_{1,A} \hookrightarrow W_A$ by Lemma 2.59. Hence, $W_A/W_{1,A}$ is crystabelline by induction, and the Hodge-Tate weights of $W_A/W_{1,A}$ are $\{k_{2,\sigma}, k_{3,\sigma}, \cdots, k_{n,\sigma}\}_{\sigma\in\mathcal{P}}$ (with multiplicity $[A : E]$) by the condition (3) of Definition 2.45. Moreover, $W'_{1,A}$ has the Hodge-Tate weights $\{k_{1,\sigma}\}_{\sigma\in\mathcal{P}}$ (with multiplicity) by (3) of Definition 2.45. Because $k_{1,\sigma} \neq k_{i,\sigma}$ for any $i \neq 1$ and there is no extension between the objects with different Hodge-Tate weights by a theorem of Tate, so W_A is a Hodge-Tate E-B-pair. Hence, W_A is crystabelline by Lemma 2.57, and we have that $[W_A] \in D_W^{\mathrm{cris}}(A)$. □

Definition 2.61. For $R = R_W$ or $R_{W,\mathcal{T}_\tau}$, we denote by

$$t(R) := \mathrm{Hom}_E(\mathfrak{m}_R/\mathfrak{m}_R^2, E)$$

the tangent space of R. We have a natural inclusion $t(R_{W,\mathcal{T}_\tau}) \hookrightarrow t(R_W)$ for each $\tau \in \mathfrak{S}_n$.

The following theorem is the main theorem of § 2, which is crucial for the applications to some problems on the Zariski density of crystalline points or modular points. This theorem was first discovered by Chenevier (Theorem 3.19 of [Ch09b]) for $K = \mathbb{Q}_p$.

Theorem 2.62. *Let W be a benign E-B-pair of rank n. We have an equality*

$$\sum_{\tau\in\mathfrak{S}_n} t(R_{W,\mathcal{T}_\tau}) = t(R_W).$$

Proof. We prove this theorem by induction on n. When $n = 1$, then the theorem is trivial. Assume that the theorem is true for the rank $n-1$. Let W be a benign E-B-pair of rank n. We take an element $\tau \in \mathfrak{S}_n$. We define a subfunctor $D_{W,\tau}$ of D_W by

$$\begin{aligned} D_{W,\tau}(A) := \{[W_A] \in D_W(A) |\ & \text{there exists a rank one} \\ & \text{sub } A\text{-}B\text{-pair } W_{1,A} \subseteq W_A \text{ such that the quotient} \\ & W_A/W_{1,A} \text{ is an } A\text{-}B\text{-pair and } W_{1,A} \otimes_A E \xrightarrow{\sim} W'_\tau\}, \end{aligned}$$

where W'_τ is defined in Definition 2.54. Then, $D_{W,\tau}^{\mathrm{cris}}$ is a subfunctor of $D_{W,\tau}$, and we can show in the same way that

$$D_{W,\tau}(E[\varepsilon]) \xrightarrow{\sim} \mathrm{H}^1(G_K, \mathrm{ad}_\tau(W)),$$

where we define

$$\mathrm{ad}_\tau(W) := \{f \in \mathrm{ad}(W) | f(W'_\tau) \subseteq W'_\tau\}.$$

By Lemma 2.60, we obtain an equality

$$\mathrm{H}^1(G_K, \mathrm{ad}_\tau(W)) + \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}_\tau}(W)) = \mathrm{H}^1(G_K, \mathrm{ad}(W)).$$

Because we have a natural inclusion $D_{W,\mathcal{T}_{\tau'}} \subseteq D_{W,\tau}$ for any $\tau' \in \mathfrak{S}_{\tau,n} := \{\tau' \in \mathfrak{S}_n | \tau'(1) = \tau(n)\}$, we have a natural map $\mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}_{\tau'}}(W)) \rightarrow \mathrm{H}^1(G_K, \mathrm{ad}_\tau(W))$ for each such τ'. Therefore, it suffices to prove that the map

$$\oplus_{\tau' \in \mathfrak{S}_{\tau,n}} \mathrm{H}^1(G_K, \mathrm{ad}_{\mathcal{T}_{\tau'}}(W)) \rightarrow \mathrm{H}^1(G_K, \mathrm{ad}_\tau(W))$$

is a surjection. We prove this surjection as follows. By the definition, we have the following short exact sequences of E-B-pairs for each $\tau' \in \mathfrak{S}_{\tau,n}$,

$$(1) \qquad 0 \rightarrow \mathrm{Hom}(W/W'_\tau, W) \rightarrow \mathrm{ad}_\tau(W) \rightarrow \mathrm{ad}(W'_\tau) \rightarrow 0,$$

and

$$(2) \qquad 0 \rightarrow \{f \in \mathrm{ad}_{\mathcal{T}_{\tau'}}(W) | f(W'_\tau) = 0\} \rightarrow \mathrm{ad}_{\mathcal{T}_{\tau'}}(W) \rightarrow \mathrm{ad}(W'_\tau) \rightarrow 0.$$

Moreover, we have

$$(3) \qquad 0 \rightarrow \mathrm{Hom}(W/W'_\tau, W'_\tau) \rightarrow \mathrm{Hom}(W/W'_\tau, W) \rightarrow \mathrm{ad}(W/W'_\tau) \rightarrow 0,$$

and

$$(4) \quad 0 \rightarrow \mathrm{Hom}(W/W'_\tau, W'_\tau) \rightarrow \{f \in \mathrm{ad}_{\mathcal{T}_{\tau'}}(W) | f(W'_\tau) = 0\} \rightarrow \mathrm{ad}_{\mathcal{T}_{\bar{\tau}'}}(W/W'_\tau) \rightarrow 0,$$

where, for $\tau' \in \mathfrak{S}_{\tau,n}$, we denote by

$$\mathcal{T}_{\bar{\tau}'} : 0 \subseteq W_{\tau',2}/W'_\tau \subseteq W_{\tau',3}/W'_\tau \subseteq \cdots \subseteq W_{\tau',n-1}/W'_\tau \subseteq W/W'_\tau$$

the triangulation of W/W'_τ induced from $\mathcal{T}_{\tau'}$. We have $\mathrm{H}^2(G_K, \mathrm{Hom}(W/W'_\tau, W'_\tau)) = 0$ by the condition (1) of Definition 2.45 and Proposition 2.10, and $\mathrm{H}^2(G_K, \mathrm{ad}_{\mathcal{T}_\tau}(W)) = 0$ by the proof of Proposition 2.41. Hence, from the short exact sequence (4) above, we obtain the equality $\mathrm{H}^2(G_K, \{f \in$

$\mathrm{ad}_{\tau'}(W)|f(W'_\tau)=0\})=0$. From this and from (1) and (2) above, it suffices to show that the map

$$\oplus_{\tau'\in\mathfrak{S}_{\tau,n}}\mathrm{H}^1(G_K,\{f\in \mathrm{ad}_{\tau'}(W)|f(W'_\tau)=0\})\to \mathrm{H}^1(G_K,\mathrm{Hom}(W/W'_\tau,W))$$

is a surjection. By (3) and (4) above and the fact that $\mathrm{H}^2(G_K,\mathrm{Hom}(W/W'_\tau, W'_\tau))=0$, this surjectivity follows from the surjectivity of the map

$$\oplus_{\tau'\in\mathfrak{S}_{\tau,n}}\mathrm{H}^1(G_K,\mathrm{ad}_{\mathcal{T}_{\bar{\tau}'}}(W/W'_\tau))\to \mathrm{H}^1(G_K,\mathrm{ad}(W/W'_\tau)),$$

which is the induction hypothesis. Hence we have finished the proof of the theorem. □

3. Construction of p-Adic Families of Two Dimensional Trianguline Representations

In this section, we generalize the Kisin's theory of the finite slope subspace for any p-adic field, and then construct p-adic families of two dimensional triangulline representations, which are crucial for the density of two dimensional crystalline representations.

3.1. Almost $\mathbb{C}_p$-representations

In the first subsection, we prove some propositions concerning Banach G_K-modules, which we need for the generalization of the Kisin's theory of the finite slope subspace.

We first recall some rings of Lubin-Tate's p-adic periods defined by Colmez [Co02] and the definition of almost $\mathbb{C}_p$-representations defined by Fontaine [Fo03].

Let π_K be a fixed uniformizer of K. Let $P(X)\in\mathcal{O}_K[X]$ be a monic polynomial of degree $q:=p^f$ such that $P(X)\equiv \pi_K X \pmod{X^2}$ and $P(X)\equiv X^q \pmod{\pi_K}$. Let $\mathcal{F}_{\pi_K}$ be the Lubin-Tate formal group law over $\mathcal{O}_K$ on which the multiplication by π_K is given by $[\pi_K]=P(X)$. Let $\chi_{\mathrm{LT}}: G_K\to\mathcal{O}_K^\times$ be the Lubin-Tate character associated to π_K. Put $\mathbf{A}_{\mathrm{inf},K}:=\mathbf{A}^+\otimes_{\mathcal{O}_{K_0}}\mathcal{O}_K$, which is equipped with the weak topology on which G_K acts continuously. The ring $\mathbf{A}_{\mathrm{inf},K}$ also has a $\mathcal{O}_K$-linear continuous $\varphi_K:=\varphi^f$-action. Any element of $\mathbf{A}_{\mathrm{inf},K}$ can be written uniquely of the form $\sum_{k=0}^{\infty}[x_k]\pi_K^k$ ($x_k\in\widetilde{\mathbf{E}}^+$). Put $\mathbf{B}_{\mathrm{inf},K}:=\mathbf{A}_{\mathrm{inf},K}[p^{-1}]$. By Lemma 8.3 of [Co02], for each $x\in\widetilde{\mathbf{E}}^+$, there exists a unique element $\{x\}\in\mathbf{A}_{\mathrm{inf},K}$ such that $\{x\}$ is a lift of x and $\varphi_K(\{x\}) = [\pi_K](\{x\})(= P(\{x\}))$. We fix a set $\{\omega_n\}_{n\geqq 0}$ such that $\omega_1 \in \mathfrak{m}_{\overline{K}}$ is a primitive $[\pi_K]$-torsion point of $\mathcal{F}_{\pi_K}(\overline{K})$ and $[\pi_K](\omega_{n+1}) = \omega_n$ for any $n \geqq 0$, then $(\bar{\omega}_n)_{n\geqq 0}$ defines an element in $\widetilde{\mathbf{E}}^+ \xrightarrow{\sim} \varprojlim_n \mathcal{O}_{\overline{K}}/\pi_K\mathcal{O}_{\overline{K}}$ where the projective limit is given by the q-th power Frobenius map. We define $\omega_K := \{(\bar{\omega}_n)_{n\geqq 0}\} \in \mathbf{A}_{\mathrm{inf},K}$. By the definition of $\{-\}$ and the uniqueness of $\{-\}$, the actions of G_K and φ_K on ω_K are given by $g(\omega_K) = [\chi_{\mathrm{LT}}(g)](\omega_K)$ for $g \in G_K$ (which converges for the weak topology) and $\varphi_K(\omega_K) = [\pi_K](\omega_K)$. Take a subset $\{\pi_n\}_{n\geqq 0} \subseteq \mathcal{O}_{\overline{K}}$ such that $\pi_0 = \pi_K$ and $\pi_{n+1}^q = \pi_n$ for any n, and put $\widetilde{\pi}_K := (\bar{\pi}_n)_{n\geqq 0} \in \widetilde{\mathbf{E}}^+$. Define $\mathbf{A}_{\max,K} := \widehat{\mathbf{A}_{\mathrm{inf},K}[\frac{[\widetilde{\pi}_K]}{\pi_K}]}$ the p-adic completion of $\mathbf{A}_{\mathrm{inf},K}[\frac{[\widetilde{\pi}_K]}{\pi_K}]$. Define $\mathbf{B}^+_{\max,K} = \mathbf{A}_{\max,K}[p^{-1}]$, which is a K-Banach space with continuous actions of G_K and φ_K. By the definition, we have a canonical isomorphism $K \otimes_{K_0} \mathbf{B}^+_{\max,\mathbb{Q}_p} \xrightarrow{\sim} \mathbf{B}^+_{\max,K}$ (Remark 7.13 of [Co02]). By Lemma 8.8 and Proposition 8.9 of [Co02], there exists a power series $F_K(X) \in K[[X]]$ which is the Lubin-Tate's logarithm such that $F_K(X) \circ [a] = aF_K(X)$ for any $a \in \mathcal{O}_K$, and $t_K := F_K(\omega)$ converges in $\mathbf{A}_{\max,K}$ such that $\varphi_K(t_K) = \pi_K t_K$, $g(t_K) = \chi_{\mathrm{LT}}(g)t_K$ for $g \in G_K$. We define $\mathbf{B}_{\max,K} := \mathbf{B}^+_{\max,K}[t_K^{-1}]$. We define $\mathbf{B}^+_{\mathrm{dR}} := \varprojlim_n \mathbf{B}^+_{\mathrm{inf},K}/(\mathrm{Ker}(\theta))^n$ which is equipped with the projective limit topology of K-Banach spaces $\{\mathbf{B}^+_{\mathrm{inf},K}/(\mathrm{Ker}(\theta))^n\}_{n\geqq 1}$ whose $\mathcal{O}_K$-lattice is defined as the image of $\mathbf{A}_{\mathrm{inf},K} \to \mathbf{B}^+_{\mathrm{inf},K}/(\mathrm{Ker}(\theta))^n$. By Proposition 7.12 of [Co02], this $\mathbf{B}^+_{\mathrm{dR}}$ is canonically topologically isomorphic to the usual $\mathbf{B}^+_{\mathrm{dR}}$ defined in §2. We define $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}^+_{\mathrm{dR}}[t^{-1}] = \mathbf{B}^+_{\mathrm{dR}}[t_K^{-1}]$.

Using these preliminaries, we define a functor from the category of φ_K-modules to the category of almost $\mathbb{C}_p$-representations defined by Fontaine. We can see this construction as a very special case of a generalization of Berger's results [Be09] to the case of Lubin-Tate period rings.

Definition 3.1. We say that D is a φ_K-module over K if D is a finite dimensional K-vector space with a K-linear isomorphism $\varphi_K : D \xrightarrow{\sim} D$.

Let D be a φ_K-module over K, we extend the action of φ_K to $\widehat{K^{\mathrm{ur}}} \otimes_K D$ by $\varphi_K(a \otimes x) := \varphi_K(a) \otimes \varphi_K(x)$, where $\widehat{K^{\mathrm{ur}}}$ is the p-adic completion of the maximal unramified extension K^{ur} of K and $\varphi_K \in \mathrm{Gal}(\widehat{K^{\mathrm{ur}}}/K)$ is the lift of q-th Frobenius in $\mathrm{Gal}(\overline{\mathbb{F}}/\mathbb{F}_q)$. The Dieudonné-Manin theorem gives a decomposition

$$\widehat{K^{\mathrm{ur}}} \otimes_K D = \oplus_{s\in\mathbb{Q}} D_s,$$

where for any $s = \frac{a}{h} \in \mathbb{Q}$ such that $(a, h) \in \mathbb{Z} \times \mathbb{Z}_{\geqq 1}$ are co-prime, D_s is zero or a finite direct sum of $D_{a,h} := \widehat{K^{\mathrm{ur}}}e_1 \oplus \widehat{K^{\mathrm{ur}}}e_2 \oplus \cdots \oplus \widehat{K^{\mathrm{ur}}}e_h$ such that $\varphi_K(e_1) = e_2, \varphi_K(e_2) = e_3, \cdots, \varphi_K(e_{h-1}) = e_h, \varphi_K(e_h) = \pi_K^a e_1$. We define the set of slopes of D as the set of $s \in \mathbb{Q}$ such that $D_s \neq 0$. We define a $\widehat{K^{\mathrm{ur}}}$-semi-linear G_K-action on $\widehat{K^{\mathrm{ur}}} \otimes_K D$ by $g(a \otimes x) := g(a) \otimes x$ for $g \in G_K$, $a \in \widehat{K^{\mathrm{ur}}}$, $x \in D$, then D_s is preserved by this G_K-action for any $s \in \mathbb{Q}$ because the actions of G_K and φ_K commute each other. For $s = \frac{a}{h}$, if we define $D'_s := \{x \in D_s | \varphi_K^h(x) = \pi_K^a x\}$, then we have $D_s = \widehat{K^{\mathrm{ur}}} \otimes_{K_h^{\mathrm{ur}}} D'_s$ and D'_s is preserved by G_K and φ_K, where K_h^{ur} is the unramified extension of K of degree h.

The notion of almost $\mathbb{C}_p$-representations was defined by Fontaine [Fo03].

DEFINITION 3.2. Let U be a $\mathbb{Q}_p$-Banach space equipped with a continuous $\mathbb{Q}_p$-linear G_K-action. We say that U is an almost $\mathbb{C}_p$-representation if there exists $\mathbb{Q}_p$-representations V_1, V_2 and an integer $d \in \mathbb{Z}_{\geqq 0}$ such that V_1 is a sub G_K-module of U and V_2 is a sub G_K-module of $\mathbb{C}_p^d$ and there exists an isomorphism $U/V_1 \xrightarrow{\sim} \mathbb{C}_p^d/V_2$ as $\mathbb{Q}_p$-Banach G_K-modules.

REMARK 3.3. By Theorem C of [Fo03], $\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+$ is an almost $\mathbb{C}_p$-representation for any $k \geqq 0$. By Theorem B of [Fo03], for any continuous $\mathbb{Q}_p$-linear G_K-morphism $f : U_1 \to U_2$ between almost $\mathbb{C}_p$-representations U_1, U_2, it is known that $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ (as $\mathbb{Q}_p[G_K]$-modules) are almost $\mathbb{C}_p$-representations and $\mathrm{Im}(f)$ is also an almost $\mathbb{C}_p$-representation which is a closed subspace of U_2.

Let D be a φ_K-module over K. We prove that $X_0(D) := (\mathbf{B}_{\max,K}^+ \otimes_K D)^{\varphi_K=1}$ is an almost $\mathbb{C}_p$-representation.

LEMMA 3.4. *Let D be a φ_K-module over K.*

(1) *$X_0(D)$ is an almost $\mathbb{C}_p$-representation.*

(2) *If any slope s of D satisfies $s > 0$, then $X_0(D) = 0$.*

PROOF. The proof is similar to that of Proposition 2.2 of [Be09]. If we denote by $\widehat{K^{\mathrm{ur}}} \otimes_K D = \oplus_{s \in \mathbb{Q}} D_s$ as above, then we have $\mathbf{B}_{\max,K}^+ \otimes_K D =$

$\oplus_{s\in\mathbb{Q}}\mathbf{B}^+_{\max,K}\otimes_{\widehat{K^{\mathrm{ur}}}} D_s$ as a φ_K-module. For $s=\frac{a}{h}$, $\mathbf{B}^+_{\max,K}\otimes_{\widehat{K^{\mathrm{ur}}}} D_s = \mathbf{B}^+_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s$ is preserved by the actions of G_K and φ_K. Hence, it suffices to show that, for any $s=\frac{a}{h}$, $(\mathbf{B}^+_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s)^{\varphi_K=1}$ is an almost $\mathbb{C}_p$-representation and is zero if $a>0$. By the definition of D'_s, we have a canonical inclusion $(\mathbf{B}^+_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s)^{\varphi_K=1}\subseteq \mathbf{B}^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s$. By 8.5 of [Co02], we have $\mathbf{B}^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K}=0$ for $a>0$, and a short exact sequence

$$0\to K^{\mathrm{ur}}_h t^{-a}_{K^{\mathrm{ur}}_h}\to \mathbf{B}^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K}\to \mathbf{B}^+_{\mathrm{dR}}/t^{-a}\mathbf{B}^+_{\mathrm{dR}}\to 0$$

for $a\leqq 0$, where $t_{K^{\mathrm{ur}}_h}\in \mathbf{B}^+_{\max,K}=\mathbf{B}^+_{\max,K^{\mathrm{ur}}_h}$ is defined from the triple $(K^{\mathrm{ur}}_h,\pi_K,\varphi_K^h)$ in the same way as in the definition of t_K defined from (K,π_K,φ_K). Moreover, because $\mathbf{B}^+_{\mathrm{dR}}/t^{-a}\mathbf{B}^+_{\mathrm{dR}}\otimes_{K^{\mathrm{ur}}_h} D'_s$ is a $\mathbf{B}^+_{\mathrm{dR}}$-representation, so this is also an almost $\mathbb{C}_p$-representation by Theorem 5.13 of [Fo03]. Hence, $\mathbf{B}^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s$ is also an almost $\mathbb{C}_p$-representation. Because we have an equality

$$\begin{aligned}&(\mathbf{B}^+_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s)^{\varphi_K=1}\\ &\qquad=\mathrm{Ker}(\varphi_K-1:\mathbf{B}^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s\to \mathbf{B}^{+,\varphi_K^h=\pi_K^{-a}}_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s),\end{aligned}$$

so $(\mathbf{B}^+_{\max,K}\otimes_{K^{\mathrm{ur}}_h} D'_s)^{\varphi_K=1}$ is also an almost $\mathbb{C}_p$-representation by Remark 3.3. □

As an application of this lemma, we obtain the following corollary. We fix an embedding $\sigma:K\hookrightarrow E$. For a K-vector space M and an E-vector space N, we denote by $M\otimes_{K,\sigma}N$ the tensor product of M and N over K, where we view N as a K-vector space by the map $\sigma:K\hookrightarrow E$.

Corollary 3.5. *Let $\alpha\in E^\times$ be a non-zero element, then $(\mathbf{B}^+_{\max,K}\otimes_{K,\sigma}E)^{\varphi_K=\alpha}$ is an almost $\mathbb{C}_p$-representation. For any positive integer k such that $k>e_Kv_p(\alpha)$, where e_K is the absolute ramified index of K, the natural map*

$$(\mathbf{B}^+_{\max,K}\otimes_{K,\sigma}E)^{\varphi_K=\alpha}\to(\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}})\otimes_{K,\sigma}E$$

is an injection, Moreover, if we denote the cokernel of this inclusion by U_k, we have the following short exact sequence of E-Banach almost $\mathbb{C}_p$-

representations,

$$0 \to (\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha} \to (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}) \otimes_{K,\sigma} E \to U_k \to 0.$$

PROOF. For $\alpha \in E^\times$, we define a φ_K-module D_α over K by $D_\alpha := Ee$ such that $\varphi_K(ae) = \alpha^{-1}ae$ for any $a \in E$. D_α has a unique slope $-e_K v_p(\alpha)$ and we have a natural isomorphism

$$X_0(D_\alpha) \xrightarrow{\sim} (\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha}.$$

Hence, $(\mathbf{B}^+_{\max,K} \otimes_K E)^{\varphi=\alpha}$ is a non-zero almost $\mathbb{C}_p$-representation by Lemma 3.4. Moreover, using Proposition 8.10 of [Co02], we can show that we have an equality

$$(\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha} \cap (t^k\mathbf{B}^+_{\mathrm{dR}} \otimes_{K,\sigma} E) = (t^k_K\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha}$$

for any $k \geqq 0$, which is isomorphic to $X_0(D_{\alpha\sigma(\pi_K)^{-k}})$. Because, $D_{\alpha\sigma(\pi_K)^{-k}}$ has a unique slope $(k - e_K v_p(\alpha))$, so we have $X_0(D_{\alpha\sigma(\pi_K)^{-k}}) = 0$ when $k > e_K v_p(\alpha)$ by Lemma 3.4. This implies that the natural map

$$(\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} E)^{\varphi_K=\alpha} \to (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}) \otimes_{K,\sigma} E$$

is an injection. Because both of these are almost $\mathbb{C}_p$-representations which are also E-Banach spaces, hence the cokernel U_k is also an E-Banach almost $\mathbb{C}_p$-representation by Remark 3.3. □

For two K-Banach spaces M_1 and M_2, we denote by $M_1 \hat{\otimes}_K M_2$ the complete tensor product of M_1 and M_2 over K. Let R be a complete topological E-algebra. We say that R is an E-Banach algebra if there exists a map $|-|_R : R \to \mathbb{R}_{\geqq 0}$ which satisfies the following,

(1) $|1|_R = 1$, $|x|_R = 0$ if and only if $x = 0$,

(2) $|x + y|_R \leqq \max\{|x|_R, |y|_R\}$,

(3) $|xy|_R \leqq |x|_R|y|_R$ and $|ax|_R = |a|_p|x|_R$

for any $x, y \in R$, $a \in E$, and if the topology of R is defined by the metric induced from $|-|_R$.

