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A New Characterization of Random Times for Specifying Information Delay

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Abstract. We introduce a stochastic process called a follower process consisting of a non-decreasing sequence of random times f_t whose values do not exceed t. It was originally introduced for representing information delay in structural credit risk models. The follower process is an extension of a time change process introduced by Guo, Jarrow and Zeng in the sense that each component of the follower process is not required to be a stopping time. We introduce a class of follower processes called idempotent, which contains natural examples including follower processes driven by renewal processes. We show that any idempotent follower process is hard to be an example of time change processes. We define a filtration modulated by the follower process and show that it is a natural extension of the continuously delayed filtration that is the filtration modulated by the time change process. We show that conditional expectations given idempotent follower filtrations have some Markov property in a binomial setting, which is useful for pricing defaultable financial instruments.

1. Introduction

The original motivation of this paper comes from the theory of credit risk models. In the credit risk theory, it is crucial to introduce a sort of incompleteness into its model when adopting a so-called structural approach. In order to make it, Guo, Jarrow and Zeng [GJZ09] introduced a process called time change.

Let $\mathcal{T}$ be a fixed time domain that has the least element 0, equipped with an adequate topology, and $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\in\mathcal{T}}, \mathbb{P})$ be a filtered probability

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space where the filtration $\mathbb{F}$ satisfies the usual condition for continuous time domains. Then the time change process is defined like the following.

Definition 1.1 [Time Change Process (Guo, Jarrow and Zeng [GJZ09])]. An $\mathbb{F}$-*time change process* is an $\mathbb{F}$-adapted stochastic process $f : \mathcal{T} \times \Omega \to \mathcal{T}$ satisfying the following conditions,

(1) $f_0 = 0$ $\mathbb{P}$-a.s.,

(2) $f_t \leq t$ $\mathbb{P}$-a.s. for all $t \in \mathcal{T}$,

(3) $t_1 \leq t_2 \;\to\; f_{t_1} \leq f_{t_2}$ $\mathbb{P}$-a.s. for all $t_1, t_2 \in \mathcal{T}$.

(4) for all $t \in \mathcal{T}$, f_t is an $\mathbb{F}$-stopping time.

The time change process f_t represents the amount of delayed time $t - f_t$. It can be read in the context of the credit risk theory that if the market knows an event at time t, then the event actually happened at time f_t (ahead of t) when managers learned it. So, it models the fact that the market would know the information possibly after the managers know it, that is, representing asymmetric information.

Guo, Jarrow and Zeng succeeded to make their credit risk model an incomplete one by using a filtration $\{\mathcal{F}_{f_t}\}_{t \in \mathcal{T}}$ called a *continuously delayed filtration*. The results were somehow consistent with empirically observed data.

Note that the continuously delayed filtration is well-defined since f_t is an $\mathbb{F}$-stopping time.

Now let us suppose a natural example representing delayed information, which we call a *renewal follower process* $\{f_t\}_{t \in \mathcal{T}}$.

Example 1.2 [Renewal Follower Processes].

(1) $X_n \sim$ i.i.d. random variables such that $0 < \mathbb{E}^{\mathbb{P}}[X_n] < \infty$ for $n = 1, 2, \ldots$,

(2) $S_n := \sum_{k=1}^{n} X_k$,

(3) $N_t := \sup\{n \mid S_n \leq t\}$,

(4) $f_t := S_{N_t}$.

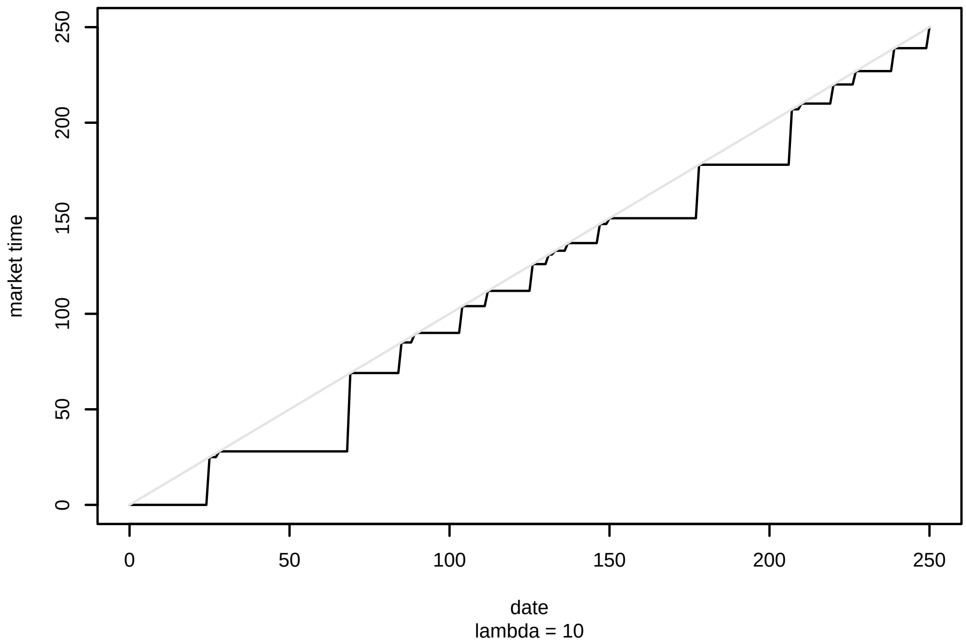

Fig. 1. Renewal follower process.

Intuitively, the random variable X_n specifies an interval time between n-th and $(n+1)$-th jumps when the follower process catch up with the current time, i.e. $f_t = t$. Figure 1 shows a sample trajectory of a renewal follower process where the random variables X_n above obey an exponential distribution $Exp(10)$.

One of the possible interpretation of this example in reality is the situation that the firm makes all its insider information available to the market only when it is under an audit activity by authorities (at the jumping time).

However, we will see in Section 2.2 that it is hard for renewal follower processes to be examples of the time change processes because of the strong condition (4) in Definition 1.1. This is the main reason we introduced the following new concept by dropping the condition.

Definition 1.3 [Follower Processes].

A *raw follower process* is a stochastic process $f : \mathcal{T} \times \Omega \to \mathcal{T}$ satisfying the following conditions,

(1) $f_0 = 0$ $\mathbb{P}$-a.s.,

(2) $f_\tau \leq \tau$ $\mathbb{P}$-a.s. for all $\tau \in \mathcal{T}^*$,

(3) $\tau_1 \leq \tau_2 \ \to \ f_{\tau_1} \leq f_{\tau_2}$ $\mathbb{P}$-a.s. for all $\tau_1, \tau_2 \in \mathcal{T}^*$.

where $\mathcal{T}^*$ is the set of all $\mathcal{T}$-valued random times.

An $\mathbb{F}$-*follower process* is a raw follower process which is $\mathbb{F}$-adapted.

Apparently, renewal follower processes are raw follower processes.

The difference of Definition 1.3 from the first three conditions of Definition 1.1 is the use of $\mathcal{T}^*$ instead of $\mathcal{T}$. This change is necessary if we consider the case when we need to make a reasoning such as $f_s \leq t$ implies $f_{f_s} \leq f_t$. Therefore, the original definition of time change processes in Definition 1.1 is also better to be rewritten with t_i varying in $\mathcal{T}^*$ instead of in $\mathcal{T}$. So in the rest of this paper, we assume that any $\mathbb{F}$-time change process is an $\mathbb{F}$-follower process.

