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On the Comparison of One Pair of Second Order Linear Differential Equations

By Zinelâabidine LATREUCH and Benharrat BELAÏDI

Abstract. The main purpose of this paper is to study the controllability of solutions of one pair of linear differential equations

$$f'' + A(z) f = 0$$

and

$$g'' + B(z) g = 0.$$

We study the growth and oscillation of $w = d_1 f + d_2 g$, where f, g are the solutions of the above equations and d_1, d_2 are entire functions of finite order.

1. Introduction and Main Results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory (see [9], [15]). Let f be a meromorphic function in the complex plane, we define

$$m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(re^{i\theta}\right) \right| d\theta,$$

$$N(r, f) = \int_0^r \frac{n(t, f) - n(0, f)}{t} dt + n(0, f) \log r,$$

and

$$T(r, f) = m(r, f) + N(r, f)$$

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is the Nevanlinna characteristic function of f, where $\log^+ x = \max(0, \log x)$ for $x \geq 0$, and $n(t, f)$ is the number of poles of $f(z)$ lying in $|z| \leq t$, counted according to their multiplicity. Also, we define

$$N\left(r, \frac{1}{f}\right) = \int_0^r \frac{n\left(t, \frac{1}{f}\right) - n\left(0, \frac{1}{f}\right)}{t} dt + n\left(0, \frac{1}{f}\right) \log r,$$

$$\overline{N}\left(r, \frac{1}{f}\right) = \int_0^r \frac{\overline{n}\left(t, \frac{1}{f}\right) - \overline{n}\left(0, \frac{1}{f}\right)}{t} dt + \overline{n}\left(0, \frac{1}{f}\right) \log r,$$

where $n\left(t, \frac{1}{f}\right)$ is the number of zeros of $f(z)$ lying in $|z| \leq t$, counted according to their multiplicity, and $\overline{n}\left(t, \frac{1}{f}\right)$ indicate the number of distinct zeros of $f(z)$ lying in $|z| \leq t$. In addition, we will use notations $\lambda(f) = \limsup\limits_{r\to+\infty} \frac{\log N\left(r, \frac{1}{f}\right)}{\log r}$ and $\overline{\lambda}(f) = \limsup\limits_{r\to+\infty} \frac{\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}$ to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of f. A meromorphic function $\varphi(z)$ is called a small function with respect to $f(z)$ if $T(r, \varphi) = o(T(r, f))$ as $r \to +\infty$ except possibly a set of r of finite linear measure. See ([9], [11], [15]) for notations and definitions.

Definition 1.1 ([9], [15]). Let f be a meromorphic function. Then the order of growth $\rho(f)$ of f is defined by

$$\rho(f) = \limsup_{r\to+\infty} \frac{\log T(r, f)}{\log r}.$$

Definition 1.2 ([7], [15]). Let f be a meromorphic function. Then the hyper-order of f is defined by

$$\rho_2(f) = \limsup_{r\to+\infty} \frac{\log \log T(r, f)}{\log r}.$$

Definition 1.3 ([9], [14]). The type of a meromorphic function f of order ρ $(0 < \rho < \infty)$ is defined by

$$\tau(f) = \limsup_{r\to+\infty} \frac{T(r, f)}{r^\rho}.$$

Definition 1.4 ([7], [15]). Let f be a meromorphic function. Then the hyper-exponent of convergence of zeros sequence of f is defined by

$$\lambda_2(f) = \limsup_{r\to+\infty} \frac{\log\log N\left(r, \frac{1}{f}\right)}{\log r}.$$

Similarly, the hyper-exponent of convergence of the sequence of distinct zeros of f is defined by

$$\overline{\lambda}_2(f) = \limsup_{r\to+\infty} \frac{\log\log \overline{N}\left(r, \frac{1}{f}\right)}{\log r}.$$

Suppose that f and g are solutions of the complex linear differential equations

$$f'' + A(z)f = 0 \tag{1.1}$$

and

$$g'' + B(z)g = 0, \tag{1.2}$$

and let the polynomial of solutions

$$w = d_1 f + d_2 g. \tag{1.3}$$

In [13], the authors have investigated the relation between the solutions of (1.1) and small functions. They study the growth and oscillation of $g_f = d_1 f_1 + d_2 f_2$, where f_1 and f_2 are two linearly independent solutions of (1.1) and have obtained the following results.

