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Regularity and Asymptotic Behavior for the Keller-Segel System of Degenerate Type with Critical Nonlinearity

By Masashi MIZUNO* and Takayoshi OGAWA†

Abstract. We discuss the large time behavior of a weak solution of the Keller-Segel system of degenerate type:

$$\begin{cases} \partial_t u - \Delta u^\alpha + \operatorname{div}(u\nabla\psi) = 0, & t>0\,,\, x\in\mathbb{R}^n, \\ \quad -\Delta\psi + \psi = u, & t>0\,,\, x\in\mathbb{R}^n, \\ \qquad u(0,x) = u_0(x) \geq 0, & \qquad x\in\mathbb{R}^n, \end{cases}$$

where $\alpha > 1$. It is known when the exponent $\alpha = 2 - \frac{2}{n}$, then the problem shows the critical situation. In this case, we show that the small data global solution decays and its asymptotic profile converges to the Barenblatt-Pattle solution $\mathcal{U}(t) = (1+t)^{-n/\sigma}\big(A - |x|^2/(1+t)^{1/\sigma}\big)_+^{1/(\alpha-1)}$ in L^1 such as

$$\|u(t) - \mathcal{U}(t)\|_1 \leq C(1+t)^{-\nu},$$

where $\nu > 0$ is depending on n and the regularity of the solution. To show this, we employ the forward self-similar transform and use the entropy dissipation term to derive the asymptotic profile due to Carrillo-Toscani [12] and Ogawa [47]. The Hölder continuity of the weak solution for the forward self-similar equation plays a crucial role. We derive the uniform Hölder continuity by using the rescaled alternative selection originated by DiBenedetto-Friedman [18, 19].

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1. Introduction

We consider a large time behavior of the global solution of the degenerate parabolic elliptic system:

$$
\begin{cases} \partial_t u - \Delta u^\alpha + \operatorname{div}(u\nabla\psi) = 0, & t > 0\,,\ x \in \mathbb{R}^n, \\ -\Delta\psi + \psi = u, & t > 0\,,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) \ge 0, & x \in \mathbb{R}^n, \end{cases} \tag{1.1}
$$

where $\alpha > 1$. This system describes the dynamics of the chemical attracted mold. The system originally consists of two reaction diffusion equations. By taking the zero relaxation time limit, one can obtain the above form as the result. For the case of $\alpha = 1$, it is a semi-linear problem and the system (1.1) is analyzed by many authors. For $\alpha > 1$, the problem (1.1) is a degenerated parabolic elliptic system and there are some work on it ([1], [2], [7], [8], [14], [15], [34], [47] ,[48], [54], [55]). On the other hand, the system has a strong relation with the variational structure and the large time behavior of the solution is really depending on the variational functional reduced from the entropy-energy inequality.

$$
W[u](t) := \frac{1}{\alpha-1}\|u(t)\|_\alpha^\alpha - \frac{1}{2}\int_{\mathbb{R}^n} u(t)\psi(t)\,dx \le W[u_0].
$$

Then it appears that there exists a critical exponent $\alpha = 2-\frac{2}{n}$ that the global behavior of the solution is changed. This exponent is considered a threshold exponent to separate the global asymptotic behavior of the weak solution. Roughly speaking, the small solution with small initial data decays as $t \to \infty$. Then the main concern for this case is its asymptotic profile. By the self-similar rescaling, one may find that there appears some particular profile in its rescaled form. On the other hand, the equation is degenerated and it has some hyperbolic like feature in its weak solution when the solution meets 0. In this case, the regularity breaks down and the behavior is governed by the hyperbolic like structure. The best possible regularity for the weak solution is generally known as the Hölder continuity. Indeed, to show the asymptotic profile of the decaying solution, the regularity of the weak solution plays an important role.

In this paper, we consider the regularity problem of the system (1.1) and apply it for the large time asymptotic behavior of the decaying solution in the critical case and show its convergence rate for the asymptotic profile if it is rescaled in the self-similar way. Since the equation is degenerated, the smoothness of the solution is not generally guaranteed and we necessarily consider the weak solution.

DEFINITION. For a nonnegative initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$, we call $(u\,,\,\psi)$ a weak solution of the system (1.1) if there exists $T > 0$ such that

1. $u(t,x) \geq 0$ for almost all $(t,x) \in [0,T) \times \mathbb{R}^n$,
2. $u \in L^\infty(0,T\,;\,L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n))$ with $\nabla u^\alpha \in L^2((0,T) \times \mathbb{R}^n)$, and
3. u satisfies the first equation of (1.1) in the sense of distribution, namely for any $\phi \in C^\infty([0,T]\,;C_0^\infty)$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n} u(t)\phi(t)\,dx - \int_{\mathbb{R}^n} u_0\phi(0)\,dx \\ &\quad = \int_0^t d\tau \int_{\mathbb{R}^n} \Big\{ u(\tau)\partial_t\phi(\tau) - \nabla u^\alpha(\tau)\cdot\nabla\phi(\tau) \\ &\qquad\qquad + u(\tau)\nabla\psi(\tau)\cdot\nabla\phi(\tau) \Big\}\,dx \end{aligned} \tag{1.2}$$

for almost all $0 < t < T$, where $\psi = (-\Delta + 1)^{-1}u$ is given by the Bessel potential.

We may obtain the time local weak solution of (1.1) by some approximating procedure. Then the existence of time global weak solution is classified by a threshold exponent $\alpha = 2 - \frac{2}{n}$. We summarize the known results for the existence and non-existence of time global weak solutions.

PROPOSITION 1.1 ([1], [2], [7], [8], [47], [48], [54], [55]). *Let $n \geq 3$, $\alpha > 1$ and we assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$. Then there exists a weak solution (u,ψ) of (1.1) that satisfies for $0 < t < T$,*

$$\|u(t)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)},$$

$$W(t)+\int_0^t\int_{\mathbb{R}^n} u(t)\left|\nabla\frac{\alpha}{\alpha-1}u^{\alpha-1}-\nabla\psi\right|^2 dtdx\le W(0), \tag{1.3}$$

where

$$W(t)=\frac{1}{\alpha-1}\|u\|^{\alpha}_{L^{\alpha}(\mathbb{R}^n)}-\frac{1}{2}\|\Lambda^{-1}u(t)\|^2_{L^2(\mathbb{R}^n)}$$

with $\Lambda=(-\Delta+1)^{\frac{1}{2}}$. *In addition:*

(1) *if* $\alpha>2-\frac{2}{n}$, *then, for any initial data* u_0 *the solution exists globally in time and the solution is uniformly bounded.*

(2) *If* $1<\alpha\le 2-\frac{2}{n}$ *and the initial data satisfying* $W(0)>0$ *with*

$$\|u_0\|^{1-\gamma}_{L^1(\mathbb{R}^n)}W(0)^{\frac{\gamma-\alpha+1}{\alpha}}<CC^{-1}_{\mathrm{HLS}}, \tag{1.4}$$

then the weak solution exists globally in time, where $C>0$ *is some constant depending only on* n,α *and* C_{HLS} *is a best possible constant of the Hardy-Littlewood-Sobolev inequality on* $\mathbb{R}^n$.

(3) *In particular, if* $\alpha=2-\frac{2}{n}$ *the above condition* (1.4) *is given by*

$$\|u_0\|^{\frac{2}{n}}_{L^1(\mathbb{R}^n)}<\frac{2n}{n-2}C^{-1}_{\mathrm{HLS}}. \tag{1.5}$$

(4) *If* $1<\alpha\le 2-\frac{2}{n}$ *and the initial data* $u_0\in L^1(\mathbb{R}^n)\cap L^{\alpha}(\mathbb{R}^n)$ *with* $|x|^2u_0\in L^1(\mathbb{R}^n)$ *satisfies* $W(0)<0$, *then the weak solution blows up in a finite time* T *in the following sense:*

$$\limsup_{t\to T}\|u(t)\|_{L^q(\mathbb{R}^n)}=\infty \quad \textit{for all } \alpha\le q\le\infty.$$

By Proposition 1.1, the weak solution to (1.1) exists globally in time when $n\ge 3$, $1<\alpha\le 2-\frac{2}{n}$ and the initial data is sufficiently small. When we consider the small data problem, then the system can be regarded as the perturbed problem from the porous medium equation:

$$\begin{cases} \partial_t w-\Delta w^{\alpha}=0, \quad t>0\,,\, x\in\mathbb{R}^n,\\ w(0,x)=w_0(x), \quad x\in\mathbb{R}^n. \end{cases} \tag{1.6}$$

For the porous medium equation, there exists an explicit solution called the Barenblatt-Pattle solution.

DEFINITION (the Barenblatt-Pattle solution). For $\alpha > 1$, we set $\sigma = n(\alpha-1)+2$. For some $A > 0$, the function $\mathcal{U}(t)$ defined by

$$\mathcal{U}(t,x) = (1+\sigma t)^{-\frac{n}{\sigma}} \left(A - \frac{\alpha-1}{2\alpha} \frac{|x|^2}{(1+\sigma t)^{\frac{2}{\sigma}}} \right)_+^{\frac{1}{\alpha-1}} \tag{1.7}$$

is called as the Barenblatt-Pattle solution, where $(f)_+ = \max\{f, 0\}$.

It is well known that the Barenblatt-Pattle solution (1.7) solves the porous medium equation (1.6) with the initial data $w_0(x) = \big(A - \frac{\alpha-2}{2\alpha}|x|^2\big)_+^{1/(\alpha-1)}$.

In the case $\alpha \le 2 - \frac{2}{n}$ and the initial data u_0 is small, then we may regard the nonlinear term $\operatorname{div}(u\nabla\psi)$ in (1.1) as a small perturbation and we speculate that the solution of (1.1) asymptotically converges to the solution of the porous medium equation. In fact, Luckhaus-Sugiyama [34] showed the asymptotic convergence of the solution in L^p spaces for $1 < \alpha \le 2 - \frac{2}{n}$, $n \ge 3$ and $1 \le p \le \infty$. Ogawa [48] showed that if $1 < \alpha < 2 - \frac{2}{n}$, then we obtain the algebraic convergence rate of the solution in L^1 space via the argument due to Carrillo-Toscani [12] and the critical Sobolev type inequality [49]. Namely, for $1 < \alpha < 2 - \frac{2}{n}$ and $W(0) > 0$ with (1.4) then there exist constants $\nu > 0$ and $C > 0$ such that

$$\|u(t) - \mathcal{U}(t)\|_{L^1(\mathbb{R}^n)} \le C(1+t)^{-\nu}, \quad t > 0, \tag{1.8}$$

where $\mathcal{U}$ is the Barenblatt-Pattle solution with $\|\mathcal{U}\|_1 = \|u_0\|_1$.

In this paper, we show the same asymptotic convergence in $L^1(\mathbb{R}^n)$ for the critical case $\alpha = 2 - \frac{2}{n}$. Our main theorem is the following:

THEOREM 1.2. *Let $\alpha = 2 - \frac{2}{n}$ and $n \ge 3$. Assume that $u_0 \in L^1(\mathbb{R}^n) \cap L^\alpha(\mathbb{R}^n)$ satisfying $W(0) > 0$, (1.4), $\|u_0\|_1 < 2$ and $|x|^a u_0 \in L^1(\mathbb{R}^n)$ for some $a > n$. Then, there exist $C > 0$ and $\nu > 0$ such that the corresponding global weak solution u of (1.1) satisfies*

$$\|u(t) - \mathcal{U}(t)\|_{L^1(\mathbb{R}^n)} \le C(1+t)^{-\nu}, \quad t > 0, \tag{1.9}$$

where $\mathcal{U}$ *is the Barenblatt-Pattle solution with* $\|\mathcal{U}(0)\|_{L^1(\mathbb{R}^n)} = \|u_0\|_{L^1(\mathbb{R}^n)}$.

To show the asymptotic behavior of the decaying solution, we necessarily consider the regularity of the solution. Indeed, the weak solution to the degenerate problem (1.1) has a hyperbolic feature in it when the solution meets zero. In this case, the equation loses the parabolic behavior and the solution behaves as if it is a solution of the hyperbolic equation. In [48], the Hölder regularity is used essential way to show the asymptotic convergence rate. To see this, we firstly introduce the forward self-similar transform, which plays an important role in studying the large time asymptotic behavior. We introduce the forward self-similar scaling (t', x') as

$$t' = \frac{1}{\sigma}\log(1+\sigma t), \quad x' = \frac{x}{(1+\sigma t)^{\frac{1}{\sigma}}},$$

where $\sigma = n(\alpha-1)+2$ and the forward self-similar transform $(v(t',x'),\, \phi(t',x'))$ as

$$v(t',x') = (1+\sigma t)^{\frac{n}{\sigma}} u(t,x), \quad \phi(t',x') = (1+\sigma t)^{\frac{n}{\sigma}} \psi(t,x).$$

Then, the forward self-similar transform (v, ϕ) satisfies the following degenerate parabolic elliptic system:

$$\begin{cases} \partial_{t'} v - \operatorname{div}_{x'}(\nabla_{x'} v^\alpha + x'v - e^{-\kappa t'} v \nabla_{x'}\phi) = 0, & t' > 0,\, x' \in \mathbb{R}^n, \\ -e^{-2t'}\Delta_{x'}\phi + \phi = v, & t' > 0,\, x' \in \mathbb{R}^n, \\ v(0,x') = u_0(x') \geq 0, & x' \in \mathbb{R}^n, \end{cases} \tag{1.10}$$

where $\kappa = n+2-\sigma = n(2-\alpha)$. The weak solution of the system (1.10) is similarly defined as in the case for (1.1).

For $1 \leq p \leq \infty$, we obtain

$$(1+\sigma t)^{\frac{n}{\sigma}(1-\frac{1}{p})} \|u(t) - \mathcal{U}(t)\|_p = \|v(t') - V\|_p \tag{1.11}$$

where

$$V(x') := \left(A - \frac{\alpha-1}{2\alpha}|x'|^2\right)_+^{\frac{1}{\alpha-1}} = (1+\sigma t)^{\frac{n}{\sigma}} \mathcal{U}(t,x)$$

is a self-similar profile of the Barenblatt-Pattle solution. If $p = 1$, then equation (1.11) is rewritten by

$$\|u(t) - \mathcal{U}(t)\|_1 = \|v(t') - V\|_1$$

For the sake of obtaining the convergence rate of the solution in L^1, we should show the convergence rate of the forward self-similar transform in L^1 space. Ogawa [48] showed that if the self-similar transformed solution v is uniformly Hölder continuous, then we obtain the exponential convergence rate of the self-similar transform v. More precisely,

PROPOSITION 1.3 ([48]). *Let $1 < \alpha \leq 2 - \frac{2}{n}$. Assume that the initial data u_0 satisfies $W(0) > 0$ and (1.4). If the corresponding forward self-similar transform v is uniformly Hölder continuous, then there exist $\nu > 0$ and $C > 0$ such that*

$$\|v(t') - V\|_{L^1(\mathbb{R}^n)} \leq Ce^{-\nu t'}, \quad t' > 0 \tag{1.12}$$

where V is the self-similar profile of the Barenblatt-Pattle solution with $\|V\|_1 = \|u_0\|_1$.

Our main concern is to obtain the algebraic convergence rate of the solution in L^1 space for the case of critical exponent $\alpha = 2 - \frac{2}{n}$. The reason why the critical case is excluded in [48] is because the uniform Hölder continuity of the rescaled weak solution $v(s, y)$ is required for proving algebraic asymptotic convergence. The Hölder continuity was obtained in [48] for $v(s, y)$ via the rescaled weak solution $u(t, x)$ and hence it was not the uniform estimate for (s, y). By this argument, the critical case has to be necessarily excluded since the decaying factor $e^{-(\kappa-2)t}$ disappears in the crucial estimates. In this case, however, the nonlinear diffusion is dominant over the nonlocal transport for large time asymptotic behavior and we expect that the asymptotic profile is analogous to the sub-critical cases $\alpha < 2-\frac{2}{n}$. This is because of the effect from the lower order linear term on the equation of ψ (cf. [34]). To cover the critical case, we necessarily derive the uniform Hölder regularity for the weak solution of the rescaled solution $v(s, y)$ directly by assuming that the moment of the solution is uniformly bounded in time. To this end, we show the following regularity result:

THEOREM 1.4. *Let (v, ϕ) be a weak solution of the scaled Keller-Segel system* (1.10) *in $(u, \phi) \in L^\infty(0, T; L^1 \cap L^\alpha) \times L^\infty(0, T; W^{2,\alpha})$ with $|x|^a u_0 \in L^1$ for some $a > n$. Then $v(t, x)$ is uniformly Hölder continuous. Namely, there exists a constant $C = C(n, \alpha, \|v\|_{L^1 \cap L^\infty})$ and $0 < \gamma < 1$ such that for any (t, x) and $(s, y) \in [0, T] \times \mathbb{R}^n$,*

$$|v(t, x) - v(s, y)| \leq C\big(|t - s|^{\gamma/2} + |x - y|^\gamma\big).$$

From the regularity result Theorem 1.4, when the small initial data has a compact support especially, we may obtain the explicit asymptotic convergence rate to the solution of the porous medium equation. That is to say, the nonlinear diffusion is more effective than the nonlocal transport for the asymptotic behavior of generic solutions. We emphasize that for fine large time asymptotic estimates in the critical situation, we need some uniform regularity estimates for rescaled solution v.

The main argument to obtain the uniform Hölder continuity is along the argument due to DiBenedetto-Friedman [18], [19] (see also [17]). In particular, we employ the maximum depending parabolic cylinder to consider the Hölder regularity. Since we have the external term in the equation (1.10), we need to treat it from the very first step of the alternative selection lemma. In fact, we show the Hölder continuity of the weak solution involving the external term div $F :=$ div $(yv - e^{-\kappa s} v \nabla\phi)$. Unfortunately, the alternative method in [18] does not work directly well for this case, we reconstruct the alternative selection lemma (Lemma 3.2 and Lemma 3.3 in the section 3 below) for proving the oscillation estimate of the weak solution. This part gives a large contrast to the classical argument found in [18], [19]. Thanks to the uniform moment bound for the weak solution, this is possible and we many derive the asymptotic convergence in the algebraic decay order.

This paper is organized as follows. In the next section, we introduce the forward self-similar transform and its properties. We show the Hölder continuity of the self-similar transform in section 3. In section 4, we give the relationship between the asymptotic stability of the solution of (1.1) and the Hölder continuity of the self-similar transform.

We introduce some notation. We denote a positive part of a by $a_+ := \max\{a, 0\}$ and a negative part of a by $a_- := \max\{-a, 0\}$. We write the oscillation of a function f in a set A as

$$\operatorname*{osc}_A f := \sup_{x\in A} f(x) - \inf_{x\in A} f(x).$$

For $\rho > 0$, we denote by $B_\rho(x_0)$ the open ball centered at x_0 with the radius $\rho > 0$. We simply write $B_\rho = B_\rho(0)$.

