J. Math. Sci. Univ. Tokyo
20 (2013), 285–315.

A Generalized Hypergeometric System

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Abstract. We give a combinatorial formula of the dimension of global solutions to a generalization of Gauss-Aomoto-Gelfand hypergeometric system, where the quadratic differential operators are replaced by higher order operators. We also derive a polynomial estimate of the dimension of global solutions for the case in 3×3 variables.

1. Introduction

A new type of hypergeometric differential equations was introduced and studied by H. Sekiguchi in [17], [18]. The investigated system of partial differential equation generalizes the Gauss-Aomoto-Gelfand system which in its turn stems from the classical set of differential relations for the solutions to a generic algebraic equation introduced by K. Mayr in [15].

Gauss-Aomoto-Gelfand systems can be expressed as the determinants of 2×2 matrices of derivations with respect to certain variables. The Gauss-Aomoto-Gelfand hypergeometric system arises in numerous problems of algebraic geometry, partial differential equations, the theory of special functions, representation theory and combinatorics. It has been in the focus of intensive research since its introduction by K. Aomoto in 1977. H. Sekiguchi generalized this construction by looking at determinants of $l \times l$ matrices of derivations with respect to certain variables.

In this paper we study the dimension of global solutions to the generalized systems of Gauss-Aomoto-Gelfand hypergeometric systems. The main results in the paper are the combinatorial formula for the dimension of global (and local) solutions of the generalized Gauss-Aomoto-Gelfand system.

1.1. Generalized Gauss-Aomoto-Gelfand hypergeometric systems

Let k and n be integers such that $0 \leq k < n$ and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$, $\beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{C}^n$. Let $M(n, \mathbb{C})$ be the space of $n \times n$ complex ma-

2010 *Mathematics Subject Classification*. Primary 33C70, 35G35; Secondary 05A17.
Key words: Generalized hypergeometric system, Global solution, Kostka number.

trices $Z = \begin{pmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{n1} & \cdots & z_{nn} \end{pmatrix}$. Consider the systems of partial differential equations, denoted by $\mathcal{M}_n^{(k)}$ and $\mathcal{N}_n(\alpha, \beta)$, for a holomorphic function $\phi(Z)$ of n^2 variables $Z = (z_{ij}) \in M(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$:

$$(\mathcal{M}_n^{(k)}) \qquad \begin{cases} \det \begin{pmatrix} \frac{\partial}{\partial z_{i_1 j_1}} & \cdots & \frac{\partial}{\partial z_{i_1 j_{k+1}}} \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial z_{i_{k+1} j_1}} & \cdots & \frac{\partial}{\partial z_{i_{k+1} j_{k+1}}} \end{pmatrix} \phi(Z) = 0, \\ 1 \leq \forall i_1 < \cdots < \forall i_{k+1} \leq n, \\ 1 \leq \forall j_1 < \cdots < \forall j_{k+1} \leq n \end{cases}$$

and

$$(\mathcal{N}_n(\alpha, \beta)) \qquad \begin{cases} \displaystyle\sum_{i=1}^{n} z_{ij} \frac{\partial}{\partial z_{ij}} \phi(Z) = \alpha_j \phi(Z), \quad j = 1, \ldots, n, \\ \displaystyle\sum_{j=1}^{n} z_{ij} \frac{\partial}{\partial z_{ij}} \phi(Z) = \beta_i \phi(Z), \quad i = 1, \ldots, n. \end{cases}$$

We will denote by $E_n^{(k)}(\alpha, \beta)$ the system of partial differential equations consisting of both $\mathcal{M}_n^{(k)}$ and $\mathcal{N}_n(\alpha, \beta)$. The system $E_n^{(1)}(\alpha, \beta)$ is called a *Gauss-Aomoto-Gelfand hypergeometric system* or *general hypergeometric system* (see section 6.4.4 of [21], Proposition 1 of [11], and [9]). Holomorphic solutions of this system $E_n^{(1)}(\alpha, \beta)$ are called *Gauss-Aomoto-Gelfand hypergeometric functions* or *general hypergeometric functions*. The hypergeometric system $E_n^{(k)}(\alpha, \beta)$ is a natural generalization of the Gauss-Aomoto-Gelfand hypergeometric system to higher order. This system is introduced by Sekiguchi, where $E_n^{(k)}(\alpha, \beta)$ is denoted by $\widetilde{\mathcal{M}}_k(\nu)$ in [17], $\widetilde{\mathcal{M}}_{n,k}(\nu)$ in [18] with $\alpha = (\nu_1 - k, \ldots, \nu_n - k)$ and $\beta = (-\nu_{n+1}, \ldots, -\nu_{2n})$.

It is well known that the space of global solutions to the system $E_n^{(k)}(\alpha, \beta)$ is finite dimensional if $E_n^{(k)}(\alpha, \beta)$ is holonomic [10]. The phenomenon that the solution space is finite dimensional even when $E_n^{(k)}(\alpha, \beta)$ is not a holonomic system was observed in connection with the theory of admissible restrictions [12], namely, in the setting where irreducible unitary representation decomposes discretely with finite multiplicities with respect to reductive subgroups. By using geometric realization of admissible restrictions, explicit computations were first carried out by H. Sekicughi [17], [18] in special cases of [12] related to the system $E_n^{(k)}(\alpha,\beta)$.

1.2. Summary of known results of the hypergeometric system $E_n^{(k)}(\alpha,\beta)$

Up to now, there are the following results on the hypergeometric system $E_n^{(k)}(\alpha,\beta)$:

(i) If $(n,l)=(2,1)$ then the system $E_2^{(1)}((\alpha_1,\alpha_2),(\beta_1,\beta_2))$ is essentially equivalent to the Gauss hypergeometric equation. Namely, any solution of $E_2^{(1)}((\alpha_1,\alpha_2),(\beta_1,\beta_2))$ can be represented in the form

$$\phi(Z)=z_{11}^{\alpha_1}z_{12}^{\alpha_2-\beta_2}z_{22}^{\beta_2}y\left(\frac{z_{12}z_{21}}{z_{11}z_{22}}\right),$$

where $y(x)$ satisfies the Gauss hypergeometric equation

$$x(1-x)\frac{d^2y(x)}{dx^2}+(c-(a+b+1)x)\frac{dy(x)}{dx}-aby(x)=0,$$

where $a=-\alpha_1$, $b=-\beta_2$ and $c=\alpha_2-\beta_2+1$.

(ii) For an arbitrary n, the hypergeometric system $E_n^{(1)}(\alpha,\beta)$ is the Aomoto-Gelfand hypergeometric system and has been studied extensively, in particular,

- The system $E_n^{(1)}(\alpha,\beta)$ is holonomic (see [3], [7]), and the dimension of solutions to the system $E_n^{(1)}(\alpha,\beta)$ near a generic point is equal to $\binom{2n-2}{n-1}$ (see [8], [20]).
- The dimension of global solutions to the system $E_n^{(1)}(\alpha,\beta)$ is at most one ([18]). Global solutions to this hypergeometric system are Louck polynomials, which are, up to constant multiples, of the form

$$P_{\alpha,\beta}(Z)=\sum_{\gamma\in H\binom{\alpha}{\beta}}\frac{1}{\gamma!}Z^{\gamma},$$

where $\gamma!=\prod_{1\le i,j\le n}\gamma_{ij}!$, $Z^{\gamma}=\prod_{1\le i,j\le n}z_{ij}^{\gamma_{ij}}$, and see section 2.1 for definition of $H\binom{\alpha}{\beta}$. These polynomial solutions of the system

$E_n^{(1)}(\alpha,\beta)$ arise naturally in the theory of representations of the group $GL(n,\mathbb{C})$, (for details see section 3.5 of [4], and [5]). There exists a generating function for the polynomials $P_{\alpha,\beta}(Z)$ (see [5], section 1.7 of [9]).

(iii) For an arbitrary pair (n,k), the space of solutions of the system $E_n^{(k)}(\alpha,\beta)$ near the origin is finite dimensional (see [17], [18]).

(iv) The system $E_n^{(k)}(\alpha,\beta)$ is not always holonomic, if $k>1$ (see [19]).

(v) A combinatorial formula and the estimates of the dimension of global solutions of $E_3^{(2)}(\alpha,\beta)$ are obtained in [18].

1.3. Summary of our results of the hypergeometric system $E_n^{(k)}(\alpha,\beta)$

In this paper, we consider the dimension of the space of global (and local) solutions of the hypergeometric system $E_n^{(k)}(\alpha,\beta)$.

Our results are stated briefly as followings:

(i) For an arbitrary n, we give a combinatorial formula of the dimension of global solutions to the system $E_n^{(n-1)}(\alpha,\beta)$ (Theorem 2.6).

(ii) We give a simple formula for the Kostka number $K_{\lambda\mu}$ in the case when the length of μ is less than or equals to 3 (Corollary 3.2)(see section 2.2 for Kostka numbers and notations).

(iii) We also give a polynomial estimate for the dimension of global solutions to the system $E_3^{(2)}(\alpha,\beta)$ (Theorem 3.4).

This paper is organized as follows: in Section 2, we consider the space of global solutions of the system of homogeneous equations $\mathcal{N}_n(\alpha,\beta)$. We show the finite dimensionality and that the compatibility condition $\alpha_1+\cdots+\alpha_n=\beta_1+\cdots+\beta_n$ on $(\alpha,\beta)\in\mathbb{Z}_{\geq 0}^{2n}$ is necessary to have a non-trivial space of global solutions. As a direct corollary, we also have the same conclusion on the system $E_n^{(k)}(\alpha,\beta)$, which has been obtained in [18].

We give an upper estimate of the dimension of the space of global solutions of the system $E_n^{(k)}(\alpha,\beta)$ by using the cardinality of some sets $H\binom{\alpha}{\beta}$ of integral matrices (Theorem 2.3). For the case $k=n-1$, we show that the system of linear equations corresponding to the equation $\mathcal{M}_n^{(n-1)}$ is an upper triangular form with respect to an appropriate linear order on $H\binom{\alpha}{\beta}$. We also have an expression for the cardinality of $H\binom{\alpha}{\beta}$ by Kostka numbers, which enables us to give a combinatorial formula of the global dimensions by Kostka numbers (Theorem 2.6).

In Section 3, we consider the case $n = 3$ with $k = 2$, that is, the space of global solutions of the system $E_3^{(2)}(\alpha, \beta)$. In this case we only need Kostka numbers $K_{\lambda\mu}$ where the length of μ is at most 3. We obtain a piecewise linear expression of such Kostka numbers (Corollary 3.2).

We give an explicit polynomial estimate of third order for the dimension of global solutions to the system $E_3^{(2)}(\alpha, \beta)$ (Theorem 3.4). This improves the estimate in [18, Theorem 4.1]. In the case $\alpha = \beta$, the upper and lower estimates coincide with each other, so we have a concise formula for the dimension (Corollary 3.8).

