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Dixmier Approximation and Symmetric Amenability for C*-*Algebras*

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Abstract. We study some general properties of tracial C*-algebras. In the first part, we consider Dixmier type approximation theorem and characterize symmetric amenability for C*-algebras. In the second part, we consider continuous bundles of tracial von Neumann algebras and classify some of them.

1. Introduction

The general study of tracial states on C*-algebras has a long history, but recently it gained a renewed interest in connection with the ongoing classification program for finite nuclear C*-algebras. In this note, we record several facts about tracial C*-algebras which may be useful in the future study. The results are two-fold. First, we consider Dixmier type approximation property for C*-algebras and relate it to symmetric amenability. The Dixmier approximation theorem (Theorem III.5.1 in [Di]) states a fundamental fact about von Neumann algebras that for any von Neumann algebra N and any element $a \in N$, the norm-closed convex hull of $\{uau^* : u \in \mathcal{U}(N)\}$ meets the center $\mathcal{Z}(N)$ of N. Here $\mathcal{U}(N)$ denotes the unitary group of N. If N is moreover a finite von Neumann algebra, then this intersection is a singleton and consists of $\mathrm{ctr}(a)$. Here ctr denotes the center-valued trace, which is the unique conditional expectation from N onto $\mathcal{Z}(N)$ that satisfies $\mathrm{ctr}(xy) = \mathrm{ctr}(yx)$. It is proved by Haagerup and Zsido ([HZ]) that the Dixmier approximation theorem holds for simple C*-algebras having at most one tracial states (and obviously does not for simple C*-algebras having more than one tracial states). Recall that a C*-algebra A has the *quotient tracial state property* (QTS property) if every non-zero quotient

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C*-algebra of A has a tracial state ([Mu]). We denote by $T(A)$ the space of the tracial states on A, equipped with the weak*-topology.

THEOREM 1. *For a unital* C*-*algebra A, the following are equivalent.*

(i) *The* C*-*algebra A has the QTS property.*

(ii) *For every $\varepsilon > 0$ and $a \in A$ that satisfy $\sup_{\tau \in T(A)} |\tau(a)| < \varepsilon$, there are k and $u_1, \dots, u_k \in \mathcal{U}(A)$ such that $\|\frac{1}{k}\sum_{i=1}^{k} u_i a u_i^*\| < \varepsilon$.*

Unlike the case for von Neumann algebras, there is no bound of k in terms of ε and $\|a\|$ that works for an arbitrary element a in a C*-algebra (see Section 3, where we study a relation between trace zero elements and commutators). Recall that a Banach algebra A is said to be *amenable* if there is a net $(\Delta_n)_n$, called an approximate diagonal, in the algebraic tensor product $A \otimes_{\mathbb{C}} A$ (we reserve the symbol $\otimes$ for the minimal tensor product) such that

(1) $\sup_n \|\Delta_n\|_\wedge < +\infty$,

(2) $(\mathrm{m}(\Delta_n))_n$ is an approximate identity,

(3) $\lim_n \|a \cdot \Delta_n - \Delta_n \cdot a\|_\wedge = 0$ for every $a \in A$.

Here $\|\cdot\|_\wedge$ is the projective norm on $A \otimes_{\mathbb{C}} A$, $\mathrm{m}\colon A \otimes_{\mathbb{C}} A \to A$ is the multiplication, and $a \cdot (\sum_i x_i \otimes y_i) = \sum_i a x_i \otimes y_i$ and $(\sum_i x_i \otimes y_i) \cdot a = \sum_i x_i \otimes y_i a$. The celebrated theorem of Connes–Haagerup ([Co, Ha1]) states that a C*-algebra A is amenable as a Banach algebra if and only if it is nuclear. The Banach algebra A is said to be *symmetrically amenable* ([Jo]) if the approximate diagonal $(\Delta_n)_n$ can be taken symmetric under the flip $x \otimes y \to y \otimes x$. We characterize symmetric amenability for C*-algebras.

THEOREM 2. *For a unital* C*-*algebra A, the following are equivalent.*

(i) *The* C*-*algebra A is nuclear and has the QTS property.*

(ii) *The* C*-*algebra A has an approximate diagonal $\Delta_n = \sum_{i=1}^{k(n)} x_i(n)^* \otimes x_i(n)$ such that $\lim_n \sum_{i=1}^{k(n)} \|x_i(n)\|^2 = 1$, $\mathrm{m}(\Delta_n) = 1$, and $\lim_n \|1 - \sum_{i=1}^{k(n)} x_i(n) x_i(n)^*\| = 0$.*

(iii) *The* C*-*algebra* A *is symmetrically amenable.*

(iv) *The* C*-*algebra* A *has a symmetric approximate diagonal* $(\Delta_n)_n$ *in*

$$\{\sum_i x_i^* \otimes x_i \in A \otimes_{\mathbb{C}} A : \sum_i \|x_i\|^2 \le 1\}.$$

Recall that a unital C*-algebra A is *strongly amenable* if there is an approximate diagonal that consists of convex combinations of $\{u^* \otimes u : u \in \mathcal{U}(A)\}$. This property is formally stronger than symmetric amenability, but it is unclear whether there is really a gap between these properties.

Second, we describe what is the C*-completion $\overline{A}^{\mathrm{u}}$ of a unital C*-algebra A under the uniform 2-norm. This work is strongly influenced by the recent works of Kirchberg–Rørdam ([KR]), Sato ([Sa2]), and Toms–White–Winter ([TWW]), who studied the central sequence algebra of a C*-algebra modulo uniformly 2-norm null sequences, in order to extend Matui–Sato's result ([MS]) from C*-algebras with finitely many extremal tracial states to more general ones. In fact, our result is very similar to theirs (particularly to Kirchberg–Rørdam's). Let A be a C*-algebra and $S \subset T(A)$ be a non-empty metrizable closed face. The reason we assume S be metrizable is because it makes the description of the boundary measures simpler. We define the uniform 2-norm on A corresponding to S by

$$\|a\|_{2,S} = \sup\{\tau(a^*a)^{1/2} : \tau \in S\}.$$

The uniform 2-norm satisfies

$$\|ab\|_{2,S} \le \min\{\|a\|\|b\|_{2,S}, \|a\|_{2,S}\|b\|\} \quad \text{and} \quad \sup_{\tau\in S}|\tau(a)| \le \|a\|_{2,S}.$$

The C*-completion $\overline{A}^{\mathrm{u}}$ is defined to be the C*-algebra of the norm-bounded uniform 2-norm Cauchy sequences, modulo the ideal of the uniform 2-norm null sequences. For $\tau \in T(A)$, we denote by π_τ the corresponding GNS representation and also $\|a\|_{2,\tau} = \tau(a^*a)^{1/2}$. Let $N = (\bigoplus_{\tau\in S} \pi_\tau)(A)''$ be the enveloping von Neumann algebra with respect to S. When $S = T(A)$, it is the finite summand A_{f}^{**} of the second dual von Neumann algebra A^{**}. The tracial state $\tau \in S$ and the GNS representation π_τ extend normally on N. For the center-valued trace $\mathrm{ctr}\colon N \to \mathcal{Z}(N)$, one has $\|a\|_{2,S} = \|\mathrm{ctr}(a^*a)\|^{1/2}$

and $\overline{A}^{\mathrm{u}}$ coincides with the closure $\overline{A}^{\mathrm{st}}$ of A in N with respect to the strict topology associated with the Hilbert $\mathcal{Z}(N)$-module (N, ctr).

Recall that the trace space $T(A)$ of a unital C*-algebra is a Choquet simplex and so is the closed face S. We denote by $\mathrm{Aff}(S)$ the space of the affine continuous functions on S and consider the function system $\mathcal{A}\mathrm{ff}(S) = \{f|_{\partial S} : f \in \mathrm{Aff}(S)\}$ in $B(\partial S)$, where $B(\partial S)$ denotes the C*-algebra of the bounded Borel functions on ∂S. For every $a \in A$, the formula $\hat{a}(\tau) = \tau(a)$ defines a function $\hat{a}$ in $\mathrm{Aff}(S)$ (or $\mathcal{A}\mathrm{ff}(S)$). We note that $\{\hat{a} : a \in A\}$ is dense in $\mathrm{Aff}(S)$ (in fact equal, see [CP]). Let $\mathcal{M}_+^1(\partial S)$ be the space of the probability measures on the extreme boundary ∂S of S. Since S is a metrizable Choquet simplex, every $\tau \in S$ has a unique representing measure $\mu_\tau \in \mathcal{M}_+^1(\partial S)$, which satisfies

$$\tau(a) = \int \lambda(a)\, d\mu_\tau(\lambda) = \int \hat{a}(\lambda)\, d\mu_\tau(\lambda)$$

for every $a \in A$ (Theorem II.3.16 in [Al]). The center $\mathcal{Z}(\mathcal{A}\mathrm{ff}(S))$ is defined to be

$$\mathcal{Z}(\mathcal{A}\mathrm{ff}(S)) = \{f \in B(\partial S) : f\, \mathcal{A}\mathrm{ff}(S) \subset \mathcal{A}\mathrm{ff}(S)\} \subset \mathcal{A}\mathrm{ff}(S).$$

When ∂S is closed (i.e., when S is a Bauer simplex), one has $\mathcal{A}\mathrm{ff}(S) = C(\partial S)$ and $\mathcal{Z}(\mathcal{A}\mathrm{ff}(S)) = C(\partial S)$. However in general, the center $\mathcal{Z}(\mathcal{A}\mathrm{ff}(S))$ can be trivial (see Section II.7 in [Al]).