Lemma 3.6. *Let R be an E-Banach algebra, and let $\alpha \in R$ be an element of R such that $\alpha - 1$ is topologically nilpotent, then there exists $u \in (\widehat{K^{\mathrm{ur}}}\hat{\otimes}_{K,\sigma}R)^{\times}$ such that $\varphi_K(u) = \alpha u$.*

Proof. The proof is the same as that of Lemma 3.6 of [Ki03]. $\square$

Here, we recall some terminologies concerning Banach modules from §2 of [Bu07]. Let R be an E-Banach algebra and let M be a topological R-module. We say that M is a Banach R-module if M is a complete topological R-module with a map $|-| : M \to \mathbb{R}_{\geqq 0}$ which satisfies the following,

(1) $|m| = 0$ if and only if $m = 0$,

(2) $|m + n| \leqq \max\{|m|, |n|\}$,

(3) $|am| \leqq |a|_R|m|$ ($|-|_R$ is a fixed E-Banach norm on R as above)

for any $m, n \in M$, $a \in R$, and if the topology on M is defined by the metric induced from $|-|$. Let M be a Banach R-module. We say that M is orthonormalizable if there exist a map $|-| : M \to \mathbb{R}_{\geqq 0}$ as above and a subset $\{e_i\}_{i\in I}$ of M such that, for any $m \in M$,

(1) there exists a unique $\{a_i\}_{i\in I}$ ($a_i \in R$) such that $a_i \to 0$ ($i \to \infty$) and $m = \sum_{i\in I} a_i e_i$,

(2) we have $|m| = \max_{i\in I}\{|a_i|_R\}$.

We say that a Banach R-module M has the property (Pr) if there exists a Banach R-module N such that $M \oplus N$ is orthonormalizable.

The following proposition is also a generalization of Corollary 3.7 of [Ki03] which will play a crucial role in the next subsection.

Proposition 3.7. *Let R be an E-Banach algebra and let $Y \in R^{\times}$ be a unit of R. Assume that there exists a finite Galois extension E' of E and $\lambda \in (R \otimes_E E')^{\times}$ such that $E'[\lambda] \subseteq R \otimes_E E'$ is an étale E'-algebra and that $Y\lambda^{-1} - 1$ is topologically nilpotent in $R \otimes_E E'$. Then, for any sufficiently large $k \in \mathbb{Z}_{>0}$, there exists a Banach R-module U_k with the property (Pr)* *which is equipped with a continuous R-linear G_K-action such that there exists a G_K-equivariant short exact sequence of Banach R-modules with the property (Pr)*

$$0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R \to U_k \to 0$$

which is compatible with any base change, i.e. for any continuous homomorphism $f : R \to R'$ of E-affinoid algebras, the complete tensor product of the above exact sequence with R' is equal to

$$0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=f(Y)} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R' \to U_k\hat{\otimes}_R R' \to 0.$$

PROOF. Decompose $E'[\lambda] = \prod_{i\in I} E_i \subseteq R \otimes_E E'$ such that each E_i is a finite extension of E', and denote by $\lambda_i \in E_i$ the image of λ in E_i, then we obtain a decomposition $R \otimes_E E' = \prod_{i\in I} R_i$ such that $E_i \subseteq R_i$ for any i. By Corollary 3.5, we have a short exact sequence of E_i-Banach spaces

$$0 \to (\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} E_i)^{\varphi_K=\lambda_i} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \otimes_{K,\sigma} E_i \to U_{k,i} \to 0.$$

for any $k \in \mathbb{Z}_{>0}$ such that $k > e_K v_p(\lambda_i)$ for any i. Hence, if we take the complete tensor product over E_i of this sequence with R_i, we obtain a short exact sequence of orthonormalizable R_i-Banach spaces

$$0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R_i)^{\varphi_K=\lambda_i} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R_i \to U_{k,i}\hat{\otimes}_{E_i}R_i \to 0.$$

By the assumption, the element $Y\lambda_i^{-1} - 1$ is topologically nilpotent in R_i, hence we have an element $u_i \in (\widehat{K^{\mathrm{ur}}}\hat{\otimes}_{K,\sigma}R_i)^\times$ such that $\varphi_K(u_i) = Y\lambda_i^{-1}u_i$ by Lemma 3.6. Multiplying u_i to the above short exact sequence, we obtain a short exact sequence

$$0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R_i)^{\varphi_K=Y} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R_i \to (U_{k,i}\hat{\otimes}_{E_i}R_i) \to 0.$$

Summing up for all $i \in I$, we obtain

$$\begin{aligned} 0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}(R\otimes_E E'))^{\varphi_K=Y} &\to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}(R\otimes_E E') \\ &\to \oplus_{i\in L}(U_{k,i}\hat{\otimes}_{E_i}R_i) \to 0, \end{aligned}$$

which we can see as an exact sequence of R-Banach modules with the property (Pr). Taking the $\mathrm{Gal}(E'/E)$-fixed part, we obtain a short exact sequence

$$0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R \to U_k \to 0$$

satisfying all the conditions in the proposition, where $U_k := (\oplus_{i\in I}(U_{k,i}\hat{\otimes}_{E_i}R_i))^{\mathrm{Gal}(E'/E)}$. Finally, the base change property of this exact sequence is clear from the proof. □

Let V be an E-representation, then we define

$$\mathbf{D}^+_{\mathrm{cris}}(V) := (\mathbf{B}^+_{\max}\otimes_{\mathbb{Q}_p} V)^{G_K},\ \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) := \mathbf{D}_{\mathrm{cris}}(V)\cap \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V),$$

where we recall that $\mathbf{B}^+_{\max} = \mathbf{B}^+_{\max,\mathbb{Q}_p}$. Then, we have a natural inclusion $\mathbf{D}^+_{\mathrm{cris}}(V) \subseteq \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)$, which is not an equality in general.

Lemma 3.8. *Let $\alpha \in E^\times$ be a non zero element. If a φ-submodule D of $\mathbf{D}_{\mathrm{cris}}(V)^{\varphi^f=\alpha}$ is contained in $\mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(V)$, then D is also contained in $\mathbf{D}^+_{\mathrm{cris}}(V)^{\varphi^f=\alpha}$.*

Proof. It suffices to show that if an element $x \in (\mathbf{B}_{\max}\otimes_{\mathbb{Q}_p} E)^{\varphi^f=\alpha}$ satisfies that $\varphi^i(x) \in \mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} E$ for any $i \in \mathbb{Z}_{\geqq 0}$, then $x \in (\mathbf{B}^+_{\max}\otimes_{\mathbb{Q}_p} E)^{\varphi^f=\alpha}$. If we write $x = \frac{a}{t^n}$ for some $a \in (\mathbf{B}^+_{\max}\otimes_{\mathbb{Q}_p} E)^{\varphi^f=\alpha p^{fn}}$ and $n \geqq 0$, then we have $\frac{\varphi^i(a)}{p^{ni}t^n} = \varphi^i(x) \in \mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} E$ for any $0 \leqq i \leqq f-1$. Hence, we have

$$\varphi^i(a) \in (\mathbf{B}^+_{\max}\otimes_{\mathbb{Q}_p} E)^{\varphi^f=\alpha p^{nf}} \cap t^n\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} E = (t^n_{K_0}\mathbf{B}^+_{\max}\otimes_{\mathbb{Q}_p} E)^{\varphi^f=\alpha p^{nf}},$$

where the last equality follows from Proposition 8.10 of [Co02]. Hence, we can write $a = \varphi^{-i}(t^n_{K_0})a_i$ for some $a_i \in \mathbf{B}^+_{\max}\otimes_{\mathbb{Q}_p} E$ for any $1 \leqq i \leqq f-1$. Then we can write by $a = (\prod_{i=0}^{f-1}\varphi^{-i}(t^n_{K_0}))a'$ for some $a' \in \mathbf{B}^+_{\max}\otimes_{\mathbb{Q}_p} E$ by Lemma 8.18 of [Co02]. Because $\prod_{i=0}^{f-1}\varphi^{-i}(t_{K_0}) \in K_0^\times t$ by Lemma 8.17 of [Co02], we have $x = \frac{a}{t^n} \in \mathbf{B}^+_{\max}\otimes_{\mathbb{Q}_p} E$. □

3.2. Construction of the finite slope subspace for general p-adic field

In this subsection, we generalize the theory of the finite slope subspace X_{fs} for any p-adic field. Using Proposition 3.7, the construction and the proof is essentially the same as those for $K = \mathbb{Q}_p$, but there is a difference that we need to consider all the embeddings $\sigma : K \hookrightarrow E$, which makes the theory more complicated. Hence, for convenience of readers (and the author), here we reprove the Kisin's theory in detail in our generalized setting.

Let X be a separated rigid analytic space (in the sense of Tate) over E. Let M be a free $\mathcal{O}_X$-module of rank d for some $d \geqq 1$ equipped with a continuous $\mathcal{O}_X$-linear G_K-action, where "continuous" means that, for any admissible open affinoid $U = \mathrm{Spm}(R)$ of X, the action of G_K on $\Gamma(U, M)$ is continuous on which the topology is naturally induced by that of R. We denote by $M^\vee := \mathrm{Hom}_{\mathcal{O}_X}(M, \mathcal{O}_X)$ the $\mathcal{O}_X$-dual of M. For $x \in X$, we denote by $\mathcal{O}_{X,x}$ the local ring at x, $\mathfrak{m}_x$ the maximal ideal at x, $E(x)$ the residue field at x which is a finite extension of E. We denote by $M(x)$ the fiber of M at x, which is a d-dimensional $E(x)$-representation of G_K.

Under this situation, we briefly recall Sen's theory [Se88] of the analytic variations of Hodge-Tate weights following [Be-Co08] and [Ch09a]. By Lemma 3.18 of [Ch09a], we can take an affinoid covering $\{U_i\}_{i\in I}$ of X such that $M_i := \Gamma(U_i, M)$ is a free $R_i := \Gamma(U_i, \mathcal{O}_X)$-module with a G_K-stable finite free R_i^0-module M_i^0 such that $M_i = M_i^0[p^{-1}]$ for a model $R_i^0 \subseteq R_i$ for any $i \in I$. Here, for an affinoid A, a model is defined as a topologically finite generated complete $\mathcal{O}_E$-subalgebra of A which generates A after inverting p. Then, we can apply the results of [Be-Co08] to M_i (and M_i^0) for any $i \in I$. By Proposition 4.1.2 of [Be-Co08], there exists a unique monic polynomial $P_{M_i}(T) \in K \otimes_{\mathbb{Q}_p} R_i[T]$ of dimension d, which is defined as the characteristic polynomial of Sen's operator on $\mathbf{D}^L_{\mathrm{Sen}}(M_i)$ for a sufficiently large finite extension L of K, such that the specialization of $P_{M_i}(T)$ at x gives the Sen's polynomial $P_{M(x)}(T) \in K \otimes_{\mathbb{Q}_p} E(x)[T]$ of $M(x)$ for any $x \in \mathrm{Spm}(R_i)$. By the uniqueness of $\mathbf{D}^L_{\mathrm{Sen}}(M_i)$, $\{P_{M_i}(T)\}_{i\in I}$ glue together to a monic polynomial $P_M(T) \in K \otimes_{\mathbb{Q}_p} \mathcal{O}_X[T]$. By the canonical decomposition $K \otimes_{\mathbb{Q}_p} E = \oplus_{\sigma\in\mathcal{P}} E : a \otimes b \mapsto (\sigma(a)b)_\sigma$, $P_M(T)$ decomposes into the σ-components $P_M(T) = (P_M(T)_\sigma)_{\sigma\in\mathcal{P}} \in \oplus_{\sigma\in\mathcal{P}} \mathcal{O}_X[T]$.

From now on, we assume that the constant term of $P_M(T)_\sigma$ is zero for any $\sigma \in \mathcal{P}$. We denote by $P_M(T)_\sigma = TQ_\sigma(T)$ for $Q_\sigma(T) \in \mathcal{O}_X[T]$.

Before stating the theorem, we recall some terminologies of rigid geometry from § 5 of [Ki03] which we need to characterize the finite slope subspace X_{fs}.

Let $X = \mathrm{Spm}(R)$ be an affinoid over E, and let U be an admissible open in X. We say that U is scheme theoretically dense in X if there exists a Zariski open $V \subseteq \mathrm{Spec}(R)$ which is dense in $\mathrm{Spec}(R)$ for the Zariski topology and $U = V^{\mathrm{an}}$, where V^{an} is the associated rigid space of V. For any rigid analytic space X over E, an admissible open U of X is said to be scheme theoretically dense in X if there exists an admissible affinoid covering $\{U_i\}_{i\in I}$ of X such that $U \cap U_i$ is scheme theoretically dense in U_i for any $i \in I$. The typical example is the following. For any $f \in \Gamma(X, \mathcal{O}_X)$, we set $X_f := \{x \in X | f(x) \neq 0\}$ which is an admissible open in X. If f is a non-zero divisor, then X_f is scheme theoretically dense in X.

Next, let $Y \in \Gamma(X, \mathcal{O}_X^{\times})$ be an invertible function on X, and let R be an affinoid algebra over E. We say that a morphism $f : \mathrm{Spm}(R) \to X$ is Y-small if there exist a finite extension E' of E and $\lambda \in (R \otimes_E E')^{\times}$ such that $E'[\lambda] \subseteq R \otimes_E E'$ is a finite étale E'-algebra and $Y\lambda^{-1} - 1 \in R \otimes_E E'$ is topologically nilpotent. A typical example of Y-small morphism is following. For any $x \in X$ and $n \in \mathbb{Z}_{\geqq 1}$, the natural map $\mathrm{Spm}(\mathcal{O}_{X,x}/\mathfrak{m}_x^n) \to X$ is Y-small for any $Y \in \Gamma(X, \mathcal{O}_X)^{\times}$.

The following theorem is the generalization of Proposition 5.4 of [Ki03] for general K, which states the existence and the characterization of the finite slope subspace X_{fs}. This theorem is the most important for the construction of p-adic families of trianguline representations in the next subsection. For an E-affinoid algebra R, we set $\mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} R := \varprojlim_k \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} R$ which is equipped with the projective limit topology.

THEOREM 3.9. *Let X be a separated rigid analytic space over E, and let M be a free $\mathcal{O}_X$-module of rank d with a continuous $\mathcal{O}_X$-linear G_K-action. Let $Y \in \Gamma(X, \mathcal{O}_X^{\times})$ be an invertible function. We assume that the constant term of $P_M(T)_\sigma$ is zero for any $\sigma \in \mathcal{P}$. Then, there exists a unique Zariski closed subspace X_{fs} of X satisfying the following conditions,*

(1) *$X_{fs,Q_\sigma(i)}$ is scheme theoretically dense in X_{fs} for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$,*

(2) *for any Y-small map $f : \mathrm{Spm}(R) \to X$ which factors through $X_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$, the following two conditions are equivalent,*

    (i) *$f : \mathrm{Spm}(R) \to X$ factors through X_{fs},*

(ii) *any R-linear G_K-equivariant map* $h : M^{\vee} \otimes_{\mathcal{O}_X} R \to \mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} R$ *factors through* $h' : M^{\vee} \otimes_{\mathcal{O}_X} R \to K \otimes_{K_0} (\mathbf{B}^+_{\max} \hat{\otimes}_{\mathbb{Q}_p} R)^{\varphi^f = Y}$.

As in [Ki03], we prove this theorem by several steps. We first prove the following lemma.

LEMMA 3.10. *Let X, M be as above. Let X' be a separated rigid analytic space over E, and let $f : X' \to X$ be a flat E-morphism. If there exists a Zariski closed subspace $X_{fs} \subseteq X$ which satisfies* (1) *and* (2) *of the above theorem, then $X'_{fs} := X_{fs} \times_X X' \subseteq X'$ also satisfies* (1) *and* (2) *for X', $M' := f^*M$ and $Y' := f^*(Y) \in \Gamma(X', \mathcal{O}^{\times}_{X'})$.*

PROOF. The condition (1) is satisfied for X'_{fs} because the notion of scheme theoretically dense is preserved by flat base changes and we have $f^*(P_M(T)) = P_{f^*M}(T)$. That X'_{fs} satisfies (2) is trivial. □

We next prove the uniqueness of X_{fs}.

LEMMA 3.11. *If two Zariski closed subspaces X_1 and X_2 of X satisfy the conditions* (1) *and* (2)*, then $X_1 = X_2$.*

PROOF. For any admissible open $U \subseteq X$ and $i = 1, 2$, $X_i \cap U \subseteq U$ satisfies (1) and (2) for U by Lemma 3.10 because the inclusion $U \hookrightarrow X$ is flat. Hence, it suffices to prove that $X_1 \cap U_i = X_2 \cap U_i$ for any $i \in I$ for an admissible covering $\{U_i\}_{i \in I}$ of X. Hence we may assume that $X = \mathrm{Spm}(R)$ is an affinoid. We denote by $X_1 = \mathrm{Spm}(R/I_1)$, $X_2 = \mathrm{Spm}(R/I_2)$ for some ideals $I_1, I_2 \subseteq R$. Set $X_3 := \mathrm{Spm}(R/I_1 \cap I_2)$, then we claim that X_3 also satisfies the conditions (1) and (2) . For (1), we have inclusions $R/I_j \hookrightarrow R/I_j[\frac{1}{Q_{\sigma}(i)}]$ for any $j = 1, 2$, $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$ by the assumption, hence we also have an inclusion $R/I_1 \cap I_2 \hookrightarrow R/I_1 \cap I_2[\frac{1}{Q_{\sigma}(i)}]$ for any $\sigma \in \mathcal{P}, i \in \mathbb{Z}_{\leqq 0}$, which proves that X_3 satisfies (1). To prove that X_3 satisfies (2), we take a Y-small morphism $f : \mathrm{Spm}(R') \to X$ which factors $f : \mathrm{Spm}(R') \to X_{Q_{\sigma}(i)}$ for any $\sigma \in \mathcal{P}, i \in \mathbb{Z}_{\leqq 0}$. Set $Y' := f^*(Y) \in R'^{\times}$. If f satisfies (ii) of (2), then f factors through X_1 and X_2 by definition, hence also factors through X_3 because we have $X_1, X_2 \subseteq X_3$. Next we assume that f satisfies (i) of

(2). We have the following canonical decompositions

$$K \otimes_{K_0} (\mathbf{B}^+_{\max}\hat{\otimes}_{\mathbb{Q}_p} R')^{\varphi^f = Y'} = ((K \otimes_{K_0} \mathbf{B}^+_{\max})\hat{\otimes}_{\mathbb{Q}_p} R')^{\varphi^f = Y'}$$
$$= (\mathbf{B}^+_{\max,K}\hat{\otimes}_{\mathbb{Q}_p} R')^{\varphi^f = Y'} = \oplus_{\sigma\in\mathcal{P}}(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y'}$$

and

$$\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p} R' = \oplus_{\sigma\in\mathcal{P}}\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R'.$$

Hence, it suffices to show that any G_K-equivariant R'-linear map $h : M^\vee \otimes_{\mathcal{O}_X} R' \to \mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R'$ factors through $M^\vee \otimes_{\mathcal{O}_X} R' \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y'}$ for any $\sigma \in \mathcal{P}$. Because $Q_\sigma(i)$ is invertible in R' for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$ by the assumption, the natural map

$$(\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} M \otimes_{\mathcal{O}_X} R')^{G_K} \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} M \otimes_{\mathcal{O}_X} R')^{G_K}$$

is isomorphism for any $\sigma \in \mathcal{P}$ and $k \in \mathbb{Z}_{\geqq 1}$ by Proposition 2.5 of [Ki03]. Hence, it suffices to show that, for some $k \in \mathbb{Z}_{\geqq 1}$, any G_K-equivariant map $h : M^\vee \otimes_{\mathcal{O}_X} R' \to (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R')$ factors through $M^\vee \otimes_{\mathcal{O}_X} R' \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y'}$. We choose $k \in \mathbb{Z}_{\geqq 1}$ sufficiently large such that there exists a short exact sequence

$$(5) \qquad 0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R')^{\varphi_K = Y'} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R' \to U_{k,\sigma} \to 0$$

of Banach R'-modules compatible with any base change as in Proposition 3.7. If we set $\mathrm{Spm}(R'_i) := f^{-1}(X_i) \subseteq \mathrm{Spm}(R')$ for $i = 1, 2$, then we have an inclusion $R' \hookrightarrow R'_1 \oplus R'_2$ because f factors through X_3. From these arguments, the above short exact sequence (5) can be embedded in the following short exact sequence

$$0 \to \oplus_{i=1}^2(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R'_i)^{\varphi_K = Y'} \to \oplus_{i=1}^2\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R'_i$$
$$\to \oplus_{i=1}^2 U_{k,\sigma}\hat{\otimes}_{R'} R'_i \to 0.$$

Then, the composition of h with $\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R' \hookrightarrow \oplus_{i=1}^2\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R'_i$ factors through $M^\vee \otimes_{\mathcal{O}_X} R' \hookrightarrow \oplus_{i=1}^2(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R'_i)^{\varphi_K = Y'}$ by the definition of X_i. Hence, h also factors through $M^\vee \otimes_{\mathcal{O}_X} R' \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R'_i)^{\varphi_K = Y'}$ by a diagram chase. Hence, X_3 also satisfies (1) and (2).

Therefore, to prove the lemma, we may assume that $X_1 \subseteq X_2$. We put $W \subseteq X_2$ the support of I_1/I_2 with the reduced structure. If $x \in X_2$

satisfies $Q_\sigma(i)(x) \neq 0$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$, then the natural map $\mathrm{Spm}(\mathcal{O}_{X_2,x}/\mathfrak{m}_x^n) \to X_2$ which is Y-small factors through $\mathrm{Spm}(\mathcal{O}_{X_2,x}/\mathfrak{m}_x^n) \to X_1$ for any $n \geqq 1$ by the definition on X_1 and X_2. This implies that there exists a map $\mathcal{O}_{X_1,x} \to \hat{\mathcal{O}}_{X_2,x}$ such that the composition of this with the natural map $\mathcal{O}_{X_2,x} \to \mathcal{O}_{X_1,x}$ is the natural map $\mathcal{O}_{X_2,x} \to \hat{\mathcal{O}}_{X_2,x}$. This implies that the natural quotient map $\mathcal{O}_{X_2,x} \to \mathcal{O}_{X_1,x}$ is isomorphism, hence we have $x \notin W$. Hence, we obtain an inclusion $W \subseteq \cup_{\sigma\in\mathcal{P}, i\in\mathbb{Z}_{\leqq 0}}\{x \in X_2 | Q_\sigma(i)(x) = 0\}$. By Lemma 5.7 [Ki03], then there exists a $Q \in \Gamma(X_2, \mathcal{O}_{X_2})$ which is a finite product of $Q_\sigma(i)$ such that $X_{2,Q} \subseteq X_2 \setminus W = X_1 \setminus W \subseteq X_1 \subseteq X_2$. Then, the condition (1) for X_2 implies that $X_1 = X_2$. We finish to prove the lemma. □

Assume that there exists an admissible affinoid covering $\{U_i\}_{i\in I}$ of X such that the subspace $U_{i,fs} \subseteq U_i$ exists for any $i \in I$. By the uniqueness of X_{fs}, $\{U_{i,fs}\}_{i\in I}$ glue to a Zariski closed subspace $X'_{fs} \subseteq X$ satisfying that $X'_{fs} \cap U_i = U_{i,fs}$ for any $i \in I$.

LEMMA 3.12. *In the above situation, $X'_{fs} \subseteq X$ satisfies the conditions* (1) *and* (2) *in the theorem, i.e. we have $X'_{fs} = X_{fs}$.*

PROOF. That X'_{fs} satisfies (1) is trivial. We show that X'_{fs} satisfies (2). Let $f : \mathrm{Spm}(R) \to X$ be a Y-small map which factors through $X_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$. Because X is separated, $f^{-1}(U_i)$ is an affinoid for any $i \in I$. Set $\mathrm{Spm}(R_i) := f^{-1}(U_i)$. First, we show that (i) implies (ii). We assume that f factors through X'_{fs}. Let $h : M^\vee \otimes_{\mathcal{O}_X} R \to \mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R$ be a R-linear G_K-equivariant map. By Proposition 2.5 of [Ki03], it suffices to show that $h : M^\vee \otimes_{\mathcal{O}_X} R \to \mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R$ factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R)^{\varphi_K = Y}$ for some $k \in \mathbb{Z}_{\geqq 1}$. We choose $k \in \mathbb{Z}_{\geqq 1}$ such that there exists a short exact sequence

$$0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R)^{\varphi_K = Y} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R \to U_{k,\sigma} \to 0$$

of Banach R-modules as in Proposition 3.7. By the base change property, this short exact sequence can be embedded into the following exact sequence

$$0 \to \prod_{i\in I}(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma} R_i)^{\varphi_K = Y} \to \prod_{i\in I}\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma} R_i \to \prod_{i\in I} U_{k,\sigma}\hat{\otimes}_R R_i \to 0.$$

By the assumption, the map $M^\vee \otimes_{\mathcal{O}_X} R \xrightarrow{h} \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R_i$ factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R_i)^{\varphi_K=Y}$ for any $i \in I$. Hence, $h : M^\vee \otimes_{\mathcal{O}_X} R \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R$ also factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y}$ by a diagram chase of the above two exact sequences.

