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Ricci Flow on Open Surface

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Abstract. In this note, we study the normalized Ricci flow with incomplete initial metric. By an approximation method initiated by Giesen and Topping very recently, we show such flow with suitable initial value always converges exponentially to a metric with constant Gaussian curvature. If moreover the initial metric is complete, the flow converges to the hyperbolic metric. Applications of Ricci flow to uniformization of Riemann surfaces are also considered.

1. Introduction

In [6], Hamilton first studied the following equation on a closed manifold M, which we refer to as Hamilton's normalized Ricci flow

$$\frac{\partial g}{\partial t}(t,x) = \frac{2}{n} r g(t,x) - 2Ric_{g_t}(x) \tag{1.1}$$

where n is the dimension of M, g is the metric, Ric_g is the Ricci tensor of g and r is the average of the scalar curvature R. He proved that if the manifold is closed, then (1.1) is always solvable for any initial smooth metric g_0. For compact surfaces, the behavior of such flow is well understood by Hamilton ([7]) and Chow ([3]). In fact, they showed the normalized Ricci flow with any initial metric will converge to a constant Gaussian curvature metric. It suggests us to study the same problem for non-compact manifolds, especially non-compact surfaces. In the case of complete manifolds with bounded curvature, Shi ([10]) proved that a complete Ricci flow $g(t)$ exists for $t \in [0, T]$, for some $T > 0$, depending on the bounds of curvature. When the manifold is of dimension two, Ji, Mazzeo and Sesum ([9]) followed by Yin ([13]) generalized these results to hyperbolic cusp surfaces and nonparabolic surfaces with additional reasonable conditions.

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The above all achievements concentrate on non-compact complete manifolds. More general situation with incomplete initial metric seems to be much more complicated. One of main obstructions comes from the absence of the maximum principle for incomplete manifolds. As remarked in [10], we have in general no short-term existence for the Ricci flow with an arbitrary initial metric g_0, even g_0 is complete.

In this paper, we will study the normalized Ricci flows with incomplete initial metrics on surfaces. For the incompleteness of initial metric g_0, we can not use the same techniques as in [9] and [13] where the authors related the question to certain Poisson equation and then solved the equation. Thus we should find other ways to tackle it.

Recently, Giesen and Topping ([5]) investigated the Ricci flow of negatively curved incomplete surfaces. They showed such a flow exists for all time. This suggests that its normalized form should be convergent. We will verify this. In other words, we will demonstrate how the initial negatively curved metric converges to a constant Gaussian curvature metric along the normalized Ricci flow.

THEOREM 1.1. *Suppose M is any Riemann surface equipped with a smooth conformal metric g_0. If its Gaussian curvature satisfies $K_{g_0} \leq -1$, then the normalized Ricci flow (2.1) converges exponentially to a conformal metric with Gaussian curvature -1. In particular, if moreover g_0 is complete, the flow converges to the standard hyperbolic metric.*

There is another motivation to study the Ricci flow on non-compact surfaces. Recall that as pointed out in [2], the normalized Ricci flow can be given an independent proof of the uniformization theorem for compact Riemann surfaces. It is therefore natural to ask if such approach goes through well for non-compact Riemann surfaces. The first result in this direction is obtained in [9] and [13]. They showed it indeed holds for hyperbolic surfaces of finite type.

THEOREM 1.2. *Let M be a Riemann surface obtained from compact Riemann surface by removing finitely many points and/or disjoint disks. If no disk is removed, then we further assume its Euler characteristic is negative. Then M must be hyperbolic.*

Here a hyperbolic surface means a smooth surface with a complete metric of Gaussian curvature -1. So in order to prove the uniformization theorem, we must construct a complete metric of constant Gaussian curvature. In the proof of above theorem, boundary properties of surfaces of finite type are used to construct a good initial metric and then solve the normalized Ricci flow. But for general surfaces, the boundary property may be very bad and can not be used. If the surface is a sub-domain of a punctured and bordered Riemann surface, then the uniformization theorem remains true from above theorem.

COROLLARY 1.3. *If $\Omega \subset M$ is sub-domain of a punctured and bordered Riemann surface with $\chi(M) < 0$, then it is hyperbolic.*

For hyperbolic planar domains, we also have uniformization theorem. Here a domain $\Omega \subset \mathbb{C}$ is hyperbolic if its complement contains at least two points.