For Lebesgue measurable set $A \subset \mathbb{R}^n$, we denote by $|A|$ the Lebesgue measure of A. For $f \in L^p(\mathbb{R}^n)$, let $\|f\|_p$ denote the norm of f in $L^p(\mathbb{R}^n)$. $L^p_s = L^p_s(\mathbb{R}^n)$ denotes the weighted Lebesgue space defined by $\{f \in L^p(\mathbb{R}^n),\ |x|^s f(x) \in L^p(\mathbb{R}^n)\}$. For an open interval $I \subset \mathbb{R}$, a domain $\Omega \subset \mathbb{R}^n$ and $1 \le p, q \le \infty$, we denote $L^p(I\,;\, L^q(\Omega))$ by $L^p(L^q)(I \times \Omega)$.

2. Forward Self-Similar Transform

In this section, we show the time decay of the global weak solution of the degenerated Keller-Segel system. This is originally shown in [54] however, we present the method of rescaling which is shown in [47].

2.1. Rescaled equation

We introduce the new scaled variables (t', x') as

$$\begin{cases} t' = \dfrac{1}{\sigma}\log(1+\sigma t), \\ x' = \dfrac{x}{(1+\sigma t)^{\frac{1}{\sigma}}}, \end{cases} \tag{2.1}$$

where $\sigma = n(\alpha - 1) + 2$ and introduce the new scaled unknown functions $v(t', x')$, $\phi(t', x')$ by

$$u(t,x) = (1+\sigma t)^{-\frac{n}{\sigma}} v\left(\frac{1}{\sigma}\log(1+\sigma t),\ \frac{x}{(1+\sigma t)^{\frac{1}{\sigma}}}\right),$$
$$\psi(t,x) = (1+\sigma t)^{-\frac{n}{\sigma}} \phi\left(\frac{1}{\sigma}\log(1+\sigma t),\ \frac{x}{(1+\sigma t)^{\frac{1}{\sigma}}}\right).$$

Or one may write as

$$v\left(t', x'\right) \equiv e^{nt'} u\left(\frac{1}{\sigma}(e^{\sigma t'} - 1), x' e^{t'}\right),$$

$$\phi\left(t', x'\right) \equiv e^{nt'}\psi\left(\frac{1}{\sigma}(e^{\sigma t'}-1), x'e^{t'}\right)$$

and the resulting scaling equation of (v, ϕ) follows by setting $\kappa = n+2-\sigma = n(2-\alpha)$,

$$\text{(2.2)}\quad \begin{cases} \partial_{t'} v - \operatorname{div}_{x'}(\nabla_{x'} v^\alpha + x'v - e^{-\kappa t'} v\nabla_{x'}\phi) = 0, & t'>0, x' \in \mathbb{R}^n, \\ -e^{-2t'}\Delta_{x'}\phi + \phi = v, & t'>0, x' \in \mathbb{R}^n, \\ v(0,x') = u_0(x'), & x' \in \mathbb{R}^n. \end{cases}$$

In this case, the vanishing exponent as before can be found as $\alpha = 2$ by

$$0 = \sigma - n - 2 = n(\alpha - 2)$$

and thus the sub-critical case is corresponding to $\alpha < 2$. Hereafter we analyze the above rescaled equation (2.2) to see the asymptotic behavior of the solution. We slightly change the outlook of the solution as follows:

The existence of the weak solution of (2.2) may be proven by a similar way to the original equation. Indeed, the scaling does not change any analytical feature of the original weak solution so that the solution can be obtained from the weak solution of (1.1) except the weighted restriction such as $v \in C((0,T); L^\alpha \cap L^1_a(\mathbb{R}^n))$ for $a \geq 2$. Similar to the original system, we consider the approximated system by the parabolic regularization:

$$\text{(2.3)}\quad \begin{cases} \begin{aligned}\partial_{t'} v - \operatorname{div}_{x'}(\nabla_{x'}(v+\varepsilon)^\alpha + x'v \\ -e^{-\kappa t'} v\nabla_{x'}\phi) = 0,\end{aligned} & t'>0, x' \in \mathbb{R}^n, \\ -e^{-2t'}\Delta_{x'}\phi + \phi = v, & t'>0, x' \in \mathbb{R}^n, \\ v(0,x') = u_0(x'), & x' \in \mathbb{R}^n. \end{cases}$$

Namely we again consider the nonnegative weak solution $v(t', x')$ as before. Note that for the construction of the weak solution, we need to use the diagonal argument obtaining the weak solutions (u, ψ) and (v, ϕ) simultaneously, since we do not know the uniqueness of the weak solution.

2.2. Rescaled uniform bounds

The following estimate is a direct consequence of the above a priori bound of the rescaled solution.

PROPOSITION 2.1. *Let $1 < \alpha \leq 2 - \frac{2}{n}$ and $(v(t'), \phi(t'))$ be a weak solution of (2.2) for the initial data $u_0 \in L^1_2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Assume that*

$$\|u_0\|_1^{1-\gamma} W(0)^{\frac{\gamma-\alpha+1}{\alpha}} < C C_{\mathrm{HLS}}^{-1} \tag{2.4}$$

for $1 < \alpha \leq 2 - \frac{2}{n}$ and $\gamma + 1 = \frac{\alpha}{\alpha-1}\frac{n-2}{n}$, where E_n is the fundamental solution to $-\Delta + 1$ in $\mathbb{R}^n$. Then

(1) *we have*

$$\|v(t)\|_q \leq C$$

for all $1 \leq q \leq \infty$.

(2) *for all $\frac{n}{n-1} < r \leq \infty$,*

$$\|\nabla \phi(t)\|_r \leq Ce^t.$$

Once we obtain the above uniform bound for the rescaled solution, we can immediately obtain the time decay estimate for the solution of the original equation.

$$\begin{aligned} \int_{\mathbb{R}^n} v^q(t', x')dx' &= \int_{\mathbb{R}^n} e^{n(q-1)t'} u^q(t, x)dx \\ &= (1+\sigma t)^{\frac{n}{\sigma}(q-1)} \int_{\mathbb{R}^n} u^q(t, x)dx \end{aligned} \tag{2.5}$$

in the original variables (t, x). Hence we obtain the following decay estimate for the original solution as a corollary of Proposition 2.1.

PROPOSITION 2.2 ([47], [54]). *Let $u_0 \in L^1_2 \cap L^\infty$ and $(u(t), \psi(t))$ be a weak solution of (1.1). If $1 < \alpha \leq 2 - \frac{2}{n}$ with small initial data (2.4), we have*

$$\|u(t)\|_q \leq C(1+\sigma t)^{-\frac{n}{\sigma}(1-\frac{1}{q})}$$

for all $1 \leq q \leq \infty$.

2.3. The moment bounds

The last part of this section, we show the second moment of the weak solution remains bounded for $t \in [0,T]$.

PROPOSITION 2.3. *Let $u_0 \in L^1 \cap L^\alpha$ with $|x|^2 u_0(x) \in L^1(\mathbb{R}^n)$. Then the weak solution (v,ϕ) of (2.2) satisfies*

$$\int_{\mathbb{R}^n} |x'|^2 v(t') dx' \le e^{-nt'} \int_{\mathbb{R}^n} |x'|^2 u_0 dx' + \frac{2(n-2)}{n} W_s(0), \tag{2.6}$$

where

$$W_s(t') := \frac{1}{\alpha-1}\int_{\mathbb{R}^n} v^\alpha(t')\,dx' + \frac{1}{2}\int_{\mathbb{R}^n} |x'|^2 v(t')\,dx' - \frac{1}{2} e^{-\kappa t'} v(t')\phi(t')\,dx'.$$

Namely, $|x'|^2 v(t') \in L^1(\mathbb{R}^n)$ for almost all t'. In addition if we assume that $u_0 \in L^1_a(\mathbb{R}^2)$ with $a > 2$, then we have

$$\int_{\mathbb{R}^2} |x'|^a v(t')\,dx \le e^{-at'} \int_{\mathbb{R}^2} |x'|^a u_0\,dx + C. \tag{2.7}$$

PROOF OF PROPOSITION 2.3. We only give the formal proof. It can be justified some appropriate cut off and approximation procedure. To show (2.6) we test $|x'|^2$ to the equation and we see

$$\begin{aligned} \frac{d}{dt'}\int_{\mathbb{R}^n} |x'|^2 v(t')\,dx' &= 2n\|v(t')\|_\alpha^\alpha - 2\int_{\mathbb{R}^n} |x'|^2 v(t')\,dx' \\ &\quad + 2e^{-\kappa t'}\int_{\mathbb{R}^n} \Big(x' v(t') \cdot \nabla\phi(t')\Big)\,dx'. \end{aligned} \tag{2.8}$$

We invoke the Pohozaev identity for the second equation. We multiply the elliptic part of the system by the generator of the dilation $x' \cdot \nabla\phi$ and integrate it by parts. Then it follows

$$\begin{aligned} \int_{\mathbb{R}^n} x' \cdot \nabla\phi(t') v(t') dx' &= e^{-2t'}\left(1-\frac{n}{2}\right)\int_{\mathbb{R}^n} |\nabla\phi(t')|^2 dx \\ &\quad - \frac{n}{2}\int_{\mathbb{R}^n} |\phi(t')|^2 dx' \\ &= \left(1-\frac{n}{2}\right)\int_{\mathbb{R}^n} v(t')\phi(t') dx' - \|\phi(t')\|_2^2. \end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9), we obtain

$$\begin{aligned}
&\frac{d}{dt'}\int_{\mathbb{R}^n}|x'|^2v(t')dx+n\int_{\mathbb{R}^n}|x'|^2v(t')dx'\\
&\quad=2n\|v(t')\|_\alpha^\alpha+(n-2)\int_{\mathbb{R}^n}|x'|^2v(t')dx'\\
&\qquad+(2-n)e^{-\kappa t}\int_{\mathbb{R}^n}v(t')\phi(t')dx'-2e^{-\kappa t'}\|\phi(t')\|_2^2\\
&\quad=2(n-2)W_s(t')\\
&\qquad+2n\left(\frac{\alpha-2+\frac{2}{n}}{\alpha-1}\right)\|v(t')\|_\alpha^\alpha-2e^{-\kappa t'}\|\phi(t')\|_2^2.
\end{aligned}\tag{2.10}$$

Thus under the condition $\alpha\le 2-\frac{2}{n}$, we see that

$$\int_{\mathbb{R}^n}|x'|^2v(t')dx'\le e^{-nt'}\int_{\mathbb{R}^n}|x'|^2u_0dx'+\frac{2(n-2)}{n}W_s(0)(1-e^{-nt'}).$$

For further weighted estimate, we modify (2.8) to have

$$\begin{aligned}
&\frac{d}{dt'}\int_{\mathbb{R}^2}|x'|^av(t')\,dx'+a\int_{\mathbb{R}^2}|x'|^av(t')\,dx'\\
&\quad=a(n-2+a)\int_{\mathbb{R}^2}|x'|^{a-2}v^\alpha(t')\,dx'\\
&\qquad+ae^{-\kappa t'}\int_{\mathbb{R}^2}|x'|^{a-2}\big(x'\cdot\nabla\phi\big)v(t')\,dx'.
\end{aligned}\tag{2.11}$$

It follows that

$$\begin{aligned}
&\frac{d}{dt'}\left[e^{at'}\int_{\mathbb{R}^2}|x'|^av(t')\,dx'\right]\\
&\quad=a(n-2+a)e^{at'}\|v(t')\|_\infty^{\alpha-1}\int_{\mathbb{R}^n}|x'|^{a-2}v(t')\,dx'\\
&\qquad+ae^{-(\kappa-a)t'}\|\nabla\phi\|_\infty\int_{\mathbb{R}^n}|x'|^{a-1}v(t')\,dx'.
\end{aligned}\tag{2.12}$$

By the uniform boundedness for $\|v(t')\|_\infty$, $e^{-t'}\|\nabla\phi\|_\infty$ (Proposition 2.1) and the lower moment bounds implies that

$$\int_{\mathbb{R}^n}|x'|^av(t')\,dx'\le e^{-at'}\int_{\mathbb{R}^n}|x'|^au_0\,dx'+C.\ \square\tag{2.13}$$

This uniform bound for the second moment of $v(t)$ naturally yields the bound for the second moment of $u(t)$ such as

$$\int_{\mathbb{R}^n} |x|^2 u(t) dx \leq \int_{\mathbb{R}^n} |x|^2 u_0 dx + \frac{2(n-2)}{n} W_s(0)(1+\sigma t)^{2/\sigma}.$$

3. The Hölder Continuity

In this section, we consider the Hölder estimate of the perturbed degenerate parabolic equation:

$$\text{(3.1)} \qquad \begin{cases} \partial_t \varphi(u) - \Delta u = \operatorname{div} F, \quad t > 0\,, x \in \mathbb{R}^n, \\ \qquad u(0,y) = u_0^{\frac{1}{\alpha}}(x) \geq 0, \end{cases}$$

where $u = u(t,x)$ is the unknown function and we put $\varphi(u) := u^{\frac{1}{\alpha}}(t,x)$ and $F = F(t,x)$ is a given function. The solution of the above problem is connected to our rescaled problem (2.2) by letting $v(t,x) = \varphi(u)(t,x)$ and

$$F := x\varphi(u) - e^{-\kappa t}\varphi(u)\nabla\phi,$$

where $\phi = (-e^{-2t}\Delta + 1)^{-1}\varphi(u)$. To state the Hölder estimate, we introduce the uniformly local L^p spaces.

DEFINITION. For $1 \leq p < \infty$, we define the uniformly local L^p spaces $L^p_{\mathrm{uloc}}(\mathbb{R}^n)$ as

$$L^p_{\mathrm{uloc}}(\mathbb{R}^n) := \Big\{ f \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \,:\, \|f\|_{L^p_{\mathrm{uloc}}(\mathbb{R}^n)} := \sup_{r \leq 1\,,\, a \in \mathbb{R}^n} \|f\|_{L^p(B_r(a))} < \infty \Big\}.$$

Now, we state the Hölder estimate of the weak solution of (3.1).

THEOREM 3.1. *Let u be a bounded nonnegative weak solution of* (3.1). *For $p > n$, we assume $F \in L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))$. Then for all $\varepsilon > 0$, there exist some constants $C = C(n,\alpha,p,\varepsilon)$ and $0 < \gamma = \gamma(n,\alpha,p) < 1$ such that*

$$\text{(3.2)} \quad |u(t,x) - u(s,y)| \\ \leq C(\|u\|_{L^\infty((0,\infty)\times\mathbb{R}^n)} + \|F\|_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))})$$

$$\times \left(\|u\|_{L^\infty((0,\infty)\times\mathbb{R}^n)}^{\frac{\gamma\beta}{2}}|t-s|^{\frac{\gamma}{2}}+|x-y|^\gamma\right)$$

for all $(t,x)\,,\,(s,y)\in(\varepsilon,\infty)\times\mathbb{R}^n$, *where* $\beta=1-\frac{1}{\alpha}$.

To show the Hölder regularity of the perturbed degenerate parabolic equation, we employ the alternative argument originally due to DiBenedetto-Friedman [18] (see also [17]). The argument is a substituted to the standard Harnack property for the degenerate case.

3.1. Alternative Lemmas

For a fixed set of the nonnegative parameters (ρ, M, θ_0) and space-time point $(t_0,x_0)\in\mathbb{R}^{n+1}$, we introduce usual parabolic cylinders Q_ρ, $Q_\rho^{\theta_0}$ and modified parabolic cylinders $Q_{\rho,,M}$, $Q_{\rho,M}^{\theta_0}$.

Definition. Let $\beta=1-\frac{1}{\alpha}$. We denote the time intervals by

$$\begin{aligned} I_{\rho,M}(t_0)&=(t_0-\frac{\rho^2}{M^\beta},t_0), \qquad I_{\rho,M}\equiv I_{\rho,M}(0),\\ I_{\rho,M}^{\theta_0}(t_0)&=(t_0-\frac{\theta_0}{2}\frac{\rho^2}{M^\beta},t_0), \quad I_{\rho,M}^{\theta_0}\equiv I_{\rho,M}^{\theta_0}(0)\end{aligned}$$

and we introduce the parabolic cylinders given by

$$\begin{aligned} Q_{\rho,M}&=I_{\rho,M}(t_0)\times B_\rho(x_0),\\ Q_{\rho,M}^{\theta_0}&=I_{\rho,M}^{\theta_0}(t_0)\times B_\rho(x_0).\end{aligned}$$

We abbreviate $Q_\rho=Q_{\rho,1}$ and $Q_\rho^{\theta_0}=Q_{\rho,1}^{\theta_0}$ for simplicity.

In what follows, we arbitrary choose the centered point (t_0,x_0) and fix it. We frequently omit to denote it. We set $p_*>1$ as

$$\frac{1}{p_*}=\frac{1}{n}-\frac{1}{p}. \tag{3.3}$$

Then we assume that two important parameters $M>0$ and $\omega>0$ are chosen so that

$$\sup_{Q_{\rho,M}} u\le M\le 4\sup_{Q_{\rho,M}} u \tag{3.4}$$

and

$$\frac{3}{4}\omega \leq \underset{Q_{\rho,M}}{\operatorname{osc}}\ u \leq \omega. \tag{3.5}$$

Namely the parameters M and ω are approximately the oscillation and the maximum of the weak solution $u(t,x)$ for (3.1) over $Q_{\rho,M}$ respectively:

$$\begin{aligned} \omega &\simeq \underset{Q_{\rho,M}}{\operatorname{osc}}\ u, \\ M &\simeq \max_{Q_{\rho,M}} u. \end{aligned} \tag{3.6}$$

We then have the following alternative lemma.

LEMMA 3.2 (1st alternative). *Assume* (3.4) *and* (3.5). *For any $p > n$ we set p_* by* (3.3) *and let $\rho > 0$ satisfy*

$$\omega^{-2}\rho^{\frac{2n}{p_*}}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \leq \delta_1$$

for some $\delta_1 > 0$. Then there exists a constant $0 < \theta_0 < 1$ depending only on n, α, p, δ_1 such that if

$$\left|Q_{\rho,M} \cap \left\{(t,x) : u(t,x) < \inf_{Q_{\rho,M}} u + \frac{\omega}{2}\right\}\right| \leq \theta_0 |Q_{\rho,M}|, \tag{3.7}$$

we obtain

$$u(t,x) \geq \inf_{Q_{\rho,M}} u + \frac{\omega}{4} \tag{3.8}$$

for all $(t,x) \in Q_{\frac{\rho}{2},M}$.