The remainder of this paper consists of three sections.

In Section 2, we begin with giving some properties and a couple of examples of follower processes. After seeing that the set of all follower processes forms a monoid, We introduce an important class of *idempotent* follower processes that satisfy $f_{f_t} = f_t$ for all $t \in \mathcal{T}$. It is obvious that the class contains all renewal follower processes. Then, we show that an idempotent follower process fails to be an example of time change processes. Next, we provide some characterizations of idempotent follower processes. In the end of Section 2, we show that an idempotent follower process consists of a sequence of honest times, and that any honest time can be represented as a limit of an idempotent follower process.

One of our motivations to introduce the concept of follower processes is to use it for modulating a given filtration, which is necessary for pricing defaultable securities. However, except the case when f_t is a stopping time, it is not generally obvious how to define the σ-field $\mathcal{F}_{f_t}$ and the filtration consisting of those σ-fields. In Section 3, we present a definition of filtrations modulated by follower processes and show that our filtration is a natural extension of the continuously delayed filtrations in the sense that they coincide each other when the underlying follower process consists of stopping times.

Once we try to apply the theory of follower processes to the credit risk theory, we would face the necessity to calculate some conditional expectations given the filtration defined in Section 3. Especially, we need a strong Markov property like the following:

$$\mathbb{E}^{\mathbb{P}}[g(Y_s) \mid \mathcal{F}_t^f] = \mathbb{E}^{\mathbb{P}}[g(Y_s) \mid f_t, Y_{f_t}]$$

where $\mathcal{F}_t^f$ is the filtration modulated by the follower process f_t. In Section 4, we prove this when f_t is idempotent in a binomial model.

2. Follower Processes

For $s, t \in \mathcal{T}$, we define $[s,t]_{\mathcal{T}} := \{u \in \mathcal{T} \mid s \leq u \leq t\}$, and similarly define $[s,t[_{\mathcal{T}}$, $]s,t]_{\mathcal{T}}$ and $]s,t[_{\mathcal{T}}$. We write $\mathcal{T}_+$ for $\mathcal{T} - \{0\}$.

For a function f whose domain is $\mathcal{T}$, $f(t-) := \lim_{s \to t-0} f(s)$ and $f(t+) := \lim_{s \to t+0} f(s)$. Note that in the case $\mathcal{T} = \{n\delta \mid n = 0, 1, \ldots\}$, $t- = t - \delta$ for $t \in \mathcal{T}_+$ and $t+ = t + \delta$ for $t \in \mathcal{T}$.

2.1. Properties and examples of follower processes

Here are some simple properties of follower processes whose proofs are left to readers.

Proposition 2.1. *Let f and g be raw follower processes. Then, so are the following processes:*

(1) $f_t \wedge g_t$,

(2) $f_t \vee g_t$,

(3) $h_t := \begin{cases} f_t & \textit{if}\, t \leq s, \\ f_s + g_{t-s} & \textit{if}\, t > s \end{cases}.$

The simplest example of follower processes is the identity process $\{t\}_{t \in \mathcal{T}}$. Other than that and renewal follower processes, we give a couple of examples in the rest of this subsection.

Example 2.2 [Constantly Delayed Follower Processes].

Lindset et al. introduced the two time lags for markets and managers in [LLP08]. Their lags are constant and not stochastically varying like ours.

Let d be a positive constant. A raw follower process $f = \{f_t\}_{t\in\mathcal{T}}$ is called a *constantly delayed follower process* with delay d if for all $t \in \mathcal{T}$,

$$f_t := \max\{t - d, 0\}.$$

Example 2.3 [Occupation Times].

This example is taken from Example 6.2 in Chapter 3 of Karatzas and Shreve [KS98]. Let $W = \{W_t\}_{t\in\mathcal{T}}$ be a Brownian motion and $B \in \mathcal{B}(\mathbb{R})$ be a Borel set. Then, the *occupation time* of B by the Brownian path up to time t is the process f define by

$$f_t := \int_0^t \mathbb{1}_B(W_s)ds.$$

Obviously, any occupation time f is a follower process. However, the occupation time will not recover to the managers' time (that is, $f_t \neq t$) once it had a chance to walk out of the Borel set B. More precisely speaking, the delay $t - f_t$ is increasing as time passes, and never shrinks. Therefore, the converse is untrue.

Similarly, for a given continuous semimartingale $X = \{X_t\}_{t\in\mathcal{T}}$, its local time $L = \{L_t\}_{t\in\mathcal{T}}$ is a follower process.

2.2. Idempotent follower processes

In this subsection, we introduce a class of follower processes whose elements are called idempotent. We show that the class contains all renewal follower processes and that every idempotent follower process is hard to be a time change process.

First, we introduce a composition operator defined on the space of follower processes.

A process f is a raw follower process if there is a raw follower process f' such that $f_\tau = f'_\tau$ for all $\tau \in \mathcal{T}^*$. Therefore, we can treat the space of follower processes as a quotient space safely.

DEFINITION 2.4 [Space of Follower Processes].

(1) $\mathcal{M}$ is the set of all raw follower processes.

(2) For $f^1, f^2 \in \mathcal{M}$, the *composite process* $f^1 \circ f^2$ is defined by for $t \in \mathcal{T}$ and $\omega \in \Omega$,

$$(f^1 \circ f^2)_t(\omega) = (f^1 \circ f^2)(t, \omega) := f^1(f^2(t, \omega), \omega).$$

(3) $\mathbb{M} := \mathcal{M}/\sim$, where $\sim$ is a binary relation on $\mathcal{M}$ defined by for any pair of f^1 and f^2 in $\mathcal{M}$, $f^1 \sim f^2$ iff $f^1_\tau = f^2_\tau$ $\mathbb{P}$-a.s. for all $\tau \in \mathcal{T}^*$.

For $f \in \mathcal{M}$, we write $f \in \mathbb{M}$ by identifying f with the equivalence class $[f]_\sim \in \mathbb{M}$ if it leads no confusion.

(4) An *identity process* is a process $\mathbb{1}^{\mathcal{M}} \in \mathcal{M}$ defined by $\mathbb{1}^{\mathcal{M}}_t(\omega) = t$ for all $t \in \mathcal{T}$ and $\omega \in \Omega$.

THEOREM 2.5. *The structure $\langle \mathbb{M}, \circ, \mathbb{1}^{\mathcal{M}} \rangle$ forms a monoid, that is, a semigroup with identity, where $\circ$ is a well-defined operator on $\mathbb{M}$ induced by the operator $\circ$ on $\mathcal{M}$.*

PROOF. Straightforward. □

DEFINITION 2.6 [Idempotent Follower Processes]. A raw follower process f is called *idempotent* if for all $\tau \in \mathcal{T}^*$,

$$f_{f_\tau} = f_\tau \ \mathbb{P}\text{-a.s.}. \tag{2.1}$$

We would see Equation 2.1 as $f \circ f = f$. The word *idempotent* comes from the fact.

You can easily verify that the identity follower process and the renewal follower process are idempotent follower processes. On the other hand, neither constantly-delayed follower processes nor occupation times are idempotent.