THEOREM 3. *Let A, S, and N be as above. Then, there is a unital $*$-homomorphism $\theta\colon B(\partial S) \to \mathcal{Z}(N)$ with ultraweakly dense range such that $\theta(\hat{a}) = \mathrm{ctr}(a)$ and*

$$\tau(\theta(f)a) = \int f(\lambda)\lambda(a)\, d\mu_\tau(\lambda) = \int f\hat{a}\, d\mu_\tau$$

for every $a \in A$ and $\tau \in S$. One has

$$\overline{A}^{\mathrm{st}} = \{x \in N : \mathrm{ctr}(xA) \subset \theta(\mathcal{A}\mathrm{ff}(S)),\ \mathrm{ctr}(x^*x) \in \theta(\mathcal{A}\mathrm{ff}(S))\}.$$

In particular,

$$\overline{A}^{\mathrm{st}} \cap \mathcal{Z}(N) = \{\theta(f) : f \in \mathcal{Z}(\mathcal{A}\mathrm{ff}(S))\}.$$

Moreover, if ∂S *is closed, then for every* $\tau \in \partial S$*, one has* $\pi_\tau(\overline{A}^{\mathrm{st}}) = \pi_\tau(N) = \pi_\tau(A)''$.

Takesaki and Tomiyama ([TT]) have studied the structure of a C^*-algebra, for which the set of pure states is closed in the state space, by using a continuous bundle of C^*-algebras (see also [Fe]). We carry out a similar study in Section 5 for a C^*-algebra A, for which ∂S is closed, in terms of a continuous W^*-bundle, and present W^*-analogues of a few results for C^*-bundles obtained in [HRW, DW]. In particular, we give a criterion for a continuous W^*-bundle over a compact space K with all fibers isomorphic to the hyperfinite II_1 factor $\mathcal{R}$ to be isomorphic to the trivial bundle $C_\sigma(K, \mathcal{R})$, the C^*-algebra of the norm-bounded and ultrastrongly continuous functions from K into $\mathcal{R}$. We denote the evaluation map at $\lambda \in K$ by $\mathrm{ev}_\lambda \colon C_\sigma(K, \mathcal{R}) \to \mathcal{R}$. As an application, we show that $\overline{A}^{\mathrm{st}} \cong C_\sigma(\partial S, \mathcal{R})$ for certain A.

Theorem 4. *Let* A *be a separable* C^**-algebra and* $S \subset T(A)$ *be a closed face. Assume that* $\pi_\tau(A)'' \cong \mathcal{R}$ *for all* $\tau \in \partial S$ *and that* ∂S *is a compact space with finite covering dimension. Then, one can coordinatize the isomorphisms* $\pi_\tau(A)'' = \mathcal{R}$ *in such a way that they together give rise to a* $*$*-homomorphism* $\pi \colon A \to C_\sigma(\partial S, \mathcal{R})$ *such that* $\pi_\tau = \mathrm{ev}_\tau \circ \pi$*. The image of* π *is dense with respect to the uniform* 2*-norm.*

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2. QTS Property and Symmetric Amenability

Proof of Theorem 1. Ad (i) $\Rightarrow$ (ii). Although the proof becomes a bit shorter if we use Theorem 5 in [HZ], we give here a more direct proof of this implication. Let $a \in A$ and $\varepsilon > 0$ be given as in condition (ii). Let $\varepsilon_0 = \sup_{\tau \in T(A)} |\tau(a)| < \varepsilon$. We decompose the second dual von Neumann algebra A^{**} into the finite summand A^{**}_{f} and the properly infinite summand A^{**}_∞. We denote the corresponding embedding of A by π_{f} and π_∞, and the center-valued trace of A^{**}_{f} by ctr. We note that $\|\operatorname{ctr}(\pi_{\mathrm{f}}(a))\| = \varepsilon_0$. By the Dixmier approximation theorem, there are $v_1, \ldots, v_k \in \mathcal{U}(A^{**}_{\mathrm{f}})$

such that $\| \mathrm{ctr}(\pi_{\mathrm{f}}(a)) - \frac{1}{k}\sum_{i=1}^{k} v_i \pi_{\mathrm{f}}(a) v_i^* \| < \varepsilon - \varepsilon_0$. On the other hand, by Halpern's theorem ([Hal]), there are $w_1, \dots , w_l \in \mathcal{U}(A_\infty^{**})$ such that $\| \frac{1}{l}\sum_{j=1}^{l} w_i \pi_\infty(a) w_i^* \| < \varepsilon$. Before giving the detail of the proof of this fact, we finish the proof of (i) ⇒ (ii). By allowing multiplicity, we may assume that $k = l$ and consider $u_i = v_i \oplus w_i \in A^{**}$. Then, $\| \frac{1}{k}\sum_{i=1}^{k} u_i a u_i^* \| < \varepsilon$ in A^{**}. For each i, take a net $(u_i(\lambda))_\lambda$ of unitary elements in A which converges to $u_i \in A^{**}$ in the ultrastrong*-topology. By the Hahn-Banach theorem, $\mathrm{conv}\{\frac{1}{k}\sum_{i=1}^{k} u_i(\lambda) a u_i(\lambda)^*\}_\lambda$ contains an element of norm less than ε.

Now, we explain how to apply Halpern's theorem. Let Z (resp. I) be the center (resp. strong radical) of A_∞^{**}. Let Λ be the directed set of all finite partitions of unity by projections in Z, and $\lambda = \{p_{\lambda,i}\}_i \in \Lambda$ be given. Applying the QTS property to the non-zero $*$-homomorphism $A \ni x \mapsto p_{\lambda,i}(\pi_\infty(x) + I) \in p_{\lambda,i}((\pi_\infty(A) + I)/I)$, one obtains a (tracial) state $\tau_{\lambda,i}$ on $\pi_\infty(A) + I$ such that $\tau_{\lambda,i}(p_{\lambda,i}) = 1$, $\tau_{\lambda,i}(I) = 0$, and $|\tau_{\lambda,i}(\pi_\infty(a))| \le \varepsilon_0$. Let $\tilde{\tau}_{\lambda,i}$ be a state extension of it on $p_{\lambda,i}A_\infty^{**}$. We define the linear map $\varphi_\lambda \colon A_\infty^{**} \to Z$ by $\varphi_\lambda(x) = \sum_i \tilde{\tau}_{\lambda,i}(x) p_{\lambda,i}$, and take a limit point $\varphi\colon A_\infty^{**} \to Z$. The map φ is a unital positive Z-linear map such that $\varphi(I) = 0$ and $\|\varphi(\pi_\infty(a))\| \le \varepsilon_0$. By Halpern's theorem (Theorem 4.12 in [Hal]), the norm-closed convex hull of the unitary conjugations of $\pi_\infty(a)$ contains $\varphi(\pi_\infty(a))$.

Ad (ii) ⇒ (i). Suppose that there is a closed two-sided proper ideal I in A such that A/I does not have a tracial state. Let e_n be the approximate unit of I. Then, one has $\tau(1 - e_n) \searrow 0$ for every $\tau \in T(A)$. By Dini's theorem, there is n such that $q = 1 - e_n$ satisfies $\tau(q) < 1/2$ for all $\tau \in T(A)$. By condition (ii), there are $u_1, \dots , u_k \in \mathcal{U}(A)$ such that $\| \frac{1}{k}\sum_{i=1}^{k} u_i q u_i^* \| < 1/2$, which is in contradiction with the fact that $\frac{1}{k}\sum_{i=1}^{k} u_i q u_i^* \in 1 + I$. □

PROOF OF THEOREM 2. The implication (iv) ⇒ (iii) is obvious and (iii) ⇒ (i) is standard: Since amenability implies nuclearity by Connes's theorem ([Co]), we only have to prove the QTS property. Let $(\Delta_n)_n$ be a symmetric approximate diagonal and define $\mathrm{m}_\Delta(a) = \sum_i x_i a y_i$ for $\Delta = \sum_i x_i \otimes y_i \in A \otimes_{\mathbb{C}} A$ and $a \in A$. Then, for any proper ideal I in A and a state φ on A such that $\varphi(I) = 0$, any limit point τ of $(\varphi \circ \mathrm{m}_{\Delta_n})_n$ is a bounded trace such that $\tau(I) = 0$ and $\tau(1) = 1$. By polar decomposition, one obtains a tracial state on A which vanishes on I.

