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# *On Fractional Whittaker Equation and Operational Calculus*

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**Abstract.** This paper is intended to investigate a fractional differential Whittaker's equation of order $2\alpha$, with $\alpha \in ]0,1]$, involving the Riemann-Liouville derivative. We seek a possible solution in terms of power series by using operational approach for the Laplace and Mellin transform. A recurrence relation for coefficients is obtained. The existence and uniqueness of solutions is discussed via Banach fixed point theorem.

## 1. Introduction

The Whittaker functions arise as solutions of the Whittaker differential equation (see [4], Vol.1)). These functions have acquired a significant increasing due to its frequent use in applications of mathematics to physical and technical problems [1]. Moreover, they are closely related to the confluent hypergeometric functions, which play an important role in various branches of applied mathematics and theoretical physics, for instance, fluid mechanics, electromagnetic diffraction theory, and atomic structure theory. This justifies the continuous effort in studying properties of these functions and in gathering information about them. As far as the authors aware, there were no attempts to study the corresponding fractional Whittaker equation (see below).

Fractional differential equations are widely used for modeling anomalous relaxation and diffusion phenomena (see [3], Ch. 5, [6], Ch. 2). A systematic development of the analytic theory of fractional differential equations with variable coefficients can be found, for instance, in the books of Samko, Kilbas and Marichev (see [13], Ch. 3).

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Among analytical methods, which are widely used to solve precisely fractional equations, there are methods involving integral transforms, for example, the Laplace, Fourier, Mellin transforms (see [2, 7, 8]). Solutions of fractional equations are usually given in the form of special functions (see [6], Ch. 2, [10], Ch. 3, [5]), such as the Mittag-Leffler function, the Wright function, the Fox H-function and the Meijer G-function. However, since some useful properties of the standard calculus cannot carry over analogously to the case of fractional calculus, the analytical solutions of fractional differential equations under certain initial and boundary conditions are difficult to obtain.

In the recent paper [12] we have an important application of fractional calculus techniques. In this paper the authors investigated the fractional differential Bessel's equation, involving Riemann-Liouville derivative, and obtained a possible solution in terms of power series by using operational calculus for the Laplace and Mellin transform.

The aim of the paper is to study the following fractional Whittaker equation

$$-4x^{2\alpha}(D_{0^+}^{2\alpha}w)(x) + (x^{2\alpha} - 4\mu x^{\alpha} - 4\tau^2 - 1)w(x) = 0,$$

where $x > 0$, $\alpha \in ]0, 1[$ and $\lambda \in \mathbb{C}$. In particular, the case $\alpha = 1$ leads us to the classical Whittaker's equation (see [4], Vol.1), whose solutions are, correspondingly, the Whittaker functions. The paper is structured as follows: in the Preliminaries we recall basic properties of the Mellin transform and necessary elements of fractional calculus. In Section 3, we study the existence and uniqueness of solutions appealing to the Banach fix point theorem. Finally in Section 4, we obtain a recurrence relation for the coefficients of the series solution associated to the considered fractional differential equation, and a Mellin transform method for solving fractional Whittaker equation will be presented.

## 2. Preliminaries

### 2.1. The Mellin transform of fractional derivatives

The resolution of differential equations with polynomial coefficients becomes more efficient considering the Mellin transform. Usually, this integral transform is defined by ([14], Sec. 2.7, [9], Ch. 10)

$$F(s) = M\{f(x); s\} = \int_0^\infty f(x) x^{s-1} dx, \tag{1}$$

where $s$ is complex, such as $\gamma_1 < Re(s) < \gamma_2$. The Mellin transform exists if $f(x)$ is piecewise continuous in every closed interval $[a, b] \subset ]0, +\infty[$ and

$$\int_0^1 |f(x)| x^{\gamma_1 - 1} dx < \infty, \qquad \int_1^\infty |f(x)| x^{\gamma_2 - 1} dx < \infty.$$