Remark 1.1. In this article, we consider holomorphic solutions of the generalized Gauss-Aomoto-Gelfand system $E_n^{(k)}(\alpha, \beta)$ on the complex manifold $M(n, \mathbb{C})$. We have the same conclusion if we consider analytic solutions of the same system on the real manifold $M(n, \mathbb{R})$.

2. A Combinatorial Formula of the Dimension of Global Solutions to the System $E_n^{(n-1)}(\alpha, \beta)$

2.1. Preliminary notations and an upper estimate of the dimension of global solutions to the system $E_n^{(k)}(\alpha, \beta)$

Let $\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n$ be complex numbers. We define

$$H\binom{\alpha}{\beta} = H\begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \beta_1 & \beta_2 & \cdots & \beta_n \end{pmatrix} = \{\gamma = (\gamma_{ij}) \in M(n, \mathbb{Z}_{\geq 0}) \mid \sum_{j=1}^{n} \gamma_{ij} = \beta_i \text{ for } i = 1, \dots, n;\ \sum_{i=1}^{n} \gamma_{ij} = \alpha_j \text{ for } j = 1, \dots, n\}.$$

From the definition of $H\binom{\alpha}{\beta}$, it is obvious that if at least one of $\alpha_1, \dots, \alpha_n$, $\beta_1, \dots, \beta_n$ is not a nonnegative integer, then $H\binom{\alpha}{\beta}$ is an empty set.

In this paper, a domain means an open connected subset of $M(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$. For each domain W of $M(n, \mathbb{C}) \simeq \mathbb{C}^{n^2}$, we denote by $\mathrm{Sol}(W, E_n^{(k)}(\alpha, \beta))$ the space of holomorphic solutions of the system $E_n^{(k)}(\alpha, \beta)$ on W, and by

$\mathrm{Sol}(W, \mathcal{N}_n(\alpha,\beta))$ that of $\mathcal{N}_n(\alpha,\beta)$. It then follows immediately that

$$\mathrm{Sol}(W, E_n^{(1)}(\alpha,\beta)) \subseteq \mathrm{Sol}(W, E_n^{(2)}(\alpha,\beta)) \subseteq \cdots \subseteq \mathrm{Sol}(W, E_n^{(n-1)}(\alpha,\beta)). \tag{2.1}$$

Note that for any permutations $\alpha' = (\alpha'_1, \ldots, \alpha'_n)$ of α and $\beta' = (\beta'_1, \ldots, \beta'_n)$ of β, we have

$$\dim \mathrm{Sol}(W, E_n^{(k)}(\alpha',\beta')) = \dim \mathrm{Sol}(W, E_n^{(k)}(\alpha,\beta)), \tag{2.2}$$

$$\mathrm{Sol}(W, E_n^{(k)}(\alpha,\beta)) \subseteq \mathrm{Sol}(W, N_n(\alpha,\beta)).$$

For each open subset W of $M(n,\mathbb{C}) \simeq \mathbb{C}^{n^2}$, we denote by $\mathrm{Sol}(W, \mathcal{N}_n(\alpha,\beta))$ the space of holomorphic solutions of the system $\mathcal{N}_n(\alpha,\beta)$ on W. We define

$$\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_Z := \varinjlim_{W \ni Z} \mathrm{Sol}(W, \mathcal{N}_n(\alpha,\beta))$$

$$(\text{or } \mathrm{Sol}(E_n^{(k)}(\alpha,\beta))_Z := \varinjlim_{W \ni Z} \mathrm{Sol}(W, E_n^{(k)}(\alpha,\beta)))$$

the germs of the solution sheaf of the system $\mathcal{N}_n(\alpha,\beta)$ (or $E_n^{(k)}(\alpha,\beta)$) at a point $Z \in M(n,\mathbb{C})$. In particular, the germs of the solutions at the origin of $M(n,\mathbb{C})$ is denoted by $\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0$ (or $\mathrm{Sol}(E_n^{(k)}(\alpha,\beta))_0$).

Lemma 2.1.

(i) *If* $\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0 \neq \{0\}$ *then* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{Z}_{\geq 0}$ *and* $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$.

(ii) *If* $\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0 \neq \{0\}$ *then*

$$\{Z^\gamma = \prod_{1 \leq i,j \leq n} z_{ij}^{\gamma_{ij}} \mid \gamma = (\gamma_{ij}) \in H\tbinom{\alpha}{\beta}\}$$

is a basis of $\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0$.

(iii) $\dim \mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0 < \infty$.

(iv) *For any domain* W *in* $M(n,\mathbb{C})$ *containing the origin, the restriction maps induce the isomorphisms*

$$\mathrm{Sol}(\mathbb{C}^{n^2}, \mathcal{N}_n(\alpha,\beta)) \xrightarrow{\sim} \mathrm{Sol}(W, \mathcal{N}_n(\alpha,\beta)) \xrightarrow{\sim} \mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0.$$

Proof. (i) Since $\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0 \neq \{0\}$, there exists a nonzero function $\phi(Z) \in \mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0$. By the first equation of the system $\mathcal{N}_n(\alpha,\beta)$,

$$\sum_{i,j=1}^{n} z_{ij}\frac{\partial}{\partial z_{ij}}\phi(Z) = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} z_{ij}\frac{\partial}{\partial z_{ij}}\phi(Z)\right) = (\sum_{j=1}^{n}\alpha_j)\phi(Z)$$

and, by the second equation of the system $\mathcal{N}_n(\alpha,\beta)$,

$$\sum_{i,j=1}^{n} z_{ij}\frac{\partial}{\partial z_{ij}}\phi(Z) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} z_{ij}\frac{\partial}{\partial z_{ij}}\phi(Z)\right) = (\sum_{i=1}^{n}\beta_i)\phi(Z).$$

Since $\phi(Z) \neq 0$, then $\sum_{j=1}^{n}\alpha_j = \sum_{i=1}^{n}\beta_i$.

Since $\phi(Z)$ is a holomorphic function at the origin, we denote the Taylor expansion by

$$\phi(Z) = \sum_{\gamma\in M(n,\mathbb{Z}_{\geq 0})} c_\gamma Z^\gamma,\ Z^\gamma = \prod_{1\leq i,j\leq n} z_{ij}^{\gamma_{ij}}.$$

Because

$$z_{ij}\frac{\partial}{\partial z_{ij}}Z^\gamma = \gamma_{ij}Z^\gamma,$$

the first equation of the system $\mathcal{N}_n(\alpha,\beta)$ is

$$\begin{aligned} 0 &= \left(\sum_{i=1}^{n} z_{ij}\frac{\partial}{\partial z_{ij}} - \alpha_j\right)\phi(Z) \\ &= \sum_{\gamma\in M(n,\mathbb{Z}_{\geq 0})}\left(\sum_{i=1}^{n} z_{ij}\frac{\partial}{\partial z_{ij}} - \alpha_j\right)c_\gamma Z^\gamma \\ &= \sum_{\gamma\in M(n,\mathbb{Z}_{\geq 0})}\left(\sum_{i=1}^{n}\gamma_{ij} - \alpha_j\right)c_\gamma Z^\gamma. \end{aligned}$$

Thus, $(\sum_{i=1}^{n}\gamma_{ij} - \alpha_j)c_\gamma = 0$ for any $\gamma \in M(n,\mathbb{Z}_{\geq 0})$ and $j = 1,\dots,n$. Since $\phi(Z) \neq 0$, there exists a nonzero coefficient c_γ of $\phi(Z)$. If $c_\gamma \neq 0$, then γ

satisfies $\sum_{i=1}^{n} \gamma_{ij} = \alpha_j$. Since all γ_{ij} are nonnegative integers, then all α_j are nonnegative integers. The proof for β_i is similar.

(ii) The proof of (i) shows that the Taylor expansion of every function in $\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0$ turns to be a polynomial of the form

$$\phi(Z) = \sum_{\gamma \in H\binom{\alpha}{\beta}} c_\gamma Z^\gamma.$$

Conversely, for any $\gamma = (\gamma_{ij})_{1\le i,j\le n} \in H\binom{\alpha}{\beta}$,

$$\sum_{i=1}^{n} z_{ij} \frac{\partial}{\partial z_{ij}} Z^\gamma = (\sum_{i=1}^{n} \gamma_{ij}) Z^\gamma = \alpha_j Z^\gamma,$$

$$\sum_{j=1}^{n} z_{ij} \frac{\partial}{\partial z_{ij}} Z^\gamma = (\sum_{j=1}^{n} \gamma_{ij}) Z^\gamma = \beta_i Z^\gamma.$$

Thus,

$$\{Z^\gamma \mid \gamma \in H\tbinom{\alpha}{\beta}\}$$

is a basis of $\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0$.

(iii) Since $H\binom{\alpha}{\beta}$ is a bounded subset of $M(n,\mathbb{Z}_{\ge 0})$, it is a finite set. By (ii), it shows that $\dim \mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0 < \infty$.

(iv) Every element in $\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0$ has a Taylor expansion, which is a finite linear combination of monomials, that is, such an element is always a polynomial. So, it extends to whole space $\mathbb{C}^{n^2}$, and, by the restriction, to domain in $\mathbb{C}^{n^2}$ containing the origin. □

Together with the inclusion (2.2), Lemma 2.1 implies the following:

COROLLARY 2.2.

(1) *If* $\mathrm{Sol}(E_n^{(k)}(\alpha,\beta))_0 \neq \{0\}$, *then* $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n \in \mathbb{Z}_{\ge 0}$ *and* $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$.

(2) $\dim \mathrm{Sol}(E_n^{(k)}(\alpha,\beta))_0 < \infty$.

(3) *For any domain* W *in* $M(n,\mathbb{C})$ *containing the origin,*

$$\mathrm{Sol}(\mathbb{C}^{n^2}, E_n^{(k)}(\alpha,\beta)) \xrightarrow{\sim} \mathrm{Sol}(W, E_n^{(k)}(\alpha,\beta)) \xrightarrow{\sim} \mathrm{Sol}(E_n^{(k)}(\alpha,\beta))_0.$$

Now, we give an upper estimate of the dimension of the space of global solutions of the system $E_n^{(k)}(\alpha,\beta)$.

THEOREM 2.3. *Let W be a domain in $M(n,\mathbb{C})$ containing the origin. Then*

$$\begin{aligned}&\dim \mathrm{Sol}(W, E_n^{(k)}(\alpha,\beta))\\ &\quad\le {}^{\#}H\binom{\alpha}{\beta} - {}^{\#}\Big\{\gamma=(\gamma_{ij})\in H\binom{\alpha}{\beta} \mid \exists \begin{pmatrix} 1\le i_1<\cdots<i_{k+1}\le n\\ 1\le j_1<\cdots<j_{k+1}\le n\end{pmatrix}\\ &\qquad\qquad \textit{such that } \gamma_{i_1j_1}>0, \gamma_{i_2j_2}>0,\dots,\gamma_{i_{k+1}j_{k+1}}>0\Big\},\end{aligned}$$

here ${}^{\#}A$ means the cardinality of a set A.