Theorem A ([13]). *Let $A(z)$ be a transcendental entire function of finite order. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that* $\max\{\rho(d_1), \rho(d_2)\} < \rho(A)$. *If f_1 and f_2 are two linearly independent solutions of* (1.1), *then the polynomial of solutions $g_f = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(g_f) = \rho(f_j) = \infty \ \ (j = 1, 2)$$

and

$$\rho_2\left(g_f\right)=\rho_2\left(f_j\right)=\rho\left(A\right)\quad\left(j=1,2\right).$$

THEOREM B ([13]). *Under the hypotheses of Theorem A, let $\varphi\left(z\right)\not\equiv 0$ be an entire function with finite order such that*

$$\psi\left(z\right)=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}\varphi^{(3)}+\phi_2\varphi''+\phi_1\varphi'+\phi_0\varphi\not\equiv 0,$$

where

$$\phi_2=\frac{3d_2^2d_1''-3d_1d_2d_2''}{h},$$

$$\phi_1=\frac{2d_1d_2d_2'A+6d_2d_1'd_2''-6d_2d_2'd_1''-2d_2^2d_1'A}{h},$$

$$\phi_0=\frac{1}{h}\Big[2d_2d_1'd_2'''-2d_1d_2'd_2'''-3d_1d_2d_2''A-3d_2d_1''d_2''+2d_1d_2d_2'A'$$

$$-4d_2d_1'd_2'A-6d_1'd_2'd_2''+3d_1\left(d_2''\right)^2+4d_1\left(d_2'\right)^2A$$

$$+3d_2^2d_1''A+6\left(d_2'\right)^2d_1''-2d_2^2d_1'A'\Big].$$

If f_1 and f_2 are two linearly independent solutions of (1.1), *then the polynomial of solutions $g_f=d_1f_1+d_2f_2$ satisfies*

$$\overline{\lambda}\left(g_f-\varphi\right)=\lambda\left(g_f-\varphi\right)=\rho\left(f_j\right)=\infty\quad\left(j=1,2\right)$$

and

$$\overline{\lambda}_2\left(g_f-\varphi\right)=\lambda_2\left(g_f-\varphi\right)=\rho_2\left(f_j\right)=\rho\left(A\right)\quad\left(j=1,2\right).$$

Before we state our results we define h and ψ by

(1.4)
$$h=\begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1''-d_1A & 2d_1' & d_2''-d_2B & 2d_2' \\ d_1'''-3d_1'A-d_1A' & d_1''-d_1A+2d_1'' & d_2'''-3d_2'B-d_2B' & d_2''-d_2B+2d_2'' \end{vmatrix}$$

and

$$\psi(z)=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}\varphi^{(3)}+\phi_2\varphi''+\phi_1\varphi'+\phi_0\varphi, \tag{1.5}$$

where $\varphi\not\equiv 0$ is an entire function of finite order and

$$\phi_2=\frac{-3d_1d_2d_2''-Ad_1d_2^2+Bd_1d_2^2+3d_2^2d_1''}{h}, \tag{1.6}$$

$$\phi_1=\frac{2Ad_1d_2d_2'+6d_2d_1'd_2''-6d_2d_2'd_1''-2Bd_2^2d_1'}{h}, \tag{1.7}$$

$$\phi_0=\frac{1}{h}\left[\left(d_1d_2d_2''-2d_1(d_2')^2\right)A\right.$$

$$+\left(-4d_1d_2d_2''-4d_2d_1'd_2'+6d_1(d_2')^2+3d_2^2d_1''\right)B$$

$$+\left(2d_1d_2d_2'-2d_2^2d_1'\right)B'-ABd_1d_2^2+B^2d_1d_2^2-2d_1d_2'd_2'''$$

$$\left.+2d_2d_1'd_2'''-3d_2d_1''d_2''-6d_1'd_2'd_2''+3d_1(d_2'')^2\right]. \tag{1.8}$$

In 1972, H. Herold ([10]) showed some criteria on the comparison between two pairs of complex differential equations. In [16] , L. Z. Yang considered the common solutions of a pair of differential equations and gave some of their applications in the uniqueness problems of entire functions. Recently, A. Asiri ([1, 2, 3]) studied some proprieties on solutions of different equations having the same zeros. It is interesting now to study the growth and oscillation of $w=d_1f+d_2g$ where f and g are solutions of the above equations, d_1 and d_2 are entire functions not all vanishing identically. We obtain the following results.