If the function satisfies Dirichlet's condition in every closed interval $[a, b] \subset ]0, +\infty[$, then it can be restored using the inverse Mellin transform formula

$$f(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} M\{f(x); s\} x^{-s}\, ds, \qquad 0 < x < \infty, \tag{2}$$

where $\gamma_1 < \gamma < \gamma_2$. Moreover, this integral transform has the following properties (see [9], Ch. 10, [11], Ch. 8)

$$\begin{aligned} &M\{x^\beta f(x); s\} = M\{f(x); s + \beta\} \equiv F(s + \beta), \\ &\gamma_1 - Re(\beta) < Re(s) < \gamma_2 - Re(\beta); \end{aligned} \tag{3}$$

$$\begin{aligned} &M\{f(\beta x); s\} = \beta^{-s} M\{f(x); s\} \equiv \beta^{-s} F(s), \\ &\gamma_1 < Re(s) < \gamma_2, \ \beta > 0; \end{aligned} \tag{4}$$

$$\begin{aligned} &M\{f(x^\beta); s\} = \frac{1}{|\beta|} M\left\{f(x); \frac{s}{\beta}\right\} \equiv \frac{1}{|\beta|}\, F\left(\frac{s}{\beta}\right), \\ &\left\{ \begin{array}{ll} \beta\gamma_1 < Re(s) < \beta\gamma_2, & \Leftarrow \beta > 0 \\ \beta\gamma_2 < Re(s) < \beta\gamma_1, & \Leftarrow \beta < 0 \end{array} \right. . \end{aligned} \tag{5}$$

Further, if additionally $f \in C^n(\mathbb{R}^+), n \in \mathbb{N}$, then

$$M\{f^{(n)}(x); s\} = \frac{\Gamma(1 + n - s)}{\Gamma(1 - s)}\, F(s - n). \tag{6}$$

We will appeal to the Mellin transform to examine fractional Whittaker's equation. Namely, we will consider the following differential properties of this transform (see [11], Ch. 8, [13], Ch. 3)

$$M\{x^{2\beta} (D_{0+}^{2\beta} y)(x); s\} = \frac{\Gamma(1 - s)}{\Gamma(1 - s - 2\beta)}\, Y(s), \tag{7}$$

$$M\{x^\beta(D^\beta_{0+}y)(x);s\} = \frac{\Gamma(1-s)}{\Gamma(1-s-\beta)}\ Y(s), \tag{8}$$

$$M\{x^{2\beta}y(x);s\} = \ Y(s+2\beta). \tag{9}$$

We denote by $L^p(a,b)$, $p \in [1,+\infty[$, the space of measurable functions $f$ on $(a,b)$ satisfying

$$\int_a^b |f(x)|^p\ dx < \infty,$$

and we write $L$ instead of $L^1$ (see [14], Sec. 1.7).

Theorem 2.1 (see [14], Sec.2.7). *Let $x^{\beta-1}f(x)$ and $x^{\beta-1}g(x)$ belong to $L(0,+\infty)$, and let*

$$h(x) = (f *_M g)(x) = \int_0^\infty f(y)\ g\left(\frac{x}{y}\right)\ \frac{dy}{y}.$$

*Then $x^{\beta-1}h(x)$ belongs to $L(0,+\infty)$, and its Mellin transform is $F(s)G(s)$, where $F(s) = M\{f(x);s\}$ and $G(s) = M\{g(x);s\}$.*

### 2.2. Fractional Calculus

In this section we recall some results about fractional calculus which will be used below. Let us take $0 \leq n-1 < \beta < n$. According to the definition of Riemann-Liouville fractional derivative (see [6], Ch. 2), we can write

$$(D^\beta_{0+}f)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\beta)} \int_0^x \frac{f(t)}{(x-t)^{\beta-n+1}}\ dt, \qquad n = [\beta]+1, \tag{10}$$

where $[\beta]$ is the integer part of $\beta$. Note that the Riemann-Liouville derivative is defined for some functions with a singularity at the origin. For example, if $f(x) = x^d, d > -1$, then