We prove the implication (i) ⇒ (ii) ⇒ (iv). Since A is nuclear, it is amenable thanks to Haagerup's theorem (Theorem 3.1 in [Ha1]). Moreover, there is an approximate diagonal $(\Delta'_n)_n$ in the convex hull of $\{x^* \otimes x : \|x\| \le 1\}$. We note that $\varepsilon_n := \|1 - \mathrm{m}(\Delta'_n)\| \to 0$. We fix n for the moment and write $\Delta'_n = \sum_i x_i^* \otimes x_i$. By replacing x_i with $x_i\,\mathrm{m}(\Delta'_n)^{-1/2}$, we may assume $\mathrm{m}(\Delta'_n) = 1$ but $\sum_i \|x_i\|^2 \le (1-\varepsilon_n)^{-1}$. Since $\tau(\sum_i x_i x_i^*) = 1$ for all $\tau \in T(A)$, Theorem 1 provides $u_1, \ldots, u_l \in \mathcal{U}(A)$ such that $\|\frac{1}{l}\sum_{j=1}^{l}\sum_i u_j x_i x_i^* u_j^*\| \le 1 + \varepsilon_n$. Thus, $\Delta_n = \frac{1}{l}\sum_{i,j} x_i^* u_j^* \otimes u_j x_i$ satisfies condition (ii). Now, rewrite Δ_n as $\sum_i y_i^* \otimes y_i$. Then, $\Delta_n^\sharp = (\sum_i \|y_i\|^2)^{-2} \sum_{i,j} y_i^* y_j \otimes y_j^* y_i$ is a symmetric approximate diagonal that meets condition (iv). □

3. Trace Zero Elements and Commutators

In this section, we consider the trace zero elements in a C*-algebra. A simple application of the Hahn–Banach theorem implies that $a \in A$ satisfies $\tau(a) = 0$ for all $\tau \in T(A)$ if and only if it belongs to the norm-closure of the subspace $[A, A]$ spanned by commutators $[b, c] = bc - cb$, $b, c \in A$. Moreover, such a can be written as a convergent sum of commutators ([CP]). There are many works as to how uniformly this happens ([PP, Fa, FH, Ma, Po] just to name a few). The following fact is rather standard.

Theorem 5. *There is a constant $C > 0$ which satisfies the following. Let A be a* C*-*algebra and $a \in A$ and $\varepsilon > 0$ be such that* $\sup_{\tau \in T(A)} |\tau(a)| < \varepsilon$. *Then, there are $k \in \mathbb{N}$ and b_i and c_i in A such that $\sum_{i=1}^{k} \|b_i\|\|c_i\| \le C\|a\|$ and $\|a - \sum_{i=1}^{k} [b_i, c_i]\| < \varepsilon$.*

Unlike the case for von Neumann algebras, there is no bound on k in terms of ε and $\|a\|$ that works for general C*-algebras. A counterexample is constructed by Pedersen and Petersen (Lemma 3.5 in [PP]: the element $x_n - y_n \in [A_n, A_n]$ constructed there has the property that $\|(x_n - y_n) - z\| \ge 1$ for any sum z of n self-commutators). This also means that k in Theorem 1 depends on the particular element a in A. Nevertheless one can bound k under some regularity condition. Recall that A is said to be *$\mathcal{Z}$-stable* if $A \cong \mathcal{Z} \otimes A$ for the Jiang–Su algebra $\mathcal{Z}$. The Jiang–Su algebra $\mathcal{Z}$ is a simple C*-algebra which is an inductive limit of prime dimension drop algebras and such that $\mathcal{Z} \cong \mathcal{Z}^{\otimes\infty}$ (Theorem 2.9 and Theorem 4 in [JS]).

Theorem 6. *There is a constant $C > 0$ which satisfies the following. Let A be an exact $\mathcal{Z}$-stable* C*-*algebra, and $\varepsilon > 0$ and $a \in A$ be such* *that* $\sup_{\tau\in T(A)}|\tau(a)|<\varepsilon$. *Then, for every* $R\in\mathbb{N}$, *there are* $b(r)$ *and* $c(r)$ *in* A *such that* $\sum_{r=1}^R\|b(r)\|\|c(r)\|\le C\|a\|$ *and* $\|a-\sum_{r=1}^R[b(r),c(r)]\|<\varepsilon+C\|a\|R^{-1/2}$.

PROOF OF THEOREM 5. Let $a\in A$. We denote by ctr the center-valued trace from the second dual von Neumann algebra A^{**} onto the center $\mathcal{Z}(A_{\mathrm{f}}^{**})$ of the finite summand A_{f}^{**} of A^{**}. One has $\|\operatorname{ctr}(a)\|=\sup_{\tau\in T(A)}|\tau(a)|<\varepsilon$ and $a':=a-\operatorname{ctr}(a)$ has zero traces. By a theorem of Fack and de la Harpe, for $C=2\cdot 12^2$ and $m=10$, there are $b_i,c_i\in A^{**}$ such that $\sum_{i=1}^m\|b_i\|\|c_i\|\le C\|a\|$ and $a'=\sum_{i=1}^m[b_i,c_i]$. See [Ma, Po] for a better estimate of C and m. By Kaplansky's density theorem, there is a net $(b_i(\lambda))_\lambda$ in A such that $\|b_i(\lambda)\|\le\|b_i\|$ and $b_i(\lambda)\to b_i$ ultrastrongly. Likewise for $(c_i(\lambda))_\lambda$. Since

$$\|\lim_\lambda(a-\sum_{i=1}^m[b_i(\lambda),c_i(\lambda)])\|=\|a-a'\|<\varepsilon,$$

there is $a''\in\operatorname{conv}\{\sum_{i=1}^m[b_i(\lambda),c_i(\lambda)]\}_\lambda$ which satisfies $\|a-a''\|<\varepsilon$. $\square$

The proof of Theorem 6 is inspired by [Ha2] and uses the free semicircular system and random matrix argument of Haagerup–Thorbjørnsen ([HT]). Let $\mathcal{O}_\infty$ be the Cuntz algebra generated by isometries $l_i(r)$ such that $l_i(r)^*l_j(s)=\delta_{i,j}\delta_{r,s}$, and let $S_i(r):=l_i(r)+l_i(r)^*$ be the corresponding semicircular system. We note that $\mathfrak{C}:=\mathrm{C}^*(\{S_i(r):i,r\})$ is $*$-isomorphic to the reduced free product of the copies of $C([-2,2])$ with respect to the Lebesgue measure (see Section 2.6 in [VDN]), and the corresponding tracial state coincides with the restriction of the vacuum state on $\mathcal{O}_\infty$ to $\mathfrak{C}$.

LEMMA 7. *Let* $b_i,c_i\in A$ *be such that* $\|b_i\|=\|c_i\|$. *Then, for every* $R\in\mathbb{N}$, *letting* $\tilde{b}(r)=\sum_{i=1}^n S_i(r)\otimes b_i$ *and* $\tilde{c}(r)=\sum_{j=1}^n S_j(r)\otimes c_j$, *one has*

$$\frac{1}{R}\sum_{r=1}^R\|\tilde{b}(r)\|\|\tilde{c}(r)\|\le 4\sum\|b_i\|\|c_i\|$$

and

$$\|1\otimes\sum_{i=1}^n[b_i,c_i]-\frac{1}{R}\sum_{r=1}^R[\tilde{b}(r),\tilde{c}(r)]\|\le\frac{6}{\sqrt{R}}\sum_i\|b_i\|\|c_i\|.$$

Proof. For every r, one has

$$\begin{aligned}\|\tilde{b}(r)\| &\le \|\sum l_i(r)\otimes b_i\| + \|\sum l_i(r)^*\otimes b_i\|\\ &= \|\sum b_i^* b_i\|^{1/2} + \|\sum b_i b_i^*\|^{1/2} \le 2(\sum \|b_i\|^2)^{1/2},\end{aligned}$$

and likewise for $\tilde{c}(r)$. It follows that $\|\tilde{b}(r)\|\|\tilde{c}(r)\| \le 4\sum\|b_i\|\|c_i\|$. Moreover,

$$\tilde{b}(r)\tilde{c}(r) = \sum_{i,j}(\delta_{i,j}1 + l_i(r)l_j(r) + l_i(r)l_j(r)^* + l_i(r)^*l_j(r)^*)\otimes b_ic_j,$$

and

$$\begin{aligned}\|\sum_{r,i,j} l_i(r)l_j(r)\otimes b_ic_j\| &= \|\sum_{r,i,j} c_j^*b_i^*b_ic_j\|^{1/2} \le R^{1/2}\sum_i \|b_i\|\|c_i\|,\\ \|\sum_{r,i,j} l_i(r)^*l_j(r)^*\otimes b_ic_j\| &= \|\sum_{r,i,j} b_ic_jc_j^*b_i^*\|^{1/2} \le R^{1/2}\sum_i \|b_i\|\|c_i\|,\\ \|\sum_{r,i,j} l_i(r)l_j(r)^*\otimes b_ic_j\| &= \max_r \|\sum_{i,j} l_i(r)l_j(r)^*\otimes b_ic_j\| \le \sum_i \|b_i\|\|c_i\|.\end{aligned}$$

Likewise for $\tilde{c}(r)\tilde{b}(r)$, and one obtains the conclusion. $\square$

Proof of Theorem 6. Let $a \in A\setminus\{0\}$ be such that $\sup_{\tau\in T(A)}|\tau(a)| < \varepsilon$. Since $\mathcal{Z}\cong\mathcal{Z}^{\otimes\infty}$, we may assume that $A = \mathcal{Z}\otimes A_0$ and $a\in A_0$. By Theorem 5, there are b_i, c_i such that $\|b_i\| = \|c_i\|$, $\sum_{i=1}^k \|b_i\|\|c_i\| \le C\|a\|$, and $\|a - \sum_{i=1}^k [b_i, c_i]\| < \varepsilon$. Recall the theorem of Haagerup and Thorbjørnsen ([HT]) which states that the C*-algebra $\mathfrak{C}$ can be embedded into $\prod\mathbb{M}_n/\bigoplus\mathbb{M}_n$. By exactness of A_0, there is a canonical $*$-isomorphism

$$(\prod\mathbb{M}_n/\bigoplus\mathbb{M}_n)\otimes A_0 \cong ((\prod\mathbb{M}_n)\otimes A_0)/(\bigoplus\mathbb{M}_n\otimes A_0).$$