LEMMA 3.3 (2nd alternative). *Assume* (3.4) *and* (3.5). *For any $p > n$ we set p_* by* (3.3). *Then for $0 < \theta_0 < 1$, there exist $\delta_2 > 0$ and $q_0 > 0$ depending only on n, α, p and θ_0 such that if*

$$\omega^{-2}\rho^{\frac{2n}{p_*}}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \leq \delta_2$$

and

$$\left|Q_{\rho,M} \cap \left\{(t,x) : u(t,x) < \inf_{Q_{\rho,M}} u + \frac{\omega}{2}\right\}\right| > \theta_0 |Q_{\rho,M}|,$$

we obtain

$$u(t,x) \le \sup_{Q_{\rho,M}} u - \frac{\omega}{2^{q_0+2}}$$

for all $(t,x) \in Q^{\theta_0}_{\frac{\rho}{2},M}$.

For simplicity, we set for a weak solution u;

$$\mu^+ := \sup_{Q_{\rho,M}} u\,, \qquad \mu^- := \inf_{Q_{\rho,M}} u \tag{3.9}$$

so that we may write $\operatorname*{osc}_{Q_{\rho,M}} u = \mu^+ - \mu^-$.

3.2. Proof of Theorem 3.1

We show Theorem 3.1 by temporary admitting the alternative lemmas, Lemma 3.2 and Lemma 3.3. We recall the parameters play in the roles in (3.6).

LEMMA 3.4 (oscillation lemma). *Let M and ω satisfy the assumptions* (3.4) *and* (3.5). *For any $p > n$ let p_* be defined in* (3.3). *Then there exist $0 < \theta_0, \delta, \eta < 1$ depending only on n, α and p such that for $\rho^{\frac{2n}{p_*}} \omega^{-2} \|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \le \delta$, it holds that*

$$\operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho}{2},M}} u \le \eta\omega.$$

PROOF OF LEMMA 3.4. By Lemma 3.2, for $\rho > 0$ with $\omega^{-2}\rho^{\frac{2n}{p_*}} \|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \le 1$, there exists $0 < \theta_0 < 1$ depending only on n, α, p such that if

$$\left| Q_{\rho,M} \cap \left\{ (t,x) : u(t,x) < \mu^- + \frac{\omega}{2} \right\} \right| \le \theta_0 |Q_{\rho,M}|,$$

we have

$$u(t,x) \ge \mu^- + \frac{\omega}{4} \tag{3.10}$$

for $(t,x) \in Q_{\frac{\rho}{2},M}$ and by Lemma 3.3, there exist $q_0, \delta_2 > 0$ depending only on n, α, p such that if

$$\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \leq \delta_2$$

and

$$\left| Q_{\rho,M} \cap \left\{ (t,x) : u(t,x) < \mu^- + \frac{\omega}{2} \right\} \right| > \theta_0 |Q_{\rho,M}|,$$

then

$$u(t,x) \leq \mu^+ - \frac{\omega}{2^{q_0+2}} \tag{3.11}$$

for $(t,x) \in Q^{\theta_0}_{\frac{\rho}{2},M}$. Letting

$$\delta := \min\{1, \delta_2\}\,,\ \eta := 1 - \frac{1}{2^{q_0+2}},$$

we obtain from (3.10) and (3.11) that

$$\operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho}{2},M}} u \leq (\mu_+ - \mu_-) - \frac{\omega}{2^{q_0+2}} = \eta\omega$$

provided

$$\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \leq \delta. \ \square$$

REMARK. We can further assume $\eta \geq \frac{3}{4}$.

PROOF OF THEOREM 3.1. We put $Q = (0,\infty) \times \mathbb{R}^n$ and $M_0 = \sup_Q u$, $\omega_0 = M_0$. Let θ_0, δ and η be as in Lemma 3.4. We choose $\rho_0 > 0$ satisfying $\rho_0^{\frac{2n}{p_*}} \leq \delta \|F\|^{-2}_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))}\omega_0^2$. We denote $Q_0 := Q_{\rho_0,M_0}$, $\mu_0^+ = \sup_{Q_0} u$ and $\mu_0^- = \inf_{Q_0} u$. Then we find

$$\begin{cases} \sup\limits_{Q_0} u \leq \sup\limits_{Q} u \leq M_0, \\ \operatorname*{osc}\limits_{Q_0} u \leq \omega_0, \\ \rho_0^{\frac{2n}{p_*}} \leq \delta\omega_0^2 \|F\|^{-2}_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))}. \end{cases}$$

We now fix every parameters as follows:

$$c_0 := \min\Big\{\eta^{\frac{p_*}{n}},\ \frac{1}{2}\Big(\frac{1}{3}\Big)^{\frac{\beta}{2}}\Big(\frac{\theta_0}{2}\Big)^{\frac{1}{2}}\Big\}$$

and we take sequences as

$$\begin{aligned} &\omega_k := \eta^k\omega_0, && \rho_k := c_0^k\rho_0, \\ &M_k := \max\{\mu_{k-1}^+,\ \omega_k\}, && Q_k := Q_{\rho_k, M_k}, \\ &\mu_k^+ := \sup_{Q_k} u, && \mu_k^- := \inf_{Q_k} u \end{aligned} \tag{3.12}$$

for $k \in \mathbb{N}$. In the following, we claim for all $k \in \mathbb{N}$ that

$$\begin{cases} \sup\limits_{Q_k} u \le \sup\limits_{Q_{k-1}} u \le M_k, \\ \operatorname*{osc}\limits_{Q_k} u \le \omega_k, \\ \rho_k^{\frac{2n}{p_*}} \le \delta\omega_k^2\|F\|^{-2}_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))} \end{cases} \tag{3.13}$$

by the inductive argument. Indeed, either if $\operatorname{osc}_{Q_0} u \le \frac{3}{4}\omega_0$, then by

$$\frac{M_1}{M_0} \ge \frac{\omega_1}{M_0} = \eta \ge \frac{3}{4}, \quad c_0 \le \min\left\{\eta^{\frac{p_*}{n}},\ \Big(\frac{3}{4}\Big)^{\frac{\beta}{2}}\right\}$$

we have $Q_1 \subset Q_0$ and hence (3.13) holds for $k = 1$. If otherwise, $\frac{3}{4}\omega_0 \le \operatorname{osc}_{Q^{(0)}} u \le \omega_0$ and since $M_0 = \omega_0 \le \frac{4}{3}\mu_0^+$, we can apply Lemma 3.4 to obtain

$$\operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho_0}{2}, M_0}} u \le \eta\omega_0.$$

Since $c_0 \le \min\{\eta^{\frac{p_*}{n}},\ \frac{1}{2}(\frac{\theta_0}{2})^{\frac{1}{2}}(\frac{3}{4})^{\frac{\beta}{2}}\}$, we have $Q_1 \subset Q^{\theta_0}_{\frac{\rho_0}{2}, M_0} \subset Q_0$ and (3.13) also holds for the case $k = 1$. We then assume (3.13) for $1, 2, \cdots, k$ and we will show the case of $k+1$. If we consider the case of $\mu_{k-1}^- \le \frac{1}{3}\mu_{k-1}^+$, then

$$\mu_{k-1}^+ \le \omega_{k-1} + \frac{1}{3}\mu_{k-1}^+$$

and it follows $\mu_{k-1}^+ \le \frac{3}{2}\omega_{k-1} = \frac{3}{2\eta}\omega_k$. In the other case, we have

$$\mu_{k-1}^+ < 3\mu_{k-1}^- \le 3\mu_k^- \le 3\mu_k^+.$$

Hence we obtain

$$\mu_{k-1}^+ \le \max\left\{\frac{3}{2\eta}\omega_k\,,\, 3\mu_k^+\right\}. \tag{3.14}$$

Using (3.14), we show (3.13) for $k+1$.

CASE 1. We assume $\operatorname{osc}_{Q_k} u \le \frac{3}{4}\omega_k$. By the definition in (3.12), either $M_k = \omega_k$ or $M_k = \mu_{k-1}^+$ holds. If $M_k = \omega_k$, then

$$\frac{M_{k+1}}{M_k} = \frac{M_{k+1}}{\omega_k} \ge \frac{\eta\omega_k}{\omega_k} = \eta \ge \frac{3}{4}.$$

Since $c_0 \le (\frac{3}{4})^{\frac{\beta}{2}}$, we obtain $Q_{k+1} \subset Q_k$. On the other hand, if $M_k = \mu_{k-1}^+$, then by (3.14), we have

$$\begin{aligned}\frac{M_{k+1}}{M_k} = \frac{M_{k+1}}{\mu_{k-1}^+} &\ge \begin{cases} \dfrac{2\eta^2}{3} & \text{if } \dfrac{3}{2\eta}\omega_k \ge 3\mu_k^+, \\ \dfrac{1}{3} & \text{if } \dfrac{3}{2\eta}\omega_k \le 3\mu_k^+, \end{cases} \\ &\ge \frac{1}{3}.\end{aligned}$$

Since $c_0 \le (\frac{1}{3})^{\frac{\beta}{2}}$, it follows $Q_{k+1} \subset Q_k$. Hence we have

$$\begin{aligned}&\sup_{Q_{k+1}} u \le \sup_{Q_k} u \le M_{k+1}, \\ &\operatorname*{osc}_{Q_{k+1}} u \le \omega_{k+1} \text{ and } \rho_{k+1}^{\frac{2n}{p_*}} \le \delta \|F\|_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))}^{-2}\omega_{k+1}^2.\end{aligned}$$

CASE 2. We assume $\frac{3}{4}\omega_k \le \operatorname{osc}_{Q_k} u \le \omega_k$. Since $\omega_k \le \frac{4}{3}\mu_k^+$, we obtain

$$\mu_{k-1}^+ \le \max\left\{\frac{3}{2\eta}\omega_k\,,\, 3\mu_k^+\right\} \le \max\left\{\frac{2}{\eta}\mu_k^+\,,\, 3\mu_k^+\right\} \le 3\mu_k^+$$

and hence

$$M_k \le \max\left\{\frac{4}{3}\mu_k^+ \,,\, 3\mu_k^+\right\} \le 3\mu_k^+.$$

Applying Lemma 3.4, we obtain

$$\operatorname*{osc}_{Q^{\theta_0}_{\frac{\rho_k}{2},M_k}} u \le \eta\omega_k.$$

Since

$$\frac{M_{k+1}}{M_k} \ge \frac{\mu_k^+}{3\mu_k^+} = \frac{1}{3} \text{ and } c_0 \le \frac{1}{2}\left(\frac{1}{3}\right)^{\frac{\beta}{2}}\left(\frac{\theta_0}{2}\right)^{\frac{1}{2}},$$

we have $Q_{k+1} \subset Q^{\theta_0}_{\frac{\rho_k}{2},M_k}$ and hence

$$\sup_{Q_{k+1}} u \le \sup_{Q_k} u \le M_{k+1}, \quad \operatorname*{osc}_{Q_{k+1}} u \le \omega_{k+1} \text{ and } \rho_{k+1}^{\frac{2n}{p_*}} \le \delta\|F\|^{-2}_{L^\infty(L^p)(Q)}\omega_{k+1}^2.$$

Remarking that $M_k \ge M_{k+1}$ for $k \in \mathbb{N}$, we have

$$\operatorname*{osc}_{Q_{\rho_k,M_0}} u \le \operatorname*{osc}_{Q_k} u \le \omega_k.$$

Choosing $0 < \gamma < 1$ such that $c_0^\gamma = \eta$, we see

$$\operatorname*{osc}_{Q_{\rho_k,M_0}} u \le \eta^k\omega_0 = \omega_0\left(\frac{\rho_k}{\rho_0}\right)^\gamma.$$

We then obtain the intermediate case $\rho \ne \rho_k$ for all $k \in \mathbb{N}$. When $\rho \le \rho_0$, there exists $k \in \mathbb{N}\cup\{0\}$ such that $\rho_k < \rho < \rho_{k-1}$ and

$$\operatorname*{osc}_{Q_{\rho,M_0}} u \le \omega_0\left(\frac{\rho_{k-1}}{\rho_0}\right)^\gamma \le \omega_0 c_0^{-\gamma}\left(\frac{\rho}{\rho_0}\right)^\gamma. \tag{3.15}$$

Taking $\rho_0^{\frac{2n}{p_*}} = \delta\|F\|^{-2}_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))}\omega_0^2$, we find from (3.15) that

$$\operatorname*{osc}_{Q_{\rho,M_0}} u \le C_{n,\alpha,p}\omega_0^{1-\frac{p_*}{n}\gamma}\|F\|^{\frac{p_*}{n}\gamma}_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))}\rho^\gamma$$

$$\leq C_{n,\alpha,p} M_0^{1-\frac{p_*}{n}\gamma} \|F\|_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))}^{\frac{p_*}{n}\gamma} \rho^\gamma.$$

On the other hand, if $\rho > \rho_0$, we immediately have

$$\operatorname*{osc}_{Q_{\rho,M_0}} u \leq M_0 \leq M_0 \Big(\frac{\rho}{\rho_0}\Big)^\gamma \leq C_{n,\alpha,p} M_0^{1-\frac{p_*}{n}\gamma} \|F\|_{L^\infty(0,\infty\,;\,L^p_{\mathrm{uloc}}(\mathbb{R}^n))}^{\frac{p_*}{n}\gamma} \rho^\gamma.$$

In the both cases, we obtain by the Young inequality

$$\operatorname*{osc}_{Q_{\rho,M_0}} u \leq C_{n,\alpha,p} (M_0 + \|F\|_{L^\infty(0,\infty;L^p_{\mathrm{uloc}}(\mathbb{R}^n))}) \rho^\gamma. \tag{3.16}$$

This concludes the Hölder continuity for the weak solution around the fix point (t_0, x_0). Since the center point (t_0, x_0) is chosen arbitrary and from the right hand side of the estimate (3.16), we obtain the uniform estimate. This completes the proof of Theorem 3.1. □

3.3. Proof of the first alternative lemma

Without loss of generality, we assume $t_0 = 0$ by using the parallel translation. And we omit the center of ball x_0. For an open interval $(a, b) \in \mathbb{R}$ and an open ball $B_\rho(x_0) \subset \mathbb{R}^n$, we call $\eta = \eta(t, x)$ a cut-off function in $Q = (a, b) \times B_\rho(x_0)$ if $\eta \in C^\infty(Q)$ satisfies

$$\begin{aligned} \partial_t \eta(t, x) \geq 0, &\quad a \leq t \leq b,\ x \in B_\rho(x_0) \\ \eta(t, x) \equiv 0, &\quad a \leq t \leq b,\ x \in \partial B_\rho(x_0) \end{aligned}$$

and

$$\eta(a, x) \equiv 0, \quad x \in B_\rho(x_0).$$

LEMMA 3.5 (the Caccioppoli estimates for sublevel sets). *For $p > n$, we set p_* as in (3.3). Let $\eta = \eta(t, x)$ be a cut-off function in $Q_{\rho,M} = I_{\rho,M} \times B_\rho$. For $\lambda < \mu^- + \frac{1}{2}\omega$, there exists $C_\alpha > 0$ such that*

$$\sup_{I_{\rho,M}} \int_{B_\rho} (u-\lambda)_-^2 \eta^2\, dx + \varphi'(\mu^+)^{-1} \iint_{Q_{\rho,M}} |\nabla (u-\lambda)_-|^2 \eta^2\, dt dx$$
$$\leq C_\alpha \Big\{ \omega \iint_{Q_{\rho,M}} (u-\lambda)_- \eta \partial_t \eta\, dx dt$$

$$
\begin{aligned}
&+\varphi'(\mu^+)^{-1}\iint_{Q_{\rho,M}}(u-\lambda)_-^2|\nabla\eta|^2\,dxdt\\
&+\varphi'(\mu^+)^{-1}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\\
&\times\left(\int_{I_{\rho,M}}|B_\rho\cap\{u(t)<\lambda\}|^{\frac{r_*}{q_*}}\,dt\right)^{\frac{2}{r_*}(1+\frac{2}{p_*})}\Bigg\},
\end{aligned}
$$

where $r_*=2(1+\frac{2}{p_*})$ *and* $q_*=r_*(1-\frac{2}{p})^{-1}$.

REMARK. The exponent r_*,q_* satisfies the condition for the smooth class of the weak solution (so called the Serrin class in the Navier-Stokes system), namely

$$\frac{2}{r_*}+\frac{n}{q_*}=\frac{n}{2}.$$

PROOF OF LEMMA 3.5. Testing the function $-(u-\lambda)_-\eta^2$ to the equation (3.1), we have

$$
\begin{aligned}
&\sup_{I_{\rho,M}}\int_{B_\rho}\left(\int_0^{(u-\lambda)_-}\varphi'(\lambda-\xi)\xi\,d\xi\right)\eta^2\,dx\\
&\qquad+\iint_{Q_{\rho,M}}|\nabla(u-\lambda)_-|^2\eta^2\,dxdt\\
&=\iint_{Q_{\rho,M}}\left(\int_0^{(u-\lambda)_-}\varphi'(\lambda-\xi)\xi\,d\xi\right)\partial_t\eta^2\,dxdt\\
&\qquad-\iint_{Q_{\rho,M}}(\nabla(u-\lambda)_-\cdot\nabla\eta^2)(u-\lambda)_-\,dxdt\\
&\qquad-\iint_{Q_{\rho,M}}F\cdot\nabla(u-\lambda)_-\eta^2\,dxdt\\
&\qquad-\iint_{Q_{\rho,M}}F\cdot\nabla\eta^2(u-\lambda)_-\,dxdt.
\end{aligned}
\tag{3.17}
$$

Since

$$\lambda\le\mu^-+\frac{\omega}{2}\le\mu^+-\frac{1}{4}\omega\le\mu^+$$

and $\varphi'(\xi)=\frac{1}{\alpha}\xi^{\frac{1}{\alpha}-1}$ is monotone decreasing, we have for the first term of left hand side of (3.17) that

$$\varphi'(\lambda-\xi)\ge\varphi'(\lambda)\ge\varphi'(\mu^+),\quad \xi\ge0$$

and hence

$$
\begin{aligned}
&\sup_{I_{\rho,M}} \int_{B_\rho} (u-\lambda)_-^2 \eta^2 \, dx + \frac{1}{4}\varphi'(\mu^+)^{-1} \iint_{Q_{\rho,M}} |\nabla (u-\lambda)_-|^2 \eta^2 \, dxdt \\
&\quad = \varphi'(\mu^+)^{-1} \iint_{Q_{\rho,M}} \Big(\int_0^{(u-\lambda)_-} \varphi'(\lambda-\xi)\xi \, d\xi \Big) \partial_t \eta^2 \, dxdt \\
&\qquad + 3\varphi'(\mu^+)^{-1} \iint_{Q_{\rho,M}} (u-\lambda)_-^2 |\nabla \eta|^2 \, dxdt \\
&\qquad + 2\varphi'(\mu^+)^{-1} \iint_{Q_{\rho,M}\cap\{u<\lambda\}} |F|^2 \eta^2 \, dxdt.
\end{aligned} \tag{3.18}
$$