PROOF. If at least one of $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n$ is not a nonnegative integer or $\alpha_1+\cdots+\alpha_n\neq\beta_1+\cdots+\beta_n$ is satisfied, then the theorem is true by Corollary 2.2. Thus, let $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n$ be nonnegative integers such that $\alpha_1+\cdots+\alpha_n=\beta_1+\cdots+\beta_n$. We denote

$$\begin{aligned}H_+\binom{\alpha}{\beta} = \Big\{\gamma=(\gamma_{ij})\in H\binom{\alpha}{\beta} \mid \exists \begin{pmatrix} 1\le i_1<\cdots<i_{k+1}\le n\\ 1\le j_1<\cdots<j_{k+1}\le n\end{pmatrix}&\\ \text{such that } \gamma_{i_1j_1}>0, \gamma_{i_2j_2}>0,\dots,\gamma_{i_{k+1}j_{k+1}}>0\Big\}.&\end{aligned}$$

Lemma 2.1 implies

$$\mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0 = \Big\{\sum_{\gamma\in H\binom{\alpha}{\beta}} c_\gamma Z^\gamma \mid c_\gamma\in\mathbb{C}\Big\}$$

and $\dim \mathrm{Sol}(\mathcal{N}_n(\alpha,\beta))_0 = {}^{\#}H\binom{\alpha}{\beta}$. To conclude the proof of the theorem, we will show that there exists a system (A) of linear equations of rank greater than or equal to ${}^{\#}H_+\binom{\alpha}{\beta}$ with ${}^{\#}H\binom{\alpha}{\beta}$ variables such that

$$\begin{aligned}\mathrm{Sol}(E_n^{(k)}(\alpha,\beta))_0 = \Big\{\sum_{\gamma\in H\binom{\alpha}{\beta}} c_\gamma Z^\gamma \mid (c_\gamma)_{\gamma\in H\binom{\alpha}{\beta}} \in \mathbb{C}^{{}^{\#}H\binom{\alpha}{\beta}}&\\ \text{is a solution of (A)}\Big\}.&\end{aligned}$$

We define

$$H_-\binom{\alpha}{\beta} = \left\{ \nu = (\nu_{ij}) \in M(n, \mathbb{Z}_{\geq 0}) \;\middle|\; \begin{array}{l} \beta_i - \sum_{j=1}^n \nu_{ij} \in \{0,1\} \\ \quad \text{for all} \quad i = 1, \dots, n, \\ \alpha_j - \sum_{i=1}^n \nu_{ij} \in \{0,1\} \\ \quad \text{for all} \quad j = 1, \dots, n, \\ \text{and } \sum_{i,j=1}^n \nu_{ij} = \sum_{j=1}^n \alpha_j - (k+1) \end{array} \right\}.$$

For $\nu \in H_-\binom{\alpha}{\beta}$, we put $I = \{i \mid \beta_i - \sum_{j=1}^n \nu_{ij} = 1\}$, and $J = \{j \mid \alpha_j - \sum_{i=1}^n \nu_{ij} = 1\}$. We introduce the numbering so that $I = \{i_1, \dots, i_{k+1}\}$ with $1 \leq i_1 < \cdots < i_{k+1} \leq n$ and $I = \{j_1, \dots, j_{k+1}\}$ with $1 \leq j_1 < \cdots < j_{k+1} \leq n$. We denote by $E_{ij} \in M(n, \mathbb{Z})$ the matrix unit. Note that for $\nu \in H_-\binom{\alpha}{\beta}$, we see that $\nu + E_{i_1 j_1} + \cdots + E_{i_{k+1} j_{k+1}} \in H\binom{\alpha}{\beta}$. This defines a map

$$\iota : H_-\binom{\alpha}{\beta} \ni \nu \mapsto \nu + E_{i_1 j_1} + \cdots + E_{i_{k+1} j_{k+1}} \in H\binom{\alpha}{\beta}.$$

We denote by S_{k+1} the symmetric group of $(k+1)$ letters, and the signature of a permutation $\sigma \in S_{k+1}$ by $\operatorname{sgn}(\sigma)$. For $\gamma \in H\binom{\alpha}{\beta}$ and $\nu \in H_-\binom{\alpha}{\beta}$, we define

$$a_{\nu\gamma} = \operatorname{sgn}(\sigma) \prod_{t=1}^{k+1} \gamma_{i_t j_{\sigma(t)}}$$

if there exist $1 \leq i_1 < \cdots < i_{k+1} \leq n$, $1 \leq j_1 < \cdots < j_{k+1} \leq n$ and $\sigma \in S_{k+1}$ such that $\gamma - \nu = E_{i_1 j_{\sigma(1)}} + \cdots + E_{i_{k+1} j_{\sigma(k+1)}}$, and define $a_{\nu\gamma} = 0$ otherwise. Using these integers $a_{\nu\gamma}$'s, we define the linear forms on $\mathbb{C}^{\# H\binom{\alpha}{\beta}}$ by

$$g_\nu(C) = \sum_{\gamma \in H\binom{\alpha}{\beta}} a_{\nu\gamma} c_\gamma,$$

for $C = (c_\gamma)_{\gamma \in H\binom{\alpha}{\beta}} \in \mathbb{C}^{\# H\binom{\alpha}{\beta}}$.

For $C = (c_\gamma)_{\gamma \in H\binom{\alpha}{\beta}} \in \mathbb{C}^{\# H\binom{\alpha}{\beta}}$, we put

$$\phi(Z; C) = \sum_{\gamma \in H\binom{\alpha}{\beta}} c_\gamma Z^\gamma \in \operatorname{Sol}(\mathcal{N}_n(\alpha, \beta))_0.$$

Then

$$\det\left(\frac{\partial}{\partial z_{i_t j_r}}\right)_{t,r=1,\dots,k+1}\phi(Z;C)=\sum_{\nu\in H_-\binom{\alpha}{\beta}} g_\nu(C)Z^\nu.$$

In fact, we compute

$$\begin{aligned}
&\det\left(\frac{\partial}{\partial z_{i_t j_r}}\right)_{t,r=1,\cdots,k+1}\phi(Z;C)\\
&=\left(\sum_{\sigma\in S_{k+1}}\operatorname{sgn}(\sigma)\frac{\partial^{k+1}}{\partial z_{i_1 j_{\sigma(1)}}\partial z_{i_2 j_{\sigma(2)}}\cdots\partial z_{i_{k+1} j_{\sigma(k+1)}}}\right)\sum_{\gamma\in H\binom{\alpha}{\beta}} c_\gamma Z^\gamma\\
&=\sum_{\gamma\in H\binom{\alpha}{\beta}}\sum_{\sigma\in S_{k+1}}\operatorname{sgn}(\sigma)c_\gamma\gamma_{i_1 j_{\sigma(1)}}\cdots\gamma_{i_{k+1} j_{\sigma(k+1)}}Z^{\gamma-E_{i_1 j_{\sigma(1)}}-\cdots-E_{i_{k+1} j_{\sigma(k+1)}}}\\
&=\sum_{\nu\in H_-\binom{\alpha}{\beta}} g_\nu(C)Z^\nu.
\end{aligned}$$

Here we have used the fact that if $\nu\in H\binom{\alpha}{\beta}$, $\sigma\in S_{k+1}$ such that $\gamma-(E_{i_1 j_{\sigma(1)}}+\cdots+E_{i_{k+1} j_{\sigma(k+1)}})\notin H_-\binom{\alpha}{\beta}$, then $\gamma_{i_1 j_{\sigma(1)}}\cdots\gamma_{i_{k+1} j_{\sigma(k+1)}}=0$. Then $\phi(Z;C)$ satisfies the system $E_n^{(k)}(\alpha,\beta)$ if and only if $g_\nu(C)=0$ for all $\nu\in H_-\binom{\alpha}{\beta}$. So, we denote by (A) the system

$$g_\nu(C)=0 \text{ for all } \nu\in H_-\tbinom{\alpha}{\beta}.$$

Now we consider the rank of the linear system (A). We define a linear order $<$ on $H\binom{\alpha}{\beta}$ as follows:
for $\eta=(\eta_{ij}),\tau=(\tau_{ij})\in H\binom{\alpha}{\beta}$, $\tau>\eta\Leftrightarrow$ if there exists $i\in\mathbb{N}$ such that for any $j<i$, $\tau_j=\eta_j$ and $\tau_i>\eta_i$, where

$$\begin{aligned}
&(\eta_1,\eta_2,\dots,\eta_{n^2})\\
&\quad=(\eta_{1n},\eta_{1n-1}\dots,\eta_{11},\eta_{2n},\eta_{2n-1}\dots,\eta_{21},\dots,\eta_{nn},\eta_{nn-1}\dots,\eta_{n1})
\end{aligned}$$

and

$$\begin{aligned}
&(\tau_1,\tau_2,\dots,\tau_{n^2})\\
&\quad=(\tau_{1n},\tau_{1n-1}\dots,\tau_{11},\tau_{2n},\tau_{2n-1}\dots,\tau_{21},\dots,\tau_{nn},\tau_{nn-1}\dots,\tau_{n1}).
\end{aligned}$$

Then we will show that

(i) $a_{\nu\gamma} \neq 0$ if $\gamma = \iota(\nu)$.

(ii) $a_{\nu\gamma} = 0$ if $\gamma < \iota(\nu)$.

PROOF OF (i). For $\gamma = \iota(\nu)$, we see that $a_{\nu\gamma} = a_{\nu,\, \nu+E_{i_1j_1}+E_{i_2j_2}+\cdots+E_{i_{k+1}j_{k+1}}} = \prod_{t=1}^{k+1}(\nu_{i_tj_t}+1) > 0$. □

PROOF OF (ii). By the way of contradiction, we suppose that for a $\gamma \in H\binom{\alpha}{\beta}$ such that $\nu + E_{i_1j_1} + E_{i_2j_2} + \cdots + E_{i_{k+1}j_{k+1}} > \gamma$, $a_{\nu,\gamma} \neq 0$. Then, there exists a permutation $\sigma \in S_{k+1}$ such that

$$\gamma = \nu + E_{i_1j_{\sigma(1)}} + E_{i_2j_{\sigma(2)}} + \cdots + E_{i_{k+1}j_{\sigma(k+1)}}.$$

Since $\nu + E_{i_1j_1} + E_{i_2j_2} + \cdots + E_{i_{k+1}j_{k+1}} > \gamma$ we have

$$\sigma \neq \begin{pmatrix} 1 & 2 & \cdots & k+1 \\ 1 & 2 & \cdots & k+1 \end{pmatrix}.$$

Thus, there exists k such that for any $i < k$, $\sigma(i) = i$ and $\sigma(k) > k$. So

$$\begin{aligned} \gamma = \nu + E_{i_1j_1} + \cdots + E_{i_{k-1}j_{k-1}} + E_{i_kj_{\sigma(k)}} + \cdots + E_{i_{k+1}j_{\sigma(k+1)}} \\ > \nu + E_{i_1j_1} + E_{i_2j_2} + \cdots + E_{i_{k+1}j_{k+1}}. \end{aligned}$$

However, it contradicts to the assumption $\nu+E_{i_1j_{k+1}}+E_{i_2j_l}+\cdots+E_{i_{k+1}j_1} > \gamma$. Therefore (ii) is proved. □

Note that the image of the map ι is $H_+\binom{\alpha}{\beta}$. Let us take a subset H' of $H_-\binom{\alpha}{\beta}$ such that the map ι is bijective from H' to $H_+\binom{\alpha}{\beta}$. Then the square submatrix $(a_{\nu\gamma})_{\nu\in H',\gamma\in H_+\binom{\alpha}{\beta}}$ of the matrix $(a_{\nu\gamma})_{\nu\in H_-\binom{\alpha}{\beta},\gamma\in H\binom{\alpha}{\beta}}$ is an upper triangular matrix whose diagonal elements never be zero. This means that the rank of the system has the rank at least $H_+\binom{\alpha}{\beta}$. The proof of the theorem is completed. □

The following Corollary has been obtained in [18], but we will give another proof.