Lemma 7, combined with this fact, implies that there are matrices $s_i^{(n)}(r)\in\mathbb{M}_n$ such that $\tilde{b}^{(n)}(r) = \sum_{i=1}^k s_i^{(n)}(r)\otimes b_i$ and $\tilde{c}^{(n)}(r) = \sum_{j=1}^k s_j^{(n)}(r)\otimes c_j$ satisfy

$$\limsup_n \frac{1}{R}\sum_{r=1}^R \|\tilde{b}^{(n)}(r)\|\|\tilde{c}^{(n)}(r)\| \le 4\sum\|b_i\|\|c_i\| \le 4C\|a\|$$

and

$$\limsup_n \|1 \otimes a - \frac{1}{R}\sum_{r=1}^{R}[\tilde{b}^{(n)}(r), \tilde{c}^{(n)}(r)]\| \le \varepsilon + \frac{6C\|a\|}{\sqrt{R}}.$$

For every relatively prime $p, q \in \mathbb{N}$, the Jiang–Su algebra $\mathcal{Z}$ contains the prime dimension drop algebra

$$I(p,q) = \{f \in C([0,1], \mathbb{M}_p \otimes \mathbb{M}_q) : f(0) \in \mathbb{M}_p \otimes \mathbb{C}1,\ f(1) \in \mathbb{C}1 \otimes \mathbb{M}_q\}$$

and hence $t\mathbb{M}_q$ and $(1-t)\mathbb{M}_p$ also, where $t \in I(p,q)$ is the identity function on $[0,1]$. It follows that there are $b(r), c(r), b'(r), c'(r) \in \mathcal{Z} \otimes A_0$ such that

$$\frac{1}{R}\sum_{r=1}^{R}(\|b(r)\|\|c(r)\| + \|b'(r)\|\|c'(r)\|) < 9C\|a\|$$

and

$$\|a - \frac{1}{R}\sum_{r=1}^{R}([b(r), c(r)] + [b'(r), c'(r)])\| < \varepsilon + \frac{7C\|a\|}{\sqrt{R}}.$$

Here, we note that $\|t \otimes x + (1-t) \otimes y\| = \max\{\|x\|, \|y\|\}$ for any x and y. $\square$

Let $(A_n)_n$ be a sequence of C*-algebras and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. We denote by

$$\prod A_n = \{(a_n)_{n=1}^{\infty} : a_n \in A_n,\ \sup_n \|a_n\| < +\infty\}$$

the ℓ_∞-direct sum of (A_n), and by

$$\prod A_n/\mathcal{U} = (\prod A_n)/\{(a_n)_{n=1}^{\infty} : \lim_{\mathcal{U}}\|a_n\| = 0\}$$

the ultraproduct of A_n. For every m, we view $\tau \in T(A_m)$ as an element of $T(\prod A_n)$ by $\tau((a_n)_n) = \tau(a_m)$. For each $(\tau_n)_n \in \prod T(A_n)$, there is a corresponding tracial state $\tau_{\mathcal{U}} := \lim_{\mathcal{U}}\tau_n$ on $\prod A_n/\mathcal{U}$, defined by

$$\tau_{\mathcal{U}}((a_n)_n) = \lim_{\mathcal{U}}\tau_n(a_n).$$

The set of tracial states that arise in this way is denoted by $\prod T(A_n)/\mathcal{U}$. We note that as soon as $\partial T(\prod A_n/\mathcal{U})$ is infinite, the inclusion $\prod T(A_n)/\mathcal{U} \subset T(\prod A_n/\mathcal{U})$ is proper (see [BF]). Moreover, if we take A_n to be the counterexamples of Pedersen and Petersen ([PP]), then $\prod T(A_n)/\mathcal{U}$ (resp. $\operatorname{conv} \bigsqcup T(A_n)$) is not weak*-dense in $T(\prod A_n/\mathcal{U})$ (resp. $T(\prod A_n)$). The following theorem is proved by Sato [Sa1] (see also [Rø]) in the case where A is a simple nuclear C*-algebra having finitely many extremal tracial states.

Theorem 8. *Let $(A_n)_n$ be a sequence of exact $\mathcal{Z}$-stable* C*-*algebras and $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then, $\prod T(A_n)/\mathcal{U}$ (resp. $\operatorname{conv} \bigsqcup T(A_n)$) is weak*-dense in $T(\prod A_n/\mathcal{U})$ (resp. $T(\prod A_n)$). In particular, for every $\tau \in T(\prod A_n/\mathcal{U})$ and every separable* C*-*subalgebra $B \subset \prod A_n/\mathcal{U}$, there is $\tau' \in \prod T(A_n)/\mathcal{U}$ such that $\tau|_B = \tau'|_B$.*

Proof of Theorem 8. Let A be either $\prod A_n$ or $\prod A_n/\mathcal{U}$, and denote by $\Sigma \subset T(A)$ either $\operatorname{conv}(\bigsqcup T(A_n))$ or $\prod T(A_n)/\mathcal{U}$ accordingly. Suppose that the conclusion of the theorem is false for $\Sigma \subset T(A)$. Then, by the Hahn–Banach theorem, there are τ in $T(A)$ and a self-adjoint element a_0 in A such that $\gamma := \tau(a_0) - \sup_{\sigma\in\Sigma} \sigma(a_0) > 0$. Let $\alpha = (|\inf_{\sigma\in\Sigma} \sigma(a_0)| - \tau(a_0)) \vee 0$, and take $b \in A_+$ such that $\tau(b) = \alpha$ and $\|b\| < \alpha + \gamma$. It follows that $a = a_0 + b$ satisfies $\sup_{\sigma\in\Sigma} |\sigma(a)| < \tau(a)$. Now, expand $a \in A$ as $(a_n)_n$. We may assume that $\|a_n\| \le \|a\|$ for all n. Let $I \in \mathcal{U}$ (or $I = \mathbb{N}$ in case $A = \prod A_n$) be such that $\varepsilon_0 := \sup_{n\in I} \sup_{\sigma\in T(A_n)} \sigma(a_n) < \tau(a)$. Let $R \in \mathbb{N}$ be such that $\varepsilon_1 := \varepsilon_0 + C\|a\|R^{-1/2} < \tau(a)$. Then, by Theorem 6, for each $n \in I$ there are $b_n(r), c_n(r) \in A_n$ such that $\sum_{r=1}^R \|b_n(r)\|\|c_n(r)\| \le C\|a\|$ and $\|a_n - \sum_{r=1}^R [b_n(r), c_n(r)]\| \le \varepsilon_1$. It follows that for $b(r) = (b_n(r))_n$ and $c(r) = (c_n(r))_n \in A$, one has

$$\tau(a) = \tau(a - \sum_{r=1}^{R} [b(r), c(r)]) \le \|a - \sum_{r=1}^{R} [b(r), c(r)]\| < \tau(a),$$

which is a contradiction. This proves the first half of the theorem.

For the second half, let τ and B be given. Take a dense sequence $(x(i))_{i=0}^{\infty}$ in B and expand them as $x(i) = (x_n(i))_n$. By the first half, for every m, there is $(\tau_n^{(m)})_n \in \prod T(A_n)$ such that $|\tau(x(i)) - \tau_{\mathcal{U}}^{(m)}(x(i))| < m^{-1}$ for $i = 0, \ldots, m$. Let

$$I_m = \{n \in \mathbb{N} : |\tau(x(i)) - \tau_n^{(m)}(x_n(i))| < m^{-1} \text{ for all } i = 0, \ldots, m\} \in \mathcal{U}$$

(so $I_0 = \mathbb{N}$), and $J_m = \bigcap_{l=0}^m I_l \in \mathcal{U}$. We define τ_n to be $\tau_n^{(m)}$ for $n \in J_m \setminus J_{m+1}$. It is not too hard to check $\tau = \tau_{\mathcal{U}}$ on B. $\square$

In passing, we record the following fact.

LEMMA 9. *Let A be a (non-separable)* C*-*algebra and $X \subset A$ be a separable subset. Then there is a separable* C*-*subalgebra $B \subset A$ that contains X and such that the restriction from $T(A)$ to $T(B)$ is onto.*

PROOF. We may assume that A is unital. We first claim that for every $x_1, \dots, x_n \in A$ and $\varepsilon > 0$, there is a separable C*-subalgebra C which satisfies the following property: for every $\tau \in T(C)$ there is $\sigma \in T(A)$ such that $\max_i |\tau(x_i) - \sigma(x_i)| < \varepsilon$. Indeed if this were not true, then for every C there is $\tau_C \in T(C)$ such that $\max_i |\tau_C(x_i) - \sigma(x_i)| \geq \varepsilon$ for all $\sigma \in T(A)$. The set of separable C*-subalgebras of A is upward directed and one can find a limit point τ of $\{\tau_C\}$. Then, we arrive at a contradiction that $\tau \in T(A)$ satisfies $\max_i |\tau(x_i) - \sigma(x_i)| \geq \varepsilon$ for all $\sigma \in T(A)$. We next claim that for every separable C*-subalgebra $B_0 \subset A$, there is a separable C*-subalgebra $B_1 \subset A$ that contains B_0 and such that $\mathrm{Res}_{B_0} T(B_1) = \mathrm{Res}_{B_0} T(A)$ in $T(B_0)$, where Res is the restriction map. Take a dense sequence $x_1, x_2, \dots$ in B_0, and let $C_0 = B_0$. By the previous discussion, there is an increasing sequence of separable C*-subalgebras $C_0 \subset C_1 \subset \cdots$ such that for every $\tau \in T(C_n)$ there is $\sigma \in T(A)$ satisfying $|\tau(x_i) - \sigma(x_i)| < n^{-1}$ for $i = 1, \dots, n$. Now, letting $B_1 = \overline{\bigcup_n C_n}$ and we are done. Finally, we iterate this construction and obtain $X \subset B_0 \subset B_1 \subset \cdots$ such that $\mathrm{Res}_{B_n} T(B_{n+1}) = \mathrm{Res}_{B_n} T(A)$. The separable C*-subalgebra $B = \overline{\bigcup B_n}$ satisfies the desired property. $\square$