Next, we assume that, for any $\sigma \in \mathcal{P}$, any R-linear G_K-equivariant map $h : M^\vee \otimes_{\mathcal{O}_X} R \to \mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R$ factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y}$. Because we have $Q_\sigma(j) \in R^\times$ for any $\sigma \in \mathcal{P}$ and $j \in \mathbb{Z}_{\leqq 0}$, the natural map

$$(\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}M \otimes_{\mathcal{O}_X} R)^{G_K} \otimes_R R_i \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}M \otimes_{\mathcal{O}_X} R_i)^{G_K}$$

is isomorphism for any $k \geqq 1$ and $i \in I$ by Corollary 2.6 of [Ki03]. Hence, any R_i-linear G_K-equivariant map $h_i : M^\vee \otimes_{\mathcal{O}_X} R \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R_i$ factors through $M^\vee \otimes_{\mathcal{O}_X} R \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R_i)^{\varphi_K=Y}$ for any $i \in I$. This implies that $f|_{\mathrm{Spm}(R_i)} : \mathrm{Spm}(R_i) \to U_i$ factors through $U_{i,fs}$ for any $i \in I$. Hence, f also factors through X'_{fs}. □

By this lemma, it suffices to construct X_{fs} when $X = \mathrm{Spm}(R)$ is affinoid. Moreover, in the same way as in (5.9) of [Ki03], we may assume that $|Y|$ satisfies $|Y||Y^{-1}| < \frac{1}{|\pi_K|_p}$ for a norm $|-| : R \to \mathbb{R}_{\geqq 0}$ which defines the topology of R as in §3.2. Then, we construct $X_{fs} \subseteq \mathrm{Spm}(R)$ as follows. We first construct an ideal of R which defines X_{fs}. Let $\lambda \in \overline{E}$ be an element such that $|Y^{-1}|^{-1} \leqq |\lambda|_p \leqq |Y|$, and let E' be a finite Galois extension of E which contains λ. By Corollary 3.5, we can take a sufficiently large $k \in \mathbb{Z}_{\geqq 1}$ such that there exists a short exact sequence of E'-Banach spaces

$$0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}E')^{\varphi_K=\sigma(\pi_K)\lambda} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}E' \to U_{k,\sigma,\lambda} \to 0$$

for any λ, E' as above and $\sigma \in \mathcal{P}$. Fix such a $k \geqq 1$ until the end of the proof of this theorem. For any $x \in \widetilde{\mathbf{E}}^+$ such that $v(x) > 0$, we define an element

$$P(x, \frac{Y}{\sigma(\pi_K)\lambda}) := \sum_{n\in\mathbb{Z}} \varphi_K^n([x])(\frac{Y}{\sigma(\pi_K)\lambda})^n \in (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R \otimes_E E')^{\varphi_K=\frac{\sigma(\pi_K)\lambda}{Y}}.$$

This element converges because we have

$$|\frac{\sigma(\pi_K)\lambda}{Y}| \leqq |\sigma(\pi_K)|_p|\lambda|_p|Y^{-1}| < |\sigma(\pi_K)|_p|\lambda|_p|Y|^{-1}|\pi_K|_p^{-1} \leqq 1$$

and $\varphi_K^n([x])(\frac{Y}{\sigma(\pi_K)\lambda})^n \to 0$ $(n \to +\infty)$ (see Corollary 4.4 of [Ki03]). For any $\sigma \in \mathcal{P}$ and any R-linear G_K-equivariant map $h : M^\vee \to \mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+\hat{\otimes}_{K,\sigma}R$, we consider the composition of this map with the maps

$$\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+\hat{\otimes}_{K,\sigma}R \to \mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+\hat{\otimes}_{K,\sigma}R \otimes_E E' : v \mapsto P(x, \frac{Y}{\sigma(\pi_K)\lambda})v$$

and $\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+\hat{\otimes}_{K,\sigma}R\otimes_E E' \to U_{k,\sigma,\lambda}\hat{\otimes}_{E'}(R\otimes_E E')$ which is the base change of the surjection $\mathbf{B}_{\mathrm{dR}}^+/t^k\mathbf{B}_{\mathrm{dR}}^+\hat{\otimes}_{K,\sigma}E' \to U_{k,\sigma,\lambda}$ in the above short exact sequence. We denote this composition by

$$h_{x,\lambda} : M^\vee \to U_{k,\sigma,\lambda}\hat{\otimes}_{E'}(R \otimes_E E').$$

Fix an orthonormalizable E'-base $\{e_i\}_{i\in I}$ of $U_{k,\sigma,\lambda}$. For any $m \in M^\vee$, then we can write uniquely by

$$h_{x,\lambda}(m) = \sum_{i\in I} a_{x,\lambda,i}(m)e_i \text{ for } \{a_{x,\lambda,i}(m)\}_{i\in I} \subseteq R \otimes_E E'.$$

We define an ideal

$$I(h, x, \lambda, m) \subseteq R \otimes_E E'$$

which is generated by $a_{x,\lambda,i}(m)$ for all $i \in I$. Because we have $\tau(I(h, x, \lambda, m)) = I(h, x, \tau(\lambda), m) \subseteq R \otimes_E E'$ for any $\tau \in \mathrm{Gal}(E'/E)$, the ideal

$$\sum_{\tau\in\mathrm{Gal}(E'/E)} I(h, x, \tau(\lambda), m) \subseteq R \otimes_E E'$$

descends to an ideal $I'(h, x, \lambda, m) \subseteq R$ and this ideal is independent of the choice of E'. We define an ideal I of R by

$$I := \sum_{h,x,\lambda,m} I'(h, x, \lambda, m) \subseteq R,$$

where the sum runs through all h, x, λ, m and $\sigma \in \mathcal{P}$ as above.

Next, we denote by $J_{(n,\{\sigma_l\}_{l=0}^n,\{i_l\}_{l=0}^n)}(\supseteq I)$ the kernel of the natural map $R \to R/I[\frac{1}{\prod_{l=1}^n Q_{\sigma_l}(i_l)}]$ for any triple $(n, \{\sigma_l\}_{l=0}^n, \{i_l\}_{l=0}^n)$ such that $n \geqq 1$,

$\sigma_l \in \mathcal{P}, i_l \in \mathbb{Z}_{\leqq 0}$. Denote by $J(\supseteq I)$ the sum (in fact a finite union) of the ideals $J_{(n,\{\sigma_l\}_{l=0}^n,\{i_l\}_{l=0}^n)}$ for all the triples $(n, \{\sigma_l\}_{l=0}^n, \{i_l\}_{l=0}^n)$ as above. Then, $\mathrm{Spm}(R/J)$ is the largest Zariski closed subspace of $\mathrm{Spm}(R/I)$ such that $\mathrm{Spm}(R/J)_{Q_\sigma(i)}$ is scheme theoretically dense in $\mathrm{Spm}(R/J)$ for any $\sigma \in \mathcal{P}, i \leqq 0$.

Finally, we prove the following lemma which claims that $X_{fs} = \mathrm{Spm}(R/J)$, hence proves the theorem.

Lemma 3.13. *The closed subspace* $\mathrm{Spm}(R/J) \subseteq \mathrm{Spm}(R)$ *satisfies the conditions* (1) *and* (2) *in the theorem.*

Proof. Because the map $R/J \to R/J[\frac{1}{Q_\sigma(i)}]$ is injective for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$ by the definition of J, $\mathrm{Spm}(R/J)$ satisfies the condition (1). We show that $\mathrm{Spm}(R/J)$ satisfies (2). Let $f : \mathrm{Spm}(R') \to \mathrm{Spm}(R)$ be a Y-small map which factors through $\mathrm{Spm}(R') \to \mathrm{Spm}(R)_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$. In this situation, we first prove that (ii) implies (i). We assume that any G_K-equivariant map $h : M^\vee \to \mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R'$ factors through $M^\vee \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=Y}$ for any $\sigma \in \mathcal{P}$. Then, for any $h : M^\vee \to \mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R$, $\lambda \in E'$ and $x \in \widetilde{\mathbf{E}}^+$ as in the construction of the ideal $I \subseteq R$, the map

$$P(x, \frac{Y}{\sigma(\pi_K)\lambda})h \otimes_R \mathrm{id}_{R'} : M^\vee \to U_{k,\sigma,\lambda}\hat{\otimes}_{E'}(R' \otimes_E E')$$

is zero because the multiplication by $P(x, \frac{Y}{\sigma(\pi_K)\lambda})$ sends $(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi=Y}$ to $(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R\otimes_E E')^{\varphi_K=\sigma(\pi_K)\lambda}$. Hence, the map $R \to R'$ factors through $R/I \to R'$ by the definition of I. Because $Q_\sigma(i) \in R'^{\times}$ for any $\sigma \in \mathcal{P}$, $i \in \mathbb{Z}_{\leqq 0}$, the map $R/I \to R'$ factors through $R/J \to R'$ by the definition of J.

We next prove that (i) implies (ii). Assume that $f : \mathrm{Spm}(R') \to \mathrm{Spm}(R)$ factors through $\mathrm{Spm}(R') \to \mathrm{Spm}(R/J) \to \mathrm{Spm}(R)$. Let $h : M^\vee \to \mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R'$ be a R'-linear G_K-equivariant map. We want to show that the map h factors through $M^\vee \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=Y}$. By Galois descent, it suffices to show that the map h factors through $M^\vee \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R' \otimes_E E')^{\varphi_K=Y}$ for a sufficiently large finite Galois extension E' of E. Hence, by the definition of Y-smallness, we may assume that there exists $\lambda \in E'$ such that $Y\lambda^{-1} - 1$ is topologically nilpotent in $R' \otimes_E E'$. Moreover, because the definitions of I and J are compatible with any base change $R \mapsto R \otimes_E E'$, we may assume that $E = E'$ and $\lambda \in E$. Under these assumptions, we have $|Y^{-1}|^{-1} \leqq |f^*(Y)^{-1}|_{R'}^{-1} = |\lambda|_p = |f^*(Y)|_{R'} \leqq |Y|$ ($|-|_{R'}$ is a norm on R'), hence λ satisfies the condition in the construction of $I \subseteq R$. By the definition of I, for any $m \in M^\vee$, $P(x, \frac{Y}{\sigma(\pi_K)\lambda})h(m)$ is an element in $(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\sigma(\pi_K)\lambda}$ for any $x \in \widetilde{\mathbf{E}}^+$ such that $v(x) > 0$. Take an element $u \in (\widehat{K^{\mathrm{ur}}}\hat{\otimes}_{K,\sigma}R')^{\times,\varphi_K=\frac{\lambda}{Y}}$ as in Lemma 3.6. Then we have

$$t_K uh(m) \in (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\sigma(\pi_K)\lambda}$$

because we have $t_K u \in (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\frac{\sigma(\pi_K)\lambda}{Y}}$, and because the R'-module generated by the sets $\{P(x, \frac{Y}{\sigma(\pi_K)\lambda})\}_{x\in\widetilde{\mathbf{E}}^+, v(x)>0}$ is dense in $(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\frac{\sigma(\pi_K)\lambda}{Y}}$, which can be proved in the same way as in Corollary 4.6 of [Ki03] by using Lemma 4.3.1 of [Ke05], and because $(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\sigma(\pi_K)\lambda}$ is closed in $\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R'$ by Proposition 3.7. Hence, we obtain

$$\begin{aligned} uh(m) &\in \frac{1}{t_K}(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\sigma(\pi_K)\lambda} \cap \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R' \\ &= (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=\sigma(\pi_K)\lambda}, \end{aligned}$$

where the last equality follows from Proposition 8.10 of [Co02]. Hence, we obtain $h(m) \in (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R')^{\varphi_K=Y}$, which proves the lemma, hence finishes to prove the theorem. □

We next prove some important general properties of X_{fs}, which is a generalization of Corollary 5.16 of [Ki03] for general K.

Let $\mathrm{Spm}(R) \subseteq X_{fs}$ be an affinoid open of X_{fs}. We assume that this inclusion is Y-small. By Proposition 3.7, there exists $k > 0$ such that, for any $\sigma \in \mathcal{P}$, there exists a short exact sequence of Banach R-modules with the property (Pr)

$$0 \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R \to U_{k,\sigma} \to 0.$$

We denote by M_R the restriction of M to $\mathrm{Spm}(R)$.

PROPOSITION 3.14. *Fix $k \geqq 1$ as above. For any $\sigma \in \mathcal{P}$, let $H_\sigma \subseteq R$ be the smallest ideal of R such that any R-linear G_K-equivariant morphism $M_R^\vee \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R$ factors through $M_R^\vee \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}H_\sigma \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R$. Set $H := \prod_{\sigma\in\mathcal{P}} H_\sigma \subseteq R$. Then the following hold:*

(1) *For any* $\sigma \in \mathcal{P}$, *the natural map*

$$((\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}M_R)^{\varphi_K=Y})^{G_K} \to (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}M_R)^{G_K}$$

is isomorphism, i.e. the natural map

$$K \otimes_{K_0} \mathbf{D}^+_{\mathrm{cris}}(M_R)^{\varphi_K=Y} \to (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}M_R)^{G_K}$$

is isomorphism.

(2) $\mathrm{Spm}(R) \setminus V(H)$ *and* $\mathrm{Spm}(R) \setminus V(H_\sigma)$ *(for any* $\sigma \in \mathcal{P}$*) are scheme theoretically dense in* $\mathrm{Spm}(R)$, *where* $V(H_*) := \mathrm{Spm}(R/H_*)$.

(3) *For any* $x \in \mathrm{Spm}(R)$, $M(x)$ *is a split trianguline* $E(x)$*-representation. More precisely, there exists a short exact sequence of* $E(x)$*-*B*-pairs*

$$0 \to W(\delta_{Y(x)}\prod_{\sigma\in\mathcal{P}}\sigma^{-k_\sigma}) \to W(M(x))$$
$$\to W(\det(M(x))\delta^{-1}_{Y(x)}\prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}) \to 0$$

for some $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}}\mathbb{Z}_{\geqq 0}$, *where, for any* $\lambda \in E(x)^\times$, *we define a homomorphism* $\delta_\lambda : K^\times \to E(x)^\times$ *such that* $\delta_\lambda(\pi_K) = \lambda$ *and* $\delta_\lambda|_{\mathcal{O}_K^\times}$ *is trivial.*

Proof. We first prove (1). Take a point $x \in \mathrm{Spm}(R)$ such that $x \in \mathrm{Spm}(R)_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$. By the characterization of X_{fs}, any G_K-map $M_R^\vee \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \otimes_{K,\sigma} \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ factor through $M_R^\vee \to (\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} \mathcal{O}_{X,x}/\mathfrak{m}_x^n)^{\varphi_K=Y}$ for any $n \geqq 1$ and $\sigma \in \mathcal{P}$. We denote by $V \subseteq \mathrm{Spm}(R)$ the set of points satisfying the above condition. By the same argument as in Lemma 3.12, it suffices to show that the natural map $R \to \prod_{x\in V, n\geqq 1}\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is an injection. Let $f \in R$ be an element in the kernel of this map. Let $W \subseteq \mathrm{Spm}(R)$ be the support of f with the reduced structure. Then we have $W \subseteq \cup_{\sigma\in\mathcal{P}, i\leqq 0}V(Q_\sigma(i))$, hence there exists $Q \in R$ a finite product of $Q_\sigma(i)$ such that $W \subseteq V(Q)$ by Lemma 5.7 of [Ki03]. Hence we have $X_Q \subseteq X \setminus W \subseteq X$. This implies that $f = 0 \in R[\frac{1}{Q}]$, and $f = 0$ in R by the condition (1) of Theorem 3.9. Hence, the map $R \to \prod_{x\in V, n\geqq 1}\mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is an injection.

We next prove (2). Let $x \in \mathrm{Spm}(R)$ such that $x \in \mathrm{Spm}(R)_{Q_\sigma(i)}$ for any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$. For any affinoid algebra R' which is a quotient of $\mathcal{O}_{X,x}$, then we have an isomorphism

$$(\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}M_R)^{G_K} \otimes_R R' \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \otimes_{K,\sigma} (M_R \otimes_R R'))^{G_K}$$

and this is a free R'-module of rank one by Corollary 2.6 of [Ki03] for any $\sigma \in \mathcal{P}$. Hence we obtain an equality $H_\sigma\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$ for any σ because we have $(\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}M_R)^{G_K} \otimes_R \mathcal{O}_{X,x}/H_\sigma\mathcal{O}_{X,x} = 0$ by the definition of H_σ, and then we also obtain an equality $H\mathcal{O}_{X,x} = \mathcal{O}_{X,x}$. This implies that we have an inclusion $V(H) \subseteq \cup_{\sigma\in\mathcal{P},i\leqq 0}V(Q_\sigma(i))$. Hence, there exists $Q \in R$ a finite product of $Q_\sigma(i)$ such that $\mathrm{Spm}(R)_Q \subseteq \mathrm{Spm}(R) \setminus V(H) \subseteq \mathrm{Spm}(R)$ by Lemma 5.7 of [Ki03]. Because $\mathrm{Spm}(R)_Q$ is scheme theoretically dense, so $\mathrm{Spm}(R) \setminus V(H)$ is also scheme theoretically dense. Because we have $V(H_\sigma) \subseteq V(H)$, $\mathrm{Spm}(R) \setminus V(H_\sigma)$ is also scheme theoretically dense for any $\sigma \in \mathcal{P}$.

Finally we prove (3). Let x be any point of $\mathrm{Spm}(R)$. By (2), for any $\sigma \in \mathcal{P}$, there exists $n_\sigma \geqq 0$ such that $H_\sigma \subseteq \mathfrak{m}_x^{n_\sigma}$ and $H_\sigma \not\subseteq \mathfrak{m}_x^{n_\sigma+1}$. By the definition of H_σ, there exists a G_K-map $h : M_R^\vee \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}H_\sigma)^{\varphi_K=Y}$ which, by composing with $\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}H_\sigma \to \mathbf{B}^+_{\max,K} \otimes_{K,\sigma} \mathfrak{m}_x^{n_\sigma}/\mathfrak{m}_x^{n_\sigma+1}$, induces a nonzero map $M_R^\vee \to (\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} \mathfrak{m}_x^{n_\sigma}/\mathfrak{m}_x^{n_\sigma+1})^{\varphi_K=Y(x)}$. Hence, by taking a suitable $E(x)$-linear projection $\mathfrak{m}_x^{n_\sigma}/\mathfrak{m}_x^{n_\sigma+1} \to E(x)$, we obtain a non zero G_K-map $M_R^\vee \to (\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} E(x))^{\varphi_K=Y(x)}$. This implies that $(\mathbf{B}^+_{\max,K} \otimes_{K,\sigma} M(x))^{\varphi_K=Y(x)} \neq 0$, and also implies that $\mathbf{D}^+_{\mathrm{cris}}(M(x))^{\varphi_K=Y(x)} \neq 0$, then $M(x)$ is a split trianguline $E(x)$-representation as in the statement of (3). □

3.3. Construction of p-adic families of two dimensional trianguline representations

In this subsection, we will apply our theory of X_{fs} in the previous subsection to the rigid analytic space associated with a universal deformation ring of mod p Galois representation of G_K, which is a slightly modified generalization of the results of §10 of [Ki03] for general K.

Let $\mathcal{C}_\mathcal{O}$ be the category of local Artin $\mathcal{O}$-algebras with the residue field $\mathbb{F}$. Let $\bar{\rho} : G_K \to \mathrm{GL}_2(\mathbb{F})$ be a continuous homomorphism, we denote by $\overline{V}$ a two dimensional $\mathbb{F}$-representation defined by $\bar{\rho}$. As in the case of E-

representations, we define a functor $D_{\bar{\rho}} : \mathcal{C}_{\mathcal{O}} \to Sets$ by

$$D_{\bar{\rho}}(A) := \{ \text{ equivalent classes of deformations of } \overline{V} \text{ over } A\}$$

for $A \in \mathcal{C}_{\mathcal{O}}$. In this paper, for simplicity, we assume that $\overline{V}$ satisfies that

$$\mathrm{H}^0(G_K, \mathrm{ad}(\overline{V})) = \mathbb{F}.$$

Then, $D_{\bar{\rho}}$ is pro-representable by a complete noetherian local $\mathcal{O}$-algebra $R_{\bar{\rho}}$ with the residue field $\mathbb{F}$. When $\overline{V}$ does not satisfy $\mathrm{H}^0(G_K, \mathrm{ad}(\overline{V})) = \mathbb{F}$, we can prove the same theorems below in almost the same way if we consider the framed deformations of Kisin. Let V^{univ} be the universal deformation over $R_{\bar{\rho}}$, which is a free $R_{\bar{\rho}}$-module of rank two with a $R_{\bar{\rho}}$-linear continuous G_K-action. Let $\mathfrak{X}(\bar{\rho})$ be the rigid analytic space over E associated to $R_{\bar{\rho}}$. Let $\widetilde{V}^{\mathrm{univ}}$ be a free $\mathcal{O}_{\mathfrak{X}(\bar{\rho})}$-module associated to V^{univ}, which is naturally equipped with an $\mathcal{O}_{\mathfrak{X}(\bar{\rho})}$-linear continuous G_K-action induced from that on V^{univ}, where "continuous" means that G_K acts continuously on $\Gamma(U, \widetilde{V}^{\mathrm{univ}})$ for any affinoid opens $U = \mathrm{Spm}(R) \subseteq \mathfrak{X}(\bar{\rho})$.

Remark 3.15. For a point $x \in \mathfrak{X}(\bar{\rho})$, the fiber V_x of $\widetilde{V}^{\mathrm{univ}}$ at x is a two dimensional $E(x)$-representation such that the reduction of a G_K-stable $\mathcal{O}_{E(x)}$-lattice of V_x is isomorphic to $\overline{V} \otimes_{\mathbb{F}} \mathcal{O}_{E(x)}/\mathfrak{m}_{E(x)}$. Because we assume that $\mathrm{End}_{\mathbb{F}[G_K]}(\bar{\rho}) = \mathbb{F}$, we also have $\mathrm{End}_{E(x)[G_K]}(V_x) = E(x)$ for any $x \in \mathfrak{X}(\bar{\rho})$.

Let $\mathcal{W}_E$ be the rigid analytic space over E which represents the functor $D_{\mathcal{W}_E}$ from the category of rigid analytic spaces over E to the category of groups, which is defined by

$$D_{\mathcal{W}_E}(Y) := \{\delta : \mathcal{O}_K^{\times} \to \Gamma(Y, \mathcal{O}_Y^{\times}) \text{ continuous homomorphisms } \}$$

for any rigid analytic space Y over E, where "continuous" is the same meaning as in the definition of $\widetilde{V}^{\mathrm{univ}}$. It is known that $\mathcal{W}_E$ is the rigid analytic space associated to the Iwasawa algebra $\mathcal{O}[[\mathcal{O}_K^{\times}]]$, which is non-canonically isomorphic to a finite (this number is equal to the number of torsion points in $\mathcal{O}_K^{\times}$) union of $[K : \mathbb{Q}_p]$-dimensional open unit disc over E. We denote by

$$\delta_0^{\mathrm{univ}} : \mathcal{O}_K^{\times} \to \Gamma(\mathcal{W}_E, \mathcal{O}_{\mathcal{W}_E}^{\times})$$

the universal continuous homomorphism, which is the composition of the map $\mathcal{O}_K^\times \to \mathcal{O}[[\mathcal{O}_K^\times]]^\times : a \mapsto [a]$ with the natural map $\mathcal{O}[[\mathcal{O}_K^\times]]^\times \to \Gamma(\mathcal{W}_E, \mathcal{O}_{\mathcal{W}_E}^\times)$. Using a fixed π_K, we extend δ_0^{univ} to $K^\times$ by

$$\delta^{\mathrm{univ}} : K^\times \to \Gamma(\mathcal{W}_E, \mathcal{O}_{\mathcal{W}_E}^\times) \text{ such that } \delta^{\mathrm{univ}}(\pi_K) = 1,\ \delta^{\mathrm{univ}}|_{\mathcal{O}_K^\times} = \delta_0^{\mathrm{univ}}.$$

By local class field theory, we can uniquely extend δ^{univ} to a character

$$\widetilde{\delta}^{\mathrm{univ}} : G_K^{\mathrm{ab}} \to \Gamma(\mathcal{W}_E, \mathcal{O}_{\mathcal{W}_E}^\times) \text{ such that } \delta^{\mathrm{univ}} = \widetilde{\delta}^{\mathrm{univ}} \circ \mathrm{rec}_K.$$

Set

$$X(\bar{\rho}) := \mathfrak{X}(\bar{\rho}) \times_E \mathcal{W}_E \times_E \mathbb{G}_{m,E}^{an}.$$

Let Y be the canonical parameter of $\mathbb{G}_{m,E}^{an}$. We denote the projections by

$$p_1 : X(\bar{\rho}) \to \mathfrak{X}(\bar{\rho}),\ p_2 : X(\bar{\rho}) \to \mathcal{W}_E,\ p_3 : X(\bar{\rho}) \to \mathbb{G}_{m,E}^{an}$$

respectively. We denote by $N := p_1^* \widetilde{V}^{\mathrm{univ}}$ and $M := N(p_2^*(\widetilde{\delta}^{\mathrm{univ}})^{-1})$, which is the twist of M by the cahacter $p_2^*(\widetilde{\delta}^{\mathrm{univ}})^{-1} : G_K^{\mathrm{ab}} \to \Gamma(X(\bar{\rho}), \mathcal{O}_{X(\bar{\rho})})^\times$. These are rank two free $\mathcal{O}_{X(\bar{\rho})}$-modules with $\mathcal{O}_{X(\bar{\rho})}$-linear continuous G_K-actions.