PROPOSITION 2.7. *A raw follower process f is idempotent iff for all $\tau_1, \tau_2 \in \mathcal{T}^*$, $\quad f_{\tau_1} \le \tau_2 \le \tau_1 \ \rightarrow \ f_{\tau_1} = f_{\tau_2}$ $\mathbb{P}$-a.s..*

Proof. *If part.* For $\tau \in \mathcal{T}^*$, we have

$$\{f_\tau \leq \tau\} \cap \{f_\tau \leq f_\tau \leq \tau \ \rightarrow \ f_\tau = f_{f_\tau}\} \subset \{f_\tau = f_{f_\tau}\}.$$

By the assumption, the probability of the left hand set is 1. Therefore, $\mathbb{P}\{f_\tau = f_{f_\tau}\} = 1$ as well.

Only if part. For any $\tau_1, \tau_2 \in \mathcal{T}^*$, define a set A by

$$A := \{f_{f_{\tau_1}} = f_{\tau_1}\} \cap \{f_{\tau_1} \leq \tau_2 \ \rightarrow \ f_{f_{\tau_1}} \leq f_{\tau_2}\} \cap \{\tau_2 \leq \tau_1 \ \rightarrow \ f_{\tau_2} \leq f_{\tau_1}\}.$$

Then, we have $\mathbb{P}(A) = 1$ since f is an idempotent raw follower process. Now, observing

$$\begin{aligned}
&A \cap \{f_{\tau_1} \leq \tau_2 \leq \tau_1\} \\
=&A \cap \{f_{\tau_1} \leq \tau_2\} \cap \{\tau_2 \leq \tau_1\} \\
\subset&\{f_{f_{\tau_1}} = f_{\tau_1}\} \cap \{f_{f_{\tau_1}} \leq f_{\tau_2}\} \cap \{f_{\tau_2} \leq f_{\tau_1}\} \\
=&\{f_{f_{\tau_1}} = f_{\tau_1}\} \cap \{f_{f_{\tau_1}} \leq f_{\tau_2} \leq f_{\tau_1}\} \\
\subset&\{f_{\tau_1} = f_{\tau_2}\},
\end{aligned}$$

we have $A \subset \{f_{\tau_1} \leq \tau_2 \leq \tau_1 \ \rightarrow \ f_{\tau_1} = f_{\tau_2}\}$. Therefore, $\mathbb{P}\{f_{\tau_1} \leq \tau_2 \leq \tau_1 \ \rightarrow f_{\tau_1} = f_{\tau_2}\} = 1$. □

Here is one of the important implications derived from Proposition 2.7.

Corollary 2.8. *Let $f = \{f_t\}_{t \in \mathcal{T}}$ be an idempotent $\mathbb{F}$-follower process where each f_t is a $\mathbb{F}$-stopping time. Then, for every pair t and s in $\mathcal{T}$ with $t \geq s$, we have $\{f_t = f_s\} \in \mathcal{F}_s$.*

Proof. Let $A \subset \Omega$ be the set defined by $A := \{f_t \leq s \leq t \ \rightarrow \ f_t = f_s\} \cap \{f_s \leq s\}$. Then, since f is a follower process and by Proposition 2.7, we get $\mathbb{P}(A) = 1$.

Now, under the assumption $s \leq t$, we have

$$A \cap \{f_t \leq s\} = A \cap (\{f_t \leq s \leq t \ \rightarrow \ f_t = f_s\} \cap \{f_t \leq s\}) \subset A \cap \{f_t = f_s\}$$

and

$$A \cap \{f_t = f_s\} = A \cap (\{f_s \leq s\} \cap \{f_t = f_s\}) \subset A \cap \{f_t \leq s\}.$$

Thus $A \cap \{f_t \leq s\} = A \cap \{f_t = f_s\}$. Therefore $\{f_t \leq s\} \triangle \{f_t = f_s\} \subset \Omega - A$. Hence $\mathbb{P}(\{f_t \leq s\} \triangle \{f_t = f_s\}) = 0$. Since $\{f_t \leq s\}$ is $\mathcal{F}_s$-measurable and $\mathcal{F}_s$ is complete, we have $\{f_t = f_s\} \in \mathcal{F}_s$. □

Let us think that s is the current time and that t is any future time. Then by Corollary 2.8, we can know if the information will have increased since now by *any future* time t, which is not realistic. So, we should conclude that requiring each random time f_t to be a stopping time is not practical in the case that f is idempotent while some of the idempotent follower processes are quite interesting both in the practical and the theoretical sense. This is our original motivation to develop a delayed theory that does not depend on stopping times.

Next, we show a characterization of idempotent follower processes.

Definition 2.9.

(1) For a random set $F \subset \mathcal{T} \times \Omega$, define a process $f^F : \mathcal{T} \times \Omega \to \mathcal{T}$ by

$$f^F_t(\omega) = \sup\{s \leq t \mid (s, \omega) \in F\}, \tag{2.2}$$

where we use the convention $\sup \emptyset = 0$.

(2) For a raw follower process f, define a random set F^f by

$$F^f := \{(t, \omega) \in \mathcal{T} \times \Omega \mid f_t(\omega) = t\}. \tag{2.3}$$

Note that f^F_t is the end of the random set $F_t := F \cap ([0, t]_{\mathcal{T}} \times \Omega)$. Here, the *end* of a random set $A \subset \mathcal{T} \times \Omega$ is the random time E_A defined by $E_A(\omega) := \sup\{t \in \mathcal{T} \mid (t, \omega) \in A\}$, where we use the convention $\sup \emptyset = 0$.

Proposition 2.10.

(1) *Let $F \subset \mathcal{T} \times \Omega$ be a random set. Then, the process f^F is an idempotent raw follower process.*

(2) *Let f be an idempotent raw follower process, Then, $f^{F^f}_\tau = f_\tau$ $\mathbb{P}$-a.s. for all $\tau \in \mathcal{T}^*$.*

Proof.

(1) It is clear that f^F is a raw follower process. So, let us show it is also idempotent. Let $\omega \in \Omega$, $\tau \in \mathcal{T}^*$ and $s := f^F_\tau(\omega)$. Then,

$$s \leq \tau(\omega) \quad \text{and} \quad (\forall u \in \mathcal{T}) u \leq \tau(\omega) \wedge (u, \omega) \in F \rightarrow u \leq s. \tag{2.4}$$

Now, it is enough to show that $\{u \leq s \mid (u, \omega) \in F\} = \{u \leq \tau(\omega) \mid (u, \omega) \in F\}$. Since $s \leq \tau(\omega)$, it is obvious that **LHS** $\subset$ **RHS**. Let $u \in$ **RHS**. Then by Equation 2.4, $u \leq s$. Therefore, $u \in$ **LHS**.

(2) For $\omega \in \Omega$, $s \in \mathcal{T}$ and $\tau \in \mathcal{T}^*$, We have

$$\begin{aligned} f^{F^f}_\tau(\omega) &= \sup\{s \leq \tau(\omega) \mid (s, \omega) \in F^f\} \\ &= \sup\{s \in \mathcal{T} \mid s \leq \tau(\omega) \wedge f_s(\omega) = s\}. \end{aligned} \tag{2.5}$$

Then by Equation 2.5, we have $\{f_\tau \leq \tau\} \cap \{f_{f_\tau} = f_\tau\} \subset \{f^{F^f}_\tau \geq f_\tau\}$. Here, the probability of the left hand set of the above equation is 1 since f is an idempotent follower process. Therefore, $\mathbb{P}\{f^{F^f}_\tau \geq f_\tau\} = 1$.