For the last term of the right hand side of (3.18), we use the Hölder inequality to have

$$
\begin{aligned}
&\iint_{Q_{\rho,M}\cap\{u<\lambda\}} |F|^2 \eta^2 \, dxdt \\
&\quad \le \int_{I_{\rho,M}} \||F|^2\|_{L^{\frac{p}{2}}(B_\rho)} \big| B_\rho \cap \{u(t)<\lambda\}\big|^{1-\frac{2}{p}} \, dt \\
&\quad \le \|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \int_{I_{\rho,M}} |B_\rho \cap \{u(t)<\lambda\}|^{1-\frac{2}{p}} \, dt \\
&\quad \le \|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \bigg(\int_{I_{\rho,M}} \big| B_\rho \cap \{u(t)<\lambda\}\big|^{\frac{r_*}{q_*}} \, dt \bigg)^{\frac{2}{r_*}(1+\frac{2}{p_*})} .
\end{aligned} \tag{3.19}
$$

To estimate the first term of the right hand side in (3.18), we note that

$$
\begin{aligned}
\int_0^{(u-\lambda)_-} \varphi'(\lambda-\xi)\xi \, d\xi &\le -(u-\lambda)_- \int_0^{(u-\lambda)_-} \frac{\partial}{\partial \xi} \varphi(\lambda-\xi) \, d\xi \\
&= (u-\lambda)_- [\varphi(\lambda) - \varphi(\lambda - (u-\lambda)_-)],
\end{aligned}
$$

hence we have

$$
\begin{aligned}
&\iint_{Q_{\rho,M}} \Big(\int_0^{(u-\lambda)_-} \varphi'(\lambda-\xi)\xi \, d\xi \Big) \partial_t \eta^2 \, dxdt \\
&\le \iint_{Q_{\rho,M}} \Big[\varphi\Big(\mu^- + \frac{\omega}{2}\Big) - \varphi(\mu^-) \Big] (u-\lambda)_- \partial_t \eta^2 \, dxdt.
\end{aligned} \tag{3.20}
$$

Either if $\mu^- \le \frac{1}{2}\mu^+$, then $\mu^+ \le 2\omega$ and since $\varphi(\xi) = \xi^{1/\alpha}$, it follows by ${\varphi'}^{-1}(\xi)\varphi(\xi) = \alpha\xi$ that

$$\varphi'(\mu^+)^{-1}\Big[\varphi\Big(\mu^- + \frac{\omega}{2}\Big) - \varphi(\mu^-)\Big] \le \varphi'(2\omega)^{-1}\varphi\Big(\frac{\omega}{2}\Big) \le C_\alpha\omega$$

and hence

$$I_1 \le C(\alpha)\omega \iint_{Q_{\rho,M}} (u-\lambda)_-\partial_t\eta^2\,dxdt.$$

Otherwise, if $\mu^- > \frac{1}{2}\mu^+$, then

$$\begin{aligned}\varphi'(\mu^+)^{-1}\Big[\varphi\Big(\mu^- + \frac{\omega}{2}\Big) - \varphi(\mu^-)\Big] &= \varphi'(\mu^+)^{-1}\frac{\omega}{2}\int_0^1 \varphi'\Big(\mu^- + \frac{\omega}{2}s\Big)\,ds \\ &\le \varphi'(\mu^+)^{-1}\frac{\omega}{2}\varphi'(\mu^-) \\ &\le \varphi'(\mu^+)^{-1}\frac{\omega}{2}\varphi'\Big(\frac{1}{2}\mu^+\Big) \\ &\le C_\alpha\omega.\end{aligned}$$

In both cases, we obtain

$$I_1 \le C_\alpha\omega \iint_{Q_{\rho,M}} (u-\lambda)_-\partial_t\eta^2\,dxdt. \tag{3.21}$$

Substituting (3.20) and (3.21) for (3.20), we obtain (3.17). □

Proof of Lemma 3.2. We consider the scale transform

$$\begin{aligned}&s = M^\beta(t-t_0), && Q_\rho = Q_{\rho,1}, && I_\rho = I_{\rho,1},\\ &\tilde u(s,x) = u(t,x), && \tilde\eta(s,x) = \eta(t,x), && \tilde F(s,x) = F(t,x).\end{aligned}$$

and rewrite the Caccioppoli estimate (3.17) as follows:

$$\begin{aligned}&\sup_{I_\rho}\int_{B_\rho}(\tilde u-\lambda)_-^2\tilde\eta^2\,dx + \frac{\varphi'(\mu^+)}{M^\beta}\iint_{Q_\rho}|\nabla(\tilde u-\lambda)_-|^2\tilde\eta^2\,dxds \\ &\le C_\alpha\Big\{\omega\iint_{Q_\rho}(\tilde u-\lambda)_-\partial_s\tilde\eta^2\,dxds + \frac{\varphi'(\mu^+)}{M^\beta}\iint_{Q_\rho}(\tilde u-\lambda)_-^2|\nabla\tilde\eta|^2\,dxds\end{aligned} \tag{3.22}$$

$$+\frac{\varphi'(\mu^+)}{M^\beta}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\biggl(\int_{I_\rho}|B_\rho\cap\{\tilde u(s)<\lambda\}|^{\frac{r_*}{q_*}}\,ds\biggr)^{\frac{2}{r_*}(1+\frac{2}{p_*})}\biggr\}.$$

We then resize the localization with the scaling for the solution by the parametrized sequence: For $k\in\mathbb{N}\cup\{0\}$, we set

$$\begin{aligned}
&\lambda_k=\mu^-+\frac{1}{4}\omega+\frac{1}{2^{k+1}}\omega, \qquad \rho_k=\frac{1}{2}\rho+\frac{1}{2^{k+1}}\rho,\\
&Y_k:=\frac{|Q_{\rho_k}\cap\{\tilde u<\lambda_k\}|}{|Q_\rho|},\\
&Z_k=\frac{\rho^2}{|Q_\rho|}\biggl(\int_{I_{\rho_k}}|B_{\rho_k}\cap\{\tilde u(s)<\lambda_k\}|^{\frac{r_*}{q_*}}\,ds\biggr)^{\frac{2}{r_*}},\\
&|\nabla\tilde\eta_k|\le\frac{2}{\rho_k-\rho_{k+1}}\le\frac{8\cdot 2^k}{\rho},\qquad \partial_s\tilde\eta_k\le\frac{2}{\rho_k^2-\rho_{k+1}^2}\le\frac{16\cdot 2^{2k}}{3\rho^2}
\end{aligned}\tag{3.23}$$

and we show that $Y_k\to 0$ as $k\to\infty$. Then, by using $\mu^+\le M\le 4\mu^+$ and $(\tilde u-\lambda_k)_-\le\frac{\omega}{2}$, we rewrite (3.22) as

$$\begin{aligned}
&\|(\tilde u-\lambda_k)_-\tilde\eta_k\|^2_{L^\infty(L^2)(Q_{\rho_k})\cap L^2(H^1)(Q_{\rho_k})}\\
&\le C_\alpha\biggl\{\omega\iint_{Q_{\rho_k}}(\tilde u-\lambda_k)_-\partial_s\tilde\eta_k{}^2\,dxds\\
&\quad+\iint_{Q_{\rho_k}}(\tilde u-\lambda_k)_-^2|\nabla\tilde\eta_k|^2\,dxds\\
&\quad+\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\biggl(\int_{I_{\rho_k}}|B_{\rho_i}\cap\{\tilde u(s)<\lambda_k\}|^{\frac{r_*}{q_*}}\,ds\biggr)^{\frac{2}{r_*}(1+\frac{2}{p_*})}\biggr\}\\
&\le C_\alpha\biggl\{\frac{2^{2k}\omega^2}{\rho^2}|\{u<\lambda_k\}|+\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\\
&\quad\times\biggl(\int_{I_{\rho_k}}|B_{\rho_k}\cap\{\tilde u(s)<\lambda_k\}|^{\frac{r_*}{q_*}}\,ds\biggr)^{\frac{2}{r_*}(1+\frac{2}{p_*})}\biggr\}\\
&\le C_\alpha\frac{\omega^2|Q_\rho|}{\rho^2}\biggl\{2^{2k}Y_k+\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\omega^{-2}\biggl(\frac{|Q_\rho|}{\rho^2}\biggr)^{\frac{2}{p_*}}Z_k^{1+\frac{2}{p_*}}\biggr\}.
\end{aligned}\tag{3.24}$$

By the Ladyženskaja inequality and the Hölder inequality, we have

$$\|(\tilde u-\lambda_k)_-\tilde\eta_k\|^2_{L^2(Q_{\rho_k})}$$

$$
\begin{aligned}
&\le \|(\tilde{u}-\lambda_k)_-\tilde{\eta}_k\|^2_{L^{2+\frac{4}{n}}(Q_{\rho_k})}\|\chi_{\{\tilde{u}<\lambda_k\}}\|^2_{L^{n+2}(Q_{\rho_k})}\\
&\le C_{\alpha,n}\omega^2|Q_\rho|Y_k^{\frac{2}{n+2}}\left\{2^{2k}Y_k+\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2}{p_*}}Z_k^{1+\frac{2}{p_*}}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\tilde{u}-\lambda_k)_-\tilde{\eta}_k\|^2_{L^{r_*}(L^{q_*})(Q_{\rho_k})}\\
&\quad\le C_{\alpha,n}\frac{\omega^2|Q_\rho|}{\rho^2}\left\{2^{2k}Y_k+\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2}{p_*}}Z_k^{1+\frac{2}{p_*}}\right\}.
\end{aligned}
$$

Since

$$
\begin{aligned}
\|(\tilde{u}-\lambda_k)_-\tilde{\eta}_k\|^2_{L^2(Q_{\rho_k})} &\ge \|(\tilde{u}-\lambda_k)_-\|^2_{L^2(Q_{\rho_{k+1}}\cap\{\tilde{u}<\lambda_{k+1}\})}\\
&\ge (\lambda_k-\lambda_{k+1})_-^2|Q_{\rho_{k+1}}\cap\{\tilde{u}<\lambda_{k+1}\}|\\
&= \frac{\omega^2}{64\cdot 2^{2k}}|Q_\rho|Y_{k+1}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|(\tilde{u}-\lambda_k)_-\tilde{\eta}_k\|^2_{L^{r_*}(L^{q_*})(Q_{\rho_k})}\\
&\quad\ge \|(\tilde{u}-\lambda_k)_-\|^2_{L^{r_*}(L^{q_*})(Q_{\rho_{k+1}}\cap\{\tilde{u}<\lambda_{k+1}\})}\\
&\quad\ge (\lambda_k-\lambda_{k+1})_-^2\left(\int_{I_{\rho_{k+1}}}|B_{\rho_{k+1}}\cap\{\tilde{u}(s)<\lambda_{k+1}\}|^{\frac{r_*}{q_*}}\,ds\right)^{\frac{2}{r_*}}\\
&\quad= \frac{\omega^2}{64\cdot 2^{2k}}\frac{|Q_\rho|}{\rho^2}Z_{k+1},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
Y_{k+1} &\le C_{\alpha,n}\left\{2^{4k}Y_k^{1+\frac{2}{n+2}}\right.\\
&\qquad\left.+2^{2k}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2}{p_*}}Y_k^{\frac{2}{n+2}}Z_k^{1+\frac{2}{p_*}}\right\}, \qquad (3.25)\\
Z_{k+1} &\le C_{\alpha,n}\left\{2^{4k}Y_k+2^{2k}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\omega^{-2}\left(\frac{|Q_\rho|}{\rho^2}\right)^{\frac{2}{p_*}}Z_k^{1+\frac{2}{p_*}}\right\}.
\end{aligned}
$$

Since $\frac{r_*}{q_*} < 1$ and $\frac{n}{2} = \frac{2}{r_*} + \frac{n}{q_*}$, we have

$$
\begin{aligned}
Z_0 &= \frac{\rho^2}{|Q_\rho|}\left(\int_{I_{\rho_0}} |B_{\rho_0} \cap \{\tilde{u}(s) < \lambda_0\}|^{\frac{r_*}{q_*}}\, ds\right)^{\frac{2}{r_*}} \\
&\leq \frac{\rho^2}{|Q_\rho|}\left(\int_{I_\rho} |B_\rho \cap \{\tilde{u}(s) < \lambda_0\}|\, ds\right)^{\frac{2}{q_*}} \rho^{\frac{4}{r_*}(1-\frac{r_*}{q_*})} \leq C_{n,p} Y_0^{\frac{2}{q_*}}.
\end{aligned}
$$

Therefore, by the assumption $\dfrac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \leq \delta_1$ and Lemma A.1 there exists $0 < \theta_0 = \theta_0(n, \alpha, p, \delta_1) < 1$ such that if $Y_0 \leq \theta_0$, then $Y_k \to 0$ as $k \to \infty$. This concludes

$$
\tilde{u}(s, x) > \mu^- + \frac{\omega}{4}
$$

for almost all $(s, x) \in Q_{\frac{\rho}{2}}$. $\square$

3.4. Proof of the second alternative lemma

We next show Lemma 3.3.

LEMMA 3.6. *Let $0 < \theta_0 < 1$. If assumption (3.7) fails i.e.*

$$
\left|Q_{\rho,M} \cap \left\{(t, x) : u(t, x) < \mu^- + \frac{\omega}{2}\right\}\right| > \theta_0 |Q_{\rho,M}|, \tag{3.26}
$$

then for all $0 < \theta < \theta_0$, there exists $-\frac{\rho^2}{M^\beta} < \tau < -\theta\frac{\rho^2}{M^\beta}$ such that

$$
\left|B_\rho \cap \left\{x : u(\tau, x) > \mu^- + \frac{\omega}{2}\right\}\right| \leq \frac{1-\theta_0}{1-\theta}|B_\rho|.
$$

PROOF OF LEMMA 3.6. By the change of variable $t = \frac{\rho^2}{M^\beta}s$, $\tilde{u}(s, x) = u(t, x)$, and the inequality (3.26), we obtain

$$
\begin{aligned}
&\int_{-1}^{0} \left|B_\rho \cap \left\{x \in B_\rho : \tilde{u}(s, x) > \mu^- + \frac{\omega}{2}\right\}\right| ds \\
&\quad = \frac{M^\beta}{\rho^2}\left|Q_{\rho,M} \cap \left\{u > \mu^- + \frac{\omega}{2}\right\}\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{M^\beta}{\rho^2}\left(|Q_{\rho,M}| - \left|Q_{\rho,M}\cap\left\{u<\mu^- + \frac{\omega}{2}\right\}\right|\right)\\
&< \frac{M^\beta}{\rho^2}(1-\theta_0)|Q_{\rho,M}| = (1-\theta_0)|B_\rho|.
\end{aligned}
$$

If $|B_\rho \cap \{\tilde{u}(s) > \mu^- + \frac{\omega}{2}\}| > \frac{1-\theta_0}{1-\theta}$ for all $-1 < s < -\theta$, then

$$
\begin{aligned}
&\int_{-1}^{0} \left|B_\rho \cap \left\{x\in B_\rho : \tilde{u}(s,x) > \mu^- + \frac{\omega}{2}\right\}\right| \, ds\\
&\quad \geq \int_{-1}^{-\theta_0} \left|B_\rho \cap \left\{x\in B_\rho : \tilde{u}(s,x) > \mu^- + \frac{\omega}{2}\right\}\right| \, ds\\
&\quad \geq (1-\theta_0)|B_\rho|,
\end{aligned}
$$

which is contradiction. □

Lemma 3.7. *There exists $r_0 = r_0(n,\alpha,\theta_0) > 0$ and $\delta_1 = \delta_1(n,\alpha,p,\theta_0) > 0$ such that for any $\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})} \leq \delta_1$, we have*

$$
\left|B_\rho \cap \left\{x : u(t,x) > \mu^+ - \frac{\omega}{2^{r_0}}\right\}\right| \leq \left(1 - \left(\frac{\theta_0}{2}\right)^2\right)|B_\rho|
$$

for $-\frac{\theta_0}{2}\frac{\rho^2}{M^\beta} < t < 0$.

Proof of Lemma 3.7. We take the cut-off function $\eta = \eta(x)$ as

$$
\eta \equiv 1 \text{ on } B_{(1-\sigma)\rho}\,,\ |\nabla\eta| \leq \frac{2}{\sigma\rho}
$$

where $\sigma > 0$ be chosen later. We rewrite the equation (3.1) by

$$
\partial_t u - (\varphi')^{-1}\Delta u = (\varphi')^{-1}\mathrm{div}\ F.
$$

Let $\lambda = \mu^- + \frac{\omega}{2}$, $c = \frac{\omega}{2^{r_0}}$, $H = \mu^+ - \lambda = \mathrm{osc}_{Q_{\rho,M}}\, u - \frac{\omega}{2}$ and

$$
g(\xi) := \log_+\left(\frac{H}{H - (\xi-\lambda)_+ + c}\right), \quad w = g(u).
$$

where the parameter $r_0 > 2$ will be chosen later. Testing the function $(g^2)'(u)\eta^2$ over $(\tau,t)\times B_\rho$ to the equation and using

$$
\nabla((\varphi')^{-1}(g^2)'\eta^2) = -\varphi''(\varphi')^{-2}(g^2)'\eta^2\nabla u + (\varphi')^{-1}(g^2)''\eta^2\nabla u
$$

$$+ (\varphi')^{-1}(g^2)'\nabla\eta^2,$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho} w^2\eta^2\,dx\Big|_\tau^t - \int_\tau^t\int_{B_\rho}\varphi''(\varphi')^{-2}(g^2)'|\nabla u|^2\eta^2\,dxdt \\
&\quad + \int_\tau^t\int_{B_\rho}(\varphi')^{-1}(g^2)''|\nabla u|^2\eta^2\,dxdt \\
&= -\int_\tau^t\int_{B_\rho}(\varphi')^{-1}(g^2)'\nabla u\cdot\nabla\eta^2\,dxdt \\
&\quad - \int_\tau^t\int_{B_\rho}\varphi''(\varphi')^{-2}(g^2)'\eta^2 F\cdot\nabla u\,dxdt \\
&\quad + \int_\tau^t\int_{B_\rho}(\varphi')^{-1}(g^2)''\eta^2 F\cdot\nabla u\,dxdt \\
&\quad + \int_\tau^t\int_{B_\rho}(\varphi')^{-1}(g^2)'F\cdot\nabla\eta^2\,dxdt \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.27}$$