Remark 3.16. In §10 of [Ki03], Kisin applied his theory of X_{fs} (for $K = \mathbb{Q}_p$) to the family $q_1^* \widetilde{V}^{\mathrm{univ}}$ on the space $Y(\bar{\rho}) := \mathfrak{X}(\bar{\rho}) \times_E \mathbb{G}_{m,E}^{an}$, where $q_1 : Y(\bar{\rho}) \to \mathfrak{X}(\bar{\rho})$ is the natural projection. This is because he applied the results to a study of the family of p-adic representations associated to Coleman-Mazur eigencurve, one of whose Hodge-Tate weights is always zero. On the other hands, in this article, we want to study all the two dimensional trianguline representations without any conditions on the Hodge-Tate weights. Hence, we use the space $X(\bar{\rho})$ and the representation $M := N(p_2^*(\widetilde{\delta}^{\mathrm{univ}})^{-1})$ instead of $Y(\bar{\rho})$ and $q_1^* \widetilde{V}^{\mathrm{univ}}$.

A point x of $X(\bar{\rho})$ can be written as a triple $x = ([V_x], \delta_x, \lambda_x)$, where V_x is an $E(x)$-representation such that the reduction of a suitable G_K-stable $\mathcal{O}_{E(x)}$-lattice of V_x is isomorphic to $\bar{V} \otimes_{\mathcal{O}_{E(x)}} \mathcal{O}_{E(x)}/\mathfrak{m}_{E(x)}$, and $\delta_x : \mathcal{O}_K^\times \to E(x)^\times$ is a continuous homomorphism, and $\lambda_x \in E(x)^\times$. We denote by

$$\begin{aligned} P_M(T) = (P_M(T)_\sigma)_{\sigma \in \mathcal{P}} = (T^2 - a_{1,\sigma} T + a_{0,\sigma})_{\sigma \in \mathcal{P}} &\in K \otimes_{\mathbb{Q}_p} \mathcal{O}_{X(\bar{\rho})}[T] \\ &= \prod_{\sigma \in \mathcal{P}} \mathcal{O}_{X(\bar{\rho})}[T] \end{aligned}$$

the Sen's polynomial of M. Let $X_0(\bar{\rho}) \subseteq X(\bar{\rho})$ be the Zariski closed subspace defined by the ideal generated by $a_{0,\sigma}$ for all $\sigma \in \mathcal{P}$. Let $M_0 := M|_{X_0(\bar{\rho})}$ be the restriction of M to $X_0(\bar{\rho})$, then we have

$$P_{M_0}(T) = (T(T - a_{1,\sigma}))_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathcal{O}_{X_0(\bar{\rho})}[T].$$

We denote by $Q_\sigma(T) := T - a_{1,\sigma} \in \mathcal{O}_{X_0(\bar{\rho})}[T]$ for each $\sigma \in \mathcal{P}$. Under this situation, we apply Theorem 3.9 to $X_0(\bar{\rho})$ and M_0 and $Y := (p_3^*Y)|_{X_0(\bar{\rho})}$, then we obtain a Zariski closed subspace

$$\mathcal{E}(\bar{\rho}) := X_0(\bar{\rho})_{fs} \subseteq X_0(\bar{\rho}).$$

For the properties of $\mathcal{E}(\bar{\rho})$, we have a following theorem, which is a modified generalization of Proposition 10.4 of [Ki03] for general K. For any $\lambda \in \overline{E}^\times$, we define a unramified continuous homomorphism $\delta_\lambda : K^\times \rightarrow \overline{E}^\times$ such that $\delta_\lambda(\pi_K) := \lambda$ and $\delta_\lambda|_{\mathcal{O}_K^\times}$ is trivial. For a point $\delta \in \mathcal{W}_E(\overline{E})$, i.e. for a continuous homomorphism $\delta : \mathcal{O}_K^\times \rightarrow \overline{E}^\times$, we denote by the same letter $\delta : K^\times \rightarrow \overline{E}^\times$ the homomorphism such that $\delta(\pi_K) = 1$ and $\delta|_{\mathcal{O}_K^\times} = \delta$.

THEOREM 3.17.

(1) *For any point* $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$, *there exist* $\{k_\sigma\}_{\sigma\in\mathcal{P}} \in \prod_{\sigma\in\mathcal{P}} \mathbb{Z}_{\geqq 0}$ *and a short exact sequence of* $E(x)$*-*B*-pairs*

$$0 \rightarrow W(\delta_1) \rightarrow W(V_x) \rightarrow W(\det(V_x)\delta_1^{-1}) \rightarrow 0$$

for $\delta_1 := \delta_x\delta_{\lambda_x} \prod_{\sigma\in\mathcal{P}} \sigma^{-k_\sigma}$.

(2) *Conversely, if a point* $x := ([V_x], \delta_x, \lambda_x) \in X(\bar{\rho})$ *satisfies the following conditions (i) and (ii),*

(i) V_x *is a split trianguline* $E(x)$*-representation with a triangulation*

$$\mathcal{T}_x : 0 \subseteq W(\delta_x\delta_{\lambda_x}) \subseteq W(V_x),$$

(ii) $(V_x, \mathcal{T}_x)$ *satisfies all the assumptions in Proposition* 2.41,

then we have $x \in \mathcal{E}(\bar{\rho})$.

PROOF. The property (1) follows from (3) of Proposition 3.14.

We prove (2). Extending the scalars from E to $E(x)$, we may assume that $E(x)=E$. Let $x:=([V_x],\delta_x,\lambda_x)\in X(\bar{\rho})$ be an E-rational point satisfying the conditions (i), (ii) in (2). Then, the trianguline deformation functor $D_{V_x,\mathcal{T}_x}$ is representable by a formally smooth quotient $R_{V_x,\mathcal{T}_x}$ of the universal deformation ring R_{V_x} of V_x by Proposition 2.41. Moreover, we have a canonical isomorphism $R_{V_x}\xrightarrow{\sim}\widehat{\mathcal{O}}_{\mathfrak{X}(\bar{\rho}),p_1(x)}$, and $V_x^{\mathrm{univ}}:=\widetilde{V}^{\mathrm{univ}}\otimes_{\mathcal{O}_{\mathfrak{X}(\bar{\rho})}}\widehat{\mathcal{O}}_{\mathfrak{X}(\bar{\rho}),p_1(x)}$ is the universal deformation of V_x by Proposition 9.5 of [Ki03]. Taking a quotient, we obtain a map

$$\widehat{\mathcal{O}}_{X(\bar{\rho}),x}\xrightarrow{\sim}\widehat{\mathcal{O}}_{\mathfrak{X}(\bar{\rho}),p_1(x)}\hat{\otimes}_E\widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)}\hat{\otimes}_E\widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},p_3(x)}\to R_{V_x,\mathcal{T}_x}\hat{\otimes}_E\widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)}\hat{\otimes}_E\widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},p_3(x)}.$$

By the definition of $R_{V_x,\mathcal{T}_x}$, there exists a continuous homomorphism $\delta_{\mathcal{T}_x}:K^\times\to R^\times_{V_x,\mathcal{T}_x}$ which gives the universal triangulation, i.e. we have the following compatible triangulation

$$\mathcal{T}_{\mathrm{univ},n}:0\subseteq W(\delta_{\mathcal{T}_x,n})\subseteq W(V_x^{\mathrm{univ}}\otimes_{R_{V_x}}R_{V_x,\mathcal{T}_x}/\mathfrak{m}^n)$$

of $V_x^{\mathrm{univ}}\otimes_{R_{V_x}}R_{V_x,\mathcal{T}_x}/\mathfrak{m}^n$ for each $n\geqq 1$, where $\mathfrak{m}$ is the maximal ideal of $R_{V_x,\mathcal{T}_x}$ and $\delta_{\mathcal{T}_x,n}$ is the composition of $\delta_{\mathcal{T}_x}$ with the natural quotient map $R_{V_x,\mathcal{T}_x}\to R_{V_x,\mathcal{T}_x}/\mathfrak{m}^n$. Set $\lambda_{\mathcal{T}_x}:=\delta_{\mathcal{T}_x}(\pi_K)\in R^\times_{V_x,\mathcal{T}_x}$. Denote by $\delta^{\mathrm{univ}}_{p_2(x)}:\mathcal{O}_K^\times\to\widehat{\mathcal{O}}^\times_{\mathcal{W}_E,p_2(x)}$ the composition of the universal homomorphism $\delta_0^{\mathrm{univ}}:\mathcal{O}_K^\times\to\Gamma(\mathcal{W}_E,\mathcal{O}_{\mathcal{W}_E})^\times$ with the natural map $\Gamma(\mathcal{W}_E,\mathcal{O}_{\mathcal{W}_E})^\times\to\widehat{\mathcal{O}}^\times_{\mathcal{W}_E,p_2(x)}$. Then, the E-algebra $\widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)}$ is topologically generated by $\{\delta^{\mathrm{univ}}_{p_2(x)}(a)-\delta_x(a)|a\in\mathcal{O}_K^\times\}$. Denote by $\overline{R}$ a quotient of $R_{V_x,\mathcal{T}_x}\hat{\otimes}_E\widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)}\hat{\otimes}_E\widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},p_3(x)}$ by the ideal generated by $\delta_{\mathcal{T}_x}(a)\otimes 1\otimes 1-1\otimes\delta^{\mathrm{univ}}_{p_2(x)}(a)\otimes 1$ (any $a\in\mathcal{O}_K^\times$) and $\lambda_{\mathcal{T}_x}\otimes 1\otimes 1-1\otimes 1\otimes Y$. Then, we can see that the composition of the map $R_{V_x,\mathcal{T}_x}\to R_{V_x,\mathcal{T}_x}\hat{\otimes}_{E(x)}\widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)}\hat{\otimes}\widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},E(x)}:z\mapsto z\otimes 1\otimes 1$ with the natural quotient map $R_{V_x,\mathcal{T}_x}\hat{\otimes}_{E(x)}\widehat{\mathcal{O}}_{\mathcal{W}_E,p_2(x)}\hat{\otimes}_{E(x)}\widehat{\mathcal{O}}_{\mathbb{G}^{an}_{m,E},p_3(x)}\to\overline{R}$ is an isomorphism $R_{V_x,\mathcal{T}_x}\xrightarrow{\sim}\overline{R}$, and, if we denote by $\overline{\delta}:\mathcal{O}_K^\times\to\overline{R}^\times$ and $\overline{Y}\in\overline{R}^\times$ the reduction of $1\otimes\delta^{\mathrm{univ}}_{p_2(x)}\otimes 1$ and $1\otimes 1\otimes Y$, then the universal triangulation $\mathcal{T}_{\mathrm{univ}}:=\{\mathcal{T}_{\mathrm{univ},n}\}_{n\geqq 1}$ on $R_{V_x,\mathcal{T}_x}$ is transformed to the following triangulation

$$\overline{\mathcal{T}}:0\subseteq W(\overline{\delta}\delta_{\overline{Y}})\subseteq W((p_1^*\widetilde{V}^{\mathrm{univ}})\otimes_{\mathcal{O}_{X(\bar{\rho})}}\overline{R})$$

(here we drop the notation $n \in \mathbb{Z}_{\geqq 1}$ to simplify the notation).

Put $V_{\overline{R}} := (p_1^* \widetilde{V}^{\mathrm{univ}}) \otimes_{\mathcal{O}_{X(\bar{\rho})}} \overline{R}$, and put $\overline{R}_n := \overline{R}/\mathfrak{m}^n$ and $V_{\overline{R}_n} := V_{\overline{R}} \otimes_{\overline{R}} \overline{R}_n$ for each $n \geqq 1$. Denote by the same notation $\overline{\delta} : G_K^{ab} \to \overline{R}^{\times}$ the character such that $\overline{\delta}|_{\mathcal{O}_K^{\times}} = \overline{\delta}$ and $\overline{\delta}(\mathrm{rec}_K(\pi_K)) = 1$. Under this situation, we first claim that the natural map $\mathrm{Spm}(\overline{R}_n) \to X(\bar{\rho})$ factors through $X_0(\bar{\rho})$ for any $n \geqq 1$. This immediately follows from the facts that $W(V_{\overline{R}_n}(\overline{\delta}^{-1}))$ has a triangulation $0 \subseteq W(\delta_{\overline{Y}_n}) \subseteq W(V_{\overline{R}_n}(\overline{\delta}^{-1}))$ and that $W(\delta_{\overline{Y}_n})$ is crystalline with the Hodge-Tate weight zero, where $\overline{Y}_n \in \overline{R}_n$ is the reduction of $\overline{Y}$.

This fact also implies that $\mathbf{D}_{\mathrm{cris}}(W(\delta_{\overline{Y}_n}))$ which is equal to $\mathbf{D}_{\mathrm{cris}}(W(\delta_{\overline{Y}_n}))^{\varphi_K = \overline{Y}_n} \cap \mathrm{Fil}^0 \mathbf{D}_{dR}(W(\delta_{\overline{Y}_n}))$ is a φ-stable $K_0 \otimes_{\mathbb{Q}_p} \overline{R}_n$-submodule of $\mathbf{D}_{\mathrm{cris}}(V_{\overline{R}_n}(\overline{\delta}^{-1}))^{\varphi_K = \overline{Y}_n}$ of rank one contained in $\mathrm{Fil}^0 \mathbf{D}_{\mathrm{dR}}(V_{\overline{R}_n}(\overline{\delta}^{-1}))$. By Lemma 3.8, then we have a natural inclusion

$$\mathbf{D}_{\mathrm{cris}}(W(\delta_{\overline{Y}_n})) \subseteq \mathbf{D}^{+}_{\mathrm{cris}}(V_{\overline{R}_n}(\overline{\delta}^{-1}))^{\varphi_K = \overline{Y}_n}. \tag{6}$$

Next, we take an affinoid open $\mathrm{Spm}(R) \subseteq X_0(\bar{\rho})$ which contains x and satisfies the condition in the construction of X_{fs} (see the paragraph after the proof of Lemma 3.12). Let J be the ideal of R which defines $\mathrm{Spm}(R)_{fs}$. We claim that the natural map $R \to \overline{R}$ factors through $R/J \to \overline{R}$, which proves that $x \in \mathcal{E}(\bar{\rho})(E)$ because x is the point corresponding to the kernel of the map $R \to \overline{R} \to \overline{R}/\mathfrak{m}$. By construction of J, it suffices to show the following lemma. □

Lemma 3.18. *In the above situation, the following hold:*

(i) *For any $k \geqq 1$ and $\sigma \in \mathcal{P}$, the natural map*

$$\varprojlim_n (\mathbf{B}^{+}_{\mathrm{max},K} \otimes_{K,\sigma} V_{\overline{R}_n}(\overline{\delta}^{-1}))^{\varphi_K = \overline{Y}, G_K} \to \varprojlim_n (\mathbf{B}^{+}_{\mathrm{dR}}/t^k \mathbf{B}^{+}_{\mathrm{dR}} \otimes_{K,\sigma} V_{\overline{R}_n}(\overline{\delta}^{-1}))^{G_K}$$

is a surjection.

(ii) *For any $\sigma \in \mathcal{P}$ and $i \in \mathbb{Z}_{\leqq 0}$, $Q_\sigma(i)$ is nonzero in $\overline{R}$.*

Proof. Because $\overline{R} \xrightarrow{\sim} R_{V_x, \mathcal{T}_x}$ is domain, (i) follows from (ii) and from the above inclusion (6) by the same argument as in the proof of Proposition

2.8 of [Ki03]. We prove (ii). On $\overline{R}$, we have $Q_\sigma(T) = T - \bar{a}_{1,\sigma}$, where $\bar{a}_{1,\sigma} \in \overline{R}$ is the image of $a_{1,\sigma} \in \mathcal{O}_{X(\bar{\rho})}$ by the natural map $R \to \overline{R}$. Hence, $\bar{a}_{1,\sigma} \in \overline{R}$ is the σ-part of the Hodge-Tate weights of $\det(V_{\overline{R}})(\overline{\delta}^{-2})$ for any $\sigma \in \mathcal{P}$. Set $\delta_0 := \det(V_x)|_{\mathcal{O}_K^\times} \cdot \delta_x^{-2} : \mathcal{O}_K^\times \to E^\times$. By Lemma 3.19 below, then $\bar{a}_{1,\sigma} \in \overline{R}$ is the image of the σ-part of the Hodge-Tate weight $a_\sigma^{\mathrm{univ}} \in R_{\delta_0}$ of the universal deformation $\delta_{\delta_0}^{\mathrm{univ}} : \mathcal{O}_K^\times \to R_{\delta_0}^\times$ by the injection $R_{\delta_0} \hookrightarrow R_{V_x,\mathcal{T}_x} \xrightarrow{\sim} \overline{R}$ induced by a morphism $f : D_{V_x,\mathcal{T}_x} \to D_{\delta_0}$ defined below, where the injectiveness follows from Lemma 3.19 below. Hence, for any $i \in \mathbb{Z}_{\leqq 0}$, we have $Q_\sigma(i) = (i - \bar{a}_{1,\sigma}) \neq 0 \in \overline{R}$ by Lemma 3.20 below. □

Let $\delta_0 : \mathcal{O}_K^\times \to E^\times$ be a continuous homomorphism. We define a functor $D_{\delta_0} : \mathcal{C}_E \to \mathit{Sets}$ by

$$D_{\delta_0}(A) := \{\delta_A : \mathcal{O}_K^\times \to A^\times : \text{ continuous homomorphisms } \delta_A \pmod{\mathfrak{m}_A} = \delta_0\}$$

for $A \in \mathcal{C}_E$. It is easy to show that this functor is pro-representable by a ring R_{δ_0} which is isomorphic to $E[[T_1, T_2, \cdots, T_d]]$ for $d := [K : \mathbb{Q}_p]$. Let W be a split trianguline E-B-pair of rank two with a triangulation $\mathcal{T} : 0 \subseteq W(\delta_1) \subseteq W$ such that $W/W(\delta_1) \xrightarrow{\sim} W(\delta_2)$ for some continuous homomorphisms $\delta_1, \delta_2 : K^\times \to E^\times$. We put $\delta_0 := (\delta_2/\delta_1)|_{\mathcal{O}_K^\times}$. We define a morphism of functors $f : D_{W,\mathcal{T}} \to D_{\delta_0}$ as follows. Let $[(W_A, \mathcal{T}_A)] \in D_{W,\mathcal{T}}(A)$ be an equivalent class of trianguline deformation of $(W, \mathcal{T})$ over A with a triangulation $\mathcal{T}_A : 0 \subseteq W(\delta_{1,A}) \subseteq W_A$ such that $W_A/W(\delta_{1,A}) \xrightarrow{\sim} W(\delta_{2,A})$ for some $\delta_{1,A}, \delta_{2,A} : K^\times \to A^\times$, then we define f by

$$f([(W_A, \mathcal{T}_A)]) := (\delta_{2,A}/\delta_{1,A})|_{\mathcal{O}_K^\times} \in D_{\delta_0}(A).$$

LEMMA 3.19. *Let W be a two dimensional split trianguline E-B-pair with a triangulation $\mathcal{T} : 0 \subseteq W(\delta_1) \subseteq W$ such that $W/W(\delta_1) \xrightarrow{\sim} W(\delta_2)$. Assume that $\mathrm{H}^2(G_K, W(\delta_1/\delta_2)) = 0$, then the morphism of functors $f : D_{W,\mathcal{T}} \to D_{\delta_0}$ defined above is formally smooth.*

PROOF. Let $A \in \mathcal{C}_E$ and I be an ideal of A such that $I\mathfrak{m}_A = 0$. Take any $[(W_{A/I}, \mathcal{T}_{A/I})] \in D_{W,\mathcal{T}}(A/I)$ and $\delta_A \in D_{\delta_0}(A)$ such that $f([(W_{A/I}, \mathcal{T}_{A/I})]) = \delta_A \otimes \mathrm{id}_{A/I} \in D_{\delta_0}(A/I)$. Then, it suffices to show that there exists a lift $[(W_A, \mathcal{T}_A)] \in D_{W,\mathcal{T}}(A)$ of $[(W_{A/I}, \mathcal{T}_{A/I})]$ such that $f([(W_A, \mathcal{T}_A)]) = \delta_A$. Denote $\mathcal{T}_{A/I} : 0 \subseteq W(\delta_{1,A/I}) \subseteq W_{A/I}$ and $W_{A/I}/W(\delta_{1,A/I}) \xrightarrow{\sim} W(\delta_{2,A/I})$. Because D_{δ_0} is formally smooth, there exists $\delta_{1,A} : K^\times \to A^\times$ such that $\delta_{1,A} \otimes_A A/I = \delta_{1,A/I}$. We take a lift $\lambda \in A^\times$ of $\delta_{2,A/I}(\pi_K) \in (A/I)^\times$, and define $\delta_{2,A} : K^\times \to A^\times$ by $\delta_{2,A}(\pi_K) = \lambda$ and $\delta_{2,A}|_{\mathcal{O}_K^\times} = \delta_A \delta_{1,A}|_{\mathcal{O}_K^\times}$, then we have the following short exact sequence

$$0 \to W(\delta_1/\delta_2) \otimes_E I \to W(\delta_{1,A}/\delta_{2,A}) \to W(\delta_{1,A/I}/\delta_{2,A/I}) \to 0.$$

This sequence implies that the natural map

$$\mathrm{H}^1(G_K, W(\delta_{1,A}/\delta_{2,A})) \to \mathrm{H}^1(G_K, W(\delta_{1,A/I}/\delta_{2,A/I}))$$

is a surjection because we have $\mathrm{H}^2(G_K, W(\delta_1/\delta_2)) = 0$ by the assumption. Hence, there exists a lift $[(W_A, \mathcal{T}_A)] \in D_{W,\mathcal{T}}(A)$ of $[(W_{A/I}, \mathcal{T}_{A/I})] \in D_{W,\mathcal{T}}(A/I)$ satisfying that $f([W_A, \mathcal{T}_A]) = (\delta_{2,A}/\delta_{1,A})|_{\mathcal{O}_K^\times} = \delta_A$. We finish the proof of the lemma. □

Let $A \in \mathcal{C}_E$, and let $\delta : \mathcal{O}_K^\times \to A^\times$ be a continuous homomorphism, then it is known that this is locally $\mathbb{Q}_p$-analytic by Proposition 8.3 of [Bu07]. Then, for any $\sigma \in \mathcal{P}$, we define the σ-component of Hodge-Tate weights of δ by $\frac{\partial \delta(x)}{\partial \sigma(x)}|_{x=1} \in A$, which is equal to the σ-part of Hodge-Tate weights of $A(\widetilde{\delta})$ where $\widetilde{\delta} : G_K^{\mathrm{ab}} \to A^\times$ is any character such that $\widetilde{\delta} \circ \mathrm{rec}_K|_{\mathcal{O}_K^\times} = \delta$ (see Proposition 3.3 of [Na11]).

Lemma 3.20. *Let $\delta_0 : \mathcal{O}_K^\times \to E^\times$ be a continuous homomorphism. Let R_{δ_0} be the universal deformation ring of D_{δ_0}. Let $\delta_0^{\mathrm{univ}} : \mathcal{O}_K^\times \to R_{\delta_0}^\times$ be the universal deformation of δ_0. For any $\sigma \in \mathcal{P}$, define by $a_\sigma^{\mathrm{univ}} := (a_{\sigma,n}) \in R_{\delta_0} = \varprojlim_n R_{\delta_0}/\mathfrak{m}^n$ the σ-part of Hodge-Tate weights of δ_0^{univ}, where we denote by $a_{\sigma,n}$ the σ-part of Hodge-Tate weights of $\delta_0^{\mathrm{univ}} \otimes \mathrm{id}_{R_{\delta_0}/\mathfrak{m}^n}$ for each $n \geqq 1$. Then, a_σ^{univ} is not constant, i.e. not contained in E, for any $\sigma \in \mathcal{P}$.*

Proof. Let $a := \{a_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} E$ be any element, then we define a deformation of δ_0 over $E[\varepsilon]$ by

$$\delta_a : \mathcal{O}_K^\times \to E[\varepsilon]^\times : \delta_a(x) := \delta_0(x)(1 + (\sum_{\sigma \in \mathcal{P}} a_\sigma \log(\sigma(x)))\varepsilon).$$

The σ-part of Hodge-Tate weights of δ_a is $\frac{\partial\delta_0(x)}{\partial\sigma(x)}|_{x=1} + a_\sigma\varepsilon$. The lemma follows from this. □

COROLLARY 3.21. *Let $x = [V_x] \in \mathfrak{X}(\bar{\rho})$ be a point such that V_x is a crystabelline $E(x)$-trianguline representation satisfying the conditions* (1) *of Definition* 2.45. *Then, the point $x_\tau := ([V_x], \delta_{\tau,1}|_{\mathcal{O}_K^\times}, \delta_{\tau,1}(\pi_K)) \in X(\bar{\rho})$ is contained in $\mathcal{E}(\bar{\rho})$ for any $\tau \in \mathfrak{S}_2$, where we denote the triangulation $\mathcal{T}_\tau$ by $0 \subseteq W(\delta_{\tau,1}) \subseteq W(V_x)$.*

PROOF. This follows from (2) of Theorem 3.17 and from Lemma 2.48. □

Next, we describe the local structure of $\mathcal{E}(\bar{\rho})$ at the points satisfying the conditions (i), (ii) in (2) of Theorem 3.17 by the universal trianguline deformation rings. We prove the following theorem, which is a generalization of Proposition 10.6 of [Ki03].