THEOREM 1.1. *Let $A(z)$ and $B(z)$ be transcendental entire functions of finite order. Let $d_j(z) \not\equiv 0$ $(j = 1, 2)$ be finite order entire functions such that $h \not\equiv 0$. If f and g are solutions of (1.1) and (1.2) respectively, then the polynomial of solutions (1.3) satisfies*

$$\rho(w) = \rho(f) = \rho(g) = \infty$$

and

$$\rho_2(w) = \max\{\rho(A), \rho(B)\}.$$

REMARK 1.1. If $\rho(A) \neq \rho(B)$, then the conclusions of Theorem 1.1 are trivial. The importance of Theorem 1.1 lies in the case when $\rho(A) = \rho(B)$. For example we can see that $f(z) = \exp(e^z)$ and $g(z) = \exp\left(e^{z^2}\right)$ satisfy respectively the following differential equations

$$f'' - \left(e^z + e^{2z}\right) f = 0$$

and

$$g'' - \left[\left(2 + 4z^2\right) e^{z^2} + 4z^2 e^{2z^2}\right] g = 0.$$

It is clear that

$$1 = \rho\left(e^z + e^{2z}\right) < \rho\left(\left(2 + 4z^2\right) e^{z^2} + 4z^2 e^{2z^2}\right) = 2.$$

On the other hand, we have

$$\rho_2(f + g) = 2.$$

REMARK 1.2. In the case when $\rho(A) = \rho(B)$, we can suppose in the statement of Theorem 1.1 that $d_j(z)$ $(j = 1, 2)$ are not all vanishing identically.

THEOREM 1.2. *Under the hypotheses of Theorem 1.1, let $\varphi(z) \not\equiv 0$ be an entire function with finite order such that $\psi(z) \not\equiv 0$. If f and g are* *solutions of* (1.1) *and* (1.2) *respectively*, *then the polynomial of solutions* (1.3) *satisfies*

$$\overline{\lambda}(w-\varphi)=\lambda(w-\varphi)=\infty \tag{1.9}$$

and

$$\overline{\lambda}_2(w-\varphi)=\lambda_2(w-\varphi)=\max\{\rho(A),\rho(B)\}. \tag{1.10}$$

In the next we give some sufficient conditions to remove the condition $h\not\equiv 0$.

Theorem 1.3. *Let* $A(z)$ *and* $B(z)$ *be transcendental entire functions satisfying* $\rho(A)=\rho(B)=\rho$ $(0<\rho<\infty)$ *and* $0<\tau(A)\neq\tau(B)<\infty$. *Let* $d_j(z)$ $(j=1,2)$ *be finite order entire functions that are not all vanishing identically such that* $\max\{\rho(d_1),\rho(d_2)\}<\rho$. *If* f *and* g *are solutions of* (1.1) *and* (1.2) *respectively*, *then the polynomial of solutions* (1.3) *satisfies*

$$\rho(w)=\rho(f)=\rho(g)=\infty$$

and

$$\rho_2(w)=\rho_2(f)=\rho_2(g)=\rho.$$

Theorem 1.4. *Under the hypotheses of Theorem 1.3, let* $\varphi(z)\not\equiv 0$ *be an entire function with finite order such that* $\psi(z)\not\equiv 0$. *If* f *and* g *are solutions of* (1.1) *and* (1.2) *respectively*, *then the polynomial of solutions* (1.3) *satisfies*

$$\overline{\lambda}(w-\varphi)=\lambda(w-\varphi)=\infty \tag{1.11}$$

and

$$\overline{\lambda}_2(w-\varphi)=\lambda_2(w-\varphi)=\rho. \tag{1.12}$$

REMARK 1.3. In the case when $A(z) = B(z)$, by choosing f and g as two linearly independent solutions we can deduce Theorem A and Theorem B.

Let now we consider f and g be solutions of the complex differential equations (1.1) and (1.2) respectively, where A and B are two non-constant polynomials of the same degree. It is clear that $\rho(f) = \rho(g) = \frac{\deg A+2}{2}$, but what about the growth and oscillation of $w = d_1 f + d_2 g$? Here we answer to this question and we obtain the following results.