Murphy ([Mu]) presents a non-separable example of a unital non-simple C*-algebra with a unique faithful tracial state and asks whether a separable example of such exists. The above lemma answers it. There is another example, which is moreover nuclear. Kirchberg ([Ki]) proves that the Cuntz algebra $\mathcal{O}_\infty$ (or any other unital separable exact C*-algebra) is a subquotient of the CAR algebra $\mathbb{M}_{2^\infty}$. Namely, there are C*-subalgebras J and B in $\mathbb{M}_{2^\infty}$ such that J is hereditary in $\mathbb{M}_{2^\infty}$ and is an ideal in B such that $B/J = \mathcal{O}_\infty$. It follows that B is a unital separable nuclear non-simple C*-algebra with a unique faithful tracial state.

4. Uniform 2-Norm and the Completion

Recall $S \subset T(A)$, $N = (\bigoplus_{\tau \in S} \pi_\tau)(A)''$, and the center-valued trace $\mathrm{ctr}\colon N \to \mathcal{Z}(N)$. Since S is a closed face of $T(A)$, any normal tracial state on N restricts to a tracial state on A which belongs to S. Hence, one has

$$\|a\|_{2,S} = \sup\{\|a\|_{2,\tau} : \tau \in S\} = \sup\{\|a\|_{2,\tau} : \tau \in \partial S\} = \|\,\mathrm{ctr}(a^*a)\|^{1/2}.$$

Since S is a metrizable closed face of the Choquet simplex $T(A)$, it is also a Choquet simplex and there is a canonical one-to-one correspondence

$$S \ni \tau \longleftrightarrow \mu_\tau \in \mathcal{M}^1_+(\partial S),\ \tau(a) = \int \lambda(a)\, d\mu_\tau(\lambda) \text{ for } a \in A.$$

By uniqueness of the representing measure μ_τ, this correspondence is an affine transformation and extends uniquely to a linear order isomorphism between their linear spans.

Lemma 10. *For every $\tau \in S$, there is a normal $*$-isomorphism $\theta_\tau\colon L^\infty(\partial S, \mu_\tau) \to \mathcal{Z}(\pi_\tau(A)'')$ such that*

$$\tau(\theta_\tau(f)a) = \int f(\lambda)\lambda(a)\, d\mu_\tau(\lambda)$$

for $a \in A$.

Proof. Let $f \in L^\infty(\partial S, \mu_\tau)$ be given. The right hand side of the claimed equality defines a tracial linear functional on A whose modulus is dominated by a scalar multiple of τ. Hence, by Sakai's Radon–Nikodym theorem, there is a unique $\theta_\tau(f) \in \mathcal{Z}(\pi_\tau(N))$ that satisfies the claimed equality. This defines a unital normal positive map θ_τ from $L^\infty(\partial S, \mu_\tau)$ into $\mathcal{Z}(\pi_\tau(A)'')$. Next, let $z \in \mathcal{Z}(\pi_\tau(N))_+$ be given. Then, the tracial linear functional $z\tau$ on A defined by $(z\tau)(a) = \tau(az)$ is dominated by $\|z\|\tau$. Hence one has $\mu_{z\tau} \leq \|z\|\mu_\tau$ and $z = \theta_\tau(d\mu_{z\tau}/d\mu_\tau)$ with $d\mu_{z\tau}/d\mu_\tau \in L^\infty(\partial S, \mu_\tau)$. This proves θ_τ is a positive linear isomorphism such that $\mu_{\theta_\tau(f)\tau} = f\mu_\tau$. Therefore, one has $\mu_{\theta_\tau(fg)\tau} = fg\mu_\tau = f\mu_{\theta_\tau(g)\tau} = \mu_{\theta_\tau(f)\theta_\tau(g)\tau}$, which proves $\theta_\tau(fg) = \theta_\tau(f)\theta_\tau(g)$. □

Proof of Theorem 3. We first find the $*$-homomorphism $\theta\colon B(\partial S) \to \mathcal{Z}(N)$ that satisfies

$$\tau(\theta(f)a) = \int f(\lambda)\lambda(a)\, d\mu_\tau(\lambda)$$

for every $a \in A$ and $\tau \in S$, or equivalently, $\pi_\tau(\theta(f)) = \theta_\tau(f)$ in $\pi_\tau(A)''$. For this, it suffices to show that the maps $\theta_\tau|_{B(\partial S)}$, given in Lemma 10, are compatible over $\tau \in S$. We recall that associated with the representation π_τ, there is a unique central projection $p_\tau \in \mathcal{Z}(N)$ such that $(1-p_\tau)N = \ker \pi_\tau$. Since $p_\tau \vee p_\sigma = p_{(\tau+\sigma)/2}$, the family $\{p_\tau : \tau \in S\}$ is upward directed and $\sup_\tau p_\tau = 1$. We will show that if τ and σ are such that $\tau \leq C\sigma$ for some $C > 1$, then $\theta_\tau(f) = p_\tau \theta_\sigma(f)$ in $\mathcal{Z}(N)$. We note that p_τ is the support projection of $d\tau/d\sigma \in \mathcal{Z}(N)$. For every $f \in B(\partial S)$, one has

$$\begin{aligned}
\sigma(\theta_\sigma(\frac{d\mu_\tau}{d\mu_\sigma}f)a) &= \int (\frac{d\mu_\tau}{d\mu_\sigma}f)(\lambda)\lambda(a)\,d\mu_\sigma(\lambda) \\
&= \int f(\lambda)\lambda(a)\,d\mu_\tau(\lambda) \\
&= \tau(\theta_\tau(f)a) \\
&= \sigma(\frac{d\tau}{d\sigma}\theta_\tau(f)a).
\end{aligned}$$

This implies $\theta_\sigma(\frac{d\mu_\tau}{d\mu_\sigma}f) = \frac{d\tau}{d\sigma}\theta_\tau(f)$ for every f. In particular, $\theta_\sigma(\frac{d\mu_\tau}{d\mu_\sigma}) = \frac{d\tau}{d\sigma}$ and $p_\tau\theta_\sigma(f) = \theta_\tau(f)$ in $\mathcal{Z}(N)$. Therefore, we may glue $\{\theta_\tau\}_{\tau\in S}$ together and obtain a globally defined $*$-homomorphism $\theta\colon B(\partial S) \to \mathcal{Z}(N)$. Since $\tau(\theta(\hat{a})) = \int \hat{a}(\lambda)\,d\mu_\tau(\lambda) = \tau(a)$ for every $\tau \in S$, one has $\theta(\hat{a}) = \mathrm{ctr}(a)$ for every $a \in A$. This proves the first part of the theorem.

For the second part, it suffices to prove

$$\overline{A}^{\mathrm{st}} \supset \{x \in N : \mathrm{ctr}(xA) \subset \mathcal{A}\mathrm{ff}(\partial S),\ \mathrm{ctr}(x^*x) \in \mathcal{A}\mathrm{ff}(\partial S)\},$$

as the converse inclusion is trivial. Take x from the set in the right hand side. We will prove a stronger assertion that if a net $(b_j)_j$ in A converges to x ultrastrongly in N, then x is contained in the strict closure of the convex hull of $\{b_j : j\}$. We note that $\mathrm{Aff}(S) \ni f \mapsto f|_{\partial S} \in \mathcal{A}\mathrm{ff}(\partial S)$ is an affine order isomorphism and that every positive norm-one linear functional μ on $\mathrm{Aff}(S)$ is given by the evaluation at a point $\tau_\mu \in S$. (Indeed by the Hahn–Banach theorem, we may regard μ as a state on $C(S)$, which is a probability measure on S by the Riesz–Markov theorem. The point $\tau_\mu = \int \lambda\,d\mu(\lambda)$ satisfies $f(\tau_\mu) = \mu(f)$ for $f \in \mathrm{Aff}(S)$.) Thus, one has $\mathrm{ctr}((b_j - x)^*(b_j - x)) \to 0$ weakly in $\mathcal{A}\mathrm{ff}(\partial S)$. Therefore, by the Hahn–Banach theorem, for every $\varepsilon > 0$ there is a finite sequence $\alpha_j \geq 0$, $\sum \alpha_j = 1$

such that $\|\sum_j \alpha_j \operatorname{ctr}((b_j - x)^*(b_j - x))\| < \varepsilon$. By reindexing, we assume $j = 1, \dots, k$. Let $b = \sum \alpha_j b_j$. We note that

$$b = \begin{bmatrix} \alpha_1^{1/2} & \cdots & \alpha_m^{1/2} \end{bmatrix} \begin{bmatrix} \alpha_1^{1/2} b_1 \\ \vdots \\ \alpha_m^{1/2} b_m \end{bmatrix} =: rc.$$

Hence, $b^*b = c^*r^*rc \le \|r\|^2 c^*c = \sum \alpha_j b_j^* b_j$. It follows that

$$\begin{aligned} \operatorname{ctr}((b-x)^*(b-x)) &= \operatorname{ctr}(b^*b - b^*x - x^*b + x^*x) \\ &\le \operatorname{ctr}(\sum \alpha_j b_j^* b_j - \sum \alpha_j b_j^* x - x^* \sum \alpha_j b_j + x^*x) \\ &= \operatorname{ctr}(\sum \alpha_j (b_j - x)^*(b_j - x)) \\ &< \varepsilon. \end{aligned}$$

This proves the claimed inclusion. The last assertion will be proved in more general setting as Theorem 11. □

5. Continuous W*-Bundles

Let K be a metrizable compact Hausdorff topological space. We call M a (tracial) *continuous* W*-*bundle over* K if the following axiom hold:

(1) There is a unital positive faithful tracial map $E \colon M \to C(K)$.