COROLLARY 2.4. *Let W be a domain in $M(n,\mathbb{C})$ containing the origin.*

(i) *If* $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \notin \mathbb{Z}_{\geq 0}^{2n}$ *or* $\alpha_1 + \cdots + \alpha_n \neq \beta_1 + \cdots + \beta_n$ *then*

$$\dim \mathrm{Sol}(W, E_n^{(1)}(\alpha, \beta)) = 0.$$

(ii) *If* $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \in \mathbb{Z}_{\geq 0}^{2n}$ *and* $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$ *then*

$$\dim \mathrm{Sol}(W, E_n^{(1)}(\alpha, \beta)) = 1.$$

PROOF. (i) It follows from Corollary 2.2.

(ii) Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n, \beta = (\beta_1, \dots, \beta_k) \in \mathbb{Z}_{\geq 0}^k$ with $\alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_k$. We denote

$$H_{kn}\binom{\alpha}{\beta} = \{\gamma = (\gamma_{ij}) \in M_{kn}(\mathbb{Z}_{\geq 0}) \mid \sum_{j=1}^n \gamma_{ij} = \beta_i \text{ for } i = 1, \dots, k; \ \sum_{i=1}^k \gamma_{ij} = \alpha_j \text{ for } j = 1, \dots, n\}.$$

In particular, we have $H_{nn}\binom{\alpha}{\beta} = H\binom{\alpha}{\beta}$. By induction of $k+n$, we prove that there exists an unique element ξ in $H_{kn}\binom{\alpha}{\beta}$ of the following form

$$\begin{pmatrix}
0 & 0 & \dots & 0 & 0 & & 0 & 0 & 0 & \dots & 0 & \star & \star & \dots & \star & \star \\
0 & 0 & \dots & 0 & 0 & & 0 & 0 & 0 & \dots & 0 & \star & 0 & \dots & 0 & 0 \\
\vdots & \vdots & \dots & \vdots & \vdots & & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\
0 & 0 & \dots & 0 & 0 & & 0 & 0 & 0 & \dots & 0 & \star & 0 & \dots & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & & 0 & \star & \star & \dots & \star & \star & 0 & \dots & 0 & 0 \\
 & & & & & \ddots & & & & & & & & & & \\
\star & \star & \star & \star & \star & & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0
\end{pmatrix}.$$

It is obvious when $n + k = 2$. Assume that there exists unique element ξ for any pair (n_1, m_1) when $n_1 + m_1 < n + k$ and prove for $n + k$.

If $\beta_1 > \alpha_n$, there exists an unique element ξ' in $H_{kn-1}\binom{\alpha'}{\beta'}$ of above form by induction, where $\alpha' = (\alpha'_1, \dots, \alpha'_{n-1})$, $\beta' = (\beta'_1, \dots, \beta'_k)$ is defined by $\alpha'_1 := \alpha_1, \dots, \alpha'_{n-1} := \alpha_{n-1}$, and $\beta'_1 := \beta_1 - \alpha_n, \beta'_2 := \beta_2, \dots, \beta'_k := \beta_k$ then $\alpha'_1 + \cdots + \alpha'_{n-1} = \beta'_1 + \cdots + \beta'_k$. Then the element ξ is $\begin{pmatrix} \xi' & \begin{smallmatrix} \alpha_n \\ 0 \end{smallmatrix} \end{pmatrix}$.

We can see that the element ξ is unique because we have only one choice $\xi_{1n} = \alpha_n, \xi_{2n} = 0, \dots, \xi_{nn} = 0$.

If $\beta_1 = \alpha_n$, there exists an unique element ξ' in $H_{k-1n-1}\binom{\alpha'}{\beta'}$ of above form by induction, where $\alpha' = (\alpha'_1, \dots, \alpha'_{n-1}), \beta' = (\beta'_1, \dots, \beta'_{k-1})$ is defined by $\alpha'_1 := \alpha_1, \dots, \alpha'_{n-1} := \alpha_{n-1}, \alpha'_{n-1} := \alpha_{n-1}$, $\beta'_1 := \beta_1, \dots, \beta'_{k-1} := \beta_{k-1}$, then $\alpha'_1 + \cdots + \alpha'_{n-1} = \beta'_1 + \cdots + \beta'_{k-1}$. Then the element ξ is $\begin{pmatrix} 0 & \alpha_n \\ \xi' & 0 \end{pmatrix}$. We can see that the element ξ is unique because we have only one choice $\xi_{1n} = \alpha_n, \xi_{2n} = 0, \dots, \xi_{nn} = 0, \xi_{11} = 0, \xi_{12} = 0, \dots, \xi_{1n-1} = 0$.

If $\beta_1 < \alpha_n$ then it will be proved analogously for the case $\beta_1 > \alpha_n$.

Since every element of $H\binom{\alpha}{\beta}$ except the unique element ξ belongs in $H_+\binom{\alpha}{\beta}$, we have

$$
\begin{aligned}
&{}^{\#}H\tbinom{\alpha}{\beta} - {}^{\#}\{\gamma = (\gamma_{ij}) \in H\tbinom{\alpha}{\beta} \mid \\
&\qquad \exists \begin{pmatrix} 1 \le i_1 < i_2 \le n \\ 1 \le j_1 < j_2 \le n \end{pmatrix} \text{such that } \gamma_{i_1 j_1} > 0, \gamma_{i_2 j_2} > 0\} = 1.
\end{aligned}
$$

Thus $\dim \mathrm{Sol}(W, E_n^{(1)}(\alpha, \beta)) \le 1$. On the other side, an easy computation shows that a Louck polynomial

$$
P_{\alpha,\beta}(Z) = \sum_{\gamma \in H\binom{\alpha}{\beta}} \frac{1}{\gamma!} Z^{\gamma},
$$

is a solution of the system $E_n^{(1)}(\alpha, \beta)$, where $\gamma! = \prod_{1 \le i,j \le n} \gamma_{ij}!$, $Z^{\gamma} = \prod_{1 \le i,j \le n} z_{ij}^{\gamma_{ij}}$. Thus $\dim \mathrm{Sol}(W, E_n^{(1)}(\alpha, \beta)) \ge 1$. $\square$

2.2. Kostka numbers and main theorem

A *partition* of an integer $m \ge 1$ is a (finite or infinite) sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of nonnegative integers such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_r > 0$, $\lambda_i = 0$ for $i > r$, and $\sum_{i=1}^{r} \lambda_i = m$. The number r is called the *length* of λ. We shall find that it is convenient not to distinguish between two such sequences which differ only by a string of zeros at the end. Thus, for example, we regard $(2, 1)$, $(2, 1, 0)$, $(2, 1, 0, 0, \dots)$ as the same partition.

We denote by $\mathcal{P}_m$ the set of all partitions of m. The *Young diagram* of a partition λ is an array of m boxes having r left-justified rows with row i containing λ_i boxes for $i = 1, 2, \dots r$. Let $\mu = (\mu_1, \mu_2, \dots)$ be a partition of m. A *Young tableau of shape* λ *and content* μ is an array of numbers which is obtained from the Young diagram of λ by replacing μ_i boxes with number i for all i, such that

i) the entries in every row of the diagram are weakly increasing,

ii) the entries in every column of the diagram are strictly increasing.

Young tableaux arise in various branches of mathematics (see [2] and [14]).

For $\mathcal{P}_m \ni \lambda, \mu$, we denote by $K_{\lambda\mu}$ the number of Young tableaux of shape λ and weight μ. Sometimes, the numbers $K_{\lambda\mu}$ are called *Kostka numbers*. It is well known that for general linear groups the dimension of the μ-weight space in the irreducible highest weight module with highest weight λ is equal to $K_{\lambda\mu}$ (see [2]). There are some combinatorial formulae, which gives Kostka numbers (for example, see [13]). To the given ordering on $\mathcal{P}_m$ corresponds a unique matrix $K = (K_{\lambda\mu})_{\lambda,\mu\in\mathcal{P}_m}$ and there are many ways to compute this matrix.

Lemma 2.5 (6.7 of [14]). *For partitions* $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n), \mu = (\mu_1, \mu_2, \dots, \mu_n) \in \mathcal{P}_m$,

$$^{\#}H\binom{\lambda}{\mu} = \sum_{\nu\in\mathcal{P}_m} K_{\nu\lambda}K_{\nu\mu}.$$

We state the following theorem which gives a combinatorial formula for the dimension of the space of global solutions of the system $E_n^{(n-1)}(\alpha, \beta)$. By (2.2), we may assume that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$.