THEOREM 3.22. *Let $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$ be a point such that the conditions (i), (ii) in* (2) *of Theorem* 3.17 *hold. Then, we have a canonical $E(x)$-algebra isomorphism*

$$\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x} \xrightarrow{\sim} R_{V_x,\mathcal{T}_x}.$$

In particular, $\mathcal{E}(\bar{\rho})$ is smooth of its dimension $3[K:\mathbb{Q}_p]+1$ at x.

PROOF. We may assume that $E = E(x)$.

In the proof of Theorem 3.17,we have already showed that the natural map $\widehat{\mathcal{O}}_{X(\bar{\rho}),x} \to \overline{R} \xrightarrow{\sim} R_{V_x,\mathcal{T}_x}$ factors through $\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x} \to \overline{R} \xrightarrow{\sim} R_{V_x,\mathcal{T}_x}$.

We prove the existence of the inverse map $R_{V_x,\mathcal{T}_x} \xrightarrow{\sim} \overline{R} \to \hat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}$. Because x is an E-rational point, we can take a Y-small affinoid neighborhood $\mathrm{Spm}(R)$ of x in $\mathcal{E}(\bar{\rho})$. By Proposition 3.7 and Proposition 3.14, for any sufficiently large $k > 0$, there exists a short exact sequence of Banach R-modules with the property (Pr)

$$0 \to (\mathbf{B}^+_{\mathrm{max},K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y} \to \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{K,\sigma}R \to U_{k,\sigma} \to 0$$

for any $\sigma \in \mathcal{P}$, and we have a natural isomorphism

$$K \otimes_{K_0} \mathbf{D}^+_{\mathrm{cris}}(V_R(\widetilde{\delta}^{\mathrm{univ}-1}))^{\varphi^f=Y} \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\hat{\otimes}_{\mathbb{Q}_p}V_R(\widetilde{\delta}^{\mathrm{univ}-1}))^{G_K}.$$

Fix such a $k > 0$, then we defined an ideal $H_\sigma \subseteq R$ for each $\sigma \in \mathcal{P}$ in Proposition 3.14 such that $\mathrm{Spm}(R) \setminus V(H)$ ($H := \prod_{\sigma\in\mathcal{P}} H_\sigma$) and $\mathrm{Spm}(R) \setminus V(H_\sigma)$ are scheme theoretically dense in $\mathrm{Spm}(R)$.

Under this situation, we prove the existence of the inverse $R_{V_x,\mathcal{T}_x} \xrightarrow{\sim} \overline{R} \to \hat{\mathcal{O}}_{\mathcal{E}(\bar\rho),x}$. First, we claim that $\mathbf{D}^+_{\mathrm{cris}}(V_x(\widetilde{\delta}_x^{-1}))^{\varphi^f=\lambda_x}$ is a free $K_0\otimes_{\mathbb{Q}_p} E$-module of rank one. By the definition and by Lemma 3.8, this module contains a submodule $\mathbf{D}_{\mathrm{cris}}(W(\delta_{\lambda_x})) = \mathbf{D}_{\mathrm{cris}}(W(\delta_{\lambda_x}))^{\varphi^f=\lambda_x} \cap \mathrm{Fil}^0\mathbf{D}_{\mathrm{dR}}(W(\delta_{\lambda_x}))$ which is of rank one. Hence $\mathbf{D}^+_{\mathrm{cris}}(V_x(\widetilde{\delta}_x^{-1}))^{\varphi^f=\lambda_x}$ is of rank one or two. If this is of rank two, then $V_x(\widetilde{\delta}_x^{-1})$ is crystalline with the Hodge-Tate weights $\{0,k_\sigma\}_{\sigma\in\mathcal{P}}$ such that $k_\sigma\in\mathbb{Z}_{\leqq 0}$ for any $\sigma\in\mathcal{P}$ with a unique φ_K-eigenvalue λ_x. These conditions imply that $\delta_2/\delta_1 = \prod_{\sigma\in\mathcal{P}}\sigma^{k_\sigma}$, which contradicts the assumption on $(V_x,\mathcal{T}_x)$. Hence $\mathbf{D}^+_{\mathrm{cris}}(V_x(\widetilde{\delta}_x^{-1}))^{\varphi^f=\lambda_x}$ is of rank one and the inclusion $\mathbf{D}_{\mathrm{cris}}(W(\delta_{\lambda_x})) \hookrightarrow \mathbf{D}^+_{\mathrm{cris}}(V_x(\widetilde{\delta}_x^{-1}))^{\varphi^f=\lambda_x}$ is isomorphism.

In the same way as in the proof of Proposition 10.6 of [Ki03], we take the blow up $\widetilde{T}$ of $\mathrm{Spm}(R)$ along H. By the definition of blow up, for any point $\widetilde{x}\in\widetilde{T}$ above $x\in\mathrm{Spm}(R)$ and for any $\sigma\in\mathcal{P}$, there exists $f_\sigma\in H_\sigma$ such that f_σ is a non zero divisor of $\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}$ and that $H_\sigma\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}} = f_\sigma\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}$. By the definition of H_σ, for any $\sigma\in\mathcal{P}$ and for any $\widetilde{x}\in\widetilde{T}$ above x, there exists a G_K-equivariant map $V_R(\widetilde{\delta}^{\mathrm{univ}-1})^\vee \to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y}$ such that the composite with the map

$$\begin{aligned}(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}R)^{\varphi_K=Y} &\to (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}f_\sigma\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}})^{\varphi_K=Y}\\ &\xrightarrow{\sim} (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}})^{\varphi_K=Y} \to (\mathbf{B}^+_{\max,K}\otimes_{K,\sigma}E(\widetilde{x}))^{\varphi_K=Y(\widetilde{x})}\end{aligned}$$

is non zero, where the isomorphism

$$(\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}f_\sigma\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}})^{\varphi_K=Y} \xrightarrow{\sim} (\mathbf{B}^+_{\max,K}\hat{\otimes}_{K,\sigma}\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}})^{\varphi_K=Y}$$

is given by $a\mapsto \frac{a}{f_\sigma}$. Using this map and using the fact that $\mathbf{D}^+_{\mathrm{cris}}(V_x(\widetilde{\delta}_x^{-1}))^{\varphi^f=\lambda_x}$ is rank one, we can see by induction on n that $\mathbf{D}^+_{\mathrm{cris}}(V_R(\widetilde{\delta}^{\mathrm{univ}-1})\otimes_R\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}/\mathfrak{m}^n_{\widetilde{x}})^{\varphi^f=Y}$ is a free $K_0\otimes_{\mathbb{Q}_p}\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}/\mathfrak{m}^n_{\widetilde{x}}$-module of rank one and that the natural base change map

$$\begin{aligned}\mathbf{D}^+_{\mathrm{cris}}(V_R(\widetilde{\delta}^{\mathrm{univ}-1})\otimes_R\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}/\mathfrak{m}^n_{\widetilde{x}})^{\varphi^f=Y}&\otimes_{\mathcal{O}_{T,x}/\mathfrak{m}^n_x}E(\widetilde{x})\\ &\xrightarrow{\sim}\mathbf{D}^+_{\mathrm{cris}}(V_x(\widetilde{\delta}_x^{-1})\otimes_E E(\widetilde{x}))^{\varphi^f=Y(x)}\end{aligned}$$

is isomorphism for any $n \geqq 1$. Because we have an equality

$$\mathrm{Fil}^1(K \otimes_{K_0} \mathbf{D}^+_{\mathrm{cris}}(V_x(\widetilde{\delta}_x^{-1}))^{\varphi^f = Y(x)}) = \mathrm{Fil}^1 \mathbf{D}_{\mathrm{dR}}(W(\delta_{\lambda_x})) = 0,$$

then $\mathbf{D}^+_{\mathrm{cris}}(V_R(\widetilde{\delta}^{\mathrm{univ}-1}) \otimes_R \widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}/\mathfrak{m}^n_{\widetilde{x}})^{\varphi^f = Y}$ is a $(\widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}/\mathfrak{m}^n_{\widetilde{x}})$-filtered φ-module of rank one such that $\mathrm{Fil}^0 = K \otimes_{K_0} \mathbf{D}^+_{\mathrm{cris}}(V_R(\delta^{\mathrm{univ}-1}) \otimes_R \widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}/\mathfrak{m}^n_{\widetilde{x}})^{\varphi^f = Y}$ and $\mathrm{Fil}^1 = 0$. By Lemma 2.23, this shows that $V_R \otimes_R \widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}$ is the projective limit of split trianguline $(\mathcal{O}_{\widetilde{T},\widetilde{x}}/\mathfrak{m}^n_{\widetilde{x}})$-representations with triangulations $0 \subseteq W(\bar{\delta}^{\mathrm{univ}}_n \delta_{\overline{Y}_n}) \subseteq W(V_R \otimes_R \widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}/\mathfrak{m}^n_{\widetilde{x}})$ which are trianguline deformations of $(V_x, \mathcal{T}_x) \otimes_E E(\widetilde{x})$ (for $n \in \mathbb{Z}_{\geqq 1}$), hence the natural map $R_{V_x} \to \widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x} \to \widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}$ factors through $R_{V_x} \to R_{V_x,\mathcal{T}_x}$ for any $\widetilde{x} \in \widetilde{T}$ above x. Moreover, because the natural map

$$\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x} \hookrightarrow \prod_{\widetilde{x} \in \widetilde{T}, p(\widetilde{x}) = x} \widehat{\mathcal{O}}_{\widetilde{T},\widetilde{x}}$$

is an injection by Lemma 10.7 of [Ki03] and by (2) of Proposition 3.14 (where $p : \widetilde{T} \to \mathrm{Spm}(R)$ is the projection), the map $R_{V_x} \to \widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}$ also factors through $R_{V_x,\mathcal{T}_x} \to \widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x}$. By this natural construction, we can easily check that this is the inverse of the map giving the above. We finish to prove the existence of the isomorphism $\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),x} \xrightarrow{\sim} R_{V_x,\mathcal{T}_x}$ for such points. Because this isomorphism is preserved by the base change from E to any finite extension E' by Lemma 3.10, the smoothness around these points follows from this isomorphism and from Lemma 2.8 of [BLR95]. □

4. Zariski Density of Two Dimensional Crystalline Representations

In this final section, as an application of Theorem 2.62 (in the two dimensional case) and of Theorem 3.22, we prove the Zariski density of two dimensional crystalline representations for any p-adic field.

We define a map $\pi : \mathcal{E}(\bar{\rho}) \to \mathcal{W}_E \times_E \mathcal{W}_E$ by $([V_x], \delta_x, \lambda_x) \mapsto (\delta_x, \det(V_x)|_{\mathcal{O}_K^\times} \cdot \delta_x^{-1})$.

Proposition 4.1. *For any point $x \in \mathcal{E}(\bar{\rho})$ which satisfies all the conditions of Theorem 3.22, the map $\pi : \mathcal{E}(\bar{\rho}) \to \mathcal{W}_E \times_E \mathcal{W}_E$ is smooth at x.*

Proof. Let $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$ be such a point. Set $\delta'_x := \det(V_x)|_{\mathcal{O}_K^\times} \cdot \delta_x^{-1}$. By the same argument as in Proposition 9.5 of [Ki03], we have a natural isomorphism isomorphism

$$\widehat{\mathcal{O}}_{\mathcal{W}_E \times_E \mathcal{W}_E, (\delta_x, \delta'_x)} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{W}_E, \delta_x} \hat{\otimes}_{E(x)} \widehat{\mathcal{O}}_{\mathcal{W}_E, \delta'_x} \xrightarrow{\sim} R_{\delta_x} \hat{\otimes}_{E(x)} R_{\delta'_x}.$$

Hence, by Theorem 3.22, the completion of π at x is the morphism

$$\pi : \mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}), x}) \to \mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{W}_E \times_E \mathcal{W}_E, (\delta_x, \delta'_x)})$$

induced by the morphism of functors

$$\pi_x : D_{V_x, \mathcal{T}_x} \to D_{\delta_x} \times D_{\delta'_x} : [(W_A, \mathcal{T}_A)] \mapsto (\delta_{1,A}|_{\mathcal{O}_K^\times}, \delta_{2,A}|_{\mathcal{O}_K^\times}),$$

where $\mathcal{T}_A : 0 \subseteq W(\delta_{1,A}) \subseteq W_A$ and $W_A / W(\delta_{1,A}) \xrightarrow{\sim} W(\delta_{2,A})$ for $A \in \mathcal{C}_{E(x)}$. Then, we can prove the formal smoothness of this morphism of functor in the same way as in the proof of Lemma 3.19. Hence, π is smooth at x by Proposition 2.41 and by Proposition 2.9 of [BLR95]. □

Let $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$ be an E-rational point such that V_x is a crystalline split trianguline E-representation with a triangulation $\mathcal{T}_x : 0 \subseteq W(\delta_x \delta_{\lambda_x}) \subseteq W(V_x)$ satisfying the condition (1) of Definition 2.45 (see Corollary 3.21). By Proposition 3.14, for any Y-small affinoid open neighborhood $U = \mathrm{Spm}(R)$ of x in $\mathcal{E}(\bar{\rho})$, there exits $k > 0$ and there exists a short exact sequence of Banach R-modules with the property (Pr)

$$0 \to K \otimes_{K_0} (\mathbf{B}^+_{\mathrm{max}} \hat{\otimes}_{\mathbb{Q}_p} R)^{\varphi^f = Y} \to \mathbf{B}^+_{\mathrm{dR}} / t^k \mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} R \to U_k \to 0$$

and we have a natural isomorphism

$$K \otimes_{K_0} \mathbf{D}^+_{\mathrm{cris}}(V_R(\widetilde{\delta}_R^{-1}))^{\varphi^f = Y} \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}} / t^k \mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} V_R(\widetilde{\delta}_R^{-1}))^{G_K}$$

and, for any $\sigma \in \mathcal{P}$, there exists the smallest ideal $H_\sigma \subseteq R$ satisfying that

$$(\mathbf{B}^+_{\mathrm{dR}} / t^k \mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} H_\sigma V_R(\widetilde{\delta}_R^{-1}))^{G_K} \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}} / t^k \mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{K,\sigma} V_R(\widetilde{\delta}_R^{-1}))^{G_K},$$

where we put $V_R := \Gamma(U, p_1^*(\widetilde{V}^{\mathrm{univ}}))$ and $\delta_R : \mathcal{O}_K^\times \to R^\times$ is the restriction of $p_2^*(\delta^{\mathrm{univ}})$ to U. Moreover, if we put $Q := \prod_{\sigma \in \mathcal{P}, 0 \leqq i \leqq k} Q_\sigma(-i) \in R$, then we have inclusions $\mathrm{Spm}(R)_Q \subseteq \mathrm{Spm}(R) \setminus V(H_\sigma) \subseteq \mathrm{Spm}(R)$ by the proof of Proposition 3.14. Moreover, shrinking U suitably, we may assume that $v_p(\lambda_y) = v_p(\lambda_x)$ for any $y = (V_y, \delta_y, \lambda_y) \in U$ and that $\pi|_U$ is smooth by Proposition 4.1.

Under this situation, we study the map $\pi|_U : U \to \mathcal{W}_E \times_E \mathcal{W}_E$ around x in detail. Because V_x is crystalline, we can write

$$\pi(x) = (\prod_{\sigma\in\mathcal{P}} \sigma^{k_{1,\sigma}}, \prod_{\sigma\in\mathcal{P}} \sigma^{k_{2,\sigma}}) \in \mathcal{W}_E \times_E \mathcal{W}_E$$

for some integers $\{k_{1,\sigma}, k_{2,\sigma}\}_{\sigma\in\mathcal{P}}$. Define a subset

$$(\mathcal{W}_E \times_E \mathcal{W}_E)_{cl,x} := \{(\prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma}, \prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma - m_\sigma}) \in \mathcal{W}_E \times_E \mathcal{W}_E | n_\sigma \in \mathbb{Z},$$
$$m_\sigma \in \mathbb{Z}_{\geqq k+1} \text{ for any } \sigma \in \mathcal{P} \text{ and } \sum_{\sigma\in\mathcal{P}} m_\sigma \geqq 2e_K v_p(\lambda_x) + [K:\mathbb{Q}_p] + 1\},$$

where e_K is the absolute ramified index of K. Then, for any admissible open neighborhood V of $\pi(x)$ in $\mathcal{W}_E \times_E \mathcal{W}_E$, there exists an affinoid open $V' \subseteq V$ which contains $\pi(x)$ such that $V'_{\mathrm{cl},x} := (\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x} \cap V'$ is Zariski dense in V'. Under this situation, we prove the following lemma.

Lemma 4.2. *Let* $y := (\prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma}, \prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma - m_\sigma})$ *be an element in* $(\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x}$ *and let* $z := ([V_z], \delta_z, \lambda_z)$ *be a point in* $U \cap \pi^{-1}(y)$, *then* V_z *is crystalline and split trianguline* $E(z)$*-representation with a triangulation* $\mathcal{T}_z : 0 \subseteq W(\delta_z\delta_{\lambda_z}) \subseteq W(V_z)$ *which satisfies the conditions* (1) *and* (2) *of Definition* 2.45.

Proof. Let z be such a point. By Corollary 2.6 of [Ki03], we have a natural isomorphism

$$(\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} V_R(\widetilde{\delta}_R^{-1}))^{G_K} \otimes_R E(z) \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_z(\widetilde{\delta}_z^{-1}))^{G_K}$$

and this is a free $K \otimes_{\mathbb{Q}_p} E(z)$-module of rank one. Because we have an isomorphism

$$K \otimes_{K_0} \mathbf{D}^+_{\mathrm{cris}}(V_R(\widetilde{\delta}_R^{-1}))^{\varphi^f = Y} \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} V_R(\widetilde{\delta}_R^{-1}))^{G_K}$$

and an injection

$$K \otimes_{K_0} \mathbf{D}^+_{\mathrm{cris}}(V_z(\widetilde{\delta}_z^{-1}))^{\varphi^f = \lambda_z} \hookrightarrow (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V_z(\widetilde{\delta}_z^{-1}))^{G_K}$$

induced from the injection $K\otimes_{K_0}(\mathbf{B}^+_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}E(z))^{\varphi^f=\lambda_z}\hookrightarrow \mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E(z)$, we obtain an isomorphism

$$K\otimes_{K_0}\mathbf{D}^+_{\mathrm{cris}}(V_z(\widetilde{\delta}_z^{-1}))^{\varphi^f=\lambda_z}\xrightarrow{\sim}(\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V_z(\widetilde{\delta}_z^{-1}))^{G_K}.$$

On the other hand, because the Hodge-Tate weights of $V_z(\widetilde{\delta}_z^{-1})$ are $\{0,-m_\sigma\}_{\sigma\in\mathcal{P}}$ and $m_\sigma\geqq k+1\geqq 1$, $(t^{k+1}\mathbf{B}^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}V_z(\widetilde{\delta}_z^{-1}))^{G_K}$ is also a free $K\otimes_{\mathbb{Q}_p}E(z)$-module of rank one. These implies that $\mathbf{D}_{\mathrm{dR}}(V_z(\widetilde{\delta}_z^{-1}))$ is a free of rank two $K\otimes_{\mathbb{Q}_p}E(z)$-module, i.e. $V_z(\widetilde{\delta}_z^{-1})$ is potentially semi-stable and split trianguline with a triangulation $\mathcal{T}'_z: 0\subseteq W(\delta_{\lambda_z})\hookrightarrow W(V_z(\widetilde{\delta}_z^{-1}))$. Moreover, if we set $\delta_2:=\det(V_z)\cdot\delta_z^{-2}\delta_{\lambda_z}^{-1}:K^\times\to E(z)^\times$, then we have $W(V_z(\widetilde{\delta}_z^{-1}))/W(\delta_{\lambda_z})\xrightarrow{\sim}W(\delta_2)$ such that $\delta_2|_{\mathcal{O}_K^\times}=\prod_{\sigma\in\mathcal{P}}\sigma^{-m_\sigma}$ because $z\in\pi^{-1}(y)$, which implies that $V_z(\widetilde{\delta}_z^{-1})$ is semi-stable. Finally, we claim that $V_z(\widetilde{\delta}_z^{-1})$ is crystalline. If we assume that $V_z(\widetilde{\delta}_z^{-1})$ is semi-stable but not crystalline, then the φ^f-eigenvalue of $W(\delta_2)$ is $\lambda_z p^f$ or $\lambda_z p^{-f}$. By the weakly admissibility of $\mathbf{D}_{\mathrm{st}}(V_z(\widetilde{\delta}_z^{-1}))$, we have an equality $t_N(V_z(\widetilde{\delta}_z^{-1}))=t_H(V_z(\widetilde{\delta}_z^{-1}))$. On the other hand, because we have an isomorphism $W(\det(V_z(\widetilde{\delta}_z^{-1})))\xrightarrow{\sim}W(\delta_{\lambda_z}\delta_2)$, we obtain the equalities

$$t_N(V_z(\widetilde{\delta}_z^{-1}))=\frac{2}{f}v_p(\lambda_z)\pm 1=\frac{2}{f}v_p(\lambda_x)\pm 1,$$
$$\text{and } t_H(V_z(\widetilde{\delta}_z^{-1}))=\frac{1}{[K:\mathbb{Q}_p]}(\sum_{\sigma\in\mathcal{P}}m_\sigma),$$

hence we obtain $t_N(V_z(\widetilde{\delta}_z^{-1}))<t_H(V_z(\widetilde{\delta}_z^{-1}))$ because $y\in(\mathcal{W}\times_E\mathcal{W}_E)_{\mathrm{cl},x}$, which is a contradiction. Hence, $V_z(\widetilde{\delta}_z^{-1})$ is crystalline, and V_z is also crystalline because $\widetilde{\delta}_z$ is crystalline. Finally, twisting $\mathcal{T}'_z$ by δ_z, we obtain a triangulation $\mathcal{T}_z: 0\subseteq W(\delta_z\delta_{\lambda_z})\subseteq W(V_z)$ which satisfies (1) and (2) of Definition 2.45. □

Lemma 4.3. *Let $z=([V_z],\delta_z,\lambda_z)$ be a point in $U\cap\pi^{-1}(y)$ as in the above lemma. Then we have a natural isomorphism $\widehat{\mathcal{O}}_{\pi^{-1}(y),z}\xrightarrow{\sim}R^{\mathrm{cris}}_{V_z}$.*

Proof. First, by Lemma 2.48 and Theorem 3.22 and Lemma 4.2, we have a triangulation $\mathcal{T}_z: 0\subseteq W(\delta_z\delta_{\lambda_z})\subseteq W(V_z)$, and the functor $D_{V_z,\mathcal{T}_z}$ is representable by $R_{V_z,\mathcal{T}_z}$, and we have an isomorphism $\widehat{\mathcal{O}}_{\mathcal{E}(\bar{\rho}),z}\xrightarrow{\sim}R_{V_z,\mathcal{T}_z}$.