THEOREM 1.5. *Let A and B be non-constant polynomials of the same degree n. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that $h \not\equiv 0$ and $\max\{\rho(d_1), \rho(d_2)\} < \frac{n+2}{2}$. If f and g are solutions of (1.1) and (1.2) respectively, then the polynomial of solutions (1.3) satisfies*

$$\rho(w) = \rho(f) = \rho(g) = \frac{n+2}{2}.$$

THEOREM 1.6. *Under the hypotheses of Theorem 1.5, let $\varphi(z) \not\equiv 0$ be an entire function with $\rho(\varphi) < \frac{n+2}{2}$ such that $\psi(z) \not\equiv 0$. If f and g are solutions of (1.1) and (1.2) respectively, then the polynomial of solutions (1.3) satisfies*

$$\overline{\lambda}(w - \varphi) = \lambda(w - \varphi) = \frac{n+2}{2}. \tag{1.13}$$

In the case when $A(z) = B(z)$, by choosing f and g as two linearly independent solutions we can deduce the following corollaries which have been proved by the authors in [13].

COROLLARY 1.1. *Let $A(z)$ be a non-constant polynomial of $\deg A = n$. Let $d_j(z)$ $(j = 1, 2)$ be finite order entire functions that are not all vanishing identically such that $h \not\equiv 0$ and $\max\{\rho(d_1), \rho(d_2)\} < \frac{n+2}{2}$. If f_1 and f_2 are two linearly independent solutions of (1.1), then the polynomial of solutions $g_f = d_1 f_1 + d_2 f_2$ satisfies*

$$\rho(g_f) = \rho(f_j) = \frac{n+2}{2} \ (j = 1, 2).$$

Corollary 1.2. *Under the hypotheses of Corollary 1.1, let* $\varphi(z) \not\equiv 0$ *be an entire function with* $\rho(\varphi) < \frac{n+2}{2}$ *such that*

$$\psi(z) = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2\varphi'' + \phi_1\varphi' + \phi_0\varphi \not\equiv 0,$$

where

$$\phi_2 = \frac{3d_2^2 d_1'' - 3d_1 d_2 d_2''}{h},$$

$$\phi_1 = \frac{2d_1 d_2 d_2' A + 6d_2 d_1' d_2'' - 6d_2 d_2' d_1'' - 2d_2^2 d_1' A}{h},$$

$$\phi_0 = \frac{1}{h}\Big[2d_2 d_1' d_2''' - 2d_1 d_2' d_2''' - 3d_1 d_2 d_2'' A - 3d_2 d_1'' d_2'' + 2d_1 d_2 d_2' A'$$

$$-4d_2 d_1' d_2' A - 6d_1' d_2' d_2'' + 3d_1\left(d_2''\right)^2 + 4d_1\left(d_2'\right)^2 A$$

$$+3d_2^2 d_1'' A + 6\left(d_2'\right)^2 d_1'' - 2d_2^2 d_1' A'\Big].$$

If f_1 *and* f_2 *are two linearly independent solutions of* (1.1), *then the polynomial of solutions* $g_f = d_1 f_1 + d_2 f_2$ *satisfies*

$$\overline{\lambda}(g_f - \varphi) = \lambda(g_f - \varphi) = \frac{n+2}{2}.$$

2. Auxiliary Lemmas

Lemma 2.1 ([4,6]). *Let* $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ *be finite order meromorphic functions.*
(i) *If* f *is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = F \tag{2.1}$$

with $\rho(f) = +\infty$, *then* f *satisfies*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty.$$

(ii) *If f is a meromorphic solution of equation* (2.1) *with* $\rho(f) = +\infty$ *and* $\rho_2(f) = \rho$, *then*

$$\overline{\lambda}(f) = \lambda(f) = \rho(f) = +\infty,\ \overline{\lambda}_2(f) = \lambda_2(f) = \rho_2(f) = \rho.$$

Here, we give a special case of the result given by T. B. Cao, Z. X. Chen, X. M. Zheng and J. Tu in [5]:

LEMMA 2.2. *Let $A_0, A_1, \cdots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If f is a meromorphic solution of equation* (2.1) *with*

$$\max\{\rho(A_j)\ \ (j = 0, 1, \cdots, k-1), \rho(F)\} < \rho(f) < +\infty,$$

then

$$\overline{\lambda}(f) = \lambda(f) = \rho(f).$$

LEMMA 2.3 ([8]). *For all non-trivial solutions f of*

$$f'' + A(z) f = 0, \tag{2.2}$$

the following hold:
(i) *If A is a polynomial with* $\deg A = n \geqslant 1$, *then we have*

$$\lambda(f - z) = \rho(f) = \frac{n+2}{2}.$$

(ii) *If A is transcendental and $\rho(A) < \infty$, then we have*

$$\lambda(f - z) = \rho(f) = \infty$$

and

$$\lambda_2(f - z) = \rho_2(f) = \rho(A).$$

LEMMA 2.4 ([12]). *Let f and g be meromorphic functions such that $0 < \rho(f), \rho(g) < \infty$ and $0 < \tau(f), \tau(g) < \infty$. Then we have*
(i) *If $\rho(f) > \rho(g)$, then we obtain*

$$\tau(f + g) = \tau(fg) = \tau(f).$$

(ii) *If $\rho(f) = \rho(g)$ and $\tau(f) \neq \tau(g)$, then we get*

$$\rho(f + g) = \rho(fg) = \rho(f) = \rho(g).$$

3. Proof of the Theorems

PROOF OF THEOREM 1.1. Suppose that f and g are solutions of (1.1) and (1.2) respectively. Then by Lemma 2.3 (ii), we have

$$\rho(f) = \rho(g) = \infty$$

and

$$\rho_2(f) = \rho(A), \ \rho_2(g) = \rho(B).$$

Without loss of generality we suppose $\rho(A) \geqslant \rho(B)$. We have

$$w = d_1 f + d_2 g. \tag{3.1}$$

Differentiating both sides of (3.1) , we obtain

$$w' = d_1' f + d_1 f' + d_2' g + d_2 g'. \tag{3.2}$$

Differentiating both sides of (3.2) , we have

$$w'' = d_1'' f + 2d_1' f' + d_1 f'' + d_2'' g + 2d_2' g' + d_2 g''. \tag{3.3}$$

Substituting $f'' = -Af$ and $g'' = -Bg$ into equation (3.3) , we obtain

$$w'' = \left(d_1'' - d_1 A\right) f + 2d_1' f' + \left(d_2'' - d_2 B\right) g + 2d_2' g'. \tag{3.4}$$

Differentiating both sides of (3.4) and by substituting $f'' = -Af$ and $g'' = -Bg$, we have

$$\begin{aligned} w''' &= \left(d_1''' - 3d_1' A - d_1 A'\right) f + \left(d_1'' - d_1 A + 2d_1''\right) f' \\ &\quad + \left(d_2''' - 3d_2' B - d_2 B'\right) g + \left(d_2'' - d_2 A + 2d_2''\right) g'. \end{aligned} \tag{3.5}$$

By (3.1) – (3.5) we get

$$\begin{cases} w = d_1 f + d_2 g, \\ w' = d_1' f + d_1 f' + d_2' g + d_2 g', \\ w'' = (d_1'' - d_1 A) f + 2d_1' f' + (d_2'' - d_2 B) g + 2d_2' g', \\ w''' = (d_1''' - 3d_1' A - d_1 A') f + (d_1'' - d_1 B + 2d_1'') f' \\ \quad + (d_2''' - 3d_2' B - d_2 B') g + (d_2'' - d_2 B + 2d_2'') g'. \end{cases} \tag{3.6}$$

By simple calculations we obtain

$$h = \begin{vmatrix} d_1 & 0 & d_2 & 0 \\ d_1' & d_1 & d_2' & d_2 \\ d_1'' - d_1 A & 2d_1' & d_2'' - d_2 B & 2d_2' \\ d_1''' - 3d_1'A - d_1 A' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2'B - d_2 B' & d_2'' - d_2 B + 2d_2'' \end{vmatrix}$$

$$= \left(-4d_1d_2d_1'd_2' - 4d_1d_2^2d_1'' + 4d_1^2d_2d_2'' - 2d_1^2(d_2')^2 + 6d_2^2(d_1')^2\right) A$$