(2) The closed unit ball of M is complete with respect to the uniform 2-norm

$$\|x\|_{2,\mathrm{u}} = \|E(x^*x)^{1/2}\|.$$

(3) $C(K)$ is contained in the center of M and E is a conditional expectation.

In case M satisfies only conditions (1) and (2), we say it is a continuous quasi-W*-bundle. If we denote by π_E the GNS representation of M on the Hilbert $C(K)$-module $L^2(M, E)$, condition (2) is equivalent to that $\pi_E(M)$ is strictly closed in $\mathbb{B}(L^2(M, E))$. For each point $\lambda \in K$, we denote by π_λ the GNS representation for the tracial state $\tau_\lambda := \mathrm{ev}_\lambda \circ E$, and also

$\|x\|_{2,\lambda} = \tau_\lambda(x^*x)^{1/2}$. We call each $\pi_\lambda(M)$ a *fiber* of M. A caveat is in order: the system $(M, K, \pi_\lambda(M))$ need not be a continuous C*-bundle because $\ker \pi_\lambda$ may not coincide with $C_0(K \setminus \{\lambda\})M$—rather it coincides with the strict closure of that. In particular, for $x \in M$, the map $\lambda \mapsto \|\pi_\lambda(x)\|$ need not be upper semi-continuous (but it is lower semi-continuous). The strict completion $\overline{A}^{\mathrm{st}}$ studied in Section 4 is a continuous quasi-W*-bundle over S, and by Theorem 3, it is a continuous W*-bundle over ∂S if ∂S is closed in S. Conversely, if each fiber $\pi_\lambda(M)$ is a factor, then K can be viewed as a closed subset of the extreme boundary of $T(M)$ and hence the closed convex hull S of K is a metrizable closed face of $T(M)$ such that $\partial S = K$.

THEOREM 11. *Let M be a continuous* W*-*bundle over K. Then, $\pi_\lambda(M) = \pi_\lambda(M)''$ for every $\lambda \in K$. Moreover, if a bounded function $f\colon K \ni \lambda \mapsto f(\lambda) \in \pi_\lambda(M)$ is continuous in the following sense: for every $\lambda_0 \in K$ and $\varepsilon > 0$, there are a neighborhood O of λ_0 and $c \in M$ such that*

$$\sup_{\lambda \in O} \|\pi_\lambda(c) - f(\lambda)\|_{2,\lambda} < \varepsilon;$$

then there is $a \in M$ such that $\pi_\lambda(a) = f(\lambda)$.

PROOF. Let $\lambda \in K$ be given. By Pedersen's up-down theorem (Theorem 2.4.4 in [Pe]), it suffices to show that $\pi_\lambda(M)$ is closed in $\pi_\lambda(M)''$ under monotone sequential limits. Let $(x_n)_{n=0}^\infty$ be an increasing sequence in $\pi_\lambda(M)_+$ such that $x_n \nearrow x$ in $\pi_\lambda(M)''$. We may assume that $\|x_n - x\|_{2,\lambda} < 2^{-n}$. We lift $(x_n)_{n=0}^\infty$ to an increasing sequence $(a_n)_{n=0}^\infty$ in M such that $a_n \le \|x\| + 1$. Let $b_n = a_n - a_{n-1}$ for $n \ge 1$. Since $\tau_\lambda(b_n^* b_n) < 4^{-n+2}$, there is $f_n \in C(K)_+$ such that $0 \le f_n \le 1$, $f_n(\lambda) = 1$, and $\|E(b_n^* b_n) f_n^2\| \le 4^{-n+2}$. It follows that the series $a_0 + \sum_{n=1}^\infty b_n f_n$ is convergent in the uniform 2-norm. Moreover, since $a_0 + \sum_{k=1}^n b_k f_k \le a_0 + \sum_{k=1}^n b_k = a_n \le \|x\| + 1$, the series is norm bounded. Therefore, the series converges in M, by the completeness of the closed unit ball of M. The limit point a satisfies $\pi_\lambda(a) = x$.

We prove the second half. Let us fix n for a while. For each λ, there is $b_\lambda \in M$ such that $\|b_\lambda\| \le \|f(\lambda)\|$ and $\pi_\lambda(b_\lambda) = f(\lambda)$. By continuity, there is a neighborhood O_λ of λ such that $\|\pi_\tau(b_\lambda) - f(\tau)\|_{2,\tau} < n^{-1}$ for $\tau \in O_\lambda$. Since K is compact, it is covered by a finite family $\{O_{\lambda_i}\}$. Let $g_i \in C(K) \subset \mathcal{Z}(M)$ be a partition of unity subordinated by it. Then, $a_n := \sum_i g_i b_{\lambda_i} \in M$ satisfies $\|a_n\| \le \|f\|_\infty$ and $\sup_\tau \|\pi_\tau(a_n) - f(\tau)\|_{2,\tau} < n^{-1}$.

It follows that (a_n) is a norm bounded and Cauchy in the uniform 2-norm. Hence it converges to $a \in M$ such that $\pi_\lambda(a) = f(\lambda)$ for every $\lambda \in K$. □

The following is a W*-analogue of the result for C*-algebras in [HRW], and is essentially the same as Proposition 7.7 in [KR].

COROLLARY 12. *Let M be a continuous* W*-*bundle over K. Assume that each fiber $\pi_\lambda(M)$ has the McDuff property and that K has finite covering dimension. Then, for every k, there is an approximately central approximately multiplicative embedding of $\mathbb{M}_k$ into M, namely a net of unital completely positive maps $\varphi_n\colon \mathbb{M}_k \to M$ such that* $\limsup_n \|\varphi_n(xy) - \varphi_n(x)\varphi_n(y)\|_{2,\mathrm{u}} = 0$ *and* $\limsup_n \|[\varphi_n(x), a]\|_{2,\mathrm{u}} = 0$ *for every $x, y \in \mathbb{M}_k$ and $a \in M$.*

PROOF. The proof is particularly easy when K is zero-dimensional: Since $\pi_\lambda(M)$ is McDuff, there is an approximately central embedding of $\mathbb{M}_k$ into $\pi_\lambda(M)$. We lift it to a unital completely positive map $\psi_\lambda\colon \mathbb{M}_k \to M$. It is almost multiplicative on a neighborhood O_λ of λ. Since K is compact and zero-dimensional, there is a partition of K into finitely many clopen subsets $\{V_i\}$ such that $V_i \subset O_{\lambda_i}$. By Theorem 11, one can define $\varphi\colon \mathbb{M}_k \to M$ by the relation $\pi_\lambda \circ \varphi = \pi_\lambda \circ \psi_{\lambda_i}$ for $\lambda \in V_i$. The case $0 < \dim K < +\infty$ is more complicated but follows from a standard argument involving order-zero maps. See Section 7 in [KR] (or [Sa2, TWW]) for the detail. □

Every separable hyperfinite von Neumann algebra with a faithful normal tracial state has a trace preserving embedding into the separable hyperfinite II_1 factor $\mathcal{R}$. We consider coordinatization of such embeddings for strictly separable fiberwise hyperfinite continuous quasi-W*-bundle. We define the C*-algebra $C_\sigma(K, \mathcal{R})$ to be the subalgebra of $\ell_\infty(K, \mathcal{R})$ which consists of those norm-bounded functions $f\colon K \to \mathcal{R}$ that are continuous from K into $L^2(\mathcal{R}, \tau_{\mathcal{R}})$.