On the other hand,

$$\begin{aligned} &\{f_\tau < s \leq \tau \ \rightarrow \ f_\tau = f_s\} \cap \{s \leq \tau \ \wedge \ f_s = s\} \cap \{f_\tau < s\} \\ =&\{f_\tau < s \leq \tau \ \rightarrow \ f_\tau = f_s\} \cap \{f_\tau < s \leq \tau\} \cap \{f_\tau < s\} \cap \{f_s = s\} \\ \subset&\{f_\tau = f_s\} \cap \{f_\tau < s\} \cap \{f_s = s\} \\ \subset&\{f_s < s\} \cap \{f_s = s\} = \emptyset. \end{aligned}$$

Therefore,

$$\begin{aligned} \{f_\tau < s \leq \tau \ \rightarrow \ f_\tau = f_s\} &\subset \{(s \leq \tau \ \wedge \ f_s = s) \ \rightarrow \ s \leq f_\tau\} \\ &\subset \{f^{F^f}_\tau \leq f_\tau\}. \end{aligned} \tag{2.6}$$

Here the last inclusion holds by Equation 2.5. The probability of the left most statement of Equation 2.6 is 1 by Proposition 2.7. Therefore, $\mathbb{P}\{f^{F^f}_\tau \leq f_\tau\} = 1$. □

We have the following characterization theorem for idempotent raw follower processes.

THEOREM 2.11. *Let $f : \mathcal{T} \times \Omega \to \mathcal{T}$ be a process. Then, f is an idempotent raw follower process iff there exists a random set $F \subset \mathcal{T} \times \Omega$ such that $f^F_\tau = f_\tau$ for all $\tau \in \mathcal{T}^*$.*

PROOF. Immediate from Proposition 2.10. □

The following is an example of idempotent follower process.

Example 2.12 [Starting Times for Excursions].
Let $B = \{B_t\}_{t\in\mathcal{T}}$ be a standard $\mathbb{F}$-Brownian motion, and define a random set Z by

$$Z = \{(t, \omega) \in \mathbb{R}_+ \times \Omega \mid B_t(\omega) = 0\}.$$

Then, the idempotent follower process f^Z picks the starting times for the excursions out of 0 of B.

PROPOSITION 2.13. *Let F be an $\mathbb{F}$-progressive set. Then, the process f^F is an idempotent $\mathbb{F}$-follower process.*

PROOF. It is enough to show that f^F is $\mathbb{F}$-adapted. Since F is $\mathbb{F}$-progressive, $F_t = F \cap ([0,t]_\mathcal{T} \times \Omega)$ is $\mathcal{B}[0,t] \otimes \mathcal{F}_t$-measurable. Then, since f^F_t is the end of F_t, it is $\mathcal{F}_t$-measurable. □

THEOREM 2.14. *Let $f : \mathcal{T} \times \Omega \to \mathcal{T}$ be a càdlàg process. Then, f is an idempotent $\mathbb{F}$-follower process iff there exists an $\mathbb{F}$-optional set $F \subset \mathcal{T} \times \Omega$ such that $f^F_\tau = f_\tau$ for all $\tau \in \mathcal{T}^*$.*

PROOF. *If part.* By Proposition 2.13 and the remark after Definition A.1.

Only if part. All we need to show is that the random set F^f defined by Equation 2.3 is $\mathbb{F}$-optional when f is $\mathbb{F}$-adapted.

For $n \in \mathbb{N}$, define processes p^n by $p^n := \mathbb{1}_{\{(t,\omega) \mid f_t(\omega) \le t < f_t(\omega) + \frac{1}{n}\}}$. Then, p^n is obviously $\mathbb{F}$-adapted and càdlàg. Therefore, $(p^n)^{-1}(1) = \{(t,\omega) \mid f_t(\omega) \le t < f_t(\omega) + \frac{1}{n}\}$ is an $\mathbb{F}$-optional set. Thus, so is $F^f = \bigcap_{n\in\mathbb{N}} (p^n)^{-1}(1)$. □

It is easily checked that the idempotent follower process in Example 2.12 is an idempotent $\mathbb{F}$-follower process.

2.3. Honest times

In Corollary 2.8, we were discouraged to make a follower process consist of stopping times when it is idempotent.

In this subsection, we revisit the issue by adopting a wider class of random times than the class of stopping times.

A random time τ is called $\mathbb{F}$-*honest* with respect to an $\mathbb{F}$-adapted process $\{\tau_t\}_{t\in\mathcal{T}_+}$ on $\mathcal{T}$ if $\tau = \tau_t$ on $\{\tau \le t\}$ for every $t \in \mathcal{T}_+$, i.e. $\tau 1\!\!1_{\{\tau\le t\}} = \tau_t 1\!\!1_{\{\tau\le t\}}$. A random time τ is called $\mathbb{F}$-*honest* if there exists an $\mathbb{F}$-adapted process $\{\tau_t\}_{t\in\mathcal{T}_+}$ such that τ is $\mathbb{F}$-honest with respect to $\{\tau_t\}_{t\in\mathcal{T}_+}$.

It is well known that every $\mathbb{F}$-stopping time is $\mathbb{F}$-honest (See e.g. page 373 of Protter [Pro04] or page 384 of Nikeghbali [Nik06]).

Here is another characterization of honest times by optional processes.

Theorem 2.15 [[Pro04] Theorem VI.16]. *A random time τ is $\mathbb{F}$-honest if and only if there exists an $\mathbb{F}$-optional set A such that $\tau = E_A$, where E_A is the end of A.*

The following is a very nice characterization of honest times developed by Yor (Yor [Yor78]).

Theorem 2.16 [Yor [Yor78]]. *A random time τ is $\mathbb{F}$-honest if and only if for every $u \in [0, s[_{\mathcal{T}}$, there exists $A \in \mathcal{F}_s$ such that $\{\tau \le u\} = A \cap \{\tau \le s\}$.*

Our first question in this subsection is for a given follower process $f = \{f_t\}_{t\in\mathcal{T}}$, if there exists an honest time τ with respect to f.

Here is a necessary and sufficient condition of the existence of such τ.

Proposition 2.17. *Let $f = \{f_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-follower process. Then, a random time $\tau : \Omega \to \mathcal{T} \cup \{\infty\}$ is $\mathbb{F}$-honest with respect to f if and only if $f_\infty(\omega) := \lim_{t\to\infty} f_t(\omega) = \tau(\omega) = f_{\tau(\omega)}(\omega)$ for every $\omega \in \Omega$.*

Proof. Note that the random time τ is $\mathbb{F}$-honest with respect to f iff for every $t \in \mathcal{T}_+$ and $\omega \in \Omega$,

$$\tau(\omega) \le t \to f_t(\omega) = \tau(\omega) \tag{2.7}$$

Only if part: Since f_t is monotonic, $\lim_{t\to\infty} f_t(\omega) = \sup_{t\in\mathcal{T}} f_t(\omega)$. Therefore, the result comes immediately by Equation 2.7.

If part: Since $\sup_{t\in\mathcal{T}} f_t(\omega) = \tau(\omega)$, $f_t(\omega) \le \tau(\omega)$ for every $t \in \mathcal{T}_+$. So, it is sufficient to show $f_t(\omega) \ge \tau(\omega)$, assuming $\tau(\omega) \le t$. But, by the monotonicity of f_t and the assumption $\tau(\omega) = f_{\tau(\omega)}(\omega)$, we have $\tau(\omega) = f_{\tau(\omega)}(\omega) \le f_t(\omega)$. □

As an implication of Proposition 2.17, we missed the possibility of making whole follower process be characterized by one honest time if the follower process is unbounded. However, we have the following theorem of asserting each f_t becomes an honest time for some follower processes including renewal follower processes.