By the Young inequality, we have

$$\begin{aligned}
I_1 &\le \int_\tau^t\int_{B_\rho}(\varphi')^{-1}w|\nabla w|^2\eta^2\,dxdt + 4\int_\tau^t\int_{B_\rho}(\varphi')^{-1}w|\nabla\eta|^2\,dxdt \\
I_2 &\le -\frac{1}{2}\int_\tau^t\int_{B_\rho}\varphi''(\varphi')^{-2}(g^2)'|\nabla u|^2\eta^2\,dxdt \\
&\quad -\frac{1}{2}\int_\tau^t\int_{B_\rho}\varphi''(\varphi')^{-2}(g^2)'|F|^2\eta^2\,dxdt \\
I_4 &\le 2\int_\tau^t\int_{B_\rho}(\varphi')^{-1}w|\nabla\eta|^2\,dxdt + 2\int_\tau^t\int_{B_\rho}(\varphi')^{-1}(g')^2 w|F|^2\eta^2\,dxdt.
\end{aligned}$$

Remarking that $g'' = (g')^2$, $(g^2)'' = 2(g')^2(1+g)$, we also have

$$\begin{aligned}
I_3 &\le \int_\tau^t\int_{B_\rho}(\varphi')^{-1}(g^2)''|\nabla u|^2\eta^2\,dxdt + \int_\tau^t\int_{B_\rho}(\varphi')^{-1}(g^2)''|F|^2\eta^2\,dxdt \\
&\le \frac{1}{2}\int_\tau^t\int_{B_\rho}(\varphi')^{-1}|\nabla w|^2\eta^2\,dxdt + \frac{1}{2}\int_\tau^t\int_{B_\rho}(\varphi')^{-1}w|\nabla w|^2\eta^2\,dxdt
\end{aligned}$$

$$+\int_\tau^t \int_{B_\rho} (\varphi')^{-1}(g^2)''|F|^2\eta^2\,dxdt.$$

Combining of those estimates, we have

$$\begin{aligned}
&\frac{1}{2}\int_{B_\rho} w^2(t)\eta^2(t)\,dx - \frac{1}{2}\int_\tau^t\int_{B_\rho} \varphi''(\varphi')^{-2}(g^2)'|\nabla u|^2\eta^2\,dxdt\\
&\quad+\frac{3}{2}\int_\tau^t\int_{B_\rho}(\varphi')^{-1}|\nabla w|^2\eta^2\,dxdt\\
&\quad+\frac{1}{2}\int_\tau^t\int_{B_\rho}(\varphi')^{-1}w|\nabla w|^2\eta^2\,dxdt\\
&\le \frac{1}{2}\int_{B_\rho} w^2(\tau)\eta^2(\tau)\,dx + 6\int_\tau^t\int_{B_\rho}(\varphi')^{-1}w|\nabla\eta|^2\,dxdt\\
&\quad-\frac{1}{2}\int_\tau^t\int_{B_\rho}\varphi''(\varphi')^{-2}(g^2)'|F|^2\eta^2\,dxdt\\
&\quad+2\int_\tau^t\int_{B_\rho}(\varphi')^{-1}(g')^2(1+2w)|F|^2\eta^2\,dxdt\\
&=: I_5+I_6+I_7+I_8.
\end{aligned}\tag{3.28}$$

For the simplicity, we put $\lambda' = \mu^+ - c = \mu^+ - \frac{\omega}{2^{r_0}}$. Since $\lambda' > \lambda$, we have

$$\begin{aligned}
\frac{1}{2}\int_{B_\rho} w^2(t)\eta^2(t)\,dx &\ge \int_{B_{(1-\sigma)\rho}\cap\{u(t)>\lambda'\}} w^2(t)\,dx\\
&\ge \int_{B_{(1-\sigma)\rho}\cap\{u(t)>\lambda'\}} \log^2\left(\frac{H}{H-(\lambda'-\lambda)+c}\right)dx\\
&\ge C(r_0-3)^2\big|B_{(1-\sigma)\rho}\cap\{u(t)>\lambda'\}\big|.
\end{aligned}\tag{3.29}$$

For I_5, we suppose that τ as in Lemma 3.6 with $\theta = \frac{\theta_0}{2}$. Then

$$w = \log_+\left(\frac{H}{H-(u-\lambda)_+ + c}\right) \le \log\left(\frac{\frac{1}{2}\omega}{\frac{1}{2^{r_0}}\omega}\right) = C(r_0-1)$$

and hence

$$\begin{aligned}
I_5 &\le \frac{1}{2}\int_{B_\rho\cap\{u(\tau)>\lambda\}} w^2(\tau)\,dx\\
&\le C(r_0-1)^2\big|B_\rho\cap\{u(\tau)>\lambda\}\big| \le C\frac{1-\theta_0}{1-\frac{\theta_0}{2}}(r_0-1)^2|B_\rho|.
\end{aligned}\tag{3.30}$$

Since $(\varphi')^{-1} = \alpha u^{1-\frac{1}{\alpha}}$ and $\mu^+ \le M$, we have

$$I_6 \le 6\alpha(\mu^+)^\beta (t-\tau)(r_0-1)\log 2\Big(\frac{2}{\sigma\rho}\Big)^2 |B_\rho| \le C\Big(\frac{r_0-1}{\sigma^2}\Big)|B_\rho|. \tag{3.31}$$

Next, we estimate I_7. Since

$$g' \le \frac{1}{c} = \frac{2^{r_0}}{\omega},\ (g^2)' \le \frac{2}{c}\log\frac{H}{c} \le C\frac{2^{r_0+1}}{\omega}(r_0-1)$$

and

$$u^{-\frac{1}{\alpha}} \le \lambda^{-\frac{1}{\alpha}} \le \Big(\frac{\omega}{2}\Big)^{-\frac{1}{\alpha}} \quad \text{for } u \ge \lambda,$$

we have

$$\begin{aligned}
I_7 &\le \log 2\Big(\frac{2^{r_0+1}}{\omega}\Big)(r_0-1)\int_\tau^t \int_{B_\rho\cap\{u(s)>\lambda\}} u^{-\frac{1}{\alpha}}|F(s)|^2\,dxds \\
&\le C_\alpha \frac{2^{r_0}}{\omega^2}(r_0-1)\omega^\beta \int_\tau^t \int_{B_\rho\cap\{u(s)>\lambda\}} |F(s)|^2\,dxds \\
&\le C_\alpha \frac{2^{r_0}}{\omega^2}(r_0-1)\omega^\beta \int_\tau^t \||F(s)|^2\|_{L^{\frac{p}{2}}(B_\rho)}|B_\rho|^{1-\frac{2}{p}}\,ds \\
&\le C_\alpha |B_1|^{-\frac{2}{p}}\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}2^{r_0}(r_0-1)\frac{\omega^\beta}{M^\beta}|B_\rho| \\
&\le C_\alpha |B_1|^{-\frac{2}{p}}\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}2^{r_0}(r_0-1)|B_\rho|,
\end{aligned}$$

where we have used that $\frac{3}{4}\omega \le \operatorname{osc}_{Q_{\rho,M}} u \le \mu^+ \le M$. As the estimate of I_7, we obtain for I_8 that

$$I_8 \le 2\alpha|B_1|^{-\frac{2}{p}}\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}2^{2r_0}(1+2(r_0-1)\log 2)|B_\rho|. \tag{3.32}$$

Combining of those estimate (3.29)–(3.32), we have

$$\big|B_{(1-\sigma)\rho}\cap\{x : u(t,x) > \lambda'\}\big| \le \Bigg\{\frac{1-\theta_0}{1-\frac{\theta_0}{2}}\Big(\frac{r_0-1}{r_0-3}\Big)^2 + \frac{C_1(\alpha)}{\sigma^2}\frac{r_0-1}{(r_0-3)^2}$$

$$+ C_2(n,\alpha,p)\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\frac{2^{r_0}(r_0-1)}{(r_0-3)^2}$$
$$\left. + C_3(n,\alpha,p)\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\frac{2^{2r_0}(1+2(r_0-1)\log 2)}{(r_0-3)^2}\right\}|B_\rho|.$$

Remarking that

$$\begin{aligned}|B_\rho\cap\{x:u(t,x)>\lambda'\}| &= |(B_\rho\setminus B_{(1-\sigma)\rho})\cap\{x:u(t,x)>\lambda'\}|\\ &\quad + |B_{(1-\sigma)\rho}\cap\{x:u(t,x)>\lambda'\}|,\end{aligned}$$

we have

$$\begin{aligned}&|B_{(1-\sigma)\rho}\cap\{x:u(t,x)>\lambda'\}|\\ &\quad\le\left\{\frac{1-\theta_0}{1-\frac{\theta_0}{2}}\left(\frac{r_0-1}{r_0-3}\right)^2+\frac{C_1(\alpha)}{\sigma^2}\frac{r_0-1}{(r_0-3)^2}+(1-(1-\sigma)^n)\right.\\ &\qquad\left.+\max\{C_2,C_3\}\frac{\rho^{\frac{2n}{p_*}}}{\omega^2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}C(r_0)\right\}|B_\rho|,\end{aligned}$$

where

$$C(r_0)=\max\left\{\frac{2^{r_0}(r_0-1)}{(r_0-3)^2},\frac{2^{2r_0}(1+2(r_0-1)\log 2)}{(r_0-3)^2}\right\}.$$

Now, we fix the parameters r_0,σ and δ_0 in the following way: Let $\sigma=\sigma(n,\theta_0)$ satisfying $1-(1-\sigma)^n\le\frac{1}{8}\theta_0^2$ and then we choose $r_0=r_0(n,\alpha,\theta_0)$ satisfying

$$\left(\frac{r_0-1}{r_0-3}\right)^2\le\left(1-\frac{\theta_0}{2}\right)(1+\theta_0)\text{ and }\frac{C_1(\alpha)}{\sigma^2}\frac{r_0-1}{(r_0-3)^2}\le\frac{1}{8}\theta_0^2.$$

Finally, we choose $\delta_1=\delta_1(n,\alpha,p,\theta_0)>0$ sufficiently small such that

$$\max\{C_2,C_3\}C(r_0)\le\delta_1\le\frac{1}{2}\theta_0^2.$$

Then, if $\rho^{\frac{2n}{p_*}}\omega^{-2}\|F\|^2_{L^\infty(L^p)(Q_{\rho,M})}\le\delta_1$, we have

$$\big|B_\rho\cap\{x:u(t,x)>\lambda'\}\big|\le\left(1-\left(\frac{\theta_0}{2}\right)^2\right)|B_\rho|.\ \Box$$

LEMMA 3.8 (the Caccioppoli estimates for super level sets). *Let $I^{\theta_0}_{\rho,M} = (-\frac{\theta_0}{2}\frac{\rho^2}{M}, 0)$ and let $\eta = \eta(t,x)$ be a cut-off function in $Q^{\theta_0}_{\rho,M}$. For $\lambda \geq \mu^+ - \frac{\omega}{2}$, we have*

$$
\begin{aligned}
&\frac{1}{2}\varphi'(M) \sup_{t \in I^{\theta_0}_{\rho,M}} \int_{B_\rho} (u(t)-\lambda)_+^2 \eta^2 \, dx \\
&\qquad + \frac{1}{4} \iint_{Q^{\theta_0}_{\rho,M}} |\nabla (u-\lambda)_+|^2 \eta^2 \, dxdt \\
&\leq \frac{1}{2}\varphi'\Big(\frac{1}{4}\mu^+\Big) \iint_{Q^{\theta_0}_{\rho,M}} (u-\lambda)_+^2 \partial_t \eta^2 \, dxdt \\
&\qquad + 3 \iint_{Q^{\theta_0}_{\rho,M}} (u-\lambda)_+^2 |\nabla \eta|^2 \, dxdt \\
&\qquad + \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \bigg(\int_{I^{\theta_0}_{\rho,M}} |B_\rho \cap \{u(t) > \lambda\}|^{\frac{r_*}{q_*}} \, dt \bigg)^{\frac{2}{r_*}(1+\frac{2}{p_*})}.
\end{aligned} \tag{3.33}
$$

PROOF OF LEMMA 3.8. The estimate is similarly obtained by Lemma 3.5. We take a test function $(u-\lambda)_+\eta^2$ to (3.1) and integrate by parts, we obtain

$$
\begin{aligned}
&\sup_{I^{\theta_0}_{\rho,M}} \int_{B_\rho} \bigg(\int_0^{(u-\lambda)_+} \varphi'(\lambda+\xi)\xi \, d\xi \bigg) \eta^2 \, dx \\
&\qquad + \iint_{Q^{\theta_0}_{\rho,M}} |\nabla (u-\lambda)_+|^2 \eta^2 \, dxdt \\
&\leq \iint_{Q^{\theta_0}_{\rho,M}} \bigg(\int_0^{(u-\lambda)_+} \varphi'(\lambda+\xi)\xi \, d\xi \bigg) \partial_t \eta^2 \, dxdt \\
&\qquad - \iint_{Q^{\theta_0}_{\rho,M}} \nabla (u-\lambda)_+ \cdot \nabla \eta^2 (u-\lambda)_+ \, dxdt \\
&\qquad + \iint_{Q^{\theta_0}_{\rho,M}} F \cdot \nabla (u-\lambda)_+ \eta^2 \, dxdt \\
&\qquad + \iint_{Q^{\theta_0}_{\rho,M}} F \cdot \nabla \eta^2 (u-\lambda)_+ \, dxdt \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.34}
$$

By the Young inequality, we have

$$
(3.35)\quad
\begin{aligned}
I_2 &\le \frac{1}{2}\iint_{Q^{\theta_0}_{\rho,M}} |\nabla(u-\lambda)_+|^2\eta^2\,dxdt \\
&\quad + 2\iint_{Q^{\theta_0}_{\rho,M}} (u-\lambda)_+^2|\nabla\eta|^2\,dxdt, \\
I_3 &\le \frac{1}{4}\iint_{Q^{\theta_0}_{\rho,M}} |\nabla(u-\lambda)_+|^2\eta^2\,dxdt + \iint_{Q^{\theta_0}_{\rho,M}\cap\{u>\lambda\}} |F|^2\eta^2\,dxdt, \\
I_4 &\le \iint_{Q^{\theta_0}_{\rho,M}} (u-\lambda)_+^2|\nabla\eta|^2\,dxdt + \iint_{Q^{\theta_0}_{\rho,M}\cap\{u>\lambda\}} |F|^2\eta^2\,dxdt.
\end{aligned}
$$

Since φ' is monotone decreasing,

$$
\varphi'(\lambda+\xi) \ge \varphi'(u) \ge \varphi'(\mu^+) \ge \varphi'(M) \quad \text{for } 0 \le \xi \le (u-\lambda)_+,
$$

and we have

$$
(3.36)\qquad \int_0^{(u-\lambda)_+} \varphi'(\lambda+\xi)\xi\,d\xi \ge \frac{1}{2}\varphi'(M)(u-\lambda)_+^2.
$$

Finally, we estimate of I_1. Since $\frac{3}{4}\omega \le \operatorname{osc}_{Q_{\rho,M}} u \le \mu^+$, we have

$$
\varphi'(\lambda+\xi) \le \varphi'(\lambda) \le \varphi'\Big(\mu^+ - \frac{\omega}{2}\Big) \le \varphi'\Big(\frac{1}{4}\mu^+\Big)
$$

and hence

$$
(3.37)\qquad I_1 \le \frac{1}{2}\varphi'\Big(\frac{1}{4}\mu^+\Big)\iint_{Q^{\theta_0}_{\rho,M}} (u-\lambda)_+^2\partial_t\eta^2\,dxdt.
$$

Combining of those estimates (3.35), (3.36), (3.37), we obtain

$$
\begin{aligned}
&\frac{1}{2}\varphi'(M)\sup_{t\in I^{\theta_0}_{\rho,M}}\int_{B_\rho}(u(t)-\lambda)_+^2\eta^2\,dx + \frac{1}{4}\iint_{Q^{\theta_0}_{\rho,M}}|\nabla(u-\lambda)_+|^2\eta^2\,dxdt \\
&\le \frac{1}{2}\varphi'\Big(\frac{1}{4}\mu^+\Big)\iint_{Q^{\theta_0}_{\rho,M}}(u-\lambda)_+^2\partial_t\eta^2\,dxdt + 3\iint_{Q^{\theta_0}_{\rho,M}}(u-\lambda)_+^2|\nabla\eta|^2\,dxdt \\
&\qquad + 2\iint_{Q^{\theta_0}_{\rho,M}\cap\{u>\lambda\}}|F|^2\eta^2\,dxdt.
\end{aligned}
$$

As the same argument of proof of Lemma 3.5, we have

$$\iint_{Q^{\theta_0}_{\rho,M}\cap\{u>\lambda\}} |F|^2\eta^2\,dxdt$$
$$\leq \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})}\left(\int_{I^{\theta_0}_{\rho,M}} |B_\rho\cap\{u(t)>\lambda\}|^{\frac{r_*}{q_*}}\,dt\right)^{\frac{2}{r_*}(1+\frac{2}{p_*})}.$$

Substituting this estimate, we obtain (3.33). □

LEMMA 3.9 (the hole filling argument). *Let* $\rho_0 = \frac{3}{4}\rho$. *For* $0 < \nu < 1$, *there exists* $q_0 = q_0(n,\alpha,p,\theta_0,\nu) \geq 0$ *such that for* $\rho^{\frac{2n}{p_*}}\omega^{-2}\|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \leq \min\{\theta_0^{-1}2^{-2q_0},\delta_1\}$, *we have*

$$\left|Q^{\theta_0}_{\rho_0,M}\cap\left\{(t,x):u(t,x)>\mu^+-\frac{\omega}{2^{q_0+1}}\right\}\right| \leq \nu|Q^{\theta_0}_{\rho_0,M}|,$$

where $r_0,\delta_1>0$ *is as in Lemma 3.7.*

PROOF OF LEMMA 3.9. We fix $t\in I^{\theta_0}_{\rho,M} = \left(-\frac{\theta_0}{2}\frac{\rho^2}{M^\beta},0\right)$ and set

$$\lambda' := \mu^+ - \frac{\omega}{2^{k+1}}\,,\ \lambda := \mu^+ - \frac{\omega}{2^k},$$

where $k\geq r_0$. By the Poincaré type inequality (Lemma A.2), we have

$$\frac{\omega}{2^{k+1}}|B_{\rho_0}\cap\{x:u(t,x)>\lambda'\}|$$
$$\leq \frac{C_n\rho_0^{n+1}}{|B_{\rho_0}\cap\{x:u(t,x)\leq\lambda\}|}\int_{B_{\rho_0}\cap\{\lambda<u(t)\leq\lambda'\}}|\nabla u(t)|\,dx.$$

Since $\lambda>\mu^+-\frac{\omega}{2^{r_0}}$ and Lemma 3.7, we have

$$|B_{\rho_0}\cap\{x:u(t,x)\leq\lambda\}| = |B_{\rho_0}| - |B_{\rho_0}\cap\{x:u(t,x)>\lambda\}| \geq \left(\frac{\theta_0}{2}\right)^2|B_{\rho_0}|$$

and hence

$$\frac{\omega}{2^{k+1}}|B_{\rho_0}\cap\{x:u(t,x)>\lambda'\}| \leq \frac{C_n\rho_0}{\theta_0^2}\int_{B_{\rho_0}\cap\{\lambda<u(t)\leq\lambda'\}}|\nabla u(t)|\,dx. \tag{3.38}$$