Then, the completion at z and $\pi(z)$ of the morphism $\pi : \mathcal{E}(\bar{\rho}) \to \mathcal{W}_E \times_E \mathcal{W}_E$ is the morphism

$$\pi_z : \mathrm{Spf}(R_{V_z,\mathcal{T}_z}) \to \mathrm{Spf}(R_{\delta_z} \hat{\otimes}_{E(z)} R_{\delta'_z})$$

induced by

$$D_{V_z,\mathcal{T}_z} \to D_{\delta_z} \times D_{\delta'_z} : [(V_A, \mathcal{T}_A)] \mapsto (\delta_{1,A}|_{\mathcal{O}_K^\times}, \delta_{2,A}|_{\mathcal{O}_K^\times}),$$

where we set $\delta'_z := \det(V_z)|_{\mathcal{O}_K^\times} \cdot \delta_z^{-1}$. Under this interpretation, we have an equality $\mathrm{Spf}(\widehat{\mathcal{O}}_{\pi^{-1}(y),z}) = \pi_z^{-1}((\delta_z, \delta'_z))$, and this corresponds to the subfunctor D' of $D_{V_z,\mathcal{T}_z}$ defined by

$$D'(A) := \{[(V_A, \mathcal{T}_A)] \in D_{V_z,\mathcal{T}_z}(A) | \delta_{1,A}|_{\mathcal{O}_K^\times} = \delta_z \otimes_{E(z)} \mathrm{id}_A, \\ \delta_{2,A}|_{\mathcal{O}_K^\times} = \delta'_z \otimes_{E(z)} \mathrm{id}_A\}$$

for $A \in \mathcal{C}_{E(z)}$. Because V_z is crystalline, this is equivalent to that V_A is crystalline by Lemma 2.57. Therefore we have $D' = D^{\mathrm{cris}}_{V_z}$, hence we obtain an isomorphism $R^{\mathrm{cris}}_{V_z} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\pi^{-1}(y),z}$. □

In the situation of Lemma 4.3, for any $y := (\prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma}, \prod_{\sigma\in\mathcal{P}} \sigma^{n_\sigma - m_\sigma}) \in (\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x}$, we set $U_y := \pi^{-1}(y) \cap U$, which is smooth over $E(y)$ by the assumption on U, and define a subset

$$U_{y,b} := \{z = ([V_z], \delta_z, \lambda_z) \in U_y | V_z \text{ is benign}\}.$$

PROPOSITION 4.4. *In the above situation, if U_y is not empty, then $U_{y,b}$ is an admissible open which is scheme theoretically dense in U_y, in particular $U_{y,b}$ is non-empty.*

PROOF. Set $U_y := \mathrm{Spm}(R')$. By Lemma 4.2, any point $z \in U_y$ satisfies the condition (1) and (2) of Definition 2.45 and V_z is crystalline with the Hodge-Tate weights $\{n_\sigma, n_\sigma - m_\sigma\}_{\sigma\in\mathcal{P}}$. Because U_y is smooth, so in particular U_y is reduced. Hence, by Corollary 6.3.3 of [Be-Co08] and Corollary 3.19 of [Ch09a],

$$\mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) := \varinjlim_n (\frac{1}{t^n} \mathbf{B}^+_{\max} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K}$$

is a locally free $K_0 \otimes_{\mathbb{Q}_p} R'$-module of rank two, and we have natural isomorphisms

$$\mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \otimes_{R'} E(z) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris}}(V_z(\widetilde{\delta}_z^{-1})) \text{ for any } z \in U_y$$

and

$$K \otimes_{K_0} \mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \xrightarrow{\sim} (\mathbf{B}_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K} = (\mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K},$$

where the last equality follows from the assumption on the Hodge-Tate weights of V_z for any $z \in U_y$. Because $U_y \subseteq U_Q$, we have an isomorphism

$$(\mathbf{B}_{\mathrm{dR}}^{+}/t^k \mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} V_R(\widetilde{\delta}_R^{-1}))^{G_K} \otimes_R R' \xrightarrow{\sim} (\mathbf{B}_{\mathrm{dR}}^{+}/t^k \mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K},$$

which is a locally free $K \otimes_{\mathbb{Q}_p} R'$-module of rank one by Corollary 2.6 of [Ki03]. Because the natural map $K \otimes_{K_0} (\mathbf{B}_{\max}^{+} \hat{\otimes}_{\mathbb{Q}_p} R')^{\varphi^f = Y} \hookrightarrow \mathbf{B}_{\mathrm{dR}}^{+}/t^k \mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} R'$ is an injection, hence we obtain an isomorphism

$$K \otimes_{K_0} \mathbf{D}_{\mathrm{cris}}^{+}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{\varphi^f = Y} \xrightarrow{\sim} (\mathbf{B}_{\mathrm{dR}}^{+}/t^k \mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K}.$$

From these facts, we can see that the natural map

$$(\mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K} \to (\mathbf{B}_{\mathrm{dR}}^{+}/t^k \mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K}$$

is a surjection. Hence, we obtain a short exact sequence

$$\begin{aligned} 0 \to \mathrm{Fil}^k \mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \to \mathbf{D}_{\mathrm{dR}}^{+}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \\ \to (\mathbf{B}_{\mathrm{dR}}^{+}/t^k \mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K} \to 0, \end{aligned}$$

where we define $\mathrm{Fil}^k \mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) := (t^k \mathbf{B}_{\mathrm{dR}}^{+} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K}$ which is a locally free $K \otimes_{\mathbb{Q}_p} R'$-module of rank one. If we set

$$D_2 := \mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))/\mathbf{D}_{\mathrm{cris}}^{+}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{\varphi^f = Y},$$

then the above facts imply that D_2 is also a locally free $K_0 \otimes_{\mathbb{Q}_p} R'$-module of rank one. By taking a sufficiently fine affinoid covering of $\mathrm{Spm}(R')$, we may assume that all these modules are free over $K_0 \otimes_{\mathbb{Q}_p} R'$ or $K \otimes_{\mathbb{Q}_p} R'$. If we decompose $\mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) = \oplus_{\tau: K_0 \to K_0} D_\tau$ etc, then we obtain a short exact sequence

$$0 \to D_\tau^{+, \varphi^f = Y} \to D_\tau \to D_{2,\tau} \to 0$$

of free R'-modules with an R'-linear φ^f-action for any τ. Define $Y_1 \in R'^{\times}$ by $\varphi^f(e) = Y_1 e$ for a R'-base e of $D_{2,\tau}$. Because Y_1 is a lift of the other Frobenius eigenvalue of $\mathbf{D}_{\mathrm{cris}}(V_z(\widetilde{\delta}_z^{-1}))$ (one is λ_z) for any $z \in U_y = \mathrm{Spm}(R')$ and because $\mathbf{D}_{\mathrm{cris}}(V_z(\widetilde{\delta}_z^{-1}))$ is weakly admissible, the condition $\sum_{\sigma\in\mathcal{P}} m_\sigma \geqq 2e_K v_p(\lambda_z) + [K:\mathbb{Q}_p] + 1$ for any $z \in U_y$ implies that

$$Y - Y_1 (\text{ and } Y - p^{\pm f} Y_1) \in R'^{\times}.$$

Then, an easy linear algebra implies that there exists a decomposition $D_\tau = R'e'_1 \oplus R'e'_2$ such that $R'e'_1 = D_\tau^{\varphi^f = Y} = D_\tau^{+,\varphi^f=Y}$ and $R'e'_2 = D_\tau^{\varphi^f = Y_1}$. Twisting these by φ^i for any $0 \leqq i \leqq f-1$, we obtain a decomposition

$$\mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) = \mathbf{D}^+_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{\varphi^f=Y} \oplus \mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{\varphi^f=Y_1}.$$

We denote by e_1 (resp. e_2) a $K_0 \otimes_{\mathbb{Q}_p} R'$-basis of $\mathbf{D}^+_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{\varphi^f=Y}$ (resp. $\mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{\varphi^f=Y_1}$). For any $\sigma \in \mathcal{P}$, we denote by $e_{1,\sigma}, e_{2,\sigma}$ the R'-basis of the σ-component of $\mathbf{D}^+_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \xrightarrow{\sim} K \otimes_{K_0} \mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))$ naturally induced from e_1, e_2. Under this situation, we write the σ-component $\mathrm{Fil}^k \mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))_\sigma$ by using the basis $e_{1,\sigma}, e_{2,\sigma}$ as follows. Because the natural map $\mathbf{D}^+_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{\varphi^f=Y} \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{dR}}/t^k\mathbf{B}^+_{\mathrm{dR}} \hat{\otimes}_{\mathbb{Q}_p} V_{R'}(\widetilde{\delta}_{R'}^{-1}))^{G_K}$ is isomorphism, the natural map

$$\mathrm{Fil}^k \mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \to K \otimes_{K_0} D_2,$$

which is the composition of the natural inclusion

$$\mathrm{Fil}^k \mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \to \mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) = K \otimes_{K_0} \mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))$$

with the natural projection $K \otimes_{K_0} \mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \to K \otimes_{K_0} D_2$, is an isomorphism. Hence, for any $\sigma \in \mathcal{P}$, we can take a R'-basis of $\mathrm{Fil}^k \mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1}))_\sigma$ of the form $e_{2,\sigma} + a_\sigma e_{1,\sigma}$ for some $a_\sigma \in R'$. Then, by the definition of benign representations, for any $z \in U_y$, V_z is benign if and only if $\prod_{\sigma\in\mathcal{P}} a_\sigma(z) \neq 0 \in E(z)$ because we have isomorphisms $\mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \otimes_{R'} E(z) \xrightarrow{\sim} \mathbf{D}_{\mathrm{dR}}(V_z(\widetilde{\delta}_z^{-1}))$ and $\mathbf{D}_{\mathrm{cris}}(V_{R'}(\delta_{R'}^{-1})) \otimes_{R'} E(z) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris}}(V_z(\widetilde{\delta}_z^{-1}))$ etc. Hence, to finish the proof of the proposition, it is enough to show that $\prod_{\sigma\in\mathcal{P}} a_\sigma$ is a non-zero divisor in R'. To prove this claim, it is enough to show that $a_\sigma \in \widehat{\mathcal{O}}_{U_y,z} \xrightarrow{\sim} R^{\mathrm{cris}}_{V_z}$ is non-zero for any $\sigma \in \mathcal{P}$ and

$z \in U_y$ because $R^{\mathrm{cris}}_{V_z}$ is domain. To prove this claim, we first note that we have isomorphisms

$$\mathbf{D}_{\mathrm{cris}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \otimes_{R'} R^{\mathrm{cris}}_{V_z} \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris}}(V_{R^{\mathrm{cris}}_{V_z}}(\widetilde{\delta}^{-1}_{R^{\mathrm{cris}}_{V_z}})) := \varprojlim_{n \geqq 1} \mathbf{D}_{\mathrm{cris}}(V_{R^{\mathrm{cris}}_{V_z}/\mathfrak{m}^n}(\widetilde{\delta}^{-1}_{R^{\mathrm{cris}}_{V_z},n}))$$

and

$$\mathbf{D}_{\mathrm{dR}}(V_{R'}(\widetilde{\delta}_{R'}^{-1})) \otimes_{R'} R^{\mathrm{cris}}_{V_z} \xrightarrow{\sim} \mathbf{D}_{\mathrm{dR}}(V_{R^{\mathrm{cris}}_{V_z}}(\widetilde{\delta}^{-1}_{R^{\mathrm{cris}}_{V_z}})) := \varprojlim_{n \geqq 1} \mathbf{D}_{\mathrm{dR}}(V_{R^{\mathrm{cris}}_{V_z}/\mathfrak{m}^n}(\widetilde{\delta}^{-1}_{R^{\mathrm{cris}}_{V_z},n}))$$

by construction and by Corollary 6.3.3 of [Be-Co08], where we denote by $\mathfrak{m}$ the maximal ideal of $R^{\mathrm{cris}}_{V_z}$ and denote by $\delta_{R^{\mathrm{cris}}_{V_z}} : \mathcal{O}_K^\times \to (R^{\mathrm{cris}}_{V_z})^\times$ the homomorphism induced from $\delta_{R'}$ and denote by $\delta_{R^{\mathrm{cris}}_{V_z},n} : \mathcal{O}_K^\times \to (R^{\mathrm{cris}}_{V_z}/\mathfrak{m}^n)^\times$ the reduction of $\delta_{R^{\mathrm{cris}}_{V_z}}$ for $n \geqq 1$. Hence, the claim follows from the following lemma. □

Lemma 4.5. *Let V be a crystalline E-representation with Hodge-Tate weights $\{0, -k_\sigma\}_{\sigma \in \mathcal{P}}$ such that $k_\sigma \in \mathbb{Z}_{\geqq 1}$ for any $\sigma \in \mathcal{P}$. Assume that $\mathbf{D}_{\mathrm{cris}}(V_{R^{\mathrm{cris}}_V}) = K_0 \otimes_{\mathbb{Q}_p} R^{\mathrm{cris}}_V e_1 \oplus K_0 \otimes_{\mathbb{Q}_p} R^{\mathrm{cris}}_V e_2$ such that $\varphi^f(e_1) = \lambda'_1 e_1$, $\varphi^f(e_2) = \lambda'_2 e_2$ for some $\lambda'_1, \lambda'_2 \in (R^{\mathrm{cris}}_V)^\times$ and that $\mathrm{Fil}^{k_\sigma}\mathbf{D}_{\mathrm{dR}}(V_{R^{\mathrm{cris}}_V})_\sigma$ is generated by $e_{2,\sigma} + a_\sigma e_{1,\sigma}$ for any $\sigma \in \mathcal{P}$. Then, we have $a_\sigma \neq 0 \in R^{\mathrm{cris}}_V$ for any $\sigma \in \mathcal{P}$.*

Proof. Denote by $\lambda_i := \overline{\lambda'}_i \in E^\times$ and $\bar{a}_\sigma \in E$, the images of λ'_i and a_σ by the natural quotient map $R^{\mathrm{cris}}_V \to E$. Then we have $\mathbf{D}_{\mathrm{cris}}(V) = K_0 \otimes_{\mathbb{Q}_p} E\bar{e}_1 \oplus K_0 \otimes_{\mathbb{Q}_p} E\bar{e}_2$ such that $\varphi^f(\bar{e}_i) = \lambda_i \bar{e}_i$ and $\mathrm{Fil}^{k_\sigma}\mathbf{D}_{\mathrm{dR}}(V)_\sigma = E(\bar{e}_{2,\sigma} + \bar{a}_\sigma \bar{e}_{1,\sigma})$. For any $b := \{b_\sigma\}_{\sigma \in \mathcal{P}} \in \prod_{\sigma \in \mathcal{P}} E$, we construct a deformation $D(b)$ of $\mathbf{D}_{\mathrm{cris}}(V)$ over $E[\varepsilon]$ by $D(b) := \mathbf{D}_{\mathrm{cris}}(V) \otimes_E E[\varepsilon]$ as a φ-module and $\mathrm{Fil}^0(K \otimes_{K_0} D(b)) = K \otimes_{K_0} D(b)$ and

$$\mathrm{Fil}^1(K \otimes_{K_0} D(b))_\sigma = \mathrm{Fil}^{k_\sigma}(K \otimes_{K_0} D(b))_\sigma := E[\varepsilon](\bar{e}_{2,\sigma} + (\bar{a}_\sigma + b_\sigma\varepsilon)\bar{e}_{1,\sigma}),$$

$\mathrm{Fil}^{k_\sigma+1}(K \otimes_{K_0} D(b))_\sigma = 0$. For any b as above, $D(b)$ is a deformation of $\mathbf{D}_{\mathrm{cris}}(V)$ over $E[\varepsilon]$. Here, we remark that $D(b)$ is automatically weakly admissible because, as an E-filtered φ-module, $D(b)$ is an extension of $\mathbf{D}_{\mathrm{crys}}(V)$ by $\mathbf{D}_{\mathrm{crys}}(V)$ and because the weakly admissibility is closed under extensions. The existence of such deformations implies that $a_\sigma \neq 0$ for any $\sigma \in \mathcal{P}$. □

Next, we will prove a proposition concerning the Zariski density of benign points in $\mathcal{E}(\bar{\rho})$. Before proving this proposition, we first prove some lemmas concerning general (maybe well-known easy) facts about rigid geometry.

LEMMA 4.6. *Let T_n be the n-dimensional closed unit disc defined over E. Then, for any admissible open U of T_n which contains the origin $0 := (0, \cdots, 0) \in T_n$, there exists $m >> 0$ such that $\{(x_1, \cdots, x_n) \in T_n \| x_i| \leqq 1/p^m$ for any $1 \leqq i \leqq n\} \subseteq U$.*

PROOF. Because U is admissibly covered by rational subdomains, we may assume that U is itself a rational subdomain, namely, we may assume that there exist $f_1, \cdots, f_d, g \in E\{\{T_1, \cdots, T_n\}\}$ such that $(f_1, \cdots, f_d, g) = E\{\{T_1, \cdots, T_n\}\}$ and $U = \{x = (x_1, \cdots, x_n) \in T_n \| f_i(x)| \leqq |g(x)|$ for any $1 \leqq i \leqq d\}$. Then, the condition $0 \in U$ means that $|f_{i,0}| \leqq |g_0|$ for any i, where $f_{i,0}, g_0 \in E$ are the constant terms of f_i and g. If $g_0 = 0$, then $f_{i,0} = 0$ for any i, and this implies that $(f_1, \cdots, f_d, g) \subseteq (T_1, T_2 \cdots, T_n)$, which is a contradiction. Hence we have $g_0 \neq 0$ and then, because the norms of coefficients of f_i and g are bounded, there exits $m >> 0$ large enough such that $|f_i(x)| \leqq \max\{|f_{i,0}|, |g_0|\} = |g_0|$ and $|g(x)| = |g_0|$ for any $x = (x_1, \cdots, x_n) \in T_n$ such that $|x_i| \leqq 1/p^m$ for any i, i.e. $\{x \in T_n \| x_i| \leqq 1/p^m$ for any $i\} \subseteq U$. □

LEMMA 4.7. *Let $x := ([V_x], \delta_x, \lambda_x) \in \mathcal{E}(\bar{\rho})$ be an E-rational point such that V_x is crystalline trianguline as in Lemma 4.2, and let $U \subseteq \mathcal{E}(\bar{\rho})$ be an admissible open neighborhood of x. Then, there exists an admissible open neighborhood $U' \subseteq U$ of x such that $U'_{\mathrm{cl},x} := U' \cap \pi^{-1}((\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x})$ is Zariski dense in U'.*

PROOF. Re-taking smaller U, we may assume that U satisfies the properties as in before Lemma 4.2 and that the morphism $\pi|_U : U \to \mathcal{W}_E \times_E \mathcal{W}_E$ is smooth and U is irreducible smooth of its dimension $3[K : \mathbb{Q}_p] + 1$ by Theorem 3.22 and Lemma 4.1. In particular, we may assume that

$\pi(U) \subseteq \mathcal{W}_E \times_E \mathcal{W}_E$ is an admissible open by Corollary 5.11 of [BL93]. By definition of $\mathcal{W}_E \times_E \mathcal{W}_E$ and $(\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x}$ and by Lemma 4.6, if we re-take U smaller, then we may assume that there exists an admissible open neighborhood V of $y := \pi(x)$ which is isomorphic to $T_n \xrightarrow{\sim} V$ where $n := 2[K : \mathbb{Q}_p]$ such that y corresponds to the origin $0 \in T_n$ and that, for any $m \geqq 1$, the set $V_{\mathrm{cl},m} := \{x \in T_n \| x_i| \leqq 1/p^m$ for any $1 \leqq i \leqq n\} \cap (\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x}$ is Zariski dense in V and that $\pi(U) \subseteq V$ and that $\pi|_U : U \to V$ factors through an étale morphism $\pi' : U \to V \times_E T_{n'}$ satisfying $\pi'(x) = (y, 0)$ for $n' := [K : \mathbb{Q}_p]+1$. Because $V_{\mathrm{cl},m}$ is Zariski dense in V for any m, the set $(V \times_E T_{n'})_{\mathrm{cl},m} := \{(y', z) \in V_{\mathrm{cl},m} \times_E T_{n'} \| z_i| \leqq 1/p^m$ for any $1 \leqq i \leqq n'\}$ is also Zariski dense in $V \times_E T_{n'}$. Because $\pi'(U)$ is an admissible open neighborhood of $(y, 0) \in V \times_E T_{n'}$, there exists $m >> 0$ such that $(V \times_E T_{n'})_{\mathrm{cl},m}$ is contained in $\pi'(U)$ by Lemma 4.6. Then, we have $\pi'^{-1}((V \times_E T_{n'})_{\mathrm{cl},m}) \subseteq \pi^{-1}(V_{\mathrm{cl},m}) \subseteq U_{\mathrm{cl},x}$, then the lemma follows from the following lemma. □

LEMMA 4.8. *Let $f : U := \mathrm{Spm}(B) \to V := \mathrm{Spm}(E\{\{T_1, \cdots, T_n\}\})$ be an étale morphism between E-affinoids for some n. We assume that U is irreducible and reduced. If $V_{\mathrm{cl}} \subseteq V$ is a Zariski dense subset of V such that $V_{\mathrm{cl}} \subseteq f(U)$, then $f^{-1}(V_{\mathrm{cl}})$ is also Zariski dense in U.*

PROOF. By the assumption, the natural map $A := E\{\{T_1, \cdots, T_n\}\} \to \prod_{x \in V_{\mathrm{cl}}} E(x)$ is an injection. To prove the lemma, it suffices to show that the kernel of the natural map $B \to \prod_{y \in f^{-1}(V_{\mathrm{cl}})} E(y)$ is zero. If I is the kernel of this map, then the map $A \to B/I \hookrightarrow \prod_{y \in f^{-1}(V_{\mathrm{cl}})} E(y)$ is equal to the map $A \hookrightarrow \prod_{x \in V_{\mathrm{cl}}} E(x) \to \prod_{y \in f^{-1}(V_{\mathrm{cl}})} E(y)$. Because we have $V_{\mathrm{cl}} \subseteq f(U)$ by the assumption, the map $\prod_{x \in V_{\mathrm{cl}}} E(x) \to \prod_{y \in f^{-1}(V_{\mathrm{cl}})} E(y)$ is an injection. Therefore, the map $A \hookrightarrow B/I$ is also an injection. Then, we have $\dim(A) \leqq \dim(B/I) (\leqq \dim(B))$ by Lemma 4.9 below. From this, we have $\dim(B/I) = \dim(B)$ because B is étale over A. Because U is irreducible and reduced, we obtain the equality $I = 0$. □

LEMMA 4.9. *Let $f : Z := \mathrm{Spm}(B') \to \mathrm{Spm}(E\{\{T_1, \cdots, T_n\}\})$ be a morphism of affinoids over E. We assume that the induced map $A := E\{\{T_1, \cdots, T_n\}\} \to B'$ is an injection. Then, we have* $\dim(A) \leqq \dim(B')$.

PROOF. Because $A \hookrightarrow B'$ is an injection, the base change $\mathrm{Frac}(A) \hookrightarrow \mathrm{Frac}(A) \otimes_A B'$ is also an injection, in particular, the generic fiber of the morphism of schemes $f_0 : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$ induced from the injection $A \hookrightarrow B'$ is not empty. We denote by x the generic point of $\mathrm{Spec}(A)$ and take a point $y \in f_0^{-1}(x)$. By Proposition 2.1.1 of [Berk93], if we denote by $\kappa(x)$ and $\kappa(y)$ the residue fields (in the sense of scheme) at x and y, then the natural inclusion $\kappa(x) \hookrightarrow \kappa(y)$ is an inclusion of valuation fields which induces an inclusion $\widetilde{\kappa}(x) \hookrightarrow \widetilde{\kappa}(y)$, where $\widetilde{\kappa}(-)$ is the residue field of the valuation field $\kappa(-)$. Form this inclusion, we obtain $(\dim(A) = n =)s(\widetilde{\kappa}(x)/E) \leqq s(\widetilde{\kappa}(y)/E)$, where $s(\widetilde{\kappa}(-)/E)$ is the transcendence degree of $\widetilde{\kappa}(-)$ over E. By Lemma 2.5.2 of [Berk93], then we also have $s(\widetilde{\kappa}(y)/E) \leqq \dim(B')$, hence we obtain $\dim(A) \leqq \dim(B')$. □

Set

$$\mathcal{E}(\bar{\rho})_{\mathrm{b}} := \{x \in \mathcal{E}(\bar{\rho}) | V_x \text{ is benign and crystalline }\}.$$

Proposition 4.10. *Let x be an E-rational point in $\mathcal{E}(\bar{\rho})$ as in Lemma 4.7, and let U be an admissible open neighborhood of x. If we take an affinoid neighborhood $U' := \mathrm{Spm}(R)$ of x as in Lemma 4.7. Then, $U'_{\mathrm{b}} := \mathcal{E}(\bar{\rho})_{\mathrm{b}} \cap U'$ is also Zariski dense in U'.*

Proof. Consider any element $f \in R$ in the kernel of the natural map $R \to \prod_{z \in U_{\mathrm{b}}} E(z)$. Then, for any $y \in (\mathcal{W}_E \times_E \mathcal{W}_E)_{\mathrm{cl},x} \cap \pi(U)$, $f|_{\pi^{-1}(y) \cap U} \in \mathcal{O}_{\pi^{-1}(y) \cap U}$ is equal to zero by Proposition 4.4 because $\mathcal{O}_{\pi^{-1}(y) \cap U}$ is reduced. Hence, we obtain $f = 0 \in R$ by Lemma 4.7. □

Corollary 4.11. *Let Y be the Zariski closure of $\mathcal{E}(\bar{\rho})_{\mathrm{b}}$ in $\mathcal{E}(\bar{\rho})$. Then, Y is a union of irreducible components of $\mathcal{E}(\bar{\rho})$.*

Proof. This follows from Proposition 4.10. □

Set

$$\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}} := \{x \in \mathfrak{X}(\bar{\rho}) | V_x \text{ is crystalline and the Hodge-Tate weights of } V_x \text{ are } \{k_{1,\sigma}, k_{2,\sigma}\}_{\sigma \in \mathcal{P}} \text{ such that } k_{1,\sigma} \neq k_{2,\sigma} \text{ for any } \sigma \in \mathcal{P}\},$$

and

$$\mathfrak{X}(\bar{\rho})_{\mathrm{b}} := \{x \in \mathfrak{X}(\bar{\rho}) | V_x \text{ is benign and crystalline }\}.$$

LEMMA 4.12. *If $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is not empty, then $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is also not empty.*

PROOF. If $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is not empty, then it is easy to show that there exists some $x \in \mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ which satisfies the condition (1) of Definition 2.45. Then, the lemma follows from Proposition 4.4. □

For a rigid analytic space Y over E and for a point $y \in Y$, we denote by

$$t_{Y,y} := \mathrm{Hom}_{E(y)}(\mathfrak{m}_y/\mathfrak{m}_y^2, E(y))$$

the tangent space at y, where $\mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$. The following three theorems are the main theorems of this article concerning the Zariski density of two dimensional crystalline representations.