Theorem 2.6. *Let* $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \beta = (\beta_1, \beta_2, \dots, \beta_n) \in \mathbb{C}^n$ *and* W *be connected domain in* $M(n, \mathbb{C})$ *containing the origin.*

(i) *If* $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \notin \mathbb{Z}_{\geq 0}^{2n}$ *or* $\alpha_1 + \cdots + \alpha_n \neq \beta_1 + \cdots + \beta_n$ *then*

$$\dim \mathrm{Sol}(W, E_n^{(n-1)}(\alpha, \beta)) = 0.$$

(ii) *If* $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \in \mathbb{Z}_{\geq 0}^{2n}$, $m = \alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$ *with* $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ *and* $\min(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) = 0$, *then*

$$\dim \mathrm{Sol}(W, E_n^{(n-1)}(\alpha, \beta)) = \sum_{\nu \in \mathcal{P}_m} K_{\nu(\alpha_1, \dots, \alpha_n)} K_{\nu(\beta_1, \dots, \beta_n)}.$$

(iii) *If* $(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) \in \mathbb{Z}_{\geq 0}^{2n}$, $m = \alpha_1 + \cdots + \alpha_n = \beta_1 + \cdots + \beta_n$ *with* $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ *and* $\min(\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) > 0$, *then*

$$\begin{aligned} &\dim \mathrm{Sol}(W, E_n^{(n-1)}(\alpha, \beta)) \\ &= \sum_{\nu \in \mathcal{P}_m} K_{\nu(\alpha_1, \dots, \alpha_n)} K_{\nu(\beta_1, \dots, \beta_n)} \\ &\quad - \sum_{\nu \in \mathcal{P}_{m-n}} K_{\nu(\alpha_1 - 1, \dots, \alpha_n - 1)} K_{\nu(\beta_1 - 1, \dots, \beta_n - 1)}. \end{aligned}$$

PROOF. We will use the notation of the proof of Theorem 2.3. Since we here consider the case $k + 1 = n$, we have $H_-\binom{\alpha}{\beta} = H(\alpha_1 - 1, \dots, \alpha_n - 1, \beta_1, \dots, \beta_n)$, and $\iota(\nu) = \nu + E_n$ for all $\nu \in H_-\binom{\alpha}{\beta}$. Especially, since ι is injective, we take $H' = H_-\binom{\alpha}{\beta}$ in the proof of Theorem 2.3. This shows the linear system (A) is of full rank, and its rank is equal to the cardinality of $H_-\binom{\alpha}{\beta}$ as well as the cardinality of $H_+(\alpha, \beta)$. □

Example 2.7. Let W be a domain in $M(4, \mathbb{C})$ containing the origin. Find the dimension of the space

$$\mathrm{Sol}(W, E_4^{(3)}((1, 1, 3, 1), (2, 1, 2, 1))).$$

By Theorem 2.6,

$$\begin{aligned} &\mathrm{Sol}(W, E_4^{(3)}((1, 1, 3, 1), (2, 1, 2, 1))) \\ &= \dim \mathrm{Sol}(W, E_4^{(3)}((3, 1, 1, 1), (2, 2, 1, 1))) \\ &= \sum_{\nu \in \mathcal{P}_6} K_{\nu(3,1,1,1)} K_{\nu(2,2,1,1)} - \sum_{\nu \in \mathcal{P}_2} K_{\nu(2,0,0,0)} K_{\nu(1,1,0,0)}. \end{aligned}$$

Using the table of given Kostka numbers in page 111 of [13], we have

$$\sum_{\nu\in\mathcal{P}_6} K_{\nu(3,1,1,1)}K_{\nu(2,2,1,1)} = K_{(6,0,0,0)(3,1,1,1)}K_{(6,0,0,0)(2,2,1,1)}$$
$$+ K_{(5,1,0,0)(3,1,1,1)}K_{(5,1,0,0)(2,2,1,1)} + K_{(4,2,0,0)(3,1,1,1)}K_{(4,2,0,0)(2,2,1,1)}$$
$$+ K_{(4,1,1,0)(3,1,1,1)}K_{(4,1,1,0)(2,2,1,1)} + K_{(3,3,0,0)(3,1,1,1)}K_{(3,3,0,0)(2,2,1,1)}$$
$$+ K_{(3,2,1,0)(3,1,1,1)}K_{(3,2,1,0)(2,2,1,1)} + K_{(3,1,1,1)(3,1,1,1)}K_{(3,1,1,1)(2,2,1,1)}$$
$$= 1\cdot 1 + 3\cdot 3 + 3\cdot 4 + 3\cdot 3 + 1\cdot 2 + 2\cdot 4 + 1\cdot 1 = 42$$

and

$$\sum_{\nu\in\mathcal{P}_2} K_{\nu(2,0,0,0)}K_{\nu(1,1,0,0)} = K_{(2,0,0,0)(2,0,0,0)}K_{(2,0,0,0)(1,1,0,0)} = 1.$$

Thus

$$\mathrm{Sol}(W, E_4^{(3)}((1,1,3,1),(2,1,2,1))) = 42 - 1 = 41.$$

3. A Combinatorial Formula and Polynomial Estimate of the Dimension of Global Solutions to the System $E_3^{(2)}(\alpha,\beta)$

3.1. A formula of Kostka numbers for the system $E_3^{(2)}(\alpha,\beta)$

Combinatorial formulae for calculation of Kostka numbers are not straightforward, rather they are quite complicated to use. Still, they are computable when $(n,k) = (3,2)$, and we may then apply Theorem 2.6 to derive a simple explicit formula for calculation of the dimension of global solutions of the system $E_3^{(2)}(\alpha,\beta)$.

Lemma 3.1. *For* $(\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3)\in\mathcal{P}_m$,

$$K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = \begin{cases} 0, & \lambda_1<\mu_1 \text{ or } \mu_3<\lambda_3; \\ 1, & \lambda_1=\mu_1 \text{ or } \lambda_3=\mu_3 \text{ or } \\ & \lambda_1=\lambda_2 \text{ or } \lambda_2=\lambda_3 \\ & \text{for } \lambda_1\ge\mu_1, \mu_3\ge\lambda_3; \\ K_{(\lambda_1-2,\lambda_2-1,\lambda_3)(\mu_1-1,\mu_2-1,\mu_3-1)}+1, & \lambda_1>\lambda_2>\lambda_3, \lambda_1>\mu_1, \\ & \mu_3>\lambda_3. \end{cases}$$

Proof. It is easily seen that if $\lambda_1 < \mu_1$ or $\mu_3 < \lambda_3$ then $K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 0$. It is also obvious if $\lambda_1 = \mu_1$ or $\lambda_3 = \mu_3$ or $\lambda_1 = \lambda_2$ or $\lambda_2 = \lambda_3$ for $\lambda_1 \geq \mu_1$, $\mu_3 \geq \lambda_3$ then $K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 1$. Hence, we can assume that $\lambda_1 > \lambda_2 > \lambda_3$, $\lambda_1 > \mu_1$ and $\mu_3 > \lambda_3$. So $\lambda_1 - 2 \geq \lambda_2 - 1 \geq \lambda_3$.

We will denote by $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$ the set of Young tableaux of shape $(\lambda_1, \lambda_2, \lambda_3)$ and content (μ_1, μ_2, μ_3). We will consider the map

$$f : \mathcal{T}_{(\lambda_1-2,\lambda_2-1,\lambda_3)(\mu_1-1,\mu_2-1,\mu_3-1)} \to \mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$$

given by for any element

$$\begin{array}{|c|c|c|c|c|c|c|c|}\hline q_{11} & q_{12} & \cdots & q_{1\lambda_3} & \cdots & q_{1\lambda_2-1} & \cdots & q_{1\lambda_1-2} \\ \hline q_{21} & q_{22} & \cdots & q_{2\lambda_3} & \cdots & q_{2\lambda_2-1} \\ \cline{1-6} q_{31} & q_{32} & \cdots & q_{3\lambda_3} \\ \cline{1-4} \end{array}$$

of $\mathcal{T}_{(\lambda_1-2,\lambda_2-1,\lambda_3)(\mu_1-1,\mu_2-1,\mu_3-1)}$,

$$f\left(\begin{array}{|c|c|c|c|c|c|c|c|}\hline q_{11} & q_{12} & \cdots & q_{1\lambda_3} & \cdots & q_{1\lambda_2-1} & \cdots & q_{1\lambda_1-2} \\ \hline q_{21} & q_{22} & \cdots & q_{2\lambda_3} & \cdots & q_{2\lambda_2-1} \\ \cline{1-6} q_{31} & q_{32} & \cdots & q_{3\lambda_3} \\ \cline{1-4} \end{array}\right)$$
$$= \begin{array}{|c|c|c|c|c|c|c|c|}\hline q'_{11} & q'_{12} & \cdots & q'_{1\lambda_3} & \cdots & q'_{1\lambda_2} & \cdots & q'_{1\lambda_1} \\ \hline q'_{21} & q'_{22} & \cdots & q'_{2\lambda_3} & \cdots & q'_{2\lambda_2} \\ \cline{1-6} q'_{31} & q'_{32} & \cdots & q'_{3\lambda_3} \\ \cline{1-4} \end{array},$$

where $q'_{1i} = q_{1i}$ for $i = 1, \ldots, \mu_1 - 1$; $q'_{1\mu_1} = 1$; $q'_{1\mu_1+1} = 2$; $q'_{1i+2} = q_{1i}$ for $i = \mu_1, \ldots, \lambda_1 - 2$; $q'_{2i} = q_{2i}$ for $i = 1, \ldots, \lambda_2 - 1$; $q'_{2\lambda_2} = 3$; $q'_{3i} = q_{3i}$ for $i = 1, \ldots, \lambda_3$. We notice that f is well-defined (the image satisfies the conditions (i) and (ii) in the definition of tableaux). From the definition of f it is clear that f is injective. Now, we show that the cardinality of image of f is less than the cardinality of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$ by one. This will conclude the proof of the lemma. For this, we consider the following four cases.

a) Let $\mu_1 \geq \lambda_2$ and $\mu_2 > \lambda_2$. Then there exists an element

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & 2 & \cdots & 2 & 3 & \cdots & 3 \\ \hline 2 & \cdots & 2 & \cdots & 2 \\ \cline{1-5} 3 & \cdots & 3 \\ \cline{1-3} \end{array}$$

of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$. The unique distinguishing features of T are that first row must contain at leat one 2 and the right end of the second row must be

2. For each element of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$ except T, there exists an inverse image.

b) Let $\mu_1 \geq \lambda_2$ and $\mu_2 < \lambda_2$. Then there exists an element

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 3 & \cdots & 3 \\ \hline 2 & \cdots & 2 & \cdots & 2 & 3 & \cdots & 3 \\ \cline{1-8} 3 & \cdots & 3 \\ \cline{1-3}\end{array}$$

of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$. The unique distinguishing features of T are that the first row doesn't contain 2 and the second row must contain at leat one 3. For each element of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$ except T, there exists an inverse image.

c) Let $\mu_1 \geq \lambda_2$ and $\mu_2 = \lambda_2$. Then there exists an element

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & \cdots & 1 & \cdots & 1 & 3 & \cdots & 3 \\ \hline 2 & \cdots & 2 & \cdots & 2 \\ \cline{1-5} 3 & \cdots & 3 \\ \cline{1-3}\end{array}$$

of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$. The unique distinguishing features of T are that the first row doesn't contain 2 and the right end of second row must be 2. For each element of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$ except T, there exists an inverse image.

d) Let $\mu_1 < \lambda_2$. Then there exists an element

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 & 2 & \cdots & 2 & 3 & \cdots & 3 \\ \hline 2 & \cdots & 2 & \cdots & 2 & 3 & \cdots & 3 & 3 & \cdots & 3 \\ \cline{1-11} 3 & \cdots & 3 \\ \cline{1-3}\end{array}$$

of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$. The $q'_{1\lambda_2+1} = 3$ and $q'_{2\lambda_2} = 3$ are the unique distinguishing features of T. For each element of $\mathcal{T}_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)}$ except T, there exists an inverse image. □

The following corollary to Lemma 3.1 gives us a simple formula for calculation of Kostka numbers for the case $E_3^{(2)}(\alpha,\beta)$.