Integrating (3.38) over $I^{\theta_0}_{\rho,M}$, we obtain

$$\begin{aligned}
&\frac{\omega}{2^{k+1}}\big|Q^{\theta_0}_{\rho_0,M}\cap\{u>\lambda'\}\big|\\
&\quad\le\frac{C_n\rho_0}{\theta_0^2}\int_{I^{\theta_0}_{\rho,M}}\int_{B_{\rho_0}\cap\{\lambda<u(t)\le\lambda'\}}|\nabla u(t)|\,dxdt\\
&\quad\le\frac{C_n\rho_0}{\theta_0^2}\|\nabla(u-\lambda)_+\|_{L^2(Q^{\theta_0}_{\rho_0,M})}\big|Q^{\theta_0}_{\rho_0,M}\cap\{\lambda<u\le\lambda'\}\big|^{\frac12}.
\end{aligned}$$

Let $\eta\in C_0^\infty(I\times\mathbb{R}^n)$ be a smooth cut off function satisfying $\eta\equiv1$ on $Q^{\theta_0}_{\rho_0,M}$. By the Caccioppoli estimate (Lemma 3.8), we have

$$\begin{aligned}
&\|\nabla(u-\lambda)_+\|^2_{L^2(Q^{\theta_0}_{\rho_0,M})}\\
&\quad\le\|\nabla(u-\lambda)_+\eta\|^2_{L^2(Q^{\theta_0}_{\rho,M})}\\
&\quad\le C_\alpha\bigg\{\iint_{Q^{\theta_0}_{\rho,M}}(u-\lambda)_+^2(|\nabla\eta|^2+\varphi'(\mu^+)\partial_t\eta^2)\,dxdt\\
&\qquad+\|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})}\int_{I^{\theta_0}_{\rho,M}}|B_\rho\cap\{u(t)>\lambda\}|^{1-\frac2p}\,dt\bigg\}\\
&\quad=:I_1+I_2.
\end{aligned}\tag{3.39}$$

Since we can further assume

$$\begin{aligned}
|\nabla\eta(t,x)|\le&\frac{8}{\rho},\\
\partial_t\eta(t,x)\le&\frac{10M^\beta}{\theta_0\rho^2},
\end{aligned}$$

we proceed to estimate I_1 by using $M\le4\mu_+$,

$$\begin{aligned}
I_1&\le C_\alpha(\mu^+-\lambda)_+^2\bigg(\frac{1}{\rho^2}+\frac{M^\beta}{\theta_0\rho^2}\varphi'(\mu^+)\bigg)|Q^{\theta_0}_{\rho_0,M}|\\
&\le C_\alpha\Big(\frac{\omega}{2^k}\Big)^2\frac{1}{\theta_0\rho^2}\Big(\frac{M}{\mu^+}\Big)^\beta|Q^{\theta_0}_{\rho_0,M}|\le C_\alpha\Big(\frac{\omega}{2^k}\Big)^2\frac{1}{\theta_0\rho^2}|Q^{\theta_0}_{\rho_0,M}|.
\end{aligned}\tag{3.40}$$

On the other hand, since

$$
\begin{aligned}
\int_{I^{\theta_0}_{\rho,M}} & |B_\rho \cap \{x : u(t,x) > \lambda\}|^{1-\frac{2}{p}}\, dt \\
&\le C_n |B_\rho|^{-\frac{2}{p}} |Q^{\theta_0}_{\rho_0,M}| \\
&\le C_{n,p} \left(\rho^{2-\frac{2n}{p}} \left(\frac{2^k}{\omega} \right)^2 \theta_0 \right) \frac{1}{\theta_0 \rho^2} \left(\frac{\omega}{2^k} \right)^2 |Q^{\theta_0}_{\rho_0,M}|,
\end{aligned}
$$

we obtain

$$
I_2 \le C_{n,\alpha,p} \Big(\|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \rho^{\frac{2n}{p*}} \omega^{-2} 2^{2k} \theta_0 \Big) \frac{1}{\theta_0 \rho^2} \left(\frac{\omega}{2^k} \right)^2 |Q^{\theta_0}_{\rho_0,M}|. \tag{3.41}
$$

Combining of those estimates (3.39), (3.40) and (3.41), we obtain

$$
\begin{aligned}
&\|\nabla(u-\lambda)_+\|^2_{L^2(Q^{\theta_0}_{\rho_0,M})} \\
&\quad \le C_{n,\alpha,p} \Big(1 + \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \rho^{\frac{2n}{p*}} \omega^{-2} 2^{2k} \theta_0 \Big) \frac{1}{\theta_0 \rho^2} \left(\frac{\omega}{2^k} \right)^2 |Q^{\theta_0}_{\rho_0,M}|
\end{aligned}
$$

and hence

$$
\begin{aligned}
&\left(\frac{\omega}{2^{k+1}} \right)^2 \big| Q^{\theta_0}_{\rho_0,M} \cap \{(t,x) : u(t,x) > \lambda'\} \big|^2 \\
&\qquad \le \frac{C_{n,\alpha,p}}{\theta_0^5} \left(\frac{\omega}{2^k} \right)^2 \Big(1 + \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \rho^{\frac{2n}{p*}} \omega^{-2} 2^{2k} \theta_0 \Big) \big| Q^{\theta_0}_{\rho_0,M} \big| \\
&\qquad\quad \cdot \big| Q^{\theta_0}_{\rho_0,M} \cap \{\lambda < u \le \lambda'\} \big|.
\end{aligned} \tag{3.42}
$$

Summing over $k = r_0 + 1, \ldots, q_0$, we have

$$
\begin{aligned}
(q_0 - r_0)&\left|Q^{\theta_0}_{\rho_0,M} \cap \left\{u > \mu^+ - \frac{\omega}{2^{q_0+1}}\right\}\right|^2 \\
&\leq \sum_{k=r_0+1}^{q_0} \left|Q^{\theta_0}_{\rho_0,M} \cap \left\{u > \mu^+ - \frac{\omega}{2^{k+1}}\right\}\right|^2 \\
&\leq \frac{C_{n,\alpha,p}}{\theta_0^5}|Q^{\theta_0}_{\rho_0,M}| \sum_{k=r_0+1}^{q_0} \left(1 + \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \rho^{\frac{2n}{p_*}} \omega^{-2} 2^{2k}\theta_0\right) \\
&\quad \times \left|Q^{\theta_0}_{\rho_0,M} \cap \left\{\mu^+ - \frac{\omega}{2^k} < u \leq \mu^+ - \frac{\omega}{2^{k+1}}\right\}\right| \\
&\leq \frac{C_{n,\alpha,p}}{\theta_0^5}|Q^{\theta_0}_{\rho_0,M}| \left(1 + \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \rho^{\frac{2n}{p_*}} \omega^{-2} 2^{2q_0}\theta_0\right) \\
&\quad \times \sum_{k=r_0+1}^{\infty} \left|Q^{\theta_0}_{\rho_0,M} \cap \left\{\mu^+ - \frac{\omega}{2^k} < u \leq \mu^+ - \frac{\omega}{2^{k+1}}\right\}\right| \\
&\leq \frac{C_{n,\alpha,p}}{\theta_0^5}|Q^{\theta_0}_{\rho_0,M}|^2 \cdot \left(1 + \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \rho^{\frac{2n}{p_*}} \omega^{-2} 2^{2q_0}\theta_0\right).
\end{aligned}
\tag{3.43}
$$

Choosing $q_0 > 0$ enough large such that

$$
\frac{C_{n,\alpha,p}}{\theta_0^5(q_0 - r_0)} \leq \nu^2,
$$

we have by the assumption

$$
\frac{\rho^{\frac{2n}{p_*}}}{\omega^2} \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \leq \min\{\theta_0^{-1} 2^{-2q_0}, \delta_1\}
$$

and (3.43) that

$$
\left|Q^{\theta_0}_{\rho_0,M} \cap \left\{u > \mu^+ - \frac{\omega}{2^{q_0+1}}\right\}\right|^2 \leq \nu^2 |Q^{\theta_0}_{\rho_0,M}|^2. \ \square
$$

PROOF OF LEMMA 3.3. Let $0 < \mu < 1$ be chosen later. We take $\delta_1 > 0$ as in Lemma 3.7 and q_0 as in Lemma 3.9. We put $\delta_2 := \min\{2^{-2q_0}\theta_0^{-1}, \delta_1\}$. Introduce the scale transform

$$
s = M^\beta(t - t_0), \qquad Q^{\theta_0}_\rho = Q^{\theta_0}_{\rho,1}, \qquad I^{\theta_0}_\rho = I^{\theta_0}_{\rho,1}
$$

$$\tilde{u}(s,x) = u(t,x), \qquad \tilde{\eta}(s,x) = \eta(t,x), \qquad \tilde{\psi}(s,x) = \psi(t,x).$$

Then we rewrite the Caccioppoli estimate (3.33) as follows:

$$\begin{aligned}
&\varphi'(M) \sup_{I_\rho^{\theta_0}} \int_{B_\rho} (\tilde{u}-\lambda)_+^2 \tilde{\eta}^2 \, dx + \frac{1}{M^\beta} \iint_{Q_\rho^{\theta_0}} |\nabla(\tilde{u}-\lambda)_+|^2 \tilde{\eta}^2 \, dxds \\
&\le C_\alpha \Bigg\{ \iint_{Q_\rho^{\theta_0}} (\tilde{u}-\lambda)_+^2 \Big\{ \varphi'(\mu^+)\partial_s \tilde{\eta}^2 + \frac{1}{M^\beta} |\nabla \tilde{\eta}|^2 \Big\} \, dxds \\
&\quad + \frac{1}{M^\beta} \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \\
&\quad \times \Big(\int_{-\frac{\theta_0}{2}\rho^2}^0 |B_\rho \cap \{\tilde{u}(s) > \lambda\}|^{\frac{r_*}{q_*}} \, ds \Big)^{\frac{2}{r_*}(1+\frac{2}{p_*})} \Bigg\}.
\end{aligned} \tag{3.44}$$

For $k \in \mathbb{N} \cup \{0\}$, we take $\rho = \rho_k$, $\lambda = \lambda_k$, $\tilde{\eta} = \tilde{\eta_k}$ satisfying $\tilde{\eta_k} \equiv 1$ on $Q^{\theta_0}_{\rho_{k+1}}$ and

$$\begin{aligned}
&\lambda_k = \mu^+ - \frac{1}{2^{q_0}}\omega + \frac{1}{2^{q_0+k+2}}\omega, \qquad \rho_k = \frac{1}{2}\rho + \frac{1}{2^{k+2}}\rho, \\
&Y_k := \frac{|Q_{\rho_k} \cap \{\tilde{u} > \lambda_k\}|}{|Q_{\rho_0}|}, \\
&Z_k = \frac{\rho_0^2}{|Q_{\rho_0}|} \Big(\int_{-\rho_k^2}^0 |B_{\rho_k} \cap \{\tilde{u}(s) > \lambda_k\}|^{\frac{r_*}{q_*}} \, ds \Big)^{\frac{2}{r_*}}, \\
&|\nabla \tilde{\eta_k}| \le \frac{2}{\rho_k - \rho_{k+1}} \le \frac{12 \cdot 2^k}{\rho_0}, \qquad \partial_s \tilde{\eta_k} \le \frac{4}{\theta_0} \frac{1}{\rho_k^2 - \rho_{k+1}^2} \le \frac{48 \cdot 2^{2k}}{\theta_0 \rho^2}.
\end{aligned}$$

Multiplying $\varphi'(M)^{-1}$ by (3.44), we have

$$\begin{aligned}
&\sup_{-\frac{\theta_0}{2}\rho_k^2 < t < 0} \int_{B_{\rho_k}} (\tilde{u}-\lambda_k)_+^2 \tilde{\eta_k}^2 \, dx + \frac{1}{\varphi'(M) M^\beta} \iint_{Q^{\theta_0}_{\rho_k}} |\nabla(\tilde{u}-\lambda_k)_+|^2 \tilde{\eta_k}^2 \, dxds \\
&\le C_\alpha \Bigg\{ \iint_{Q^{\theta_0}_{\rho_k}} (\tilde{u}-\lambda_k)_+^2 \Big\{ \frac{\varphi'(\mu^+)}{\varphi'(M)} \frac{1}{\theta_0} + \frac{1}{\varphi'(M) M^\beta} \Big\} \frac{2^{2k}}{\rho_0^2} \, dxds \\
&\quad + \frac{1}{\varphi'(M) M^\beta} \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \Big(\int_{-\frac{\theta_0}{2}\rho_k^2}^0 |B_{\rho_k} \cap \{\tilde{u}(s) > \lambda_k\}|^{\frac{r_*}{q_*}} \, ds \Big)^{\frac{2}{r_*}(1+\frac{2}{p_*})} \Bigg\} \\
&\le C_{\alpha,\theta_0} \Bigg\{ 2^{2k-2q_0} \omega^2 \frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0} Y_k + \|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})} \Big(\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0} \Big)^{1+\frac{2}{p_*}} Z_k^{1+\frac{2}{p_*}} \Bigg\}
\end{aligned}$$

since $(\tilde{u}-\lambda_k)_+ \le \frac{\omega}{2^{q_0+1}}$ and $M \le 4\mu^+$. Remarking that

$$\begin{aligned}\|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})}\omega^{-2}\left(\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0}\right)^{\frac{2}{p_*}} &\le C_{n,p,\theta_0}\|F\|^2_{L^\infty(L^p)(Q^{\theta_0}_{\rho,M})}\omega^{-2}\rho_0^{\frac{2n}{p_*}}\\ &\le C_{n,p,\theta_0}\delta_2 \le C_{n,p,\theta_0}2^{-2q_0},\end{aligned}$$

we obtain

$$\begin{aligned}&\|(\tilde{u}-\lambda_k)_+\tilde{\eta_k}\|^2_{L^\infty(L^2)(Q^{\theta_0}_{\rho_k})\cap L^2(H^1)(Q^{\theta_0}_{\rho_k})}\\ &\le C_{n,\alpha,p,\theta_0}2^{-2q_0}\omega^2\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0}\Big\{2^{2k}Y_k + Z_k^{1+\frac{2}{p_*}}\Big\}\end{aligned}$$

By the Ladyženskaja inequality and the Hölder inequality, we have

$$\begin{aligned}\|(\tilde{u}-\lambda_k)_+\tilde{\eta_k}\|^2_{L^2(Q^{\theta_0}_{\rho_k})} &\le \|(\tilde{u}-\lambda_k)_+\tilde{\eta_k}\|^2_{L^{2+\frac{4}{n}}(Q^{\theta_0}_{\rho_k})}\|\chi_{\{\tilde{u}>\lambda_k\}}\|^2_{L^{n+2}(Q^{\theta_0}_{\rho_k})}\\ &\le C_{n,\alpha,p,\theta_0}2^{-2q_0}\omega^2\frac{|Q^{\theta_0}_{\rho_0}|^{1+\frac{2}{n+2}}}{\rho_0^2}Y_k^{\frac{2}{n+2}}\Big\{2^{2k}Y_k + Z_k^{1+\frac{2}{p_*}}\Big\}\end{aligned}$$

and

$$\|(\tilde{u}-\lambda_k)_+\tilde{\eta_k}\|^2_{L^{r_*}(L^{q_*})(Q^{\theta_0}_{\rho_k})} \le C_{n,\alpha,p,\theta_0}2^{-2q_0}\omega^2\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0^2}\Big\{2^{2k}Y_k + Z_k^{1+\frac{2}{p_*}}\Big\}.$$

Since

$$\begin{aligned}\|(\tilde{u}-\lambda_k)_+\tilde{\eta_k}\|^2_{L^2(Q^{\theta_0}_{\rho_k})} &\ge \|(\tilde{u}-\lambda_k)_+\|^2_{L^2(Q^{\theta_0}_{\rho_{k+1}}\cap\{\tilde{u}>\lambda_{k+1}\})}\\ &\ge (\lambda_{k+1}-\lambda_k)^2|Q^{\theta_0}_{\rho_{k+1}}\cap\{\tilde{u}>\lambda_{k+1}\}|\\ &= \left(\frac{\omega}{2^{q_0+k+3}}\right)^2|Q^{\theta_0}_{\rho_0}|Y_{k+1}\end{aligned}$$

and

$$\begin{aligned}\|(\tilde{u}-\lambda_k)_+\tilde{\eta_k}\|^2_{L^{r_*}(L^{q_*})(Q^{\theta_0}_{\rho_k})} &\ge \|(\tilde{u}-\lambda_k)_+\|^2_{L^{r_*}(L^{q_*})(Q^{\theta_0}_{\rho_{k+1}}\cap\{\tilde{u}>\lambda_{k+1}\})}\\ &\ge \left(\frac{\omega}{2^{q_0+k+3}}\right)^2\frac{|Q^{\theta_0}_{\rho_0}|}{\rho_0^2}Z_{k+1},\end{aligned}$$

we obtain

$$Y_{k+1} \le C_{n,\alpha,p,\theta_0}\Big\{2^{4k}Y_k^{1+\frac{2}{n+2}} + 2^{2k}Y_k^{\frac{2}{n+2}}Z_k^{1+\frac{2}{p_*}}\Big\}$$

and

$$Z_{k+1} \le C_{n,\alpha,p,\theta_0}\Big\{2^{4k}Y_k + 2^{2k}Z_k^{1+\frac{2}{p_*}}\Big\}.$$

Since $\frac{r_*}{q_*} < 1$ and $\frac{n}{2} = \frac{2}{r_*} + \frac{n}{q_*}$, we have

$$\begin{aligned} Z_0 &= \frac{\rho_0^2}{|Q_{\rho_0}^{\theta_0}|}\left(\int_{-\rho_0^2}^0 |B_{\rho_0} \cap \{\tilde{u}(s) > \lambda_0\}|^{\frac{r_*}{q_*}}\,ds\right)^{\frac{2}{r_*}} \\ &\le \frac{\rho_0^2}{|Q_{\rho_0}^{\theta_0}|}\left(\int_{-\rho_0^2}^0 |B_{\rho_0} \cap \{\tilde{u}(s) < \lambda_0\}|\,ds\right)^{\frac{2}{q_*}} \rho^{\frac{4}{r_*}(1-\frac{r_*}{q_*})} \le C_{n,p,\theta_0} Y_0^{\frac{2}{q_*}}. \end{aligned}$$

By Lemma A.1, there exists $0 < \nu = \nu(n,\alpha,p,\theta_0) < 1$ such that if $Y_0 \le \nu$, then $Y_k \to 0$ as $k \to \infty$, i.e.

$$\tilde{u}(s,x) < \mu^+ - \frac{\omega}{2^{q_0+2}} \quad \text{a.e. } (s,x) \in Q_{\frac{\rho}{2}}^{\theta_0}. \ \square$$

4. The Asymptotic Profile

In this section, we show the asymptotic convergence of the weak solution $u(t,x)$ of (1.1) to the Barenblatt-Pattle solution by using the uniform Hölder estimate.