We denote by $\overline{\mathfrak{X}(\bar{\rho})}_{\mathrm{b}}$ the Zariski closure of $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ in $\mathfrak{X}(\bar{\rho})$. The following is a generalization of Corollary 1.10 of [Ki10] for general K.

THEOREM 4.13. *If $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is non empty, then $\overline{\mathfrak{X}(\bar{\rho})}_{\mathrm{b}}$ is non empty and a union of irreducible components of $\mathfrak{X}(\bar{\rho})$.*

PROOF. By Lemma 4.12, $Z := \overline{\mathfrak{X}(\bar{\rho})}_{\mathrm{b}}$ is non empty.

To show that Z is a union of irreducible components of $\mathfrak{X}(\bar{\rho})$, we first claim that the dimension of any irreducible component of $\mathfrak{X}(\bar{\rho})$ is at most $4[K:\mathbb{Q}_p]+1$. Take any point $x := [V_x] \in \mathfrak{X}(\bar{\rho})$. Under the assumption that $\mathrm{End}_{\mathbb{F}}(\bar{\rho}) = \mathbb{F}$, we also have $\mathrm{End}_{E(x)[G_K]}(V_x) = E(x)$ and we have a canonical isomorphism $R_{V_x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathfrak{X}(\rho),x}$ by Proposition 9.5 of [Ki03]. Under the condition $\mathrm{End}_{E(x)[G_K]}(V_x) = E(x)$, it is easy to show that the dimension of $\mathrm{H}^2(G_K, \mathrm{ad}(V_x)) \xrightarrow{\sim} \mathrm{H}^0(G_K, \mathrm{ad}(V_x)(\chi_p))^\vee$ is at most one. By deformation theory, then the dimension of R_{V_x} is $4[K:\mathbb{Q}_p]+1$, from which the claim follows.

By this claim, it suffices to show that the dimension of any irreducible component of Z is at least $4[K:\mathbb{Q}_p]+1$. Let Z' be an irreducible component of Z. Because the singular locus $Z'_{\mathrm{sing}} \subsetneq Z'$ is a proper Zariski closed set in Z', there exists a benign point $x \in \mathfrak{X}(\bar{\rho})_{\mathrm{b}} \cap Z'$ such that Z' is smooth at x. By the definition of benign representation and by Theorem 3.17, there exist the different two points

$$x_1 := ([V_x], \delta_{x_1}, \lambda_{x_1}), x_2 := ([V_x], \delta_{x_2}, \lambda_{x_2}) \in \mathcal{E}(\bar{\rho})$$

such that $p_1(x_i)=x$ and satisfy the property (ii) in the Theorem 3.17. We denote by Y_i' an irreducible component of $p_1^{-1}(Z)$ containing x_i for $i=1,2$ respectively. These are also irreducible components of $\mathcal{E}(\bar{\rho})$ by Corollary 4.11, and Y_i' is unique for each $i=1,2$ by Theorem 3.22. Because the natural morphism $p_1|_{Y_i'}: Y_i' \rightarrow \mathfrak{X}(\bar{\rho})$ factors through Z' for $i=1,2$, we obtain a map

$$t_{\mathcal{E}(\bar{\rho}),x_i} = t_{Y_i',x_i} \rightarrow t_{Z',x} \hookrightarrow t_{\mathfrak{X}(\bar{\rho}),x}$$

for $i=1,2$. Hence, we obtain a map

$$\bigoplus_{i=1,2} t_{\mathcal{E}(\bar{\rho}),x_i} \rightarrow t_{Z',x} \hookrightarrow t_{\mathfrak{X}(\bar{\rho}),x}.$$

By Theorem 2.62 and Theorem 3.22, this map is surjective, hence we obtain an equality

$$t_{Z',x} = t_{\mathfrak{X}(\bar{\rho}),x}.$$

Because x is smooth at Z', hence Z' has dimension $4[K:\mathbb{Q}_p]+1$, which proves the theorem. □

Concerning the assumption that $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ is non empty, in this paper we prove the following (maybe well-known) lemma.

LEMMA 4.14. *If* $\bar{\rho}\otimes_{\mathbb{F}}\bar{\mathbb{F}} \not\sim \begin{pmatrix} \omega & * \\ 0 & 1 \end{pmatrix} \otimes \chi$ *and* $\bar{\rho}\otimes_{\mathbb{F}}\bar{\mathbb{F}} \not\sim \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \otimes \chi$ *for any character* $\chi: G_K \rightarrow \bar{\mathbb{F}}^{\times}$, *where* ω *is the mod* p *cyclotomic character. Then,* $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ *is non empty.*

PROOF. First, we prove the lemma for the absolutely reducible case. Extending $\mathbb{F}$, we may assume that $\bar{\rho}$ is reducible. Because any character $\chi: G_K \rightarrow \mathbb{F}^{\times}$ has a crystalline lift, we may assume that $\bar{\rho} = \begin{pmatrix} \eta & * \\ 0 & 1 \end{pmatrix}$ for a character $\eta: G_K \rightarrow \mathbb{F}^{\times}$ such that $\eta \neq 1$ and $\eta \neq \omega$. Using twists of a Lubin-Tate character of K by $\sigma \in \mathcal{P}$ and a unramified character, we can take a crystalline lift $\widetilde{\eta}: G_K \rightarrow \mathcal{O}^{\times}$ of η whose Hodge-Tate weights are $\{k_\sigma\}_{\sigma\in\mathcal{P}}$ such that $k_\sigma \geqq 1$ for any $\sigma \in \mathcal{P}$. Under the assumption $\eta \neq 1, \omega$, $\mathrm{H}^1(G_K, \mathcal{O}(\widetilde{\eta}))$ is a free $\mathcal{O}$-module of rank $[K:\mathbb{Q}_p]$ and the natural map $\mathrm{H}^1(G_K, \mathcal{O}(\widetilde{\eta})) \rightarrow \mathrm{H}^1(G_K, \mathbb{F}(\eta))$ is a surjection. Because $k_\sigma \geqq 1$ for any $\sigma \in \mathcal{P}$, we have an equality $\mathrm{H}^1_f(G_K, E(\widetilde{\eta})) = \mathrm{H}^1(G_K, E(\widetilde{\eta}))$. These imply that any extension class in $\mathrm{H}^1(G_K, \mathbb{F}(\eta))$ lifts to an extension class in $\mathrm{H}^1(G_K, \mathcal{O}(\widetilde{\eta}))$ which is crystalline.

Next, we prove the lemma for the absolutely irreducible case. Denote by K_2 the unramified extension of K of degree 2, and denote by $\chi_2 : G_{K_2}^{\mathrm{ab}} \to \mathbb{F}_{p^{2f}}^\times$ the reduction of the Lubin-Tate character $\chi_{2,\mathrm{LT}} : G_{K_2}^{ab} \to \mathcal{O}_{K_2}^\times$ of K_2 associated to the uniformizer $\pi_{K_2} := \pi_K$ of K_2. Then, it is known that there exists an isomorphism $\bar{\rho} \xrightarrow{\sim} (\mathrm{Ind}_{G_{K_2}}^{G_K} \chi_2^i) \otimes \chi$ (possibly, after extending scalars) for a character $\chi : G_K \to \mathbb{F}^\times$ and for some $i \in \mathbb{Z}$ such that $i \not\equiv 0 \pmod{p^f + 1}$, where we also denote by the same letter $\chi_2 : G_{K_2}^{ab} \to \mathbb{F}_{p^{2f}}^\times \hookrightarrow \mathbb{F}^\times$ for a fixed embedding $\mathbb{F}_{p^{2f}}^\times \hookrightarrow \mathbb{F}^\times$. Hence, it suffices to show that $\mathrm{Ind}_{G_{K_2}}^{G_K} \chi_2^i$ has a crystalline lift. Because χ_2 is the reduction of $\chi_{2,\mathrm{LT}}$, we can take a lift of χ_2^i of the form $\prod_{\sigma \in \mathcal{P}} \widetilde{\sigma}(\chi_{2,\mathrm{LT}})^{k_\sigma}$ such that $k_\sigma \geqq 1$ for all $\sigma \in \mathcal{P}$, where $\widetilde{\sigma} : K_2 \hookrightarrow \overline{E}$ is an extension of σ. Then, $\mathrm{Ind}_{G_{K_2}}^{G_K} \chi_2^i$ has a crystalline lift $\mathrm{Ind}_{G_{K_2}}^{G_K} (\prod_{\sigma \in \mathcal{P}} \widetilde{\sigma}(\chi_{2,\mathrm{LT}})^{k_\sigma})$ whose Hodge-Tate weights are $\{0, k_\sigma\}_{\sigma \in \mathcal{P}}$. □

Finally, we prove the following two theorems on the density of $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ in $\mathfrak{X}(\bar{\rho})$ under the following assumptions. In particular, we need to exclude the case $p = 2$. Under these conditions, we will show below that $\mathfrak{X}(\bar{\rho})$ is a finite union of smooth irreducible components. Let $\zeta_p \in \overline{K}$ be a primitive root of unity. The difficulty of the proof to the theorems depends on whether $\zeta_p \in K$ or not, which corresponds to whether $\mathfrak{X}(\bar{\rho})$ is irreducible or not respectively.

We first prove the density when $\zeta_p \notin K$,

THEOREM 4.15. *Assume that $\zeta_p \notin K$. Moreover, assume the following conditions,*

(0) $\mathrm{End}_{G_K}(\bar{\rho}) = \mathbb{F}$,

(1) $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ *is not empty,*

(2) *if $\bar{\rho}$ is absolutely reducible, then* $\bar{\rho} \otimes_{\mathbb{F}} \overline{\mathbb{F}} \not\sim \begin{pmatrix} 1 & * \\ 0 & \omega \end{pmatrix} \otimes \chi$ *for any* $\chi : G_K \to \overline{\mathbb{F}}^\times$,

(3) *if $\bar{\rho}$ is absolutely irreducible, then* $[K(\zeta_p) : K] \neq 2$ *or* $\bar{\rho}|_{I_K} \otimes_{\mathbb{F}} \overline{\mathbb{F}} \not\sim \begin{pmatrix} \chi_2^i & 0 \\ 0 & \chi_2^{ip^f} \end{pmatrix}$ *for any i such that* $\chi_2^{i(p^f-1)}|_{I_K} = \omega|_{I_K}$,

then we have an equality $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}} = \mathfrak{X}(\bar{\rho})$.

PROOF. We claim that $\mathfrak{X}(\bar{\rho})$ is isomorphic to $(4[K:\mathbb{Q}_p]+1)$-dimensional open unit disc under the above conditions, from which the theorem follows by Theorem 4.13.

To show the claim, it suffices to show that $\mathrm{H}^2(G_K, \mathrm{ad}(\bar{\rho})) = 0$, hence suffices to show that $\mathrm{Hom}_{G_K}(\bar{\rho}, \bar{\rho}\otimes\omega) = 0$ by the Tate duality. When $\bar{\rho}$ is absolutely reducible, it is easy to see that the conditions (0), (2) imply that $\mathrm{Hom}_{G_K}(\bar{\rho}, \bar{\rho}\otimes\omega) = 0$. When $\bar{\rho}$ is absolutely irreducible, then $\bar{\rho}$ is of the form $\mathrm{Ind}_{G_{K_2}}^{G_K}(\chi_2^i)\otimes\chi$ for some i and χ after extending scalars. If $\mathrm{Hom}_{G_K}(\bar{\rho}, \bar{\rho}\otimes\omega) \neq 0$, then there exists an isomorphism $\bar{\rho} \xrightarrow{\sim} \bar{\rho}\otimes\omega$ by Schur's lemma. The latter implies that $\det(\bar{\rho}) = \det(\bar{\rho})\omega^2$ and $\begin{pmatrix} \chi_2^i|_{I_K} & 0 \\ 0 & \chi_2^{ip^f}|_{I_K} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} \chi_2^i\omega|_{I_K} & 0 \\ 0 & \chi_2^{ip^f}\omega|_{I_K} \end{pmatrix}$. Because we assume that $\zeta_p \notin K$, these imply that $[K(\zeta_p):K] = 2$ and $\chi_2^{i(p^f-1)}|_{I_K} = \omega|_{I_K}$, which proves the claim, hence proves the theorem. □

Finally, we prove the theorem on the density when $\zeta_p \in K$ and $p \neq 2$ under the following assumptions.

THEOREM 4.16. *Assume that* $\zeta_p \in K$ *and* $p \neq 2$. *Moreover, assume the following conditions,*

(0) $\mathrm{End}_{G_K}(\bar{\rho}) = \mathbb{F}$,

(1) $\mathfrak{X}(\bar{\rho})_{\mathrm{reg-cris}}$ *is not empty,*

then we have an equality $\overline{\mathfrak{X}(\bar{\rho})_{\mathrm{b}}} = \mathfrak{X}(\bar{\rho})$.

PROOF. If $\zeta_p \in K$, then $\mathfrak{X}(\bar{\rho})$ never becomes irreducible. Hence, we first need to know how to decompose $\mathfrak{X}(\bar{\rho})$ into irreducible components under the above assumptions.

Let $P \subset \mathcal{O}_K^\times$ be the subgroup of $\mathcal{O}_K^\times$ consisting of all the p-th power roots of unity, and let p^n be the order of P. Fix $\zeta_{p^n} \in \mathcal{O}_K^\times$ a generator of P, i.e. a primitive p^n-th root of unity. For each $0 \leqq i \leqq p^n - 1$, we define a subfunctor $D_{\bar{\rho},i}$ of $D_{\bar{\rho}}$ by

$$D_{\bar{\rho},i}(A) := \{[V_A] \in D_{\bar{\rho}}(A) | \det(V_A)(\mathrm{rec}_K(\zeta_{p^n})) = \iota_A(\zeta_{p^n})^i\}$$

for $A \in \mathcal{C}_{\mathcal{O}}$, where $\iota_A : \mathcal{O} \to A$ is the morphism which gives an $\mathcal{O}$-algebra structure to A. It is easy to see that the canonical inclusion $D_{\bar{\rho},i} \hookrightarrow D_{\bar{\rho}}$ is relatively representable, i.e. this satisfies the conditions (1) and (2) and (3) in the proof of Proposition 2.37. For each i, let $R_{\bar{\rho},i}$ be the quotient of $R_{\bar{\rho}}$ which represents $D_{\bar{\rho},i}$, and let $\mathfrak{X}(\bar{\rho})_i \subseteq \mathfrak{X}(\bar{\rho})$ be the Zariski closed rigid analytic space associated to $R_{\bar{\rho},i}$. Then it is easy to see that, as rigid analytic space, $\mathfrak{X}(\bar{\rho})$ is the disjoint union of $\mathfrak{X}(\bar{\rho})_i$ for $0 \leqq i \leqq p^n - 1$,

$$\mathfrak{X}(\bar{\rho}) = \coprod_{0 \leqq i \leqq p^n - 1} \mathfrak{X}(\bar{\rho})_i.$$

We claim that each $\mathfrak{X}(\bar{\rho})_i$ is isomorphic to the $(4[K : \mathbb{Q}_p] + 1)$-dimensional open unit disc. To prove this claim, it suffices to show that the functor $D_{\bar{\rho},i}$ is formally smooth.

We prove the formal smoothness of $D_{\bar{\rho},i}$ as follows. Let A be an object of $\mathcal{C}_{\mathcal{O}}$ and $I \subseteq A$ be a non zero ideal such that $I\mathfrak{m}_A = 0$. Let $[V_{A/I}] \in D_{\bar{\rho},i}(A/I)$ be a deformation of $\bar{\rho}$ over A/I. Then, it suffices to show that $[V_{A/I}]$ lifts to $D_{\bar{\rho},i}(A)$. Fixing a A/I-basis of $V_{A/I}$, we represent $V_{A/I}$ by a continuous homomorphism $\rho_{A/I} : G_K \to \mathrm{GL}_2(A/I)$. Because the obstruction of the liftings of $\det(\bar{\rho})$ comes only from that of $\det(\bar{\rho})|_{\mathrm{rec}_K(P)}$, we can take a continuous character $c_A : G_K^{\mathrm{ab}} \to A^{\times}$ which is a lift of $\det(\rho_{A/I})$ and $c_A(\mathrm{rec}_K(\zeta_{p^n})) = \iota_A(\zeta_{p^n})^i$. We take a continuous lift $\widetilde{\rho}_A : G_K \to \mathrm{GL}_2(A)$ of $\rho_{A/I}$ such that $\det(\widetilde{\rho}_A(g)) = c_A(g)$ for any $g \in G_K$ and then we define a 2-cocycle $f : G_K \times G_K \to I \otimes_{\mathbb{F}} \mathrm{ad}(\bar{\rho})$ by

$$\widetilde{\rho}_A(g_1 g_2)\widetilde{\rho}_A(g_2)^{-1}\widetilde{\rho}_A(g_1)^{-1} := 1 + f(g_1, g_2) \in 1 + I \otimes_A \mathrm{M}_2(A) = 1 + I \otimes_{\mathbb{F}} \mathrm{ad}(\bar{\rho}).$$

Because $\det(\widetilde{\rho}_A) = c_A$ is a homomorphism, $f(g_1, g_2)$ is contained in $I \otimes_{\mathbb{F}} \mathrm{ad}^0(\bar{\rho})$, where we denote by $\mathrm{ad}^0(\bar{\rho}) := \{a \in \mathrm{ad}(\bar{\rho}) | \mathrm{trace}(a) = 0\}$. Hence, we obtain a class of 2-cocycle $[f] \in \mathrm{H}^2(G_K, \mathrm{ad}^0(\bar{\rho}))$. Under the assumption (0) and the assumption that $\zeta_p \in K$ and $p \neq 2$, we have

$$\mathrm{H}^2(G_K, \mathrm{ad}^0(\bar{\rho})) \xrightarrow{\sim} \mathrm{H}^0(G_K, \mathrm{ad}^0(\bar{\rho})(\omega))^{\vee} = \mathrm{H}^0(G_K, \mathrm{ad}^0(\bar{\rho}))^{\vee} = 0$$

(we remark that we have $\mathrm{H}^0(G_K, \mathrm{ad}^0(\bar{\rho})) = \mathrm{H}^0(G_K, \mathrm{ad}(\bar{\rho})) = \mathbb{F}$ when $p = 2$). Hence, twisting $\widetilde{\rho}_A$ by using a suitable continuous one cochain $d : G_K \to I \otimes_{\mathbb{F}} \mathrm{ad}^0(\bar{\rho})$, we obtain a continuous homomorphism $\rho_A : G_K \to \mathrm{GL}_2(A)$ such that ρ_A is a lift of $\rho_{A/I}$ and $\det(\rho_A) = c_A$, which proves the formally smoothness of $D_{\bar{\rho},i}$.

By this claim and by Theorem 4.13, to prove the theorem, it suffices to show that $\mathfrak{X}(\bar{\rho})_i \cap \mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is non empty for any i under the assumption (1). We prove this claim as follows. First, there exists some i such that $\mathfrak{X}(\bar{\rho})_i \cap \mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is non empty by the assumption (1). We take a point $x = [V_x] \in \mathfrak{X}(\bar{\rho})_i \cap \mathfrak{X}(\bar{\rho})_{\mathrm{b}}$.

The twist $V_x(\chi_{\mathrm{LT}}^{j(p^f-1)})$ of V_x for any $j \in \mathbb{Z}$ is contained in $\mathfrak{X}(\bar{\rho})_{\mathrm{b}} \cap \mathfrak{X}(\bar{\rho})_{i_j}$, where we define i_j such that $0 \leqq i_j \leqq p^n - 1$ and $i_j \equiv i + 2j(p^f - 1) \pmod{p^n}$. Because we assume $p \neq 2$, i_j runs through all $0 \leqq i' \leqq p^n - 1$, hence $\mathfrak{X}(\bar{\rho})_{\mathrm{b}} \cap \mathfrak{X}(\bar{\rho})_{i'}$ is non empty for any i'. Hence, $\mathfrak{X}(\bar{\rho})_{\mathrm{b}}$ is Zariski dense in $\mathfrak{X}(\bar{\rho})$. □

REMARK 4.17. We remark that, from § 3.3, we assume $\mathrm{End}_{G_K}(\bar{\rho}) = \mathbb{F}$. However, even if $\mathrm{End}(\bar{\rho}) \neq \mathbb{F}$, it may be possible to prove Theorem 3.17 and Theorem 3.22 and Theorem 4.13 without any additional difficulties if we use the universal framed deformations instead of usual deformations. But, up to now, the author does not know whether the density is satisfied or not when $\mathrm{End}_{G_K}(\bar{\rho}) \neq \mathbb{F}$.

5. Appendix: Continuous Cohomology of B-Pairs

In [Na09], we defined a cohomology $\mathrm{H}^i(G_K, W)$ by using continuous cochains of G_K which we review below. On the other hand, Liu [Li08] defined another cohomology which we write by $\mathrm{H}^i_{\mathrm{Liu}}(G_K, W) := \mathrm{H}^i_{\varphi,\Gamma}(D(W))$ by using a complex defined from the (φ, Γ)-module $D(W)$ associated to W (see 2.1 of [Li08] for the definition). Moreover, he proved that this cohomology satisfies the Euler-Poincaré formula and the Tate duality. In this appendix, we first prove that $\mathrm{H}^i(G_K, W)$ also satisfies the Euler-Poincaré formula and the Tate duality, and finally prove that $\mathrm{H}^i(G_K, W)$ is canonically isomorphic to $\mathrm{H}^i_{\mathrm{Liu}}(G_K, W)$.