COROLLARY 1.4. *Any hyperbolic planar domain Ω is covered by the unit disc $\mathbb{D}$. In particular, any simply connected planar hyperbolic domain is biholomorphic to the unit disc $\mathbb{D}$.*

The second part of Corollary 1.4 is exactly the content of the classical Riemann mapping theorem ([1]).

The paper is organized as follows. In Section 2, we review some facts about Ricci flow on surfaces. In Section 3, we discuss the normalized Ricci flow with negatively curved initial metric. All results above will also be proved there.

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2. Ricci Flow on Surfaces

In this section, when referring a surface, we mean to be a Riemann surface without boundary. While a bordered surface denotes the interior of a Riemann surface with non-empty boundary.

Let (M, g_0) be any surface equipped with a smooth conformal Riemannian metric g_0. Consider the normalized Ricci flow

$$\begin{cases} \partial_t g(t) = (r - R(t))g(t) \\ g(0) = g_0 \end{cases} \tag{2.1}$$

where $R(t)$ is the scalar curvature of $g(t)$ and r is a constant. When M is compact and r is the average of the scalar curvature, Hamilton([7]) and Chow([3]) proved it exists for all time and converges to the unique Kahler-Einstein metric with any initial metric. Recently, Ji, Mazzeo and Sesum ([9]) and Yin ([13]) generalized the above result to certain complete surfaces. The basic idea in their proofs is to solve the Poisson equation $\Delta u = R - r$ which is initiated by Hamilton. This seems to be impossible for general surfaces. To overcome such a difficulty, we first construct a sequence of complete Ricci flows with initial complete metrics approximating g_0. Then we show they converge to a solution of equation (2.1). Without loss of generality, we choose $r = -2$ through the whole paper.

On a surface M, the scalar curvature of any conformal metric $g = e^{2u}|dz|^2$ takes the simple form $R = 2K_g = -2e^{-2u}\Delta u$, where $\Delta = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}$ is the Laplacian operator and K_g is the Gaussian curvature of g. Thus the Ricci flow which coincides with the Yamabe flow preserves the conformal class of the initial metric. We can write $g(t) = e^{2u(t)}g_0$, $u(0) = 0$ and reformulate (2.1) as following:

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = e^{-2u(t)}\Delta_0 u(t) - 1 - e^{-2u(t)}K_0 \\ u(0) = 0 \end{cases} \tag{2.2}$$

where a subscript 0 indicates that the relevant quantity is associated to g_0.

The next lemma is well-known ([7]).

LEMMA 2.1. *If $g(t)$ is a solution to (2.1), then its Gaussian curvature $K(t) = K_{g(t)}$ evolves by the equation*

$$\partial_t K = \Delta_t K + 2K^2 + 2K.$$

3. Proof of Main Results

We come to the proofs of main results.

Proof of Theorem 1.1. Thanks to Theorem 1.2, we may construct a sequence of hyperbolic sub-surfaces to exhaust an arbitrary Riemann surface. Then we solve normalized Ricci flow (2.1) on each sub-surface and get a sequence of $g_i(t)$. The final step is to verify $g_i(t)$ converges to a smooth flow $g_\infty(t)$ which satisfies (2.1).

We may assume M is non-compact. By an elementary fact in surface topology ([8]), we know that there exists a sequence M_1, M_2, M_3, $\cdots$ of compact bordered surfaces contained in M such that

(1) $\overline{M_k} \subset M_{k+1}$,

(2) $M = \bigcup_{k=1}^{\infty} M_k$.

Hence each M_k admits a complete conformal metric g_k of Gaussian curvature $-k^2$. Let $u_k \in C^\infty(M_k)$ be the unique function such that $g_k = e^{2u_k} g_0$. Then the following Schwarz-Yau lemma enables us to compare these u_k's(see [14]).