THEOREM 4.1. *Let $1 < \alpha \le 2 - \frac{2}{n}$. Then for any positive initial data $u_0 \in L_a^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ with $a > n$, the corresponding global weak solution $u(t,x)$ satisfies the following asymptotic behavior: For $M = \|u_0\|_1$, with the condition* (1.5)*, there exists a constant $C > 0$ such that*

$$\|u(t) - \mathcal{U}(t)\|_1 \le C(1+\sigma t)^{-\nu},$$

for $t > 0$, where $\sigma = n(\alpha-1)+2$ and $0 < \nu < 2$ with $\|\mathcal{U}(t)\|_1 = M$.

To show the asymptotic convergence, we consider as we mentioned in the introduction that the self-similar transform of the system and consider the weak solution of the rescaled system:

$$
(4.1)\qquad \begin{cases} \partial_{t'} v - \mathrm{div}_{x'}(\nabla_{x'} v^{\alpha} + x' v - e^{-\kappa t'} v \nabla_{x'} \phi) = 0, & t' > 0\,,\ x' \in \mathbb{R}^n, \\ -\, e^{-2t'} \Delta_{x'} \phi + \phi = v, & t' > 0\,,\ x' \in \mathbb{R}^n, \\ v(0, x') = u_0(x') \geq 0, & x' \in \mathbb{R}^n, \end{cases}
$$

where $\kappa = n + 2 - \sigma = n(2 - \alpha)$.

In what follows, we only treat the scaled system (2.2) and hence we use a simpler notations as $t' \to t$ and $x' \to x$ if it does not cause any confusion.

Applying the method of the transport equation or the Fokker-Planck equation due to Carrillo-Toscani [12], we compute the time derivative of the free energy functional: For a weak solution v and ϕ of (2.2) , we let for $\kappa = n(2 - \alpha)$,

$$
\begin{aligned}
H(v(t)) &:= \frac{1}{\alpha - 1} \int_{\mathbb{R}^n} v^{\alpha}(t) dx + \frac{1}{2} \int_{\mathbb{R}^n} |x|^2 v(t) dx, \\
J(v(t)) &:= \int_{\mathbb{R}^n} v(t) \left| \nabla \left(\frac{\alpha}{\alpha - 1} v^{\alpha - 1}(t) + \frac{|x|^2}{2} \right) \right|^2 dx, \\
I(v(t)) &:= \int_{\mathbb{R}^n} v(t) \left| \nabla \left(\frac{\alpha}{\alpha - 1} v^{\alpha - 1}(t) + \frac{|x|^2}{2} - e^{-\kappa t} \phi(t) \right) \right|^2 dx.
\end{aligned}
$$

The key idea to show the asymptotic convergence is to consider the decay of the dissipative flux term $I(v)$ in t. We firstly observe that the entropy functional has a certain relation:

Proposition 4.2. *For a weak solution v and ϕ of* (2.2) *, we have*

$$
(4.2)\qquad \begin{aligned}
H(v(t)) + \frac{1}{2} e^{-\kappa t} (e^{-2t} \|\nabla \phi(t)\|_2^2 + \|\phi(t)\|_2^2) + \int_s^t J(v(\tau)) d\tau \\
\leq H(v(s)) + \frac{1}{2} e^{-\kappa s} (e^{-2s} \|\nabla \phi(s)\|_2^2 + \|\phi(s)\|_2^2) \\
+ \int_s^t e^{-\kappa \tau} \left[\frac{2 - \kappa}{2} e^{-2\tau} \|\nabla \phi(\tau)\|_2^2 - \frac{\kappa}{2} \|\phi(\tau)\|_2^2 \right] d\tau \\
+ \int_s^t e^{-2\kappa \tau} \int_{\mathbb{R}^n} v(\tau) |\nabla \phi(\tau)|^2 dx d\tau,
\end{aligned}
$$

where $\kappa = n(2-\alpha)$. In particular, for $1 < \alpha \le 2 - \frac{2}{n}$, we have that $H(v(t))$ is uniformly bounded in t under the smallness condition (2.4)

$$\begin{aligned} H(v(t)) \le H(u_0) &- \frac{1}{2}\int_{\mathbb{R}^n} \phi(0)v(0)\,dx \\ &+ C\sup_{\tau>0}\left[e^{-2\kappa\tau}\|v(\tau)\|_\infty \|\nabla\phi(\tau)\|_2^2\right] \end{aligned} \tag{4.3}$$

for any $t > 0$.

PROOF OF PROPOSITION 4.2. Decomposing $W_s(t)$ into $H(t)$ and terms with ϕ, we see formally that

$$\begin{aligned} &\frac{d}{dt}\Big[H(v(t)) + \frac{1}{2}e^{-\kappa t}(\|\phi(t)\|_2^2 + e^{-2t}\|\nabla\phi(t)\|_2^2)\Big] + J(v(t)) \\ =&e^{-2\kappa t}\int_{\mathbb{R}^n} v(t)|\nabla\phi(t)|^2 dx + e^{-\kappa t}\Big[\frac{2-\kappa}{2}e^{-2t}\|\nabla\phi(t)\|_2^2 - \frac{\kappa}{2}\|\phi(t)\|_2^2\Big]. \end{aligned} \tag{4.4}$$

Integrate (4.4) over $[s,t]$ we obtain (4.2). Under the condition $1 < \alpha \le 2-\frac{2}{n}$, we have $\kappa \ge 2$ and by Proposition 2.1 $\|v(t)\|_\infty \le C$ and $e^{-t}\|\nabla\phi(t)\|_2 \le C$. Therefore it follows

$$\begin{aligned} H(v(t)) &+ \int_0^t J(v(\tau))d\tau \\ &\le H(v(0)) + \frac{1}{2}\Big[\|\nabla\phi(0)\|_2^2 + \|\phi(0)\|_2^2\Big] \\ &\quad + C\sup_{t>0}\Big(e^{-2t}\|v(t)\|_\infty\|\nabla\phi(t)\|_2^2\Big) \\ &\le H(v(0)) - \frac{1}{2}\int_{\mathbb{R}^n} v(0)\phi(0)dx + C\sup_{t>0}\Big(e^{-2t}\|v(t)\|_\infty\|\nabla\phi(t)\|_2^2\Big) \end{aligned}$$

for all $t > 0$. □

For a solution v and ϕ of (2.2), we let

$$I(v(t)) \equiv \int_{\mathbb{R}^n} v(t,x)\,|K(x,v(t),\phi(t))|^2\,dx.$$

where $K(x,v,\phi) = x + \frac{\alpha}{\alpha-1}\nabla v^{\alpha-1}(t,x) - e^{-\kappa t}\nabla\phi(t,x)$ and $\kappa = n(2-\alpha)$. It is not so difficult to see that the asymptotic profile is given by $J(v(t)) \to 0$

from the above inequality. However to obtain the convergence rate for a weak solution in the weighted class $L^1_2(\mathbb{R}^n)\cap L^\infty(\mathbb{R}^n)$, we derive that $I(v(t))$ is exponentially decaying. To this end, we observe the time derivative of the functional $I(v(t))$. We assume that $\kappa > 0$ namely $\alpha < 2$.

Following [12], we formally have

$$
\begin{aligned}
\frac{d}{dt} I(v(t)) = & -2 \int_{\mathbb{R}^n} v |K(x,v,\phi)|^2 \, dx \\
& - 2(\alpha - 1) \int_{\mathbb{R}^n} v^\alpha \Big| \mathrm{div}\ K(x,v,\phi) \Big|^2 \, dx \\
& - 2 \int_{\mathbb{R}^n} v^\alpha \left| \nabla K(x,v,\phi) \right|^2 \, dx \\
& + 2e^{-\kappa t} \int_{\mathbb{R}^n} v K_i(x,v,\phi) K_j(x,v,\phi) \left(D^2_{ij} \phi \right) \, dx \\
& + 2e^{-\kappa t} \int_{\mathbb{R}^n} \mathrm{div}\ (v K(x,v,\phi)) \partial_t \phi \, dx \\
& - 2\kappa e^{-\kappa t} \int_{\mathbb{R}^n} v K(x,v,\phi) \nabla \phi \, dx.
\end{aligned} \tag{4.5}
$$

Since the weak solution does not have enough regularity, the above identity is not necessarily valid and the actual estimate should be obtained in the form of the integral inequality. This is justified by an appropriate approximation: Let (v, ϕ) be a solution of the regularized system:

$$
\begin{cases}
\partial_t v - \mathrm{div}\ \big((v+\varepsilon) K_\varepsilon(x,v,\phi) \big) \\
\quad = -\varepsilon \big(e^{-(\kappa-2)t} (v - \phi) + n \big), \qquad t > 0, x \in \mathbb{R}^n, \\
- e^{-2t} \Delta \phi + \phi = v, \qquad t > 0, x \in \mathbb{R}^n, \\
\quad v(0,x) = u_0(x), \qquad x \in \mathbb{R}^n,
\end{cases} \tag{4.6}
$$

where

$$
K_\varepsilon(x,v,\phi) \equiv \frac{\alpha}{\alpha - 1} \nabla (v+\varepsilon)^{\alpha-1}(t,x) + x - e^{-\kappa t} \nabla \phi(t,x).
$$

Note that the above system (4.6) is equivalent to (2.3). The existence of the smooth and sufficiently fast decaying solution at $|x| \to \infty$ of (4.6) is obtained in a similar manner in [54].

Proposition 4.3. *Let $\zeta = \zeta(x)$ be a smooth cut off function such that $\zeta = 1$ in B_R and whose derivatives are supported in $B_{2R} \backslash B_R$. For a solution*

v and ϕ of (4.6) *belonging to L^1, we let*

$$I_\varepsilon(v(t)) \equiv \int_{\mathbb{R}^n} v(t) \left| K_\varepsilon(x, v(t), \phi(t)) \right|^2 \zeta^2 dx,$$

where $\kappa = n(2-\alpha)$. Then we have

$$\begin{aligned}
\frac{d}{dt} I_\varepsilon(v(t)) \leq &- 2\int_{\mathbb{R}^n} (v+\varepsilon)|K_\varepsilon(x,v,\phi)|^2 \zeta dx \\
&- 2(\alpha-1)\int_{\mathbb{R}^n} (v+\varepsilon)^\alpha \Big| \mathrm{div}\ K_\varepsilon(x,v,\phi) \Big|^2 \zeta dx \\
&- 2\int_{\mathbb{R}^n} (v+\varepsilon)^\alpha |\nabla K_\varepsilon(x,v,\phi)|^2 \zeta dx \\
&+ 2e^{-\kappa t} \int_{\mathbb{R}^n} (v+\varepsilon) K_{\varepsilon,i}(x,v,\phi) K_{\varepsilon,j}(x,v,\phi) \left(D_{ij}^2 \phi \right) \zeta dx \\
&+ 2e^{-(\kappa-2)t} \int_{\mathbb{R}^n} \left| v K_\varepsilon(x,v,\phi) \right|^2 \zeta dx \\
&- 2\kappa e^{-\kappa t} \int_{\mathbb{R}^n} v K_\varepsilon(x,v,\phi) \cdot \nabla\phi \zeta dx \\
&\quad + E_I(x,v,\phi,\varepsilon,\nabla\zeta),
\end{aligned} \tag{4.7}$$

where $E_I(x,v,\phi,\varepsilon,\nabla\zeta)$ denotes the error term and it will be vanishing when we take the limit $R \to \infty$ and $\varepsilon \to 0$.

The derivation and rigorous treatment of (4.7) is given in appendix in [48]. We proceed to the following.

PROPOSITION 4.4. *Let (v,ϕ) be a weak solution of* (2.2). *We set*

$$I(v(t)) \equiv \int_{\mathbb{R}^n} v(t) |K(x,v,\phi)|^2 dx$$

with $K(x,v,\phi) \equiv \frac{\alpha}{\alpha-1} v^{\alpha-1} + x - e^{-\kappa t}\nabla\phi$. Then under the condition $1 < \alpha \leq 2 - \frac{2}{n}$ and the solution v has uniform estimate $\sup_{t>0} \|v(t)\|_\infty \leq C_*$ for some constant, there exist $T_0, \nu > 0$ such that for any $T_0 < t$,

$$I(v(t)) + \nu \int_{T_0}^t I(v(\tau))\, d\tau \leq I(u(T_0)). \tag{4.8}$$

In particular, we have

$$I(v(t)) \le Ce^{-\nu t}, \quad t > T_0$$

where the constant C is depending on the initial data u_0 and T_0.

To obtain the above proposition, we need the following two ingredients. First one is the Sobolev type inequality in the critical type originally due to Brezis-Gallouet [9]. This is the generalized version obtained in Ogawa-Taniuchi [49].

PROPOSITION 4.5 ([9], [29], [49]). *There exists a constant C depending only on n such that for $f \in L^2(\mathbb{R}^n) \cap C^\gamma(\mathbb{R}^n)$, the following inequality holds:*

$$\|f\|_\infty \le C\left(1 + \|f\|_{BMO} \log\left(e + \|f\|_2 + \|f\|_{C^\gamma}\right)\right). \tag{4.9}$$

PROOF OF PROPOSITION 4.4. To avoid the complexity of the notation, we treat the estimate only for the essential parts in rather formal way, namely dropping the parameter ε and cut off function ζ without integration in t variable. The actual estimate are done for the approximated solution. The rigorous procedure requires that all those estimates are proceeded before passing to the limit $R \to \infty$ and $\varepsilon \to 0$ and the rigorous treatment can be found in [48]. Observing the estimate (4.7), we need to estimate the last four terms in the right-hand side. The fourth error term $E_I(x, v, \phi, \varepsilon, \nabla\zeta)$ is handled in [48, Appendix A] since it does not give any effect for the estimation of the other terms. Firstly, the sixth term of the right hand side of (4.7) can be estimated as follows.

$$\begin{aligned}
&-2\kappa e^{-\kappa t} \int_{\mathbb{R}^n} vK(x, v, \phi)\nabla\phi dx \\
&\qquad \le 2\kappa e^{-\kappa t}\|\nabla\phi(t)\|_\infty \left(\int_{\mathbb{R}^n} v dx\right)^{1/2} \left(\int_{\mathbb{R}^n} v|K(x, v, \phi)|^2 dx\right)^{1/2} \\
&\qquad \le 2\kappa\varepsilon^{-1} e^{-2\kappa t}\|\nabla\phi(t)\|_\infty^2 \int_{\mathbb{R}^n} v\, dx + \frac{\varepsilon}{2}\int_{\mathbb{R}^n} v|K(x, v, \phi)|^2 dx
\end{aligned}$$

where $\varepsilon > 0$ is a small parameter. Hence from Proposition 4.3, we obtain that

$$\begin{aligned}\frac{d}{dt}I(v(t)) \le& -(2-\varepsilon)I(v(t)) - 2(\alpha-1)\int_{\mathbb{R}^n} v^\alpha \Big|\mathrm{div}\ K(x,v,\phi)\Big|^2 dx \\ &- 2\int_{\mathbb{R}^n} v^\alpha \left|\nabla K(x,v,\phi)\right|^2 dx \\ &+ 2e^{-(\kappa-2)t}\int_{\mathbb{R}^n} v^2 \left|K(x,v,\phi)\right|^2 dx \\ &+ 2e^{-\kappa t}\int_{\mathbb{R}^n} vK_i(x,v,\phi)K_j(x,v,\phi)\left(D^2_{ij}\phi\right) dx \\ &+ C\varepsilon^{-1}e^{-2(\kappa-1)t}\|v(t)\|_1 \sup_t(e^{-2t}\|\nabla\phi(t)\|_\infty^2),\end{aligned} \tag{4.10}$$

where $K(x,v,\phi) = x + \frac{\alpha}{\alpha-1}\nabla v^{\alpha-1}(t,x) - e^{-\kappa t}\nabla\phi(t,x)$. We now turn into how to treat the following term:

$$\int_{\mathbb{R}^n} vK_i(x,v,\phi)K_j(x,v,\phi)D^2_{ij}\phi dx.$$

Applying the logarithmic interpolation inequality of Brezis-Gallouet type (4.9), we see

$$\|D^2\phi(t)\|_\infty \le C\Big(1 + \|D^2\phi(t)\|_{BMO}\log\big(e + \|D^2\phi(t)\|_2 + \|D^2\phi(t)\|_{C^\gamma}\big)\Big).$$

By the Calderon-Zygmund inequality, we have

$$\|D^2\phi\|_2 \le C\|\Delta\phi\|_2 \le Ce^{2t}(\|v\|_2 + \|\phi\|_2) \le Ce^{2t}\|v\|_2.$$

By the Schauder estimate, we obtain

$$\|D^2\phi\|_{C^\gamma} \le Ce^{2t}(\|v\|_{C^\gamma} + \|\phi\|_{C^\gamma}) \le Ce^{2t}.$$

Finally, by the Calderon-Zygmund inequality again, we have

$$\|D^2\phi\|_{BMO} \le C\|\Delta\phi\|_{BMO} \le C\|\Delta(-e^{-2t}\Delta+1)^{-1}v\|_{BMO}. \tag{4.11}$$

We notice that the corresponding Fourier multiplier of the operator appearing the right-hand side of (4.11) is given by

$$\frac{|\xi|^2}{e^{-2t}|\xi|^2+1} = \frac{e^{2t}|\xi|^{2-\gamma}}{|\xi|^2+e^{2t}}|\xi|^\gamma = e^{(2-\gamma)t}\frac{e^{\gamma t}|\xi|^{2-\gamma}}{|\xi|^2+e^{2t}}|\xi|^\gamma$$

and the multiplier satisfies the condition so that the operator $e^{(\gamma-2)t}|\nabla|^{2-\gamma}(-e^{-2t}\Delta+1)^{-1}$ is bounded in BMO. Therefore

$$\|D^2\phi\|_{BMO} \le Ce^{(2-\gamma)t}\||\nabla|^\gamma v\|_{BMO}.$$

From the uniform Hölder estimate Theorem 3.1 $\||\nabla|^\gamma v\|_{BMO}$ is bounded uniformly in t. This enable us to proceed the estimate as

$$\begin{aligned} &2e^{-\kappa t}\int_{\mathbb{R}^n} vK_i(x,v,\phi)K_j(x,v,\phi)\partial_i\partial_j\phi\,dx \\ &\le 2e^{-\kappa t}\|D^2\phi(t)\|_\infty\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2\,dx \\ &\le Ce^{-(\kappa-2+\gamma')t}\int_{\mathbb{R}^n} v|K(x,v,\phi)|^2\,dx \end{aligned} \tag{4.12}$$

for some $\gamma'>0$. Combining (4.10) and (4.12), we obtain that if $\kappa=2$ i.e. $\alpha=2-\frac{2}{n}$,

$$\frac{d}{dt}I(v(t)) \le -(2-\varepsilon)I(v(t)) + C\sup_t\|v(t)\|_\infty I(v(t)) + C\varepsilon^{-1}e^{-2t} \tag{4.13}$$

Note that at this stage, the inequality (4.13) does not include the higher order terms so that it is possible to justify it for the weak solution. Since $2(\kappa-2)=0$ when $\alpha=2-\frac{2}{n}$, we can choose $\nu,\eta>0$ such that for some large $T_0>0$, which depends on C, for any $t\ge T_0$,

$$\frac{d}{dt}(e^{\nu t}I(v(t)) \le Ce^{-\eta t}. \tag{4.14}$$

Immediately we obtain that

$$I(v(t)) \le e^{-\nu t}\bigg(I(v(T_0)) + C\int_{T_0}^\infty e^{\eta\tau}\,d\tau\bigg).$$

Since T_0 is only depending on C we may conclude that $I(v(t))\le C(T_0)$ for $0\le t\le T_0$ and this concludes the desired estimate. □

The proof of the asymptotic profile in Theorem 1.2 completes after proving the convergence of the rescaled solution and rescaling.