THEOREM 13. *Let M be a strictly separable continuous quasi-*W*-*bundle over K such that $\pi_\lambda(M)''$ is hyperfinite for every $\lambda \in K$. Then, there are an embedding $\theta\colon M \hookrightarrow C_\sigma(K, \mathcal{R})$ and embeddings $\iota_\lambda\colon \pi_\lambda(M) \hookrightarrow \mathcal{R}$ such that $\mathrm{ev}_\lambda \circ\theta = \iota_\lambda \circ \pi_\lambda$. If M is moreover a continuous* W*-*bundle, then one has*

$$\theta(M) = \{f \in C_\sigma(K, \mathcal{R}) : f(\lambda) \in (\iota_\lambda \circ \pi_\lambda)(M)''\}.$$

Recall the fact that if (A,τ) is a separable hyperfinite von Neumann algebra with a distinguished tracial state, then a trace-preserving embedding of A into the tracial ultrapower $\mathcal{R}^\omega$ of the hyperfinite II_1 factor is unique up to unitary conjugacy (see [Ju]). For every n-tuples $x_1,\dots,x_n\in P$ and $y_1,\dots,y_n\in Q$ in hyperfinite II_1 factors P and Q, we define

$$d(\{x_i\}_{i=1}^n,\{y_i\}_{i=1}^n)=\inf_{\pi,\rho}\max_i\|\pi(x_i)-\rho(y_i)\|_2,$$

where the infimum runs over all trace-preserving embeddings of P and Q into $\mathcal{R}^\omega$. Then, d is a pseudo-metric and it depends on $(W^*(\{x_1,\dots,x_n\}),\tau)$, i.e., the joint distribution of $\{x_1,\dots,x_n\}$ with respect to τ_P, rather than the specific embedding of $W^*(\{x_1,\dots,x_n\})$ into P. Once $*$-isomorphisms $P\cong Q\cong\mathcal{R}$ are fixed, P and Q are embedded into $\mathcal{R}^\omega$ as constant sequences and

$$d(\{x_i\}_{i=1}^n,\{y_i\}_{i=1}^n)=\inf_{U\in\mathcal{U}(\mathcal{R}^\omega)}\max_i\|\operatorname{Ad}_U(x_i)-y_i\|_2.$$

It follows that

$$d(\{x_i\}_{i=1}^n,\{y_i\}_{i=1}^n)=\inf_{\pi}\max_i\|\pi(x_i)-y_i\|_2,$$

where infimum runs over all trace-preserving $*$-homomorphisms π from $W^*(\{x_1,\dots,x_n\})$ into Q, or over all $*$-isomorphisms π from P onto Q. If M is a continuous quasi-W*-bundle, then for every $a_1,\dots,a_n\in M$, the map

$$K\ni\lambda\mapsto\{\pi_\lambda(a_i)\}_{i=1}^n$$

is continuous with respect to d.

LEMMA 14. *Let $N=C_\sigma(K,\mathcal{R})$ or any other continuous* W**-bundle over K such that $\mathrm{ev}_\lambda(N)\cong\mathcal{R}$ for every $\lambda\in K$ and such that for every $k\in\mathbb{N}$ there is an approximately central approximately multiplicative embedding of $\mathbb{M}_k$ into N. Let M be a continuous quasi-*W**-bundle over K such that $\pi_\lambda(M)''$ is hyperfinite for every $\lambda\in K$, and let $F_0\subset F_1$ be finite subsets in the unit ball of M and $\varepsilon>0$. Assume that there is a map θ_0 from F_0 into the unit ball of N such that*

$$\sup_{\lambda\in K}d(\{\pi_\lambda(a)\}_{a\in F_0},\{\mathrm{ev}_\lambda(\theta_0(a))\}_{a\in F_0})<\varepsilon.$$

Then, for every $\delta > 0$*, there is a map* θ_1 *from* F_1 *into the unit ball of* N *such that*

$$\sup_{\lambda \in K} d(\{\pi_\lambda(a)\}_{a \in F_1}, \{\mathrm{ev}_\lambda(\theta_1(a))\}_{a \in F_1}) < \delta$$

and

$$\max_{a \in F_0} \|\theta_1(a) - \theta_0(a)\|_{2,\mathrm{u}} < \varepsilon.$$

Here the symbol ev_λ, instead of π_λ, is used for the N side to make a distinction from the M side.

Proof. For each λ, there is a trace-preserving embedding ρ_λ: $\pi_\lambda(M) \to \mathrm{ev}_\lambda(N)$. By the remarks preceding this lemma, we may assume that

$$\max_{a \in F_0} \|\rho_\lambda(\pi_\lambda(a)) - \mathrm{ev}_\lambda(\theta_0(a))\|_2 < \varepsilon.$$

For each $a \in F_1$, we lift $(\rho_\lambda \circ \pi_\lambda)(a) \in \mathrm{ev}_\lambda(N)$ to $a^\lambda \in N$ with $\|a^\lambda\| \leq 1$. There is a neighborhood O_λ of λ such that $\tau \in O_\lambda$ implies

$$d(\{\pi_\tau(a)\}_{a \in F_1}, \{\mathrm{ev}_\tau(a^\lambda)\}_{a \in F_1}) < \delta$$

and

$$\max_{a \in F_0} \| \mathrm{ev}_\tau(a^\lambda) - \mathrm{ev}_\tau(\theta_0(a))\|_2 < \varepsilon.$$

By compactness, K is covered by a finite family $\{O_{\lambda_j}\}$. Take a partition of unity $g_j \in C(K)$ subordinated by $\{O_{\lambda_j}\}$. Let $h_0 = 0$ and $h_j = \sum_{i=1}^{j} g_i$. For each k, take an approximately central approximately multiplicative embedding $\varphi_{k,n}$ of $\mathbb{M}_k$ into N. Since the closed unit ball of $\mathbb{M}_k$ is norm-compact, one has

$$\forall a \in N \quad \limsup_n \sup\{\|[\varphi_{k,n}(x), a]\|_{2,\mathrm{u}} : x \in \mathbb{M}_k,\ \|x\| \leq 1\} = 0.$$

For $t \in [0, 1]$, we define $p_t \in \mathbb{M}_k$ to be $\mathrm{diag}(1, \ldots, 1, t - \lfloor t \rfloor, 0, \ldots, 0)$, with 1s in the first $\lfloor t \rfloor$ diagonal entries, $t - \lfloor t \rfloor$ in the $(\lfloor t \rfloor + 1)$-th entry, and 0s in the rest. It follows that $t \mapsto p_t$ is continuous, $0 \le p_t \le 1$, $\mathrm{tr}(p_t) = t$, and $\tau(p_t - p_t^2) \le (4k)^{-1}$. We write $p_{[s,t]} = p_t - p_s$. With the help of Theorem 11, we define $f_{k,n,j} \in N$ to be the element such that

$$\mathrm{ev}_\lambda(f_{k,n,j}) = \mathrm{ev}_\lambda(\varphi_{k,n}(p_{[h_{j-1}(\lambda),h_j(\lambda)]})).$$

For $a \in F_1$, we define $\theta_1^{k,n}(a) \in N$ by $\theta_1^{k,n}(a) = \sum_j f_{k,n,j}^{1/2} a^{\lambda_j} f_{k,n,j}^{1/2}$. Since $F_1' := F_1 \cup \{a^{\lambda_j} : a \in F_1,\ j\}$ is finite, it is not too hard to see

$$\limsup_k \limsup_n \max_{a \in F_0} \|\theta_1^{k,n}(a) - \theta_0(a)\|_{2,\mathrm{u}} < \varepsilon.$$

It remains to estimate

$$d(\{\pi_\tau(a)\}_{a\in F_1}, \{\mathrm{ev}_\tau(\theta_1^{k,n}(a))\}_{a\in F_1}).$$

Let k be fixed for the moment. Since $(\varphi_{k,n})_n$ is approximately multiplicative, there are unital $*$-homomorphisms $\psi_{k,n}^\tau \colon \mathbb{M}_k \to \mathrm{ev}_\tau(N)$ such that

$$\limsup_n \sup_\tau \sup_{x\in\mathbb{M}_k,\, \|x\|\le 1} \|\,\mathrm{ev}_\tau \circ\varphi_{k,n}(x) - \psi_{k,n}^\tau(x)\|_2 = 0.$$

Let $E_{k,n}^\tau$ be the trace-preserving conditional expectation from $\mathrm{ev}_\tau(N)$ onto the relative commutant $\psi_{k,n}^\tau(\mathbb{M}_k)' \cap \mathrm{ev}_\tau(N)$, which is given by $E_{k,n}^\tau(b) = |G|^{-1}\sum_{u\in G} \psi_{k,n}^\tau(u) b \psi_{k,n}^\tau(u)^*$ for the group G of permutation matrices in $\mathcal{U}(\mathbb{M}_k)$. It follows that

$$\limsup_n \sup_\tau \|\,\mathrm{ev}_\tau(b) - E_{k,n}^\tau(\mathrm{ev}_\tau(b))\|_2 = 0$$

for every $b \in N$. This implies

$$\limsup_n \sup_{j,\,\tau\in O_{\lambda_j}} d(\{\pi_\tau(a)\}_{a\in F_1}, \{E_{k,n}^\tau(\mathrm{ev}_\tau(a^{\lambda_j}))\}_{a\in F_1}) < \delta,$$

$$\limsup_n \sup_{j,\,\tau\in O_{\lambda_j}} d(\{\mathrm{ev}_\tau(a^{\lambda_j})\}_{a\in F_1}, \{E_{k,n}^\tau(\mathrm{ev}_\tau(a^{\lambda_j}))\}_{a\in F_1}) = 0,$$

and

$$\limsup_n \sup_{j,\,\tau\in O_{\lambda_j}} d(\{\mathrm{ev}_\tau(\theta_1^{k,n}(a))\}_{a\in F_1},$$
$$\{\sum_j \psi_{k,n}^\tau(p_{[h_{j-1}(\lambda),h_j(\lambda)]}) E_{k,n}^\tau(\mathrm{ev}_\tau(a^{\lambda_j}))\}_{a\in F_1}) = 0.$$