Corollary 3.2. *For* $(\lambda_1,\lambda_2,\dots),(\mu_1,\mu_2,\mu_3) \in \mathcal{P}_m$,

$$K_{(\lambda_1,\lambda_2,\dots)(\mu_1,\mu_2,\mu_3)} = \begin{cases} 0, & \lambda_4 \neq 0 \text{ or } \lambda_1 < \mu_1 \\ & \text{or } \mu_3 < \lambda_3; \\ \min(\lambda_1-\lambda_2,\lambda_2-\lambda_3,\lambda_1-\mu_1,\mu_3-\lambda_3)+1, & \lambda_4 = 0,\ \lambda_1 \geq \mu_1 \\ & \text{and } \mu_3 \geq \lambda_3. \end{cases}$$

PROOF. A proof is straightforward from Lemma 3.1. □

Example 3.3. Let W be a domain in $M(3,\mathbb{C})$ containing the origin. Find the dimension of global solutions of the system

$$E_3^{(2)}((5,4,3),(7,3,2))$$

using corollary 3.2.

$$\begin{aligned}&\dim \mathrm{Sol}(W, E_3^{(2)}((5,4,3),(7,3,2)))\\&= \sum_{\nu\in\mathcal{P}_{12}} K_{\nu(5,4,3)}K_{\nu(7,3,2)} - \sum_{\nu\in\mathcal{P}_9} K_{\nu(4,3,2)}K_{\nu(6,2,1)}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\sum_{\nu\in\mathcal{P}_{12}} K_{\nu(5,4,3)}K_{\nu(7,3,2)} = K_{(12,0,0)(5,4,3)}K_{(12,0,0)(7,3,2)}\\&+ K_{(11,1,0)(5,4,3)}K_{(11,1,0)(7,3,2)} + K_{(10,2,0)(5,4,3)}K_{(10,2,0)(7,3,2)}\\&+ K_{(10,1,1)(5,4,3)}K_{(10,1,1)(7,3,2)} + K_{(9,3,0)(5,4,3)}K_{(9,3,0)(7,3,2)}\\&+ K_{(9,2,1)(5,4,3)}K_{(9,2,1)(7,3,2)} + K_{(8,4,0)(5,4,3)}K_{(8,4,0)(7,3,2)}\\&+ K_{(8,3,1)(5,4,3)}K_{(8,3,1)(7,3,2)} + K_{(8,2,2)(5,4,3)}K_{(8,2,2)(7,3,2)}\\&+ K_{(7,5,0)(5,4,3)}K_{(7,5,0)(7,3,2)} + K_{(7,4,1)(5,4,3)}K_{(7,4,1)(7,3,2)}\\&+ K_{(7,3,2)(5,4,3)}K_{(7,3,2)(7,3,2)} = 1\cdot 1 + 2\cdot 2 + 3\cdot 3 + 1\cdot 1 + 4\cdot 3\\&+ 2\cdot 2 + 4\cdot 2 + 3\cdot 2 + 1\cdot 1 + 3\cdot 1 + 3\cdot 1 + 2\cdot 1 = 54\end{aligned}$$

and

$$\begin{aligned}&\sum_{\nu\in\mathcal{P}_9} K_{\nu(4,3,2)}K_{\nu(6,2,1)} = K_{(9,0,0)(4,3,2)}K_{(9,0,0)(6,2,1)}\\&+ K_{(8,1,0)(4,3,2)}K_{(8,1,0)(6,2,1)} + K_{(7,2,0)(4,3,2)}K_{(7,2,0)(6,2,1)}\\&+ K_{(7,1,1)(4,3,2)}K_{(7,1,1)(6,2,1)} + K_{(6,3,0)(4,3,2)}K_{(6,3,0)(6,2,1)}\\&+ K_{(6,2,1)(4,3,2)}K_{(6,2,1)(6,2,1)} = 1\cdot 1 + 2\cdot 2 + 3\cdot 2 + 1\cdot 1 + 3\cdot 1 + 2\cdot 1 = 17.\end{aligned}$$

Thus

$$\dim \mathrm{Sol}(W, E_3^{(2)}((5,4,3),(7,3,2))) = 54 - 17 = 37.$$

3.2. A polynomial estimate of the dimension of the global solutions to the system $E_3^{(2)}(\alpha,\beta)$

It follows from Corollary 2.2 that if $(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3) \notin \mathbb{Z}_{\geq 0}^6$ or $\alpha_1+\alpha_2+\alpha_3 \neq \beta_1+\beta_2+\beta_3$ then the dimension of global solutions to the system $E_3^{(2)}(\alpha,\beta)$ is equal to zero. Therefore, we can assume $(\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3) \in \mathbb{Z}_{\geq 0}^6$ and $\alpha_1+\alpha_2+\alpha_3 = \beta_1+\beta_2+\beta_3$. Now, we give a polynomial estimate of the dimension of global solutions for the case $E_3^{(2)}(\alpha,\beta)$.

THEOREM 3.4. *Let W be a domain in $M(3,\mathbb{C})$ containing the origin, λ_1, λ_2, λ_3, μ_1, μ_2, μ_3 be nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \lambda_3$, $\mu_1 \geq \mu_2 \geq \mu_3$ and $\lambda_1+\lambda_2+\lambda_3 = \mu_1+\mu_2+\mu_3$. Then*

$$\begin{aligned} f_1(e_1, m-e_1-e_3, e_3) &\leq \dim \mathrm{Sol}(W, E_3^{(2)}((\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3))) \\ &\leq f_2(\lambda_1,\lambda_2,\lambda_3) + f_2(\mu_1,\mu_2,\mu_3) + f_3(e_1, m-e_1-e_3, e_3), \end{aligned}$$

where we put $e_1 = \max(\lambda_1,\mu_1)$, $e_3 = \min(\lambda_3,\mu_3)$, $m = \lambda_1+\lambda_2+\lambda_3$,

$$f_1(x,y,z) = \begin{cases} \frac{1}{3}(z+1)(3yz - z^2 + 3y + z + 3), & y+z \leq x; \\ \frac{1}{3}(z+1)(3yz - z^2 + 3y + z + 3) - \binom{y+z-x+2}{3}, & y+z \geq x, \end{cases}$$

$$f_2(x,y,z) = \frac{1}{6} z(z+1)(3y - z + 1) - f_{22}(y+z-x),$$

$$f_{22}(t) = \begin{cases} 0, & t \leq 1; \\ \frac{1}{24} t(t+2)(2t-1), & t \geq 1, t \text{ is even}; \\ \frac{1}{24} t(t+2)(2t-1) - \frac{1}{8}, & t \geq 1, t \text{ is odd}, \end{cases}$$

and

$$f_3(x,y,z) = (y+1)(z+1) - f_{32}(y+z-x),$$

$$f_{32}(t) = \begin{cases} 0, & t \leq 0; \\ \frac{1}{4} t(t+2), & t \geq 0, t \text{ is even}; \\ \frac{1}{4}(t+1)^2, & t \geq 0, t \text{ is odd}. \end{cases}$$

Note that f_1, f_2, f_3 are maps from $\{(x,y,z) \in \mathbb{Z}_{\geq 0}^3 \mid x \geq y \geq z\}$ to $\mathbb{Z}_{\geq 0}$. For a partition $(\lambda_1,\lambda_2,\lambda_3)$, we have $f_1(\lambda_1,\lambda_2,\lambda_3) \leq f_1(\lambda_2+\lambda_3,\lambda_2,\lambda_3)$, and $f_1(\lambda_1,\lambda_2,\lambda_3) \geq f_1(\lambda_2,\lambda_2,\lambda_3) \geq f_1(\lambda_3,\lambda_3,\lambda_3)$.

Before going into the proof of the theorem let us introduce lemmas.

Lemma 3.5. *For a partition $(\mu_1, \mu_2, \mu_3) \in \mathcal{P}_m$, the number of all partitions $(\lambda_1, \lambda_2, \lambda_3)$ of $\mathcal{P}_m$ such that $K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 1$ is equal to*

$$\begin{cases} 1 + \mu_2 + \mu_3, & \mu_1 \geq \mu_2 + \mu_3; \\ \frac{\mu_1+\mu_2+\mu_3}{2} + 1, & \mu_1 \leq \mu_2 + \mu_3, \mu_1 + \mu_2 + \mu_3 \textit{ is even}; \\ \frac{\mu_1+\mu_2+\mu_3+1}{2}, & \mu_1 \leq \mu_2 + \mu_3, \mu_1 + \mu_2 + \mu_3 \textit{ is odd.} \end{cases}$$

Proof. Let $(\lambda_1, \lambda_2, \lambda_3)$ be a partition of $\mathcal{P}_m$ such that $\lambda_1 \geq \mu_1$ and $\mu_3 \geq \lambda_3$. Since by Corollary 3.2, $K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 1$ is equivalent to the condition $\lambda_1 = \lambda_2$, $\lambda_2 = \lambda_3$, $\lambda_1 = \mu_1$ or $\lambda_3 = \mu_3$, it is sufficient to compute the cases $\lambda_1 = \lambda_2$, $\lambda_2 = \lambda_3$, $\lambda_1 = \mu_1$ or $\lambda_3 = \mu_3$. We introduce notations

$$\begin{aligned} M &= \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{P}_m \mid K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 1\}, \\ M_1 &= \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{P}_m \mid \lambda_1 = \mu_1, K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 1\}, \\ M_2 &= \{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{P}_m \mid \lambda_1 \neq \mu_1, \lambda_3 = \mu_3, K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 1\}, \\ M_3 &= \left\{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{P}_m \mid \begin{array}{c} \lambda_1 \neq \mu_1, \lambda_3 \neq \mu_3, \lambda_2 = \lambda_3, \\ K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 1 \end{array}\right\}, \\ M_4 &= \left\{(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{P}_m \mid \begin{array}{c} \lambda_1 \neq \mu_1, \lambda_3 \neq \mu_3, \lambda_2 \neq \lambda_3, \lambda_1 = \lambda_2, \\ K_{(\lambda_1,\lambda_2,\lambda_3)(\mu_1,\mu_2,\mu_3)} = 1 \end{array}\right\}, \end{aligned}$$

we have ${}^{\#}M = {}^{\#}M_1 + {}^{\#}M_2 + {}^{\#}M_3 + {}^{\#}M_4$.