Lemma 3.1. *Let (X, h_1) be a Riemann surface with a complete conformal metric h_1 whose Gaussian curvature K_{h_1} has lower bound $-a_1 \leq 0$. Let (Y, h_2) be another Riemann surface with a conformal metric h_2 of Gaussian curvature $K_{h_2} \leq -a_2 < 0$. Then for any conformal mapping $f : X \to Y$, we have*

$$f^* h_2 \leq \frac{a_1}{a_2} h_1.$$

Setting $X = M_k$, $Y = M_{k+1}$ and f=the inclusion map. We find that $(k+1)^2 g_{k+1} \leq k^2 g_k$ which is equivalent to $u_k - u_{k+1} \geq \ln \frac{k+1}{k} > 0$. So u_k is pointwisely decreasing as $k \to \infty$. Similarly, if we set $X = M_k$, $Y = M_{k+l}$ and f again the inclusion map, we get $u_{k+l} \leq u_k - \ln \frac{k+l}{k}$. Particularly, this implies for any fixed k and $x \in M_k$, $\lim_{l\to\infty} u_{k+l}(x) = -\infty$.

For every $\epsilon > 0$, we define a smooth function $\Psi_\epsilon : \mathbb{R} \to \mathbb{R}$ with the properties that $\Psi_\epsilon(s) = 0$ for $s \leq -\epsilon$, $\Psi_\epsilon(s) = s$ for $s \geq \epsilon$ and $\Psi_\epsilon''(s) \geq 0$ for all $s \in \mathbb{R}$. So $0 \le \Psi'(s) \le 1$ and $\Psi(s) \ge s$ for all $s \in \mathbb{R}$. We use this function to define new metrics on M_k by

$$\tilde{g}_k = e^{2\Psi_\epsilon(u_k)} g_0.$$

LEMMA 3.2. *$\tilde{g}_k \ge g_k$ and $\lim_{k\to\infty} \tilde{g}_k(x) = g_0(x)$. Thus $\tilde{g}_k$ is a complete metric on M_k. Furthermore*

$$-C(k) \le K_{\tilde{g}_k} \le -e^{-2\epsilon},$$

for suitable $C(k) > 0$.

PROOF. The first assertion follows easily from the facts $\Psi(s) \ge s$ and $\lim_{l\to\infty} u_{k+l}(x) = -\infty$ for $x \in M_k$. For the second assertion, we note that $u_k(x) \to +\infty$ as $x \to \partial M_k$ for the completeness of g_k. On the other hand, by definition of Gaussian curvature we have

$$\begin{aligned} K_{\tilde{g}_k} &= -e^{-2\Psi_\epsilon(u_k)-2u_0} \Delta\{\Psi_\epsilon(u_k) + u_0\} \\ &= -e^{-2\Psi_\epsilon(u_k)-2u_0} \{\Psi''_\epsilon(u_k)|\partial_z u_k|^2 + \Psi'_\epsilon(u_k)\Delta u_k + \Delta u_0\}. \end{aligned}$$

When $x \in M_k$ such that $u_k(x) \ge \epsilon$, then $\Psi_\epsilon(u_k(x)) = u_k(x)$, $\Psi'_\epsilon(u_k(x)) = 1$ and $\Psi''_\epsilon(u_k(x)) = 0$. So $K_{\tilde{g}_k}(x) = K_{g_k}(x) = -k^2$. When x lies in the sub-level set $\{x \in M_k : u_k(x) \le \epsilon\}$, we have obviously uniform lower bound (possibly depending on k) for $K_{\tilde{g}_k}$ for the compactness reason. In any cases, $K_{\tilde{g}_k}$ has a k-dependent lower bound on M_k.

We now argue its upper bound. Because $\Psi''_\epsilon \ge 0$ and $0 \le \Psi'_\epsilon \le 1$, by above computations