THEOREM 2.18. *If $f = \{f_t\}_{t\in\mathcal{T}}$ is an idempotent $\mathbb{F}$-follower process, then for every $t \in \mathcal{T}$, f_t is an $\mathbb{F}$-honest time.*

PROOF. Define a random field $\{\tau_s^t\}_{t,s\in\mathcal{T}}$ by $\tau_s^t := f_{t\wedge s}$.

Then, it is obvious that τ_s^t is $\mathcal{F}_s$-measurable. So, all we need to show is $\tau_s^t = f_t$ on $\{f_t \le s\}$.

If $s \ge t$, we have $\tau_s^t = f_t$ on Ω. Hence, we concentrate on the case $s < t$. Now for any $\omega \in \{f_t \le s\}$, $f_t(\omega) \le s < t$. Then, since f is idempotent, we get $f_t(\omega) = f_{f_t(\omega)}(\omega) \le f_s(\omega) \le f_t(\omega)$. Therefore, $f_t(\omega) = f_s(\omega) = \tau_s^t(\omega)$. □

Here is another characterization of honest times by using idempotent follower processes.

THEOREM 2.19. *A random time $\tau : \Omega \to \bar{\mathcal{T}}$ is $\mathbb{F}$-honest if and only if there exists an idempotent $\mathbb{F}$-follower process f such that for every $t \in \mathcal{T}_+$, $\tau = f_t$ on $\{\tau \le t\}$, i.e. $\tau = f_\infty$.*

PROOF. *If part.* Immediate by the definition of honest times.

Only if part. By Theorem 2.15, there exists an $\mathbb{F}$-optional set F such that $\tau = E_F$ since τ is $\mathbb{F}$-honest.

Let $f := f^F$. Then, by Theorem 2.14, f is an idempotent $\mathbb{F}$-follower process.

On the other hand, for $\omega \in \{\tau \le t\}$, we have

$$\begin{aligned} f_t(\omega) &= \sup\{s \le t \mid (s,\omega) \in F\} \\ &= \sup\{s \in \mathcal{T} \mid (s,\omega) \in F\} \quad \text{since} \quad s \le \tau(\omega) \le t \\ &= E_F(\omega). \end{aligned}$$

Therefore, $f_t = E_F = \tau$. □

3. Follower Filtrations

As stated in Section 1, one of our motivations to introduce the concept of follower processes is to use it for modulating a given filtration. Here is a definition to make it.

Definition 3.1 [Follower Filtrations]. Let $f = \{f_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-follower process. The *follower filtration* modulated by the $\mathbb{F}$-follower process is the filtration $\mathbb{F}^f = \{\mathcal{F}^f_t\}_{t\in\mathcal{T}}$ defined by for $t \in \mathcal{T}$,

$$\mathcal{F}^f_t = \bigvee_{s\in[0,t]_{\mathcal{T}}} \mathcal{F}_{f_s}. \tag{3.1}$$

In Definition 3.1, $\mathcal{F}_{f_s}$ is the σ-field defined in Definition A.2.

Theorem 3.2. *Let $f = \{f_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-follower process. Then the follower filtration $\mathbb{F}^f$ is a subfiltration of $\mathbb{F}$.*

Proof. It is obvious that $\mathbb{F}^f$ is a filtration. So all we need to show is that $\mathcal{F}^f_t \subset \mathcal{F}_t$ for any $t \in \mathcal{T}$. But for any $s \le t$, since $f_s \le f_t \le t$, we have $\mathcal{F}_{f_s} \subset \mathcal{F}_t$ by Theorem A.4. Therefore, $\mathcal{F}^f_t = \bigvee_{s\in[0,t]_{\mathcal{T}}} \mathcal{F}_{f_s} \subset \mathcal{F}_t$. □

The following theorem shows that our follower filtration is a natural extension of the continuously delayed filtration of Guo, Jarrow and Zeng [GJZ09].

Theorem 3.3. *Let $f = \{f_t\}_{t\in\mathcal{T}}$ be an $\mathbb{F}$-follower process where each f_t is an $\mathbb{F}$-stopping time. Then, $\mathcal{F}^f_t = \mathcal{F}_{f_t}$.*

Proof. Let $s, t \in \mathcal{T}$ with $s \le t$. Then $f_s \le f_t$.

First, we want to show $\mathcal{F}_{f_s} \subset \mathcal{F}_{f_t}$. Let $A \in \mathcal{F}_{f_s}$. Then, by Theorem A.5, for any $u \in \mathcal{T}$, we have $A \cap \{f_s \le u\} \in \mathcal{F}_u$.

On the other hand, since $f_s \le f_t$, we have

$$A \cap \{f_t \le u\} = (A \cap \{f_s \le u\}) \cap \{f_t \le u\}$$

The first term of the right hand side belongs to $\mathcal{F}_u$ by the assumption, while the second term is also in $\mathcal{F}_u$ since f_t is an $\mathbb{F}$-stopping time. So again by Theorem A.5, we get $A \in \mathcal{F}_{f_t}$.

Then, we have $\mathcal{F}_t^f = \bigvee_{s\in[0,t]_\mathcal{T}} \mathcal{F}_{f_s} = \mathcal{F}_{f_t}$. □

Since a constant time is considered as a stopping time, we have the following corollary.

COROLLARY 3.4. *Assume that an $\mathbb{F}$-follower process f is deterministic, i.e. there exists a deterministic function $g : \mathcal{T} \to \mathcal{T}$ such that for all $t \in \mathcal{T}$ and $\omega \in \Omega$, $f_t(\omega) = g(t)$. Then, we have for all $t \in \mathcal{T}$, $\mathcal{F}_t^f = \mathcal{F}_{g(t)}$.*

Next, we investigate the shape of follower filtrations when the underlying follower processes are idempotent.

LEMMA 3.5. *Let f be an idempotent $\mathbb{F}$-follower process which is càdlàg. Then for every pair of $s, t \in \mathcal{T}$ with $s < t$, f_s is $\mathcal{F}_{f_t}$-measurable.*

PROOF. Let $s \in \mathcal{T}$ and $B \in \mathcal{B}(\mathcal{T})$ be fixed. For any $n \in \mathbb{N}$, define processes p^n and $q^n : \mathcal{T} \times \Omega \to \mathbb{R}$ by

$$\begin{aligned} p^n &:= \mathbb{1}_{\{(u,\omega)\in\mathcal{T}\times\Omega | f_s(\omega)\in B,\ f_s(\omega)\le u < f_s(\omega)+\frac{1}{n}\}}, \\ q^n &:= \mathbb{1}_{\{(u,\omega)\in\mathcal{T}\times\Omega | f_s(\omega)\in B,\ u\ge s+\frac{1}{n}\}}. \end{aligned}$$

Then, since f_s is càdlàg, $\mathcal{F}_s$-adapted and $\mathcal{F}_{f_s}$-adapted by Proposition A.3, both p^n and q^n are $\mathbb{F}$-adapted and càdlàg. Therefore, $P^n, Q^n \in \mathcal{F}_{f_t}$ where

$$\begin{aligned} P^n &:= \left(p^n_{f_t}\right)^{-1}(1) = \{f_s \in B,\ f_s \le f_t < f_s + \frac{1}{n}\}, \\ Q^n &:= \left(q^n_{f_t}\right)^{-1}(1) = \{f_s \in B,\ f_t \ge s + \frac{1}{n}\}. \end{aligned}$$