Case 1. Let $\mu_1 \geq \mu_2 + \mu_3$. Then

$$\begin{aligned} M_1 &= \{(\mu_1, \mu_2 + i, \mu_3 - i) \in \mathcal{P}_m \mid 0 \leq i \leq \mu_3\}, \\ M_2 &= \{(\mu_1 + \mu_2 - \mu_3 - i, \mu_3 + i, \mu_3) \in \mathcal{P}_m \mid 0 \leq i < \mu_2 - \mu_3\}, \\ M_3 &= \{(\mu_1 + \mu_2 + \mu_3 - 2i, i, i) \in \mathcal{P}_m \mid 0 \leq i < \mu_3\}, \\ M_4 &= \emptyset. \end{aligned}$$

Hence

$${}^{\#}M = (\mu_3 + 1) + (\mu_2 - \mu_3) + \mu_3 + 0 = 1 + \mu_2 + \mu_3.$$

Case 2. Let $\mu_1 \le \mu_2 + \mu_3$. Then

$$M_1 = \{(\mu_1, \mu_2 + i, \mu_3 - i) \in \mathcal{P}_m \mid 0 \le i \le \mu_1 - \mu_2\},$$
$$M_2 = \{(\mu_1 + \mu_2 - \mu_3 - i, \mu_3 + i, \mu_3) \in \mathcal{P}_m \mid 0 \le i < \mu_2 - \mu_3\},$$
$$M_3 = \{(\mu_1 + \mu_2 + \mu_3 - 2i, i, i) \in \mathcal{P}_m \mid 0 \le i < \mu_3\}.$$

This shows ${}^{\#}M_1 + {}^{\#}M_2 + {}^{\#}M_3 = (\mu_1 - \mu_2 + 1) + (\mu_2 - \mu_3) + \mu_3 = \mu_1 + 1$. If $\mu_1 + \mu_2 + \mu_3$ is even, then

$$M_4 = \{(\mu_1 + i, \mu_1 + i, \mu_2 + \mu_3 - \mu_1 - 2i) \in \mathcal{P}_m \mid 0 < i \le \frac{1}{2}(\mu_2 + \mu_3 - \mu_1)\}.$$

If $\mu_1 + \mu_2 + \mu_3$ is odd, then

$$M_4 = \{(\mu_1 + i, \mu_1 + i, \mu_2 + \mu_3 - \mu_1 - 2i) \in \mathcal{P}_m \mid 0 < i \le \frac{1}{2}(\mu_2 + \mu_3 - \mu_1 - 1)\}.$$

Hence

$$\begin{aligned} {}^{\#}M &= (\mu_1 + 1) + \begin{cases} \frac{1}{2}(\mu_2 + \mu_3 - \mu_1), & \mu_1 + \mu_2 + \mu_3 \text{ is even} \\ \frac{1}{2}(\mu_2 + \mu_3 - \mu_1 - 1), & \mu_1 + \mu_2 + \mu_3 \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{\mu_1+\mu_2+\mu_3}{2} + 1, & \mu_1 + \mu_2 + \mu_3 \text{ is even;} \\ \frac{\mu_1+\mu_2+\mu_3+1}{2}, & \mu_1 + \mu_2 + \mu_3 \text{ is odd.} \end{cases} \quad \square \end{aligned}$$

Lemma 3.6. *For a partition* $(\mu_1, \mu_2, \mu_3) \in \mathcal{P}_m$ *such that* $\mu_3 \ge 1$, *we have*

$$\sum_{\lambda \in \mathcal{P}_{m-3}} K_{\lambda(\mu_1 - 1, \mu_2 - 1, \mu_3 - 1)} = f_2(\mu_1, \mu_2, \mu_3).$$

Proof. For a partition $(\mu_1, \mu_2, \mu_3) \in \mathcal{P}_m$, we denote by $M_{d,(\mu_1,\mu_2,\mu_3)}$ the set of all partitions $(\nu_1, \nu_2, \nu_3) \in \mathcal{P}_m$ such that $K_{(\nu_1,\nu_2,\nu_3)(\mu_1,\mu_2,\mu_3)} = d$. From Lemma 3.1, we can define the bijective map

$$f : M_{d,(\mu_1,\mu_2,\mu_3)} \to M_{1,(\mu_1-d+1,\mu_2-d+1,\mu_3-d+1)}$$

given by

$$f((\nu_1, \nu_2, \nu_3)) = (\nu_1 - 2(d-1), \nu_2 - (d-1), \nu_3).$$

Therefore, ${}^{\#}M_{d,(\mu_1,\mu_2,\mu_3)} = {}^{\#}M_{1,(\mu_1-d+1,\mu_2-d+1,\mu_3-d+1)}$.

By Lemma 3.5,

$$
{}^{\#}M_{d,(\mu_1,\mu_2,\mu_3)} = \begin{cases} \mu_2+\mu_3-2d+3, & \mu_1 \geq \mu_2+\mu_3-d+1; \\ \frac{1}{2}(\mu_1+\mu_2+\mu_3-3d+5), & \mu_1 \leq \mu_2+\mu_3-d+1, \\ & \mu_1+\mu_2+\mu_3-3d+3 \text{ is even;} \\ \frac{1}{2}(\mu_1+\mu_2+\mu_3-3d+4), & \mu_1 \leq \mu_2+\mu_3-d+1, \\ & \mu_1+\mu_2+\mu_3-3d+3 \text{ is odd.} \end{cases}
$$

CASE 1. Let $\mu_1 > \mu_2+\mu_3-1$. Then

$$
\begin{aligned}
&\sum_{\nu\in\mathcal{P}_{m-3}} K_{\nu(\mu_1-1,\mu_2-1,\mu_3-1)} = \sum_{i=1}^{\mu_3} i \times {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)} \\
&= \sum_{i=1}^{\mu_3} i(\mu_2+\mu_3-2i+1) = \frac{1}{6}\mu_3(\mu_3+1)(3\mu_2-\mu_3+1) \\
&= f_2(\mu_1,\mu_2,\mu_3).
\end{aligned}
$$

CASE 2. Let $\mu_1 \leq \mu_2+\mu_3-1$. Then

$$
\begin{aligned}
\sum_{\nu\in\mathcal{P}_{m-3}} K_{\nu(\mu_1-1,\mu_2-1,\mu_3-1)} &= \sum_{i=1}^{\mu_2+\mu_3-\mu_1-1} i \times {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)} \\
&\quad + \sum_{i=\mu_2+\mu_3-\mu_1}^{\mu_3} i \times {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{i=\mu_2+\mu_3-\mu_1}^{\mu_3} i \times {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)} = \sum_{i=\mu_2+\mu_3-\mu_1}^{\mu_3} i(\mu_2+\mu_3-2i+1) \\
&= \frac{1}{6}(\mu_1-\mu_2-1) \\
&\quad \times(5\mu_1\mu_2+9\mu_1\mu_3-4\mu_1^2-\mu_2^2-3\mu_2\mu_3-6\mu_3^2-5\mu_1+6\mu_3+5\mu_2).
\end{aligned}
$$

Let us compute the sum $\sum_{i=1}^{\mu_2+\mu_3-\mu_1-1} i \times {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)}$ for the two alternative cases of even m and odd m. First, suppose m is even. Then $\mu_2+\mu_3-\mu_1-1 = 2m''+1$ for a nonnegative integer m'' and

$$\begin{aligned}
&\sum_{i=1}^{\mu_2+\mu_3-\mu_1-1} i \times {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)} \\
= &\sum_{i=0}^{m''}(2i+1) \times {}^{\#}M_{2i+1,(\mu_1-1,\mu_2-1,\mu_3-1)} + \sum_{i=0}^{m''} 2i \times {}^{\#}M_{2i,(\mu_1-1,\mu_2-1,\mu_3-1)} \\
= &\sum_{i=0}^{m''}(2i+1)\left(\frac{m-6i-2}{2}\right) + \sum_{i=0}^{m''} 2i\left(\frac{m-6i+2}{2}\right) \\
= &\frac{1}{8}(\mu_2+\mu_3-\mu_1)((\mu_2+\mu_3)(8\mu_1-2\mu_2-2\mu_3+7)-6\mu_1^2-11\mu_1-6).
\end{aligned}$$

Suppose m is odd. Then $\mu_2+\mu_3-\mu_1-1 = 2m''$ for an integer m'' and

$$\begin{aligned}
&\sum_{i=1}^{\mu_2+\mu_3-\mu_1-1} i \times {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)} \\
= &\sum_{i=0}^{m''-1}(2i+1) \times {}^{\#}M_{2i+1,(\mu_1-1,\mu_2-1,\mu_3-1)} \\
&+ \sum_{i=0}^{m''} 2i \times {}^{\#}M_{2i,(\mu_1-1,\mu_2-1,\mu_3-1)} \\
= &\sum_{i=0}^{m''-1}(2i+1)\left(\frac{m-6i-1}{2}\right) + \sum_{i=0}^{m''} 2i\left(\frac{m-6i+1}{2}\right) \\
= &\frac{1}{8}(\mu_2+\mu_3-\mu_1-1) \\
&\times((\mu_2+\mu_3)(8\mu_1-2\mu_2-2\mu_3+5)-6\mu_1^2-5\mu_1-1).
\end{aligned}$$

Thus

$$\begin{aligned}
&\sum_{\nu\in\mathcal{P}_{m-3}} K_{\nu(\mu_1-1,\mu_2-1,\mu_3-1)} \\
= &\sum_{i=1}^{\mu_2+\mu_3-\mu_1-1} {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)}i + \sum_{i=\mu_2+\mu_3-\mu_1}^{\mu_3} {}^{\#}M_{i,(\mu_1-1,\mu_2-1,\mu_3-1)}i
\end{aligned}$$

$$= \begin{cases} -\frac{1}{24}(2M-1)M(M+2)+N, & m \text{ is even;} \\ -\frac{1}{24}(2M-1)M(M+2)+N+\frac{1}{8}, & m \text{ is odd,} \end{cases}$$

where $M = \mu_2 + \mu_3 - \mu_1$ and $N = \frac{1}{6}\mu_3(\mu_3+1)(3\mu_2 - \mu_3 + 1)$. This is equal to $f_2(\mu_1, \mu_2, \mu_3)$. □

LEMMA 3.7. *For a partition $(\lambda_1, \lambda_2, \lambda_3) \in \mathcal{P}_m$, we have*

$$\sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m, \\ \nu_1\geq\lambda_1,\nu_3\leq\lambda_3}} 1 = f_3(\lambda_1, \lambda_2, \lambda_3).$$

PROOF. For integers $y \geq x \geq 0$ we define

$$P_y(x) = {}^{\#}\{(\nu_1, \nu_2) \in \mathbb{Z}^2 \mid \nu_1 \geq \nu_2 \geq 0, \nu_1 \geq x, \nu_1 + \nu_2 = y\}.$$

Then we have

$$P_y(x) = \begin{cases} y - x + 1, & 2x \geq y; \\ \frac{1}{2}(y-1), & 2x \leq y, y \text{ is odd;} \\ \frac{1}{2}y - 1, & 2x \leq y, y \text{ is even.} \end{cases}$$

This shows

$$\sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m, \\ \nu_1\geq\lambda_1,\nu\geq\lambda_3}} 1 = \sum_{i=0}^{\lambda_3} P_{m-3i}(\lambda_1 - i)$$

$$= \begin{cases} (\lambda_2+1)(\lambda_3+1), & 2\lambda_1 \geq m; \\ \frac{1}{4}(4+m(3m+2)) - m\lambda_2 - \lambda_1^2 - \lambda_1\lambda_3 - \lambda_3^2, & 2\lambda_1 \leq m, m \text{ is even;} \\ \frac{1}{4}(3+m(3m+2)) - m\lambda_2 - \lambda_1^2 - \lambda_1\lambda_3 - \lambda_3^2, & 2\lambda_1 \leq m, m \text{ is odd.} \end{cases}$$

This is equal to $f_3(\lambda_1, \lambda_2, \lambda_3)$. □

PROOF OF THEOREM 3.4. We consider the following three cases.