PROPOSITION 4.6. *Let $1 < \alpha \le 2 - \frac{2}{n}$ and (v, ϕ) be a weak solution to* (2.2). *If the initial data satisfies the condition* (1.5), *then we have for some $\nu > 0$ that*

$$\|v(t) - V\|_1 \le Ce^{-\nu t},$$

where

$$V(x) = \left[A - \frac{\alpha - 1}{2\alpha}|x|^2\right]_+^{1/(\alpha-1)}$$

and the constant A is chosen as $\|V\|_1 = \|u_0\|_1$.

PROOF OF PROPOSITION 4.6. Due to the result from Proposition 4.4, we immediately obtain that

$$\lim_{t\to\infty} I(v(t)) = 0. \tag{4.15}$$

On the other hand, since by Proposition 2.1,

$$\begin{aligned} J(v(t)) &\le 2I(v(t)) + 2e^{-2\kappa t}\int_{\mathbb{R}^n} v(t)|\nabla\phi(t)|^2 dx \\ &\le 2I(v(t)) + 2e^{-2\kappa t}\|v(t)\|_\infty \|\nabla\phi(t)\|_2^2 \\ &\le 2I(u_0)e^{-\nu t} + 2e^{-(2\kappa-2)t}C(v, \phi), \end{aligned} \tag{4.16}$$

we conclude from (4.2) in Proposition 4.2 and Proposition 2.1 that for any $s < t$,

$$\begin{aligned} &\Big|H(v(t)) - H(v(s))\Big| \\ &\le |e^{-\kappa t}(e^{-2t}\|\nabla\phi(t)\|_2^2 + \|\phi(t)\|_2^2) - e^{-\kappa s}(e^{-2s}\|\nabla\phi(s)\|_2^2 + \|\phi(s)\|_2^2)| \\ &\quad + \int_s^t \left(\frac{\kappa - 2}{2}e^{-(\kappa+2)\tau}\|\nabla\phi(\tau)\|_2^2 + \frac{\kappa}{2}e^{-\kappa\tau}\|\phi(\tau)\|_2^2 + J(v(\tau))\right) d\tau \\ &\quad + \int_s^t e^{-2\kappa\tau}\left(\int_{\mathbb{R}^n} v(\tau)|\nabla\phi(\tau)|^2\, dx\right) d\tau \\ &\le C(\kappa)e^{-\kappa s}\sup_{\tau>0}\big(e^{-2\tau}\|\nabla\phi(\tau)\|_2^2 + \|\phi(\tau)\|_2^2\big) \\ &\quad + 2I(v(u_0))e^{-\nu s} + 2C(\kappa)e^{-2(\kappa-1)s} \\ &\quad + e^{-2(\kappa-1)s}\sup_{\tau>0}\big(e^{-2\tau}\|v(\tau)\|_\infty\|\nabla\phi(\tau)\|_2^2\big) \\ &\le Ce^{-\nu s} \to 0, \quad \text{as } s, t \to \infty \end{aligned} \tag{4.17}$$

and this shows that $\{H(v(t_n))\}_n$ is the Cauchy sequence in $t_n \to \infty$. Moreover since $v^\alpha \in L^1$ with $\nabla v^\alpha \in L^1$ by

$$\int_{\mathbb{R}^n} |\nabla v^\alpha| dx \leq \frac{\alpha^2}{(\alpha - 1/2)^2} \left(\int_{\mathbb{R}^n} u_0 dx \right)^{1/2} \left(\int_{\mathbb{R}^n} |\nabla v^{\alpha - \frac{1}{2}}|^2 dx \right)^{1/2} \leq C.$$

Besides the moment bound (2.7) in Proposition 2.3, $|x|^a v \in L^1$ for some $a > 2$. Therefore by the compactness $W^{1,1} \cap L^1_a(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \cap L^1_2(\mathbb{R}^n)$, we have a subsequence $v(t_n)$ such that it converges strongly to $V(x) \in L^\alpha(\mathbb{R}^n) \cap L^1_2(\mathbb{R}^n)$. The similar argument found in [12, Theorem 3.1] works for our case and we see that there exists a limit function V in $L^1_2(\mathbb{R}^n)$ such that

$$v(t_n) \to V, \quad t_n \to \infty$$

in $L^1(\mathbb{R}^n)$. It turns out that the limit function is also nonnegative and bounded. While by (4.15), the moment bound Proposition 2.3 and the natural regularity of the weak solution, we see that

$$J(v(t)) \to J(V) = \int_{\mathbb{R}^n} V \left| \frac{\alpha}{\alpha - 1} \nabla V^{\alpha - 1} + x \right|^2 dx = 0$$

and we obtain either $V = 0$ or $\nabla V^{\alpha-1} = -\frac{\alpha-1}{\alpha} x$ almost everywhere. This concludes by recalling $M = \|u_0\|_1$,

$$V(x) = \left[A - \frac{\alpha - 1}{2\alpha} |x|^2 \right]_+^{\frac{1}{\alpha - 1}},$$

where A is chosen such that the L^1 norm of $V(x)$ is normalized as 1. Again the estimate (4.2) in Proposition 4.2 and (4.17) gives

$$|H(v(t)) - H(V)| \leq C e^{-\nu t} \tag{4.18}$$

and the desired estimate follows from the argument in [12, Theorem 4.5]. Namely we see firstly that

$$\begin{aligned} &\int_{v<V} |v(t) - V| dx \\ &\leq \left(\frac{1}{\alpha} |H(\chi_{B_M} v(t)) - H(V)| \right)^{1/2} \left(\int_{B_M} V(x)^{\frac{2}{n}} dx \right)^{1/2} \end{aligned} \tag{4.19}$$

by the special structure of the Barenblatt solution, where $B_M = \operatorname{supp} B \equiv \{|x| \le \frac{2\alpha A}{\alpha-1}\}$ and χ_{B_M} is the characteristic function on B_M. While by $M = \|V\|_1 = \|v(t)\|_1$ and $V \ge 0$, we see

$$\int_{v\ge V} |v(t)-V|dx = \int_{v<V} (V - v(t))dx = \int_{v<V} |v(t)-V|dx. \tag{4.20}$$

We note that over B_M^c, V is vanishing and by [12, Lemma 4.4]

$$\begin{aligned} &\frac{1}{\alpha-1}\int_{|x|^2>C} v^\alpha(t)dx + \frac{1}{2}\int_{|x|^2>C} (|x|^2 - D)v(t)dx \\ &\le |H(v(t)) - H(V)|, \\ &D\int_{|x|^2>C} v(t)dx \le Ce^{-\gamma t}. \end{aligned} \tag{4.21}$$

Combining (4.19), (4.20) and (4.21) with (4.18) we conclude that

$$\|v(t) - V\|_1 \le Ce^{-\nu' t},$$

for some $\nu' > 0$. □

Appendix A. Some Fundamental Calculus

Their results are well-known, however we give proofs for self-containedness.

LEMMA A.1 (cf. Ladyženskaja-Solonnikov-Ural'ceva [33, pp.96 Lemma 5.7]). *Let $C, \varepsilon, \delta > 0$, $b \ge 1$ and let $\{Y_n\}_{n=0}^\infty$, $\{Z_n\}_{n=0}^\infty \subset (0,\infty)$ satisfy*

$$\begin{aligned} Y_{n+1} &\le Cb^n(Y_n^{1+\delta} + Y_n^\delta Z_n^{1+\varepsilon}) \\ Z_{n+1} &\le Cb^n(Y_n + Z_n^{1+\varepsilon}). \end{aligned} \tag{A.1}$$

Set

$$d := \min\left\{\delta, \frac{\varepsilon}{1+\varepsilon}\right\}, \lambda = \min\left\{(2C)^{-\frac{1}{\delta}} b^{-\frac{1}{\delta d}}, (2C)^{-\frac{1+\varepsilon}{\varepsilon}} b^{-\frac{1}{\varepsilon d}}\right\}.$$

Then if $Y_0 \le \lambda$ and $Z_0 \le \lambda^{\frac{1}{1+\varepsilon}}$, we obtain

$$Y_n \le \lambda b^{-\frac{n}{d}}, Z_n \le (\lambda b^{-\frac{n}{d}})^{\frac{1}{1+\varepsilon}}. \tag{A.2}$$

Proof of Lemma A.1. Inequality (A.2) are valid for $n=0$. We prove (A.2) by induction. If (A.2) hold for n, then by (A.1), we have

$$Y_{n+1} \le 2C\lambda^{1+\delta} b^{n(1-\frac{1+\delta}{d})}\,,\; Z_{n+1} \le 2C\lambda b^{n(1-\frac{1}{d})}.$$

Since $\lambda \le (2C)^{-\frac{1}{\delta}} b^{-\frac{1}{\delta d}}$ and $d \le \delta$, we have

$$2C\lambda^{1+\delta} b^{n(1-\frac{1+\delta}{d})} \le \lambda b^{-\frac{1}{d}} b^{-\frac{n}{d}+n(1-\frac{\delta}{d})} \le \lambda b^{-\frac{n+1}{d}}.$$

Similarly, since $\lambda \le (2C)^{-\frac{1+\varepsilon}{\varepsilon}} b^{-\frac{1}{\varepsilon d}}$, we obtain

$$2C\lambda b^{n(1-\frac{1}{d})} = 2C\lambda^{\frac{\varepsilon}{1+\varepsilon}} \lambda^{\frac{1}{1+\varepsilon}} b^{-\frac{n+1}{(1+\varepsilon)d}} b^{n(1-\frac{1}{d})+\frac{n+1}{(1+\varepsilon)d}} \le (\lambda b^{-\frac{n+1}{d}})^{\frac{1}{1+\varepsilon}} b^{n(1-\frac{\varepsilon}{(1+\varepsilon)d})}.$$

Since $d \le \frac{\varepsilon}{1+\varepsilon}$, we find $1-\frac{\varepsilon}{(1+\varepsilon)d} \le 0$ and hence we have (A.2) for $n+1$. $\square$

Lemma A.2 (cf. Ladyženskaja-Solonnikov-Ural'ceva [33, p.91]). *Let $w \in W^{1,1}(B_\rho)$ and let $l > k$. Then there exists a constant C depending on n only such that*

$$(l-k)|A^+_{\rho;l}| \le \frac{C\rho^{n+1}}{|B_\rho| - |A^+_{\rho;k}|} \int_{A^+_{\rho;k}\setminus A^+_{\rho;l}} |\nabla w|\,dx$$

where

$$A^+_{\rho;k} := \{x \in B_\rho\,;\, w(x) > k\}.$$

For the proof of Lemma A.2, we need the following weighted Poincaré inequality:

Lemma A.3 (cf. Ladyženskaja-Solonnikov-Ural'ceva [33, p.89]). *Let g be a nonnegative function in $W^{1,1}(B_\rho)$ and let $N_0 := \{g = 0\}$. Then*

$$\int_{B_\rho} g(x)\,dx \le \frac{C_n \rho^{n+1}}{|N_0|} \int_{B_\rho} |\nabla g(x)| \eta(x)\,dx.$$

Proof of Lemma A.3. We only consider the case $n \ge 2$. For $x \in B_\rho,\ x' \in N_0$, we have

$$g(x) = g(x) - g(x') = -\int_0^{|x'-x|} \frac{d}{dr} g(x+r\omega)\,dr \le \int_0^{|x'-x|} |\nabla g(x+r\omega)|\,dr$$

where $\omega = \frac{x'-x}{|x'-x|}$. Integrating over $x \in B_\rho$ and $x' \in N_0$, we have

$$|N_0| \int_{B_\rho} g(x)\,dx \le \int_{B_\rho} dx \int_{N_0} dx' \int_0^{|x'-x|} |\nabla g(x+r\omega)|\,dr.$$

Let $g(x)$ be zero on $x \in \mathbb{R}^n \setminus B_\rho$. Introducing the polar coordinate, we obtain

$$\begin{aligned}
&\int_{N_0} dx' \int_0^{|x'-x|} |\nabla g(x+r\omega)|\,dr \\
&\le \int_{B_{2\rho(x)}} dx' \int_0^{|x'-x|} |\nabla g(x+r\omega)|\,dr \\
&\le \int_{B_{2\rho(x)}} dx' \int_0^{|x'-x|} |\nabla g(x+r\omega)|\,dr \\
&= \int_0^{2\rho} s^{n-1}\,ds \int_{\mathbb{S}^{n-1}} d\sigma \int_0^s \frac{|\nabla g(x+r\omega)|}{r^{n-1}} r^{n-1}\,dr \qquad \begin{pmatrix} x' = s\sigma + x, \\ \sigma \in \mathbb{S}^{n-1}, \\ 0 < s < 2\rho \end{pmatrix} \\
&= \int_0^{2\rho} s^{n-1}\,ds \int_{B_s(x)} \frac{|\nabla g(y)|}{|x-y|^{n-1}}\,dy \qquad (y = x + r\sigma) \\
&\le \frac{(2\rho)^n}{n} \int_{B_\rho} \frac{|\nabla g(y)|}{|x-y|^{n-1}}\,dy,
\end{aligned}$$

where $\mathbb{S}^{n-1}$ is the $(n-1)$-dimensional unit sphere. Therefore,

$$\begin{aligned}
|N_0| \int_{B_\rho} g(x)\,dx &\le \frac{(2\rho)^n}{n} \int_{B_\rho} dx \int_{B_\rho} \frac{|\nabla g(y)|}{|x-y|^{n-1}}\,dy \\
&= C_n \rho^n \int_{B_\rho} |\nabla g(y)|\,dy \int_{B_\rho} \frac{1}{|x-y|^{n-1}}\,dx.
\end{aligned}$$

Since

$$\int_{B_\rho} \frac{1}{|x-y|^{n-1}}\,dx \le \int_{B_{2\rho(y)}} \frac{1}{|x-y|^{n-1}}\,dx \le \int_{\mathbb{S}^{n-1}} d\sigma \int_0^{2\rho} \frac{r^{n-1}}{r^{n-1}}\,dr = C_n \rho,$$

we obtain

$$|N_0| \int_{B_\rho} g(x)\,dx \le C_n \rho^{n+1} \int_{B_\rho} |\nabla g(y)|\,dy. \ \square$$

Proof of Lemma A.2. Let

$$g(x) := \max\{l-k\,,\,(w-k)_+\} \in W^{1,1}(B_\rho), \quad N_0 := \{w < k\}.$$

Then, by Lemma A.3, we have

$$\int_{B_\rho} g(x)\,dx \le \frac{C_n \rho^{n+1}}{|N_0|}\int_{B_\rho} |\nabla g(x)|\,dx,$$

hence

$$(l-k)|\{w>l\}| \le \frac{C_n\rho^{n+1}}{|\{w<k\}|}\int_{\{k<w\le l\}} |\nabla w(x)|\,dx. \ \square$$

Lemma A.4 (scaled Bessel potential). *Let $f \in L^q(\mathbb{R}^n)$. Then for all $\lambda \ge 0$ and $1 \le r \le \frac{n}{n-2}$, we have:*

$$\|(-\lambda^{-2}\Delta+1)^{-1}f\|_p \le \lambda^{\frac{n}{q}}(1+|\lambda|^{-\frac{n}{r}})\|G_1\|_r\|f\|_q, \tag{A.3}$$

where G_1 is the Bessel potential for $(-\Delta+1)^{-1}$ and $\frac{1}{p} = \frac{1}{q}+\frac{1}{r}-1$.

Proof of Lemma A.4. Let $G_\lambda(x)$ be the Bessel kernel for the Bessel potential $(-\frac{1}{\lambda^2}\Delta+1)^{-1}$. Then since

$$\begin{aligned}
(-\lambda^{-2}\Delta+1)^{-1}f =& c_n\int_{\mathbb{R}^n} e^{ix\cdot\xi}\frac{1}{|\xi/\lambda|^2+1}\hat{f}(\xi)d\xi \\
=& c_n\int_{\mathbb{R}^n} e^{ix\cdot\lambda\eta}\frac{1}{|\eta|^2+1}\hat{f}(\lambda\eta)\lambda^n d\eta \\
=& \lambda^n c_n\int_{\mathbb{R}^n} e^{i\lambda x\cdot\eta}\frac{1}{|\eta|^2+1}\hat{\tilde{f}}(\eta)d\eta \\
=& \lambda^n G_1(\lambda\cdot) * \big(\lambda^{-n}f(\lambda^{-1}\cdot)\big) \\
=& G_1(\lambda\cdot) * \big(f(\lambda^{-1}\cdot)\big)
\end{aligned}$$

where $\tilde{f}(x) = \lambda^{-n}f(\lambda^{-1}x)$. Hence

$$\begin{aligned}
\|(-\lambda^{-2}\Delta+1)^{-1}f\|_p =& \big\|G_1(\lambda\cdot) * \big(f(\lambda^{-1}\cdot)\big)\big\| \\
\le& \|G_1(\lambda\cdot)\|_r\|f(\lambda^{-1}\cdot)\|_q \\
=& \lambda^{\frac{n}{q}}\|G_1(\lambda\cdot)\|_r\|f\|_q,
\end{aligned}$$

where

$$\frac{1}{p} = \frac{1}{q} - \left(1 - \frac{1}{r}\right)$$

and $r \leq \frac{n}{n-2}$. Namely $1 - \frac{1}{r} \leq \frac{2}{n}$ and hence we obtain the result. $\square$

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Masashi Mizuno
Department of Mathematics
College of Science and Technology
Nihon University
Tokyo, Japan
E-mail: mizuno@math.cst.nihon-u.ac.jp

Takayoshi Ogawa
Mathematical Institute
Tohoku University
Sendai, Japan
E-mail: ogawa@math.tohoku.ac.jp