Then, we have

$$\begin{aligned} P &:= \bigcap_{n\in\mathbb{N}} P^n = \{f_s \in B,\ f_s = f_t\} \in \mathcal{F}_{f_t}, \\ Q &:= \bigcup_{n\in\mathbb{N}} Q^n = \{f_s \in B,\ f_t > s\} \in \mathcal{F}_{f_t}. \end{aligned}$$

Therefore

$$P \cup Q = \{f_s \in B\} \cap \big(\{f_s = f_t\} \cup \{f_t > s\}\big) \in \mathcal{F}_{f_t}.$$

On the other hand, under the assumption $s \leq t$, Proposition 2.7 implies that the two sets $\{f_t \leq s\}$ and $\{f_s = f_t\}$ are identical by ignoring a null-measured difference. Hence

$$\{f_s \in B\} \cap \big(\{f_t \leq s\} \cup \{f_t > s\}\big) = \{f_s \in B\} \in \mathcal{F}_{f_t}.$$

Therefore, f_s is $\mathcal{F}_{f_t}$-measurable. □

Theorem 3.6. *Let f be an idempotent $\mathbb{F}$-follower process which is càdlàg. Then, for every $t \in \mathcal{T}$, we have $\mathcal{F}^f_t = \mathcal{F}_{f_t}$.*

Proof. Immediate by Lemma 3.5 and Theorem A.4. □

4. Follower Processes in a Binomial Model

When we apply the theory of follower processes to the credit risk theory, we need to calculate some conditional expectations given a follower filtration in order to valuate defaultable financial instruments. In doing so, it would be quite welcome if the follower filtration has a sort of strong Markov property such as

$$\mathbb{E}^{\mathbb{P}}[g(Y_s) \mid \mathcal{F}^f_t] = \mathbb{E}^{\mathbb{P}}[g(Y_s) \mid f_t, Y_{f_t}]. \tag{4.1}$$

However, it seems a difficult task to prove Equation 4.1 for an arbitrary time domain.

In this section we show this when f_t is idempotent in a binomial model, and leave the continuous time domain case to future work.

4.1. The setup

In this subsection, we define a binomial model.

We fix the time domain $\mathcal{T} := \{n\delta \mid n = 0, 1, 2, \ldots, N\}$, where δ is a given positive number. We denote its horizon $N\delta$ by T.

We define the set $\Omega := \{\mathfrak{H}, \mathfrak{T}\}^{\mathcal{T}_+}$ and $\omega(0) := \perp$ for $\omega \in \Omega$, where $\mathfrak{H}$, $\mathfrak{T}$ and $\perp$ are distinct constants. For $t \in \mathcal{T}$, we define a binary relation $\sim_t$ on

Ω by $\omega \sim_t \omega'$ iff $\omega(s) = \omega'(s)$ for all $s \in]0,t]_{\mathcal{T}}$. Then, define a σ-field $\mathcal{F}_t$ by $\sigma(\Omega/\sim_t)$. We also define $\mathcal{F} := \mathcal{F}_T$.

We sometimes see the set Ω as a topological space equipped with the discrete topology. In other words, any subset of Ω is an open set.

We define a probability measure $\mathbb{P}$ on Ω by $\mathbb{P}(A) := \sum_{\omega\in A} p^{\#\omega}(1-p)^{N-\#\omega}$. for $A \in \mathcal{F}$, where $p \in]0,1[$ is a given number and $\#\omega$ is the cardinality of $\omega^{-1}(\mathfrak{H})$.

Throughout this section, all discussions are under the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}, \mathbb{P})$. We also fix a state space $(E, \mathcal{E})$ satisfying $\{x\} \in \mathcal{E}$ for all $x \in E$. Note that both $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and $(\mathcal{T}, 2^{\mathcal{T}})$ satisfy this condition.

Now it is easy to show that a function $X : \Omega \to E$ is $\mathcal{F}_t$-measurable iff $\omega \sim_t \omega'$ implies $X(\omega) = X(\omega')$ for any $\omega, \omega' \in \Omega$. Consequently, we can conclude that a process $Z : \mathcal{T} \times \Omega \to E$ is $\mathbb{F}$-adapted iff $\omega \sim_t \omega'$ implies $Z(t,\omega) = Z(t,\omega')$ for all $t \in \mathcal{T}$ and $\omega, \omega' \in \Omega$.

Definition 4.1 [The Universal Process].

(1) $\Omega^* := \cup_{t\in\mathcal{T}}\{\mathfrak{H}, \mathfrak{T}\}^{]0,t]_{\mathcal{T}}}$, where $\{\mathfrak{H}, \mathfrak{T}\}^{\emptyset} := \{\perp\}$.

(2) For $\omega \in \Omega$ and $t \in \mathcal{T}$, a function $\omega|t \in \{\mathfrak{H}, \mathfrak{T}\}^{]0,t]_{\mathcal{T}}}$ is defined by $\omega|t := \omega|_{]0,t]_{\mathcal{T}}}$ whose domain is expanded to $[0,t]_{\mathcal{T}}$ by defining $(\omega|t)(0) := \perp$,

(3) The *universal process* is a process $\pi : \mathcal{T}\times\Omega \to \Omega^*$ defined by $\pi(t,\omega) := \omega|t$.

The following theorem says that the universal process has a so-called universal property.

Theorem 4.2. *Let $Z : \mathcal{T} \times \Omega \to E$ be any $\mathbb{F}$-adapted process.*

(1) *There exists a unique function $g : \Omega^* \to E$ such that $Z = g \circ \pi$,*

(2) *For any $t \in \mathcal{T}$, $\sigma(Z_t) \subset \sigma(\pi_t)$.*

Proof. Left to readers. □

4.2. Follower filtrations in a binomial model

In the rest of this section, we assume that $f : \mathcal{T} \times \Omega \to \mathcal{T}$ is an arbitrary but fixed idempotent $\mathbb{F}$-follower process.

Proposition 4.3. *$\mathcal{O}^{\mathbb{F}} = \sigma\{\{t\} \times [\omega]_{\sim_t} \mid t \in \mathcal{T}, \omega \in \Omega\}$, where $\mathcal{O}^{\mathbb{F}}$ is the optional σ-field defined in Definition A.1.*

Proof. Since Ω is equipped with the discrete topology, any function whose domain is Ω is continuous. Therefore,

$$\begin{aligned}\mathcal{O}^{\mathbb{F}} :=& \sigma\{Z \mid Z \text{ is an } \mathbb{F}\text{-adapted càdlàg process. }\}\\ =& \sigma\{Z \mid Z \text{ is an } \mathbb{F}\text{-adapted process. }\}.\end{aligned}$$

Then by Theorem 4.2 (2), we have $\mathcal{O}^{\mathbb{F}} = \sigma(\pi)$ since π itself is $\mathbb{F}$-adapted.