We first recall the definition of $\mathrm{H}^i(G_K, W)$. Let G be a topological group. For a continuous G-module M and $i \in \mathbb{Z}_{\geqq 0}$, we define the group of i-th continuous cochains of G with values in M by

$$\mathrm{C}^i(G, M) := \{c : G^{\times i} \to M | c \text{ is a continuous map }\}.$$

As usual, we define the boundary map

$$\partial^i : \mathrm{C}^i(G, M) \to \mathrm{C}^{i+1}(G, M)$$

by

$$\partial^i(c)(g_1, g_2, \cdots, g_{i+1}) := g_1 c(g_2, \cdots, g_{i+1}) + (-1)^{i+1} c(g_1, g_2, \cdots, g_i) \\ + \sum_{s=1}^{i} (-1)^i c(g_1, \cdots, g_{s-1}, g_s g_{s+1}, g_{s+2}, \cdots, g_{i+1}).$$

Let $W = (W_e, W_{\mathrm{dR}}^+)$ be a B-pair. Set $W_{\mathrm{dR}} := W_e \otimes_{\mathbf{B}_e} \mathbf{B}_{\mathrm{dR}}$. For W, we define a complex $\mathrm{C}^\bullet(G_K, W)$ of $\mathbb{Q}_p$-vector spaces as the mapping cone of the map

$$\mathrm{C}^\bullet(G_K, W_e) \oplus \mathrm{C}^\bullet(G_K, W_{\mathrm{dR}}^+) \to \mathrm{C}^\bullet(G_K, W_{\mathrm{dR}}) : (c_e, c_{\mathrm{dR}}) \mapsto c_e - c_{\mathrm{dR}},$$

i.e. defined by

$$\mathrm{C}^0(G_K, W) := \mathrm{C}^0(G_K, W_e) \oplus \mathrm{C}^0(G_K, W_{\mathrm{dR}}^+)$$

and

$$\mathrm{C}^i(G_K, W) := \mathrm{C}^i(G_K, W_e) \oplus \mathrm{C}^i(G_K, W_{\mathrm{dR}}^+) \oplus \mathrm{C}^{i-1}(G_K, W_{\mathrm{dR}})$$

for $i \geqq 1$ and the differential

$$\partial^0 : \mathrm{C}^0(G_K, W_e) \oplus \mathrm{C}^0(G_K, W_{\mathrm{dR}}^+) \\ \to \mathrm{C}^1(G_K, W_e) \oplus \mathrm{C}^1(G_K, W_{\mathrm{dR}}^+) \oplus \mathrm{C}^0(G_K, W_{\mathrm{dR}})$$

is defined by

$$\partial^0(c_e, c_{\mathrm{dR}}) := (\partial^0(c_e), \partial^0(c_{\mathrm{dR}}), c_e - c_{\mathrm{dR}})$$

and, for $i \geqq 1$, the differential

$$\partial^i : \mathrm{C}^i(G_K, W_e) \oplus \mathrm{C}^i(G_K, W_{\mathrm{dR}}^+) \oplus \mathrm{C}^{i-1}(G_K, W_{\mathrm{dR}}) \\ \to \mathrm{C}^{i+1}(G_K, W_e) \oplus \mathrm{C}^{i+1}(G_K, W_{\mathrm{dR}}^+) \oplus \mathrm{C}^i(G_K, W_{\mathrm{dR}})$$

is defined by

$$\partial^i(c_e, c_{\mathrm{dR}}, c) = (\partial^i(c_e), \partial^i(c_{\mathrm{dR}}), c_e - c_{\mathrm{dR}} - \partial^{i-1}(c)).$$

We define the cohomology of W by

$$\mathrm{H}^i(G_K, W) := \mathrm{H}^i(\mathrm{C}^\bullet(G_K, W)),$$

and also define

$$\mathrm{H}^i(G_K, W_e) := \mathrm{H}^i(\mathrm{C}^\bullet(G_K, W_e))$$

and

$$\mathrm{H}^i(G_K, W_{\mathrm{dR}}^+) := \mathrm{H}^i(\mathrm{C}^\bullet(G_K, W_{\mathrm{dR}}^+)),\ \mathrm{H}^i(G_K, W_{\mathrm{dR}}) := \mathrm{H}^i(\mathrm{C}^\bullet(G_K, W_{\mathrm{dR}})).$$

By these definitions, we have the following long exact sequence,

$$\cdots \to \mathrm{H}^{i-1}(G_K, W_{\mathrm{dR}}) \to \mathrm{H}^i(G_K, W) \to \mathrm{H}^i(G_K, W_e) \oplus \mathrm{H}^i(G_K, W_{\mathrm{dR}}^+) \to \cdots .$$

Before proving the Euler-Poincaré formula, we recall some results of [Be09] on the relationship between B-pairs and almost $\mathbb{C}_p$-representations. Let U be an almost $\mathbb{C}_p$-representation. Let V_1 and V_2 be $\mathbb{Q}_p$-representations of G_K of dimension d_1 and d_2 respectively and $d \geqq 0$ be an integer such that we have $V_1 \subseteq U$ and $V_2 \subseteq \mathbb{C}_p^{\oplus d}$ and $U/V_1 \xrightarrow{\sim} \mathbb{C}_p^{\oplus d}/V_2$. Then, we define the dimension of U by

$$\dim_{\mathcal{C}(G_K)}(U) := d$$

and the height of U by

$$\mathrm{ht}(U) := d_1 - d_2,$$

which are independent of the choice of V_1, V_2 and are additive with respect to exact sequences ([Fo03]). For a B-pair $W := (W_e, W_{\mathrm{dR}}^+)$, we define

$$X_0(W) := W_e \cap W_{\mathrm{dR}}^+ \text{ and } X_1(W) := W_{\mathrm{dR}}/(W_e + W_{\mathrm{dR}}^+).$$

Concerning $X_0(W)$ and $X_1(W)$, Berger [Be09] proved the following theorem.

THEOREM 5.1. *Let W be a B-pair of rank d, then*

(1) *$X_0(W)$ and $X_1(W)$ are almost $\mathbb{C}_p$-representations,*

(2) *if W is pure of slope $s \leqq 0$, then we have $\dim_{\mathcal{C}(G_K)}(X_0(W)) = -sd$, $\mathrm{ht}(X_0(W)) = d$ and $X_1(W) = 0$,*

(3) *if W is pure of slope $s > 0$, then we have $X_0(W) = 0$ and $\dim_{\mathcal{C}(G_K)}(X_1(W)) = sd$, $\mathrm{ht}(X_1(W)) = -d$.*

PROOF. See Theorem 3.1 of [Be09]. □

LEMMA 5.2. *Let U be an almost $\mathbb{C}_p$-representation, then $\mathrm{H}^i(G_K, U)$ is finite dimensional over $\mathbb{Q}_p$ for $i = 0, 1, 2$ and zero for $i \geqq 3$.*

PROOF. This follows from the definition of almost $\mathbb{C}_p$-representations and the facts that $\mathrm{H}^i(G_K, V) = 0$ for $i \geqq 3$ for any $\mathbb{Q}_p$-representation V of G_K and that $\mathrm{H}^i(G_K, \mathbb{C}_p) = 0$ for $i \geqq 2$ and that $\mathrm{H}^i(G_K, V)$ and $\mathrm{H}^i(G_K, \mathbb{C}_p)$ are finite dimensional over $\mathbb{Q}_p$. □

For an almost $\mathbb{C}_p$-representation U, set $\chi(U) := \sum_{i=0}^{2}(-1)^i \cdot \dim_{\mathbb{Q}_p}\mathrm{H}^i(G_K, U)$.

LEMMA 5.3. $\chi(U) = -[K : \mathbb{Q}_p]\mathrm{ht}(U)$.

PROOF. This follows from the definition of almost $\mathbb{C}_p$-representations and the Euler-Poincaré formula for $\mathbb{Q}_p$-representations of G_K and the fact that $\chi(\mathbb{C}_p) = 0$. □

LEMMA 5.4. *Let $W = (W_e, W^+_{\mathrm{dR}})$ be a B-pair, then the following equalities hold,*

(1) $\mathrm{C}^\bullet(W_e) = \varinjlim_n \mathrm{C}^\bullet(W_e \cap \frac{1}{t^n} W^+_{\mathrm{dR}})$,

(2) $\mathrm{C}^\bullet(W_{\mathrm{dR}}) = \varinjlim_n \mathrm{C}^\bullet(G_K, \frac{1}{t^n} W^+_{\mathrm{dR}})$.

PROOF. For any n, $\frac{1}{t^n} W^+_{\mathrm{dR}}$ is closed in $\frac{1}{t^{n+1}} W^+_{\mathrm{dR}}$ and the topology on $\frac{1}{t^n} W^+_{\mathrm{dR}}$ is the topology induced from $\frac{1}{t^{n+1}} W^+_{\mathrm{dR}}$. Hence, by Proposition 5.6 of [Schn01], we obtain the equality (2). For W_e, if we fix an isomorphism $W_e \xrightarrow{\sim} \mathbf{B}_e^{\oplus d}$ as $\mathbf{B}_e$-module, the topology on W_e is defined by the direct sum topology of $\mathbf{B}_e$. Because we have an equality

$$t\mathbf{B}^{+,\varphi=p^n}_{\mathrm{max}} = \cap_{m \geqq 0} \mathrm{Ker}(\theta \circ \varphi^m : \mathbf{B}^{+,\varphi=p^{n+1}}_{\mathrm{max}} \to \mathbb{C}_p)$$

by Proposition 8.10 (2) of [Co02], $\frac{1}{t^n}\mathbf{B}^{+,\varphi=p^n}_{\mathrm{max}}$ is closed in $\frac{1}{t^{n+1}}\mathbf{B}^{+,\varphi=p^{n+1}}_{\mathrm{max}}$ and the topology on $\frac{1}{t^n}\mathbf{B}^{+,\varphi=p^n}_{\mathrm{max}}$ is the topology induced from $\frac{1}{t^{n+1}}\mathbf{B}^{+,\varphi=p^{n+1}}_{\mathrm{max}}$. Hence, by Proposition 5.6 of [Schn01], we have $\mathrm{C}^\bullet(G_K, W_e) = \varinjlim_n \mathrm{C}^\bullet(G_K, (\frac{1}{t^n}\mathbf{B}^{+,\varphi=p^n}_{\mathrm{max}})^{\oplus d}) = \varinjlim_n \mathrm{C}^\bullet(G_K, W_e \cap \frac{1}{t^n} W^+_{\mathrm{dR}})$. □

LEMMA 5.5. *Let W_{dR}^{+} be a finite free $\mathbf{B}_{\mathrm{dR}}^{+}$-module with a continuous semi-linear G_K-action. Then the canonical map $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}) \to \varprojlim_n \mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}/t^n W_{\mathrm{dR}}^{+})$ is isomorphism.*

PROOF. Because we have $\mathrm{C}^{\bullet}(G_K, W_{\mathrm{dR}}^{+}) \xrightarrow{\sim} \varprojlim_n \mathrm{C}^{\bullet}(G_K, W_{\mathrm{dR}}^{+}/t^n W_{\mathrm{dR}}^{+})$, for any $i \geqq 0$, we have the following short exact sequence

$$0 \to \mathbb{R}^1 \varprojlim_n \mathrm{H}^{i-1}(G_K, W_{\mathrm{dR}}^{+}/t^n W_{\mathrm{dR}}^{+}) \to \mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}) \to \varprojlim_n \mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}/t^n W_{\mathrm{dR}}^{+})) \to 0.$$

Because $\mathrm{H}^{i-1}(G_K, W_{\mathrm{dR}}^{+}/t^n W_{\mathrm{dR}}^{+})$ is finite dimensional over $\mathbb{Q}_p$, Mittag-Leffler condition implies that $\mathbb{R}^1 \varprojlim_n \mathrm{H}^{i-1}(G_K, W_{\mathrm{dR}}^{+}/t^n W_{\mathrm{dR}}^{+}) = 0$. The lemma follows from this. □

COROLLARY 5.6. *Let W_{dR}^{+} be as above. Let $\{h_1, h_2, \cdots, h_d\}$ be the generalized Hodge-Tate weights of $W_{\mathrm{dR}}^{+}/tW_{\mathrm{dR}}^{+}$. Let $k \geqq 1$ be any integer such that $k + h_j \geqq 0$ for any $h_j \in \mathbb{Z}$. Then the natural map $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}) \to \mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}/t^k W_{\mathrm{dR}}^{+})$ is isomorphism and $\mathrm{H}^i(G_K, t^{k+1} W_{\mathrm{dR}}^{+}) = 0$ for any i.*

PROOF. By the assumption on k, we have $\mathrm{H}^i(G_K, t^l W_{\mathrm{dR}}^{+}/t^{l+1} W_{\mathrm{dR}}^{+}) = 0$ for any $l \geqq k+1$. Then the corollary follows from Lemma 5.5. □

COROLLARY 5.7. *Let W_{dR}^{+} be as above, then $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}) = \mathrm{H}^i(G_K, W_{\mathrm{dR}}) = 0$ for $i \geqq 2$ and $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+})$ and $\mathrm{H}^i(G_K, W_{\mathrm{dR}})$ are finite dimensional over $\mathbb{Q}_p$ for $i = 0, 1$.*

PROOF. Because $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}/t^n W_{\mathrm{dR}}^{+}) = 0$ for $i \geqq 2$ and $\mathrm{H}^i(G_K, W_{\mathrm{dR}}^{+}/t^n W_{\mathrm{dR}}^{+})$ is finite dimensional for $i = 0, 1$, we obtain the corollary for W_{dR}^{+} by Lemma 5.6. We prove the corollary for W_{dR}. Because we have an isomorphism $\mathrm{C}^{\bullet}(G_K, W_{\mathrm{dR}}) \xrightarrow{\sim} \varinjlim_n \mathrm{C}^{\bullet}(G_K, \frac{1}{t^n} W_{\mathrm{dR}}^{+})$ by Lemma 5.4 (2), we obtain an isomorphism $\mathrm{H}^i(G_K, W_{\mathrm{dR}}) \xrightarrow{\sim} \varinjlim_n \mathrm{H}^i(G_K, \frac{1}{t^n} W_{\mathrm{dR}}^{+})$. Then we can show that for n large enough the natural map $\mathrm{H}^i(G_K, \frac{1}{t^{n+j}} W_{\mathrm{dR}}^{+}) \to \mathrm{H}^i(G_K, \frac{1}{t^{n+j+1}} W_{\mathrm{dR}}^{+})$ is isomorphism for any $j \geqq 0$, then the natural map $\mathrm{H}^i(G_K, \frac{1}{t^n} W_{\mathrm{dR}}^{+}) \to \mathrm{H}^i(G_K, W_{\mathrm{dR}})$ is isomorphism, the corollary for W_{dR} follows from this. □

LEMMA 5.8. *Let* $W = (W_e, W^+_{\mathrm{dR}})$ *be a* B*-pair. Then we have* $\mathrm{H}^i(G_K, W_e) = 0$ *for* $i \geqq 3$.

PROOF. Because we have $\mathrm{C}^\bullet(G_K, W_e) = \varinjlim_n \mathrm{C}^\bullet(G_K, W_e \cap \frac{1}{t^n} W^+_{\mathrm{dR}})$ by Lemma 5.4 (1), we have an isomorphism $\mathrm{H}^i(G_K, W_e) \xrightarrow{\sim} \varinjlim_n \mathrm{H}^i(G_K, W_e \cap \frac{1}{t^n} W^+_{\mathrm{dR}})$. For any n, because $W_e \cap \frac{1}{t^n} W^+_{\mathrm{dR}}$ is an almost $\mathbb{C}_p$-representation by Theorem 5.1, we have $\mathrm{H}^i(G_K, W_e \cap \frac{1}{t^n} W^+_{\mathrm{dR}}) = 0$ for $i \geqq 3$ by Lemma 5.2. The lemma follows from these facts. □

THEOREM 5.9. *Let* W *be a* B*-pair, then the following hold,*

(1) $\mathrm{H}^i(G_K, W)$ *is zero for* $i \geqq 3$ *and* $\mathrm{H}^i(G_K, W)$ *is finite dimensional over* $\mathbb{Q}_p$ *for* $i = 0, 1, 2$,

(2) *(Euler-Poincaré characteristic formula)*

$$\sum_{i=0}^{2} \dim_{\mathbb{Q}_p} (-1)^i \mathrm{H}^i(G_K, W) = -[K : \mathbb{Q}_p] \mathrm{rank}(W).$$

PROOF. We first prove that $\mathrm{H}^i(G_K, W) = 0$ for $i \geqq 3$. Because there is an exact sequence

$$\cdots \to \mathrm{H}^{i-1}(G_K, W_{dR}) \to \mathrm{H}^i(G_K, W) \to \mathrm{H}^i(G_K, W_e) \oplus \mathrm{H}^i(G_K, W^+_{\mathrm{dR}}) \to \cdots,$$

the claim follows from Corollary 5.7 and Lemma 5.8. Next we prove that $\mathrm{H}^i(G_K, W)$ is finite dimensional over $\mathbb{Q}_p$. By slope filtration theorem, it suffices to show this claim when W is pure. Let W be a B-pair pure of slope s. When $s \leqq 0$, we have the following short exact sequence,

$$0 \to W_e \cap W^+_{\mathrm{dR}} \to W_e \oplus W^+_{\mathrm{dR}} \to W_{\mathrm{dR}} \to 0$$

by Theorem 5.1 (2). Hence the natural map $\mathrm{H}^i(G_K, W_e \cap W^+_{\mathrm{dR}}) \to \mathrm{H}^i(G_K, W)$ is isomorphism. Because $W_e \cap W^+_{\mathrm{dR}}$ is an almost $\mathbb{C}_p$-representation by Theorem 5.1, $\mathrm{H}^i(G_K, W_e \cap W^+_{\mathrm{dR}})$ is finite dimensional by Lemma 5.2, which proves the claim for $s \leqq 0$. When $s > 0$, then we have the following short exact sequence

$$0 \to W_e \oplus W^+_{\mathrm{dR}} \to W_{\mathrm{dR}} \to W_{\mathrm{dR}}/(W_e + W^+_{\mathrm{dR}}) \to 0$$

by Theorem 5.1 (3). Hence we obtain a natural isomorphism $\mathrm{H}^i(G_K, W) \xrightarrow{\sim} \mathrm{H}^{i-1}(G_K, W_{\mathrm{dR}}/(W_e + W^+_{\mathrm{dR}}))$. Because $W_{\mathrm{dR}}/(W_e + W^+_{\mathrm{dR}})$ is an almost $\mathbb{C}_p$-representation by Theorem 5.1, $\mathrm{H}^{i-1}(G_K, W_{\mathrm{dR}}/(W_e + W^+_{\mathrm{dR}}))$ is finite dimensional, the claim for $s > 0$ follows from this.

Next we prove (2). For W a B-pair or an almost $\mathbb{C}_p$-representation, set $\chi(W) := \sum_{i=0}^{2}(-1)^i \dim_{\mathbb{Q}_p} \mathrm{H}^i(G_K, W)$. It suffices to show (2) when W is pure of slope s. When $s \leqq 0$, then we have $\chi(W) = \chi(X_0(W))$ by the above proof. By Lemma 5.3 and by Theorem 5.1 (2), we have equalities

$$\chi(X^0(W)) = -[K : \mathbb{Q}_p]\mathrm{ht}(W) = -[K : \mathbb{Q}_p]\mathrm{rank}(W).$$

When $s > 0$, then we have $\chi(W) = -\chi(X^1(W))$ by the above proof. By Lemma 5.3 and by Theorem 5.1 (3), we have equalities

$$\chi(X^1(W)) = -[K; \mathbb{Q}_p]\mathrm{ht}(X^1(W)) = [K : \mathbb{Q}_p]\mathrm{rank}(W),$$

which proves (2). □

Next, we define the cup product pairing for B-pairs $W := (W_e, W^+_{\mathrm{dR}})$ and $W' := (W'_e, W'^+_{\mathrm{dR}})$ as follows. First, for two continuous cochains $c \in \mathrm{C}^i(G_K, W_?)$ and $c' \in \mathrm{C}^j(G_K, W'_?)$ for $W_? = W_e, W^+_{\mathrm{dR}}, W_{\mathrm{dR}}$, we define a continuous cochain

$$c \cup c' \in \mathrm{C}^{i+j}(G_K, W_? \otimes_{\mathbf{B}_?} W'_?)$$

by

$$c \cup c'(g_1, \cdots, g_{i+j}) := c(g_1, \cdots, g_i) \otimes g_1 g_2 \cdots g_i c'(g_{i+1}, \cdots, g_{i+j})$$

where $\mathbf{B}_? = \mathbf{B}_e, \mathbf{B}^+_{\mathrm{dR}}, \mathbf{B}_{\mathrm{dR}}$ when $W_? = W_e, W^+_{\mathrm{dR}}, W_{\mathrm{dR}}$ respectively. Then, $c \cup c'$ satisfies

$$\partial^{i+j}(c \cup c') = \partial^i(c) \cup c' + (-1)^i c \cup \partial^j(c').$$

For

$$c = (c_e, c^+_{\mathrm{dR}}, c_{\mathrm{dR}}) \in \mathrm{C}^i(G_K, W_e) \oplus \mathrm{C}^i(G_K, W^+_{\mathrm{dR}}) \oplus \mathrm{C}^{i-1}(G_K, W_{\mathrm{dR}})$$

and

$$c' = (c'_e, c'^+_{\mathrm{dR}}, c'_{\mathrm{dR}}) \in \mathrm{C}^j(G_K, W'_e) \oplus \mathrm{C}^j(G_K, W'^+_{\mathrm{dR}}) \oplus \mathrm{C}^{j-1}(G_K, W'_{\mathrm{dR}}),$$

and for a parameter $\gamma \in \mathbb{Q}_p$, we define

$$c \cup_\gamma c' \in \mathrm{C}^{i+j}(G_K, W_e \otimes_{\mathbf{B}_e} W'_e) \oplus \mathrm{C}^{i+j}(G_K, W^+_{\mathrm{dR}} \otimes_{\mathbf{B}^+_{\mathrm{dR}}} W'^+_{\mathrm{dR}}) \oplus \mathrm{C}^{i+j-1}(G_K, W_{\mathrm{dR}} \otimes_{\mathbf{B}_{\mathrm{dR}}} W'_{\mathrm{dR}})$$

by

$$c \cup_\gamma c' := (c_e \cup c'_e, c^+_{\mathrm{dR}} \cup c'^+_{\mathrm{dR}}, c_{\mathrm{dR}} \cup (\gamma c'_e + (1-\gamma) c'^+_{\mathrm{dR}}) + (-1)^i((1-\gamma)c_e + \gamma c^+_{\mathrm{dR}}) \cup c'_{\mathrm{dR}})).$$

Then, we can check that if $\partial^i(c) = \partial^j(c') = 0$ then $\partial^{i+j}(c \cup_\gamma c') = 0$, and if $\partial^i c = 0$ and $c' = \partial^{j-1}(c'')$ (or $c = \partial^{i-1}(c'')$ and $\partial^j(c') = 0$) then $c \cup_\gamma c' \in \mathrm{Im}(\partial^{i+j-1})$. Therefore, this paring induces a $\mathbb{Q}_p$-bi-linear paring

$$\cup_\gamma : \mathrm{H}^i(G_K, W) \times \mathrm{H}^j(G_K, W') \to \mathrm{H}^{i+j}(G_K, W \otimes W').$$

Moreover, we can check that $\cup_\gamma$ doesn't depend on the choice of a parameter γ, so we just write $\cup$ instead of $\cup_\gamma$.

We define the paring

$$\cup : \mathrm{H}^i(G_K, W) \times \mathrm{H}^{2-i}(G_K, W^\vee(\chi_p)) \to \mathbb{Q}_p$$

as the composition the following maps

$$\mathrm{H}^i(G_K, W) \times \mathrm{H}^{2-i}(G_K, W^\vee(\chi_p)) \xrightarrow{\cup} \mathrm{H}^2(G_K, W \otimes W^\vee(\chi_p)) \to \mathrm{H}^2(G_K, W(\mathbb{Q}_p(\chi_p))) \xrightarrow{\sim} \mathrm{H}^2(G_K, \mathbb{Q}_p(\chi_p)) \xrightarrow{\sim} \mathbb{Q}_p,$$

where the second map is induced from the evaluation map $W \otimes W^\vee(\chi_p) \to W(\mathbb{Q}_p(\chi_p))$ and the third isomorphism is the natural comparison isomorphism and the fourth isomorphism is Tate's trace map. The Tate duality theorem for B-pairs is following.

Theorem 5.10. *For $i = 0, 1, 2$, the paring*

$$\cup : \mathrm{H}^i(G_K, W) \times \mathrm{H}^{2-i}(G_K, W^\vee(\chi_p)) \to \mathbb{Q}_p$$

is a perfect paring.

PROOF. We can prove this theorem in the same way as in the proof of Theorem 4.7 of [Li08] if we use the Euler-Poincaré formula Theorem 5.9 and the facts that $\mathrm{H}^0(G_K, W(\prod_{\sigma\in\mathcal{P}}\sigma)) = 0$ and $\mathrm{H}^0(G_K, W(|\prod_{\sigma\in\mathcal{P}}\sigma|)) = 0$, which are proved in Proposition 2.10. □

Finally, we prove that our continuous cohomology is canonically isomorphic to Liu's cohomology. We first define an isomorphism between H^0 by the functoriality

$$\mathrm{H}^0_{\mathrm{Liu}}(G_K, W) \xrightarrow{\sim} \mathrm{Hom}_{\varphi,\Gamma}(R, D(W)) \xrightarrow{\sim} \mathrm{Hom}(W(\mathbb{Q}_p), W) \xrightarrow{\sim} \mathrm{H}^0(G_K, W),$$

where $D(W)$ is the (φ,Γ)-module associated to D and R is the trivial (φ,Γ)-module, where the second isomorphism follows from the equivalence of categories between B-pairs and (φ,Γ)-modules.

THEOREM 5.11. *The above isomorphism* $\mathrm{H}^0_{\mathrm{Liu}}(G_K, W) \xrightarrow{\sim} \mathrm{H}^0(G_K, W)$ *extends uniquely to an isomorphism of* δ*-functors* $\mathrm{H}^i_{\mathrm{Liu}}(G_K, W) \xrightarrow{\sim} \mathrm{H}^i(G_K, W)$.

PROOF. This follows from weakly effaceabilities of functors $\mathrm{H}^i_{\mathrm{Liu}}(G_K, -)$ and $\mathrm{H}^i(G_K, -)$. For $\mathrm{H}^i_{\mathrm{Liu}}$, these facts are proved in the proof of Theorem 8.1 of [Ke09]. For $\mathrm{H}^i(G_K, -)$, we can also prove in the same way as in Theorem 8.1 of [Ke09] because we have already proved the Euler-Poincaré formula and the Tate duality for $\mathrm{H}^{\mathrm{i}}(G_K, -)$ and we have a natural isomorphism $\mathrm{H}^1(G_K, W) \xrightarrow{\sim} \mathrm{Ext}^1(W(\mathbb{Q}_p), W)$ which is proved in Proposition 2.2 of [Na09]. □

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