$$(D^\beta_{0+}f)(x) = \frac{\Gamma(d+1)}{\Gamma(d+1-\beta)}\ x^{d-\beta}, \tag{11}$$

so that $D^\beta_{0+}f = 0$ if $f(x) = x^{\beta-1}$. The Caputo derivative of fractional order $\beta$ of a function $f(x)$ is defined, in turn, as (see [6], Ch. 2)

$${}^C D^\beta_{0+}f(x) = \frac{1}{\Gamma(n-\beta)} \int_0^x (x-\tau)^{n-\beta-1} f^{(n)}(\tau)\ d\tau, \tag{12}$$

where $n-1<\beta<n$ and $n\in\mathbb{N}$. The Mellin transform (1) of the Riemann-Liouville derivative (10) is equal to the following formula (see ([6], Ch. 2))

$$M\{(D_{0^+}^{\beta}f)(x);s\} = \sum_{k=0}^{n-1}\frac{\Gamma(1+k-s)}{\Gamma(1-s)}\,(D_{0^+}^{\beta-n}f)(x)x^{s-k-1}\Big|_0^{+\infty} + \frac{\Gamma(1-s+\beta)}{\Gamma(1-s)}F(s-\beta).$$

If $f$ is such that all our integrands terms are vanished, then it takes more simple form

$$M\{(D_{0^+}^{\beta}f)(x);s\} = \frac{\Gamma(1-s+\beta)}{\Gamma(1-s)}\,F(s-\beta). \tag{13}$$

In what follows $n=[\beta]+1$ for $\beta\notin\mathbb{N}_0$ and $n=\beta$ for $\beta\in\mathbb{N}_0$. Moreover, for $n\in\mathbb{N}$ we will denote by $AC^n([a,b])$ the space of complex-valued functions $f(x)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(x)\in AC([a,b])$, where $AC([a,b])$ is the space of all functions absolutely continuous on $[a,b]$ (see [6], Sec. 1.1).

Theorem 2.2 (see [6], Ch. 2). *Let $\beta\geq 0$ and $v(x)\in AC^n([a,b])$. Then $D_{a^+}^{\beta}v$ exists almost everywhere and may be represented in the form*

$$D_{a^+}^{\beta}v = \sum_{k=0}^{n-1}\frac{v^{(k)}(a)}{\Gamma(1+k-\beta)}(x-a)^{k-\beta} + \frac{1}{\Gamma(n-\beta)}\int_a^x \frac{v^{(n)}(t)}{(x-t)^{\beta-n+1}}\,dt. \tag{14}$$

We remark that fractional derivatives (10) verify the following relation

$${}^{C}(D_{a^+}^{\beta}v)(x) = (D_{a^+}^{\beta}v)(x) - \sum_{k=0}^{n-1}\frac{v^{(k)}(a)}{\Gamma(k-\beta+1)}(x-a)^{k-\beta}. \tag{15}$$

Theorem 2.3 (see [6], Ch. 2). *Let $\beta\geq 0$. If $v(x)\in AC^n([a,b])$, then the Caputo fractional derivative ${}^{C}D_{a^+}^{\beta}v$ exists almost everywhere on $[a,b]$, and if $\beta\notin\mathbb{N}_0$, ${}^{C}D_{a^+}^{\beta}v$ is represented by*

$${}^{C}D_{a^+}^{\beta}v(x) = \frac{1}{\Gamma(n-\beta)}\int_a^x \frac{v^{(n)}(t)}{(x-t)^{\beta-n+1}}\,dt := (I_{a^+}^{n-\beta}D^n v)(x), \tag{16}$$