Now remind that any element of Ω^* can be represented as $\omega|t$ for $\omega \in \Omega$ and $t \in \mathcal{T}_+$. Then, we have the desired equation since $\pi^{-1}(\omega|t) = \{t\} \times [\omega]_{\sim_t}$ for any $\omega \in \Omega$ and $t \in \mathcal{T}_+$. □

Corollary 4.4. *A process $Z : \mathcal{T} \times \Omega \to E$ is $\mathbb{F}$-optional iff it is $\mathbb{F}$-adapted.*

Proposition 4.5. *$\mathcal{F}^f_t = \sigma(\pi_{f_t})$.*

Proof. By Theorem 3.6, Corollary 4.4 and Theorem 4.2 (2). □

Now we investigate the shape of the set $\pi^{-1}_{f_t}(x)$ for $x \in \Omega^*$ in order to characterize $\mathcal{F}^f_t$.

Definition 4.6. For a random time τ, a *neighborhood* of $\omega \in \Omega$ at τ is the set $N_\tau(\omega) := [\omega]_{\sim_{\tau(\omega)}}$.

Lemma 4.7. *For $\omega, \omega_0 \in \Omega$, $\omega \in N_{f_t}(\omega_0)$ implies $f_t(\omega) \geq f_t(\omega_0)$.*

Proof. Since f is $\mathbb{F}$-adapted and $\omega \sim_t \omega_0$,

$$f_{f_t(\omega_0)}(\omega) = f_{f_t(\omega_0)}(\omega_0) = f_t(\omega_0).$$

The right most equality holds because f is idempotent. On the other hand, we have $f_t(\omega_0) \leq t$. Therefore, $f_{f_t(\omega_0)}(\omega) \leq f_t(\omega)$. □

Definition 4.8. Let τ be a random time, and $\omega_0 \in \Omega$.

(1) $K_\tau(\omega_0) := \{\omega \in N_\tau(\omega_0) \mid \tau(\omega) > \tau(\omega_0)\}$,

(2) $\underline{\mathrm{K}}_\tau(\omega_0) := \{\omega \in K_\tau(\omega_0) \mid (\forall \omega' \in K_\tau(\omega_0))(N_\tau(\omega) \subset N_\tau(\omega') \to N_\tau(\omega) = N_\tau(\omega'))\}$.

PROPOSITION 4.9. *Let $t \in \mathcal{T}$, $\omega_0 \in \Omega$ and $x_0 := \pi_{f_t}(\omega_0)$. Then,*

$$\pi_{f_t}^{-1}(x_0) = N_{f_t}(\omega_0) - \cup\big\{N_{f_t}(\omega) \mid \omega \in \underline{K}_{f_t}(\omega_0)\big\}. \tag{4.2}$$

PROOF. Let $\omega \in \pi_{f_t}^{-1}(x_0)$. Then, $\pi_{f_t}(\omega) = \pi_{f_t}(\omega_0)$. Thus, $\omega|f_t(\omega) = \omega_0|f_t(\omega_0)$. Therefore, $f_t(\omega) = f_t(\omega_0)$ and $\omega \sim_{f_t(\omega_0)} \omega_0$, which implies $\omega \in N_{f_t}(\omega_0)$.

Now, we show that $\omega' \in \underline{\mathrm{K}}_{f_t}(\omega_0)$ implies $\omega \notin N_{f_t}(\omega')$. Since $\omega' \in K_{f_t}(\omega_0)$, we have $\omega' \in N_{f_t}(\omega_0)$ and $f_t(\omega') > f_t(\omega_0)$. Suppose $\omega \in N_{f_t}(\omega')$. Then by Lemma 4.7, $f_t(\omega) \geq f_t(\omega') > f_t(\omega_0)$, which contradicts to $f_t(\omega) = f_t(\omega_0)$. Therefore, we conclude $\omega \notin N_{f_t}(\omega')$ and $LHS \subset RHS$.

Next, we show the opposite inclusion. Let $\omega \in N_{f_t}(\omega_0) - \cup\big\{N_{f_t}(\omega) \mid \omega \in \underline{\mathrm{K}}_{f_t}(\omega_0)\big\}$. We want to show $\omega \in \pi_{f_t}^{-1}(x_0)$.

Since $\omega \in N_{f_t}(\omega_0)$, we have $f_t(\omega) \geq f_t(\omega_0)$ by Lemma 4.7. Suppose $f_t(\omega) > f_t(\omega_0)$. Then, $\omega \in K_{f_t}(\omega_0)$. We can pick $\omega' \in \underline{\mathrm{K}}_{f_t}(\omega_0)$ such that $N_{f_t}(\omega') \supset N_{f_t}(\omega)$. Therefore, $\omega \in N_{f_t}(\omega) \subset N_{f_t}(\omega')$. But this contradicts to the way of the selection of ω. Hence, we have $f_t(\omega) = f_t(\omega_0)$.

On the other hand, we have $\omega|f_t(\omega_0) = \omega_0|f_t(\omega_0)$ since $\omega \in N_{f_t}(\omega_0)$. Therefore,

$$\pi_{f_t}(\omega) = \omega|f_t(\omega) = \omega|f_t(\omega_0) = \omega_0|f_t(\omega_0) = \pi_{f_t}(\omega_0) = x_0. \ \square$$

COROLLARY 4.10. $\mathcal{F}_t^f = \sigma\{N_{f_t}(\omega) \mid \omega \in \Omega\}$.

4.3. Conditional expectations given a follower filtration

We keep assuming that f is an idempotent $\mathbb{F}$-follower process throughout this subsection.

THEOREM 4.11. *Let Y be a random variable and X be an $\mathcal{F}_{f_t}$-measurable random variable. Then, $\mathbb{E}^{\mathbb{P}}[Y \mid \mathcal{F}_t^f] = X$ iff $\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} Y] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} X]$ for all $\omega_0 \in \Omega$.*

Proof. The only-if part is trivial. So we assume the right hand side. Let

$$\mathcal{G} := \{A \in \mathcal{F}_t^f \mid \int_A Y d\mathbb{P} = \int_A X d\mathbb{P}\}.$$

Then, all we need to show is $\mathcal{G} = \mathcal{F}_t^f$.

By the assumption, for any $\omega_0 \in \Omega$, we have $N_{f_t}(\omega_0) \in \mathcal{G}$. Now seeing Equation 4.2 and noticing that the following relations are satisfied for any $\omega_1, \omega_2 \in \underline{\mathrm{K}}_{f_t}(\omega_0)$,

(1) $N_{f_t}(\omega_1) \subset N_{f_t}(\omega_0)$,

(2) $N_{f_t}(\omega_1) = N_{f_t}(\omega_2) \quad \text{or} \quad N_{f_t}(\omega_1) \cap N_{f_t}(\omega_2) = \emptyset$,

we have the following equation where all unions are disjoint-sum:

$$N_{f_t}(\omega_0) = \pi_{f_t}^{-1}(\pi_{f_t}(\omega_0)) \cup \big(\bigcup \{N_{f_t}(\omega) \mid \omega \in \underline{\mathrm{K}}_{f_t}(\omega_0)\}\big).$$

Therefore, by the assumption, we have

$$\mathcal{H} := \{\pi_{f_t}^{-1}(\pi_{f_t}(\omega_0)) \mid \omega_0 \in \Omega\} \subset \mathcal{G}.$$

Again, the elements of $\mathcal{H}$ are disjoint each other, and obviously $\cup\mathcal{H} = \Omega$. So, any element of $\mathcal{F}_t^f = \sigma(\pi_{f_t})$ can be represented as a disjoint sum of the elements of $\mathcal{H}$, which concludes $\mathcal{F}_t^f \subset \mathcal{G}$. □

Now we define a process Y appeared in Equation 4.1. As a proxy of Brownian motion, we define a process M by $M_t(\omega) := \sum_{s \in]0,t]_{\mathcal{T}}} X_s(\omega)$ for $t \in \mathcal{T}$, where $\{X_t\}_{t \in \mathcal{T}_+}$ is a Bernoulli process defined by

$$X_t(\omega) = \begin{cases} \sqrt{\delta} & \text{if } \omega(t) = \mathfrak{H} \\ -\sqrt{\delta} & \text{if } \omega(t) = \mathfrak{T}. \end{cases} \tag{4.3}$$

Then, we define the process Y by

$$Y_t(\omega) := y_0 + \nu t + \sigma M_t(\omega) \tag{4.4}$$

where y_0, ν and $\sigma \geq 0$ are constants.