$$\begin{aligned} K_{\tilde{g}_k} &= -e^{-2\Psi_\epsilon(u_k)-2u_0} \{\Psi''_\epsilon(u_k)|\partial_z u_k|^2 + \Psi'_\epsilon(u_k)\Delta u_k + \Delta u_0\} \\ &\le -e^{-2\Psi_\epsilon(u_k)-2u_0} \{\Psi'_\epsilon(u_k)\Delta(u_k + u_0) + (1 - \Psi'_\epsilon(u_k))\Delta u_0\} \\ &= e^{-2\Psi_\epsilon(u_k)-2u_0} \{\Psi'_\epsilon(u_k) e^{2u_k+2u_0} K_{g_k} + (1 - \Psi'_\epsilon(u_k)) e^{2u_0} K_{g_0}\} \\ &= e^{-2(\Psi_\epsilon(u_k)-u_k)} \Psi'_\epsilon(u_k) K_{g_k} + (1 - \Psi'_\epsilon(u_k)) e^{-2\Psi_\epsilon(u_k)} K_{g_0} \\ &\le -e^{-2(\Psi_\epsilon(u_k)-u_k)} - (1 - \Psi'_\epsilon(u_k)) e^{-2\Psi_\epsilon(u_k)} \\ &\le -e^{-2\epsilon}. \end{aligned}$$

In the last second line we have used that $K_{g_k} = -k^2 \le -1$ and $K_{g_0} \le -1$. In the last line we have used that $\Psi_\epsilon(s) - s \le \epsilon$ when $\Psi'_\epsilon(s) \ne 0$ and $\Psi_\epsilon(s) \le \epsilon$ when $\Psi'_\epsilon(s) \ne 1$. □

Up to now, we have constructed a sequence of complete conformal metrics g_k with bounded Gaussian curvatures on M_k. Let $g_k(t)$ be the Ricci flow with initial value $g_k(0) = \tilde{g}_k$. From [10], there is a maximal existence interval $[0, T_k)$ for each flow, where $T_k > 0$ depends only on k and ϵ. Furthermore, these flows are complete with bounded curvatures. Apply the ODE-PDE comparison principle (see [4]) to the equation

$$\partial_t K = \Delta_t K + 2K^2 + 2K.$$

It yields for $0 < t < T_k$

$$-\frac{1}{e^{2t}-1} \leq K_{g_k(t)} + 1 \leq \frac{e^{2\epsilon}-1}{e^{2\epsilon}+e^{2t}-1}.$$

Integrating the flow equation gives bounds for the metric $g_k(t)$. All higher derivatives of R can be obtained as well by considering lemma 2.1. We have showed the maximal existence interval is $(0, \infty)$. Furthermore, above inequalities imply $K_{g_k(t)} \to -1$ as $k \to \infty$ exponentially and consequently the metric $g_k(t)$ converges to a metric $g_k(\infty)$ of Gaussian curvature -1.

Set $u_k(t, x) \in C^\infty(\mathbb{R}_+ \times M_k)$ to be the conformal factor of $g_k(t)$ with respect to g_0 so that $g_k(t, x) = e^{2u_k(t,x)} g_0(x)$. By the curvature equation we have

$$\frac{\partial u_k(t)}{\partial t} = -K_{g_k(t)} \geq \frac{e^{2t}}{e^{2\epsilon}+e^{2t}-1} > 0,$$

which implies $u_k(t, x) \geq u_k(0, x) = \Psi_\epsilon(u_k) \geq 0$. By taking the limit $k \to \infty$, $g_k(\infty) \geq g_0$.

Lemma 3.3. *$g_\infty = \lim_{k\to\infty} g_k(\infty)$ exists and $g_\infty \geq g_0$.*

Proof. It suffices to verify $g_k(\infty)$ is decreasing in k. We deduce it from a comparison result similar to [5]. In fact, we will prove $u_{k+1}(t) \leq u_k(t)$ for all t. Set

$$u_{k,\varepsilon}(t, x) = u_k\left(\frac{1}{\varepsilon}\ln(\varepsilon t + 1), x\right) + \frac{1}{2}\ln(\varepsilon t + 1).$$

An easy computation shows that $u_{k,\varepsilon}(0,x) = u_k(x)$ and

$$\begin{aligned}
&\{\partial_t u_{k,\varepsilon} - e^{-2u_{k,\varepsilon}}\Delta u_{k,\varepsilon} + 1 + e^{-2u_{k,\varepsilon}}K_0\}(t,x) \qquad (*)\\
=&\frac{1}{\varepsilon t+1}\{\partial_t u_k - e^{-2u_k}\Delta u_k + 1 + e^{-2u_k}K_0\}(\frac{1}{\varepsilon}\ln(\varepsilon t+1),x)\\
&+\frac{\varepsilon}{2(\varepsilon t+1)} + 1 - \frac{1}{\varepsilon t+1}\\
\geq&\frac{\varepsilon}{2(\epsilon t+1)}\\
>&0.
\end{aligned}$$