$$+ \left(-4d_1d_2d_1'd_2' + 4d_1d_2^2d_1'' - 4d_1^2d_2d_2'' + 6d_1^2(d_2')^2 - 2d_2^2(d_1')^2\right) B$$

$$+ \left(2d_1d_2^2d_1' - 2d_1^2d_2d_2'\right) A' + \left(-2d_1d_2^2d_1' + 2d_1^2d_2d_2'\right) B' + d_1^2d_2^2 (A - B)^2$$

$$+2d_1d_2d_1'd_2''' + 2d_1d_2d_2'd_1''' - 6d_1d_2d_1''d_2'' - 6d_1d_1'd_2'd_2'' + 3d_1^2(d_2'')^2 + 3d_2^2(d_1'')^2$$

$$-6d_2d_1'd_2'd_1'' + 6d_1(d_2')^2d_1'' + 6d_2(d_1')^2d_2'' - 2d_2^2d_1'd_1''' - 2d_1^2d_2'd_2'''. \tag{3.7}$$

Since $h \not\equiv 0$, then by Cramer's method we have

$$f = \frac{\begin{vmatrix} w & 0 & d_2 & 0 \\ w' & d_1 & d_2' & d_2 \\ w'' & 2d_1' & d_2'' - d_2 B & 2d_2' \\ w''' & d_1'' - d_1 A + 2d_1'' & d_2''' - 3d_2'B - d_2 B' & d_2'' - d_2 B + 2d_2'' \end{vmatrix}}{h}$$

$$= \frac{2\left(d_1d_2d_2' - d_2^2d_1'\right)}{h} w^{(3)} + \phi_2 w'' + \phi_1 w' + \phi_0 w, \tag{3.8}$$

where ϕ_j $(j = 0, 1, 2)$ are meromorphic functions of finite order which are defined in (1.6) – (1.8). Suppose now $\rho(w) < \infty$. Then by (3.8) we obtain $\rho(f) < \infty$ which is a contradiction. Hence $\rho(w) = \infty$. By (3.1) we have $\rho_2(w) \leqslant \max\{\rho(A), \rho(B)\} = \rho(A)$. Suppose that $\rho_2(w) < \rho(A)$. Then by (3.8) we obtain $\rho_2(f) < \rho(A)$ which is a contradiction. Hence $\rho_2(w) = \max\{\rho(A), \rho(B)\}$. $\square$

Proof of Theorem 1.2. By Theorem 1.1 we have $\rho(w) = \infty$ and $\rho_2(w) = \max\{\rho(A), \rho(B)\}$. Set $\Phi(z) = d_1 f + d_2 g - \varphi$. Since $\rho(\varphi) < \infty$, then we have $\rho(\Phi) = \rho(w) = \infty$ and $\rho_2(\Phi) = \rho_2(w) = \max\{\rho(A), \rho(B)\}$. In order to prove $\overline{\lambda}(w-\varphi) = \lambda(w-\varphi) = \infty$ and $\overline{\lambda}_2(w-\varphi) = \lambda_2(w-\varphi) = \max\{\rho(A), \rho(B)\}$ we need to prove $\overline{\lambda}(\Phi) = \lambda(\Phi) = \infty$ and $\overline{\lambda}_2(\Phi) = \lambda_2(\Phi) = \max\{\rho(A), \rho(B)\}$. By $w = \Phi + \varphi$ we get from (3.8)

$$f = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\Phi^{(3)} + \phi_2 \Phi'' + \phi_1 \Phi' + \phi_0 \Phi + \psi, \tag{3.9}$$

where

$$\psi = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi.$$

Substituting (3.9) into equation (1.1), we obtain

$$\frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\Phi^{(5)} + \sum_{j=0}^{4} \beta_j \Phi^{(j)} = -\left(\psi'' + A\psi\right) = F,$$

where β_j $(j = 0, \cdots, 4)$ are meromorphic functions of finite order. Since $\psi \not\equiv 0$ and $\rho(\psi) < \infty$, it follows that ψ is not a solution of (1.1) , which implies that $F \not\equiv 0$. Then by applying Lemma 2.1 we obtain (1.9) and (1.10) . □