PROPOSITION 4.12. *Let $g : \mathbb{R} \to \mathbb{R}$ be a given function. Then, for any $s \geq t$,*

$$\mathbb{E}^{\mathbb{P}}[g(Y_s) \mid \mathcal{F}_t^f] = h(s - f_t, Y_{f_t}) \tag{4.5}$$

where the function $h : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$\begin{aligned} h(0, y) &:= g(y), \\ h(t+, y) &:= ph(t, y + \nu\delta + \sigma\sqrt{\delta}) + (1 - p)h(t, y + \nu\delta - \sigma\sqrt{\delta}). \end{aligned}$$

PROOF. By Theorem 4.11, since $h(s - f_t, Y_{f_t})$ is $\mathcal{F}_t^f$-measurable, all we need to show is for all $\omega_0 \in \Omega$,

$$\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} g(Y_s)] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} h(s - f_t, Y_{f_t})].$$

Thinking about the shape of the set $N_{f_t}(\omega_0)$, we can prove it by showing for all $C \in \mathbb{R}$,

$$\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} g(Y_{f_t+u} + C)] = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} h(u, Y_{f_t} + C)] \tag{4.6}$$

by induction on $u \in [0, s - f_t]_{\mathcal{T}}$.

When $u = 0$, it is trivial. Assume Equation 4.6 holds at $u \in [0, s - f_t[_{\mathcal{T}}$. Then, we have

$$\begin{aligned} &\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} g(Y_{f_t+u+} + C)] \\ =&\mathbb{E}^{\mathbb{P}}[\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} g(Y_{f_t+u+} + C) \mid \mathcal{F}_{f_t+u}]] \\ =&\mathbb{E}^{\mathbb{P}}[p\mathbb{1}_{N_{f_t}(\omega_0)} g(Y_{f_t+u} + \nu\delta + \sigma\sqrt{\delta} + C) \\ &+ (1 - p)\mathbb{1}_{N_{f_t}(\omega_0)} g(Y_{f_t+u} + \nu\delta - \sigma\sqrt{\delta} + C)] \\ =&\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} \big(ph(u, (Y_{f_t} + C) + \nu\delta + \sigma\sqrt{\delta}) \\ &+ (1 - p)h(u, (Y_{f_t} + C) + \nu\delta - \sigma\sqrt{\delta})\big)] \\ =&\mathbb{E}^{\mathbb{P}}[\mathbb{1}_{N_{f_t}(\omega_0)} h(u+, Y_{f_t} + C)]. \end{aligned}$$

Therefore, Equation 4.6 holds at $u+$ as well, which completes the proof. □

COROLLARY 4.13. *Let g be a given function. Then, for any $s \geq t$,*

$$\mathbb{E}^{\mathbb{P}}[g(Y_s) \mid \mathcal{F}_t^f] = \mathbb{E}^{\mathbb{P}}[g(Y_s) \mid f_t, Y_{f_t}]. \tag{4.7}$$

Note 4.14. Practically, Corollary 4.13 is enough to price defaultable securities under a follower-process based model since we can make it as accurate as possible by making δ smaller.

Suppose $p = \frac{1}{2}$. Then, as $\delta \to 0$, the process M converges to a standard Brownian motion in distribution by the Central Limit Theorem, and the process Y will satisfy the equation $Y_t = y_0 + \nu t + \sigma B_t$. The function h defined in Proposition 4.12 will be specified with an appropriate partial differential equation, and Equation 4.7 may hold at this continuous case.

Appendix A.

This appendix consists of the known results that are necessary for the discussions in the main text.

A process $X = \{X_t\}_{t\in\mathcal{T}}$ is called $\mathbb{F}$-*progressive* if for every $t \in \mathcal{T}$, $X|_{[0,t]_{\mathcal{T}}\times\Omega}$ is $\mathcal{B}[0,t]\otimes\mathcal{F}_t$-measurable. A random set is called $\mathbb{F}$-*progressive* if its indicator function is $\mathbb{F}$-progressive.

Every right continuous $\mathbb{F}$-adapted process is $\mathbb{F}$-progressive (See [RW00] Lemma VI.3.3).

Definition A.1 [Optional Processes]. The *optional σ-field* with respect to $\mathbb{F}$ is the σ-field $\mathcal{O}^{\mathbb{F}}$ defined on $\mathcal{T}\times\Omega$ such that

$$\mathcal{O}^{\mathbb{F}} := \sigma\{X \mid X = \{X_t\}_{t\in\mathcal{T}} \text{ is an } \mathbb{F}\text{-adapted càdlàg process.}\}. \tag{A.1}$$

An element of $\mathcal{O}^{\mathbb{F}}$ is called an $\mathbb{F}$-*optional* set. A process $X = \{X_t\}_{t\in\mathcal{T}}$ is called $\mathbb{F}$-*optional* if the map $(t,\omega)\mapsto X_t(\omega)$ is $\mathcal{O}^{\mathbb{F}}$-measurable.

Every $\mathbb{F}$-optional process is an $\mathbb{F}$-progressive process, and every $\mathbb{F}$-optional set is an $\mathbb{F}$-progressive set.

The following is one of the standard σ-fields generated by arbitrary random times. See Definition XX.25 in [DMM92].

Definition A.2. Let τ be a random time. The σ-field $\mathcal{F}_\tau$ is defined by

$$\mathcal{F}_\tau := \sigma\{Z_\tau \mid Z = \{Z_t\}_{t\in\mathcal{T}} \text{ is an } \mathbb{F}\text{-optional process.}\}.$$

The σ-field $\mathcal{F}_\tau$ consists of events which depend on what happens up to and including time τ.

Proposition A.3. *Every random time τ is $\mathcal{F}_\tau$-measurable.*

Proof. Let Z be a process defined by $Z(t,\omega) = t$ for all $t \in \mathcal{T}$ and $\omega \in \Omega$. Then Z is obviously optional and $Z_\tau = \tau$. □

Theorem A.4 [[DMM92] Théorème XX.27]. *Let τ_1 and τ_2 be two random times such that $\tau_1 \leq \tau_2$. If τ_1 is $\mathcal{F}_{\tau_2}$-measurable, we have $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$.*

Theorem A.5 [[RW00] Lemma VI.17.5]. *If τ is an $\mathbb{F}$-stopping time, then*

$$\mathcal{F}_\tau = \big\{A \in \mathcal{F}_\infty \mid (\forall u \in \mathcal{T}) A \cap \{\tau \leq u\} \in \mathcal{F}_u\big\}. \tag{A.2}$$

Especially, if there exists a constant $t \in \mathcal{T}$ such that for any $\omega \in \Omega$, $\tau(\omega) = t$, then $\mathcal{F}_\tau = \mathcal{F}_t$.

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