We assume $u_{k,\varepsilon}(t,x) < u_{k+1}(t,x)$ at somewhere (t,x) in $[0,\infty)\times M_k$. Since g_k is complete on M_k, for every time $t\in[0,\infty)$ we have

$$u_{k,\varepsilon}(t,x) - u_{k+1}(t,x) \to +\infty \ \text{ as } \ x\to\partial M_k.$$

Hence $u_{k,\epsilon}(t,\cdot) - u_{k+1}(t,\cdot)$ attains its infimum in M_k. Let $(t_0,x_0)\in[0,\infty)\times M_k$ be one of the points at which $u_{k,\varepsilon}-u_{k+1}$ first becomes negative. Maximal principle implies the followings:

$$\begin{aligned}
&u_{k,\varepsilon}(t_0,x_0) = u_{k+1}(t_0,x_0), \quad \Delta(u_{k,\varepsilon}-u_{k+1})(t_0,x_0)\geq 0,\\
&\partial_t(u_{k,\varepsilon}-u_{k+1})(t_0,x_0)\leq 0.
\end{aligned}$$

At this point (t_0,x_0), we subtracting the normalized Ricci flow equation (2.2) from $(*)$ and get

$$\begin{aligned}
0 &< \partial_t u_{k,\varepsilon} - e^{-2u_{k,\varepsilon}}\Delta u_{k,\varepsilon} + e^{-2u_{k,\varepsilon}}K_0\\
&\quad - \{\partial_t u_{k+1} - e^{-2u_{k+1}}\Delta u_{k+1} + e^{-2u_{k+1}}K_0\}\\
&= \partial_t(u_{k,\varepsilon}-u_{k+1}) - e^{-2u_{k+1}}\Delta(u_k - u_{k+1})\\
&\leq 0,
\end{aligned}$$

which is a contradiction. This proves our previous assertion. □

From above lemma, we find $g_\infty(t) = \lim_{k\to\infty} g_k(t)$ exists. It also satisfies (2.1) by parabolic regularity theory. Its limit metric $g_\infty(\infty)$ is defined on the whole surface M and larger than g_0. By standard regularity theory again we see $g_\infty(\infty)$ has constant Gaussian curvature -1. This is just what we want. □

PROOF OF COROLLARY 1.3. We construct a sequence of bordered Riemann surfaces Ω_k as above. By Theorem 1.2, we may find a hyperbolic metric g_k by Ricci flow on each Ω_k. Let g_0 be the restriction of the hyperbolic metric on M. Write $g_k = e^{2u_k} g_0$. The Schwarz-Yau lemma implies $u_k \geq u_{k+1}$ on Ω_k and $u_k \geq 0$. Set $u(x) = \lim_{k\to\infty} u_k(x)$. Then it is a well-defined function on Ω. Since each u_k satisfies the curvature equation, a standard regularity for elliptic operator shows u satisfies the same equation and consequently $g(x) = e^{2u} g_0$ has Gaussian curvature -1 on Ω. For the completeness of g, we apply Schwarz-Yau lemma once more. We use absurdity discussion. Let $\gamma : [0, T) \to \Omega$ be a maximal geodesic with unit tangent vector as g and $T < \infty$. For $g \geq g_0$, it has finite length for g_0. Thus we may find a point $p \in \partial\Omega$ such that $\gamma(t) \to p$ as $t \to T$. Since $\chi(M) < 0$, the open surface $M - \{p\}$ by Theorem 1.2 has also a hyperbolic metric h. Consider the inclusion map

$$i_k : (\Omega_k, g_k) \hookrightarrow (M - \{p\}, h).$$

We see that $g_k \geq h$ on Ω_k for all k and hence $g \geq h$. This implies γ has infinite length for g. □

Proof of Corollary 1.4. We note that by Theorem 1.2 $\mathbb{C} - \{p, q\} = S^2 - \{\infty, p, q\}$ is hyperbolic. The remainder is exactly the same as above. □

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