Proof of Theorem 1.3. By the same reasoning as in Theorem 1.1 we have

$$\begin{cases} w = d_1 f + d_2 g, \\ w' = d_1' f + d_1 f' + d_2' g + d_2 g', \\ w'' = (d_1'' - d_1 A) f + 2d_1' f' + (d_2'' - d_2 B) g + 2d_2' g', \\ w''' = (d_1''' - 3d_1' A - d_1 A') f + (d_1'' - d_1 B + 2d_1'') f' \\ \quad + (d_2''' - 3d_2' B - d_2 B') g + (d_2'' - d_2 B + 2d_2'') g'. \end{cases}$$

To solve this system of equations, we need first to prove that $h \not\equiv 0$. By simple calculations we obtain

$$h = \left(-4d_1 d_2 d_1' d_2' - 4d_1 d_2^2 d_1'' + 4d_1^2 d_2 d_2'' - 2d_1^2 (d_2')^2 + 6d_2^2 (d_1')^2\right) A$$

$$+\left(-4d_1d_2d_1'd_2'+4d_1d_2^2d_1''-4d_1^2d_2d_2''+6d_1^2(d_2')^2-2d_2^2(d_1')^2\right)B$$

$$+\left(2d_1d_2^2d_1'-2d_1^2d_2d_2'\right)A'+\left(-2d_1d_2^2d_1'+2d_1^2d_2d_2'\right)B'+d_1^2d_2^2\left(A-B\right)^2$$

$$+2d_1d_2d_1'd_2'''+2d_1d_2d_2'd_1'''-6d_1d_2d_1''d_2''-6d_1d_1'd_2'd_2''+3d_1^2(d_2'')^2+3d_2^2(d_1'')^2$$

$$-6d_2d_1'd_2'd_1''+6d_1(d_2')^2d_1''+6d_2(d_1')^2d_2''-2d_2^2d_1'd_1'''-2d_1^2d_2'd_2'''.$$

Since

$$\max\left\{\rho\left(d_1\right),\rho\left(d_2\right)\right\}<\rho\left(A\right)=\rho\left(B\right)=\rho\ \ (0<\rho<\infty)$$

and $0<\tau\left(A\right)\neq\tau\left(B\right)<\infty$, then by applying Lemma 2.4 we have $\rho\left(h\right)=\rho>0$, which implies that $h\not\equiv 0$. Now by Cramer's method we have

$$f=\frac{\begin{vmatrix} w & 0 & d_2 & 0 \\ w' & d_1 & d_2' & d_2 \\ w'' & 2d_1' & d_2''-d_2B & 2d_2' \\ w''' & d_1''-d_1A+2d_1'' & d_2'''-3d_2'B-d_2B' & d_2''-d_2B+2d_2'' \end{vmatrix}}{h}$$

$$=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}w^{(3)}+\phi_2w''+\phi_1w'+\phi_0w, \tag{3.10}$$

where $\phi_j\ (j=0,1,2)$ are meromorphic functions of finite order which are defined in (1.6) – (1.8). Suppose now $\rho\left(w\right)<\infty$. Then by (3.10) we obtain $\rho\left(f\right)<\infty$ which is a contradiction. Hence $\rho\left(w\right)=\infty$. By (3.1) we have $\rho_2\left(w\right)\leqslant\rho\left(A\right)$. Suppose that $\rho_2\left(w\right)<\rho\left(A\right)$. Then by (3.10) we obtain $\rho_2\left(f\right)<\rho\left(A\right)$ which is a contradiction. Hence $\rho_2\left(w\right)=\rho$. □

PROOF OF THEOREM 1.4. By Theorem 1.3 we have $\rho\left(w\right)=\infty$ and $\rho_2\left(w\right)=\rho$. Set $\Phi\left(z\right)=d_1f+d_2g-\varphi$. Since $\rho\left(\varphi\right)<\infty$, then we have $\rho\left(\Phi\right)=\rho\left(w\right)=\infty$ and $\rho_2\left(\Phi\right)=\rho_2\left(w\right)=\rho$. In order to prove $\overline{\lambda}\left(w-\varphi\right)=$