CASE 1. Let $\lambda_3 \neq 0$ and $\mu_3 \neq 0$. We have

$$\sum_{\nu\in\mathcal{P}_m} K_{\nu(\lambda_1,\lambda_2,\lambda_3)} K_{\nu(\mu_1,\mu_2,\mu_3)}$$

$$= \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m, \\ \nu_1>\nu_2>\nu_3}} K_{(\nu_1,\nu_2,\nu_3)(\lambda_1,\lambda_2,\lambda_3)} K_{(\nu_1,\nu_2,\nu_3)(\mu_1,\mu_2,\mu_3)}$$

$$
\begin{aligned}
&+ \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m,\\ \nu_1=\nu_2 \text{ or } \nu_2=\nu_3}} K_{(\nu_1,\nu_2,\nu_3)(\lambda_1,\lambda_2,\lambda_3)} K_{(\nu_1,\nu_2,\nu_3)(\mu_1,\mu_2,\mu_3)} \\
= &\sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m,\\ \nu_1>\nu_2>\nu_3,\\ \nu_1\geq e_1,\nu_3\leq e_3}} (K_{(\nu_1-2,\nu_2-1,\nu_3)(\lambda_1-1,\lambda_2-1,\lambda_3-1)} + 1) \\
&\times (K_{(\nu_1-2,\nu_2-1,\nu_3)(\mu_1-1,\mu_2-1,\mu_3-1)} + 1) \\
&+ \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m,\\ \nu_1\geq e_1,\nu_3\leq e_3,\\ \nu_1=\nu_2 \text{ or } \nu_2=\nu_3}} 1 \\
= &\sum_{\nu\in\mathcal{P}_{m-3}} K_{\nu(\lambda_1-1,\lambda_2-1,\lambda_3-1)} K_{\nu(\mu_1-1,\mu_2-1,\mu_3-1)} \\
&+ \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_{m-3},\\ \nu_1\geq e_1-2,\nu_3\leq e_3}} \big(K_{(\nu_1,\nu_2,\nu_3)(\lambda_1-1,\lambda_2-1,\lambda_3-1)} + K_{(\nu_1,\nu_2,\nu_3)(\mu_1-1,\mu_2-1,\mu_3-1)}\big) \\
&+ f_3(e_1, m-e_1-e_3, e_3).
\end{aligned}
$$

Here we use Lemma 3.1 and Corollary 3.2 for the second equality, and Lemma 3.7 for the last equality. Then by using Theorem 2.6 we have

$$
\begin{aligned}
&\dim \mathrm{Sol}(W, E_3^{(2)}((\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3))) \\
= &\sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_{m-3},\\ \nu_1\geq e_1-2,\nu_3\leq e_3}} \big(K_{(\nu_1,\nu_2,\nu_3)(\lambda_1-1,\lambda_2-1,\lambda_3-1)} + K_{(\nu_1,\nu_2,\nu_3)(\mu_1-1,\mu_2-1,\mu_3-1)}\big) \\
&+ f_3(e_1, m-e_1-e_3, e_3).
\end{aligned}
$$

Now we give the estimates of the truncated sums of Kostka numbers appearing above.

By Lemma 3.6,

$$
\begin{aligned}
&\dim \mathrm{Sol}(W, E_3^{(2)}((\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3))) \\
&\leq \sum_{\nu\in\mathcal{P}_{m-3}} \big(K_{\nu(\lambda_1-1,\lambda_2-1,\lambda_3-1)} + K_{\nu(\mu_1-1,\mu_2-1,\mu_3-1)}\big) \\
&\quad + f_3(e_1, m-e_1-e_3, e_3) \\
&= f_2(\lambda_1,\lambda_2,\lambda_3) + f_2(\mu_1,\mu_2,\mu_3) + f_3(e_1, m-e_1-e_3, e_3).
\end{aligned}
$$

This gives the upper estimate.

Now we will prove the lower estimate. By Corollary 3.2,

$$K_{(\nu_1,\nu_2,\nu_3)(\lambda_1,\lambda_2,\lambda_3)} \geq K_{(\nu_1,\nu_2,\nu_3)(\mu_1,\mu_2,\mu_3)}$$

for $(\lambda_1,\lambda_2,\lambda_3), (\mu_1,\mu_2,\mu_3), (\nu_1,\nu_2,\nu_3) \in \mathcal{P}_{m-3}$ such that $\lambda_1 \leq \mu_1$ and $\lambda_3 \geq \mu_3$. Then

$$\begin{aligned} & \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_{m-3}, \\ \nu_1\geq e_1-2,\nu_3\leq e_3}} K_{(\nu_1,\nu_2,\nu_3)(\lambda_1-1,\lambda_2-1,\lambda_3-1)} \\ \geq & \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_{m-3}, \\ \nu_1\geq e_1-2,\nu_3\leq e_3}} K_{(\nu_1,\nu_2,\nu_3)(e_1-1,m-e_1-e_3-1,e_3-1)} \\ = & \sum_{\nu\in\mathcal{P}_{m-3}} K_{\nu(e_1-1,m-e_1-e_3-1,e_3-1)} \\ = & \ f_2(e_1, m-e_1-e_3, e_3), \end{aligned}$$

where the last equality follows from Lemma 3.6. This shows

$$\begin{aligned} & \dim \mathrm{Sol}(W, E_3^{(2)}((\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,\mu_3))) \\ \geq & \ 2f_2(e_1,m-e_1-e_3,e_3)+f_3(e_1,m-e_1-e_3,e_3) = f_1(e_1,m-e_1-e_3,e_3). \end{aligned}$$

Here we compute

$$2f_2(x,y,z)+f_3(x,y,z) = \frac{1}{3}(z+1)(3yz-z^2+3y+z+3)-(2f_{22}+f_{32})(y+z-x)$$

and explicitly, $(2f_{22}+f_{32})(t) = \begin{cases} 0, & t\leq 0; \\ \binom{t+2}{3}, & t \geq 0. \end{cases}$

Case 2. Let $\lambda_3 \neq 0$ and $\mu_3 = 0$. We have

$$\begin{aligned} & \dim \mathrm{Sol}(W, E_3^{(2)}((\lambda_1,\lambda_2,\lambda_3),(\mu_1,\mu_2,0))) \\ = & \sum_{\nu\in\mathcal{P}_m} K_{\nu(\lambda_1,\lambda_2,\lambda_3)} K_{\nu(\mu_1,\mu_2,0)} \\ = & \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m, \\ \nu_1>\nu_2>\nu_3}} K_{(\nu_1,\nu_2,\nu_3)(\lambda_1,\lambda_2,\lambda_3)} K_{(\nu_1,\nu_2,\nu_3)(\mu_1,\mu_2,0)} \\ & + \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m, \\ \nu_1=\nu_2 \text{ or } \nu_2=\nu_3}} K_{(\nu_1,\nu_2,\nu_3)(\lambda_1,\lambda_2,\lambda_3)} K_{(\nu_1,\nu_2,\nu_3)(\mu_1,\mu_2,0)} \end{aligned}$$

$$= \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m,\\ \nu_1>\nu_2>\nu_3,\\ \nu_1\geq e_1,\nu_3\leq e_3}} (K_{(\nu_1-2,\nu_2-1,\nu_3)(\lambda_1-1,\lambda_2-1,\lambda_3-1)}+1) + \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m,\\ \nu_1\geq e_1,\nu_3\leq e_3,\\ \nu_1=\nu_2 \text{ or } \nu_2=\nu_3}} 1$$

$$= \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_{m-3},\\ \nu_1\geq e_1-2,\nu_3\leq e_3}} K_{(\nu_1,\nu_2,\nu_3)(\lambda_1-1,\lambda_2-1,\lambda_3-1)}$$

$$+ \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m,\\ \nu_1>\nu_2>\nu_3,\\ \nu_1\geq e_1,\nu_3\leq e_3}} 1 + \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_m,\\ \nu_1\geq e_1,\nu_3\leq e_3,\\ \nu_1=\nu_2 \text{ or } \nu_2=\nu_3}} 1$$

$$= \sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_{m-3},\\ \nu_1\geq e_1-2,\nu_3\leq e_3}} K_{(\nu_1,\nu_2,\nu_3)(\lambda_1-1,\lambda_2-1,\lambda_3-1)} + f_3(e_1, m-e_1-e_3, e_3).$$

Since $f_2(e_1, m-e_1, 0) = f_2(\mu_1, \mu_2, 0) = 0$ and

$$\sum_{\substack{(\nu_1,\nu_2,\nu_3)\in\mathcal{P}_{m-3},\\ \nu_1\geq e_1-2,\nu_3\leq e_3}} K_{(\nu_1,\nu_2,\nu_3)(\lambda_1-1,\lambda_2-1,\lambda_3-1)} \leq f_2(\lambda_1, \lambda_2, \lambda_3),$$

the theorem is true.

Case 3. Let $\lambda_3 = \mu_3 = 0$. We have

$$\begin{aligned} &\dim \mathrm{Sol}(W, E_3^{(2)}((\lambda_1, \lambda_2, 0), (\mu_1, \mu_2, 0))) \\ &= \sum_{\nu\in\mathcal{P}_m} K_{\nu(\lambda_1,\lambda_2,0)} K_{\nu(\mu_1,\mu_2,0)} \\ &= \sum_{\substack{(\nu_1,\nu_2)\in\mathcal{P}_m,\\ \nu_1\geq e_1}} 1 = f_3(e_1, m-e_1, 0). \end{aligned}$$

Since $f_1(e_1, m-e_1, 0) = f_2(\lambda_1, \lambda_2, 0) + f_2(\mu_1, \mu_2, 0) + f_3(e_1, m-e_1, 0) = f_3(e_1, m-e_1, 0)$, the theorem is true. $\square$

In the case $\lambda = \mu$, the upper estimate and the lower estimate in Theorem 3.4 are equal to $f_1(\lambda_1, \lambda_2, \lambda_3)$ since $e_1 = \lambda_1$, $e_3 = \lambda_3$ and $m - e_1 - e_3 = \lambda_2$. Then we obtain the following:

Corollary 3.8. *Let W be a domain in $M(3, \mathbb{C})$ containing the origin.*

Then for nonnegative integers $a \geq b \geq c$,

$$\dim \mathrm{Sol}(W, E_3^{(2)}((a,b,c),(a,b,c))) = \begin{cases} \frac{1}{3}(c+1)(3bc - c^2 + 3b + c + 3), & b + c \leq a; \\ \frac{1}{3}(c+1)(3bc - c^2 + 3b + c + 3) - \binom{b+c-a+2}{3}, & b + c \geq a. \end{cases}$$

In particular,

$$\dim \mathrm{Sol}(W, E_3^{(2)}((a,a,a),(a,a,a))) = \frac{1}{2}(a+1)(a^2 + 2a + 2).$$

Acknowledgement. The authors thank Professors T. Kobayashi and A. Mekei for helpful discussions.

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(Received September 27, 2012)

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