If we view $\mathrm{ev}_\tau(N) = \mathbb{M}_k(\psi^\tau_{k,n}(\mathbb{M}_k)' \cap \mathrm{ev}_\tau(N))$, then $a' = E^\tau_{k,n}(\mathrm{ev}_\tau(a))$ looks like $\mathrm{diag}(a', a', \ldots, a')$, and $\psi^\tau_{k,n}(p_t)$ looks like $\mathrm{diag}(1, \ldots, 1, t - \lfloor t \rfloor, 0 \ldots, 0)$. Hence, one has

$$\begin{aligned}\sup_\tau d(\{\pi_\tau(a)\}_{a\in F_1}, \{\sum_j \psi^\tau_{k,n}(p_{[h_{j-1}(\lambda),h_j(\lambda)]}) E^\tau_{k,n}(\mathrm{ev}_\tau(a^{\lambda_j}))\}_{a\in F_1})^2 \\ < \frac{2|\{O_{\lambda_j}\}|}{k} + \sum_j g_j(\tau) d(\{\pi_\tau(a)\}_{a\in F_1}, \{E^\tau_{k,n}(\mathrm{ev}_\tau(a^{\lambda_j}))\}_{a\in F_1})^2.\end{aligned}$$

Altogether, one has

$$\limsup_k \limsup_n \sup_\tau d(\{\pi_\tau(a)\}_{a\in F_1}, \{\mathrm{ev}_\tau(\theta_1^{k,n}(a))\}_{a\in F_1}) < \delta.$$

Therefore, for some k, n, the map $\theta_1 = \theta_1^{k,n}$ satisfies the desired properties. □

Proof of Theorem 13. Let $(a_n)_{n=1}^\infty$ be a strictly dense sequence in the unit ball of M. We use Lemma 14 recursively and obtain sequences $(\{\theta_n(a_i)\}_{i=1}^n)_{n=1}^\infty$ in $C_\sigma(K, \mathcal{R})$ such that

$$\sup_\lambda d(\{\mathrm{ev}_\lambda(\theta_n(a_i))\}_{i=1}^n, \{\pi_\lambda(a_i)\}_{i=1}^n) < 2^{-n}$$

and

$$\max_{i=1,\ldots,n-1} \|\theta_n(a_i) - \theta_{n-1}(a_i)\|_{2,\mathrm{u}} < 2^{-(n-1)}.$$

Then, each sequence $(\theta_n(a_i))_{n=i}^\infty$ converges to an element $\theta(a_i) \in C_\sigma(K, \mathcal{R})$. The map θ extends to a $*$-homomorphism from M into $C_\sigma(K, \mathcal{R})$, and $\mathrm{ev}_\lambda \circ \theta$ factors through π_λ. This proves the first assertion. The second follows from Theorem 11. □

We give a criterion for a continuous $\mathcal{R}$-bundle to be a trivial bundle.

Theorem 15. *Let M be a strictly separable continuous* W*-*bundle over K such that $\pi_\lambda(M) \cong \mathcal{R}$ for every $\lambda \in K$. Then, the following are equivalent.*

(i) *$M \cong C_\sigma(K, \mathcal{R})$ as a continuous* W*-*bundle.*

(ii) *There is a sequence* $(p_n)_n$ *in* M *such that* $0 \le p_n \le 1$, $\|p_n - p_n^2\|_{2,\mathrm{u}} \to 0$, $\|E(p_n) - 1/2\| \to 0$, *and* $\|[p_n, a]\|_{2,\mathrm{u}} \to 0$ *for all* $a \in M$.

(iii) *For every* k, *there is an approximately central approximately multiplicative embedding of* $\mathbb{M}_k$ *into* M.

PROOF. The implication (i) $\Rightarrow$ (ii) is obvious. For (ii) $\Rightarrow$ (iii), we we observe that since $\pi_\lambda(M)$'s are all factors, the central sequence $(p_n)_n$ satisfies $\|E(p_n a) - E(p_n)E(a)\| \to 0$ for every $a \in M$. Indeed, let $a \in M$ and $\varepsilon > 0$ be given. By the Dixmier approximation theorem and the proof of Theorem 3, there are $u_1, \dots, u_k \in \mathcal{U}(M)$ such that $\|E(a) - \frac{1}{k}\sum_{i=1}^k u_i a u_i^*\|_{2,\mathrm{u}} < \varepsilon$. It follows that

$$\begin{aligned}\limsup_{n\to\infty} \|E(p_n)E(a) - E(p_n a)\| &= \limsup_{n\to\infty} \|E(p_n E(a)) - \frac{1}{k}\sum_{i=1}^k E(u_i p_n a u_i^*)\| \\ &= \limsup_{n\to\infty} \|E(p_n(E(a) - \frac{1}{k}\sum_{i=1}^k u_i a u_i^*))\| \\ &< \varepsilon.\end{aligned}$$

Let $m \in \mathbb{N}$ be arbitrary. For a given finite sequence $(p_n)_{n=1}^m$, $0 \le p_i \le 1$, and $\nu \in \{0,1\}^m$, we define $q_\nu \in M$ by

$$q_\nu = r_1^{1/2} \cdots r_{m-1}^{1/2} r_m r_{m-1}^{1/2} \cdots r_1^{1/2} \in M,$$

where $r_i = p_i$ or $1 - p_i$ depending on $\nu(i) \in \{0,1\}$. We note that $q_\nu \ge 0$ and $\sum q_\nu = 1$. By choosing $(p_n)_{n=1}^m$ appropriately, we obtain an approximately central approximately multiplicative embedding of $\ell_\infty(\{0,1\}^m)$ into M. Now, condition (iii) follows by choosing at the local level approximately central approximately multiplicative embeddings of $\mathbb{M}_k$ into $\pi_\lambda(M)$ and glue them together, as in the proof of Lemma 14, by an approximately central approximately projective partition of unity.

The proof of (iii) $\Rightarrow$ (i) is similar to that of Theorem 13. Let $(a_n)_{n=1}^\infty$ (resp. $(b_n)_{n=1}^\infty$) be a strictly dense sequence in the unit ball of M (resp. $C_\sigma(K, \mathcal{R})$). We recursively construct finite subsets $F_1 \subset F_2 \subset \cdots$ of M and maps $\theta_n \colon F_n \to C_\sigma(K, \mathcal{R})$ such that $\{a_1, \dots, a_n\} \subset F_n$,

$$\sup_\lambda d(\{\mathrm{ev}_\lambda(\theta_n(a))\}_{a \in F_n}, \{\pi_\lambda(a)\}_{a \in F_n}) < 2^{-n},$$

$$\max_{a\in F_{n-1}} \|\theta_n(a)-\theta_{n-1}(a)\|_{2,\mathrm{u}} < 2^{-(n-1)},$$

and $\{b_1,\dots,b_n\} \subset \theta_n(F_n)$. Let $F_0=\emptyset$ and suppose that we have constructed up to $n-1$. Let $F_n' = F_{n-1}\cup\{a_n\}$. We use Lemma 14 and obtain a map $\theta_n'\colon F_n'\to C_\sigma(K,\mathcal{R})$ such that

$$\sup_\lambda d(\{\mathrm{ev}_\lambda(\theta_n'(a))\}_{a\in F_n'}, \{\pi_\lambda(a)\}_{a\in F_n'}) < 2^{-(n+1)}$$

and

$$\max_{a\in F_{n-1}} \|\theta_n'(a)-\theta_{n-1}(a)\|_{2,\mathrm{u}} < 2^{-(n-1)}.$$

We may assume that θ_n' is injective and $\theta_n'(F_n')$ does not contain any of $b_1,\dots,b_n$. We use Lemma 14 again but this time to $\theta_n'(F_n')\subset\tilde{F}:=\theta_n'(F_n')\cup\{b_1,\dots,b_n\}$ and $(\theta_n')^{-1}$. Then, there is $\psi\colon\tilde{F}\to M$ such that

$$\sup_\lambda d(\{\pi_\lambda(\psi(b))\}_{b\in\tilde{F}}, \{\mathrm{ev}_\lambda(b)\}_{b\in\tilde{F}}) < 2^{-(n+1)}$$

and

$$\max_{a\in F_n'} \|a-\psi(\theta_n'(a))\|_{2,\mathrm{u}} < 2^{-(n+1)}.$$

Now, we set $F_n = F_n'\cup\{\psi(b_1),\dots\psi(b_n)\}$ (which can be assumed to be a disjoint union) and define $\theta_n\colon F_n\to C_\sigma(K,\mathcal{R})$ by $\theta_n=\theta_n'$ on F_n' and $\theta_n(\psi(b_i))=b_i$. One has

$$\begin{aligned}
&\sup_\lambda d(\{\mathrm{ev}_\lambda(\theta_n(a))\}_{a\in F_n}, \{\pi_\lambda(a)\}_{a\in F_n})\\
&\quad\le \sup_\lambda (d(\{\mathrm{ev}_\lambda(b)\}_{b\in\tilde{F}}, \{\pi_\lambda(\psi(b))\}_{b\in\tilde{F}}) + \max_{a\in F_n'}\|\pi_\lambda(\psi(\theta_n'(a)))-\pi_\lambda(a)\|_2)\\
&\quad< 2^{-n}
\end{aligned}$$

as desired. By taking the limit of $(\theta_n)_n$, one obtains a $*$-isomorphism θ from M onto $C_\sigma(K,\mathcal{R})$. □

By combining Corollary 12 and Theorem 15, one obtains the following W*-analogue of Theorem 1.1 in [DW]. This also implies Theorem 4. It is unclear whether the finite-dimensionality assumption is essential.

COROLLARY 16. *Let M be a strictly separable continuous* W*-*bundle over K. If every fiber $\pi_\lambda(M)$ is isomorphic to $\mathcal{R}$ and K has finite covering dimension, then $M \cong C_\sigma(K, \mathcal{R})$ as a continuous* W*-*bundle.*

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