$\lambda(w-\varphi)=\infty$ and $\overline{\lambda}_2(w-\varphi)=\lambda_2(w-\varphi)=\rho$ we need to prove $\overline{\lambda}(\Phi)=\lambda(\Phi)=\infty$ and $\overline{\lambda}_2(\Phi)=\lambda_2(\Phi)=\rho$. By $w=\Phi+\varphi$ we get from (3.10)

$$f=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}\Phi^{(3)}+\phi_2\Phi''+\phi_1\Phi'+\phi_0\Phi+\psi, \tag{3.11}$$

where

$$\psi=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}\varphi^{(3)}+\phi_2\varphi''+\phi_1\varphi'+\phi_0\varphi.$$

Substituting (3.11) into equation (1.1), we obtain

$$\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}\Phi^{(5)}+\sum_{j=0}^{4}\beta_j\Phi^{(j)}=-\left(\psi''+A\psi\right)=F,$$

where β_j $(j=0,\cdots,4)$ are meromorphic functions of finite order. Since $\psi\not\equiv 0$ and $\rho(\psi)<\infty$, it follows that ψ is not a solution of (1.1) , which implies that $F\not\equiv 0$. Then by applying Lemma 2.1 we obtain (1.11) and (1.12) . □

Proof of Theorem 1.5. Suppose that f and g are solutions of (1.1) and (1.2) respectively. Then by Lemma 2.3 (i), we have

$$\rho(f)=\rho(g)=\frac{n+2}{2}.$$

By the same reasonings as in Theorem 1.1 and since $h\not\equiv 0$, then by Cramer's method we have

$$f=\frac{\begin{vmatrix} w & 0 & d_2 & 0\\ w' & d_1 & d_2' & d_2\\ w'' & 2d_1' & d_2''-d_2B & 2d_2'\\ w''' & d_1''-d_1A+2d_1'' & d_2'''-3d_2'B-d_2B' & d_2''-d_2B+2d_2''\end{vmatrix}}{h}$$

$$=\frac{2\left(d_1d_2d_2'-d_2^2d_1'\right)}{h}w^{(3)}+\phi_2w''+\phi_1w'+\phi_0w, \tag{3.12}$$

where ϕ_j $(j=0,1,2)$ are meromorphic functions defined in (1.6)−(1.8) such that $\rho(\phi_j)<\frac{n+2}{2}$ $(j=0,1,2)$. By (3.1) we have $\rho(w)\leqslant\frac{n+2}{2}$. Suppose now

$\rho(w) < \frac{n+2}{2}$. Then by (3.12) we obtain $\rho(f) < \frac{n+2}{2}$ which is a contradiction. Hence $\rho(w) = \frac{n+2}{2}$. □

Proof of Theorem 1.6. By Theorem 1.5, we have $\rho(w) = \frac{n+2}{2}$. Set $\Phi(z) = d_1 f + d_2 g - \varphi$. Since $\rho(\varphi) < \frac{n+2}{2}$, then we have $\rho(\Phi) = \rho(w) = \frac{n+2}{2}$. In order to prove $\overline{\lambda}(w-\varphi) = \lambda(w-\varphi) = \frac{n+2}{2}$, we need to prove only $\overline{\lambda}(\Phi) = \lambda(\Phi) = \frac{n+2}{2}$. By $w = \Phi + \varphi$ we get from (3.12)

$$f = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\Phi^{(3)} + \phi_2 \Phi'' + \phi_1 \Phi' + \phi_0 \Phi + \psi, \tag{3.13}$$

where

$$\psi = \frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\varphi^{(3)} + \phi_2 \varphi'' + \phi_1 \varphi' + \phi_0 \varphi.$$

Substituting (3.13) into equation (1.1), we obtain

$$\frac{2\left(d_1 d_2 d_2' - d_2^2 d_1'\right)}{h}\Phi^{(5)} + \sum_{j=0}^{4} \beta_j \Phi^{(j)} = -\left(\psi'' + A\psi\right) = F,$$

where β_j $(j = 0, \cdots, 4)$ are meromorphic functions with $\rho(\beta_j) < \frac{n+2}{2}$ $(j = 0, \cdots, 4)$. Since $\psi \not\equiv 0$ and $\rho(\psi) < \frac{n+2}{2}$, it follows that ψ is not a solution of (1.1), which implies that $F \not\equiv 0$. Then by applying Lemma 2.2 we obtain (1.13). □

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Department of Mathematics
Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem-(Algeria)
E-mail: z.latreuch@gmail.com
belaidi@univ-mosta.dz