*where* $D = \frac{d}{dx}$ *is the ordinary derivative.*

If $\beta \notin \mathbb{N}_0$ and $n = [\beta] + 1$, then (see [6], Ch. 2)

$$(17) \qquad \left|(I_{a^+}^{n-\beta} D^n v)(x)\right| \leq \frac{\|v^{(n)}\|_C}{\Gamma(n-\beta)\ (n-\beta+1)} (x-a)^{n-\beta},$$

where the Riemann-Liouville fractional integral $I_{a^+}^{n-\beta}$ is defined by (see [6], Sec. 2.1)

$$(I_{a^+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \qquad x > a, \quad \mathrm{Re}(\alpha) > 0.$$

LEMMA 2.4 (see [6], Ch. 2). *Let* $\beta > 0$*. If* $v(x) \in AC^n([a,b])$ *or* $v(x) \in C^n([a,b])$*, then*

$$\left(I_{a^+}^{\alpha}\ {}^C D_{a^+}^{\beta} v\right)(x) = v(x) - \sum_{k=0}^{n-1} \frac{v^{(k)}(a)}{k!} (x-a)^k.$$

LEMMA 2.5 (see [13], Ch. 3). *If the series* $f(x) = \sum_{n=0}^{\infty} f_n(x)$*,* $f_n(x) \in C([a,b])$*, is uniformly convergent on* $[a,b]$*, then its termwise fractional integration is admissible:*

$$(18) \qquad \left(I_{a^+}^{\beta} \sum_{n=0}^{\infty} f_n\right)(x) = \sum_{n=0}^{\infty} \left(I_{a^+}^{\beta} f_n\right)(x), \qquad \beta > 0, \quad a < x < b,$$

*the series on the right-hand side being also uniformly convergent on* $[a,b]$*.*

LEMMA 2.6 (see [13], Ch. 3). *If the fractional derivatives* $D_{a^+}^{\beta} f_n$ *exist for all* $n = 0, 1, 2, \ldots$ *and let the series* $\sum_{n=0}^{\infty} f_n$ *and* $\sum_{n=0}^{\infty} D_{a^+}^{\beta} f_n$ *uniformly convergent on every sub-interval* $[a+\epsilon, b]$*,* $\epsilon > 0$*. Then, the former series admits termwise fractional differentiation using the formula*

$$(19) \qquad \left(D_{a^+}^{\beta} \sum_{n=0}^{\infty} f_n\right)(x) = \left(\sum_{n=0}^{\infty} D_{a^+}^{\beta} f_n\right)(x), \qquad \beta > 0, \quad a < x < b,$$

## 3. Existence and Uniqueness of Solutions

Here we will use Banach's fixed point theorem to study the existence and uniqueness of solution for the fractional Whittaker equation

$$-4x^{2\alpha}(D_{0+}^{2\alpha}w)(x) + (x^{2\alpha} - 4\mu x^{\alpha} - 4\tau^2 - 1)w(x) = 0, \qquad \alpha \in ]0,1[, \tag{20}$$

with $x \in [0, X_0]$, $X_0 > 1$, and $\mu, \tau \in \mathbb{C}$ under initial conditions

$$w(0) = w_0, \qquad\qquad w'(0) = w_0^*. \tag{21}$$

Let $I = [a, b]$ $(a < b,\ a, b \in \mathbb{R})$ and $m \in \mathbb{N}_0$. Denote by $C^m$ the usual space of functions $v$ which are $m$ times continuously differentiable on $I$ with the norm

$$\|v\|_{C^m} = \sum_{k=0}^{m} \|v^{(k)}\|_C = \sum_{k=0}^{m} \max_{x\in I} |v^{(k)}(x)|.$$

In particular, for $m = 0$, $C^0(I) \equiv C(I)$ is the space of continuous functions $v$ on $I$ with the norm $\|v\|_C = \max_{x\in I} |v(x)|$.

THEOREM 3.1. *The fractional problem (20-21), has a unique solution for every $\mu \in \mathbb{C}$ and $\alpha \in ]0,1[$, if*

$$|\tau|^2 > \frac{1}{4}\left[X_0\left(X_0 + 4|\mu| + \frac{X_0}{\Gamma(2(1-\alpha))\ (3-2\alpha)}\right) - 1\right]. \tag{22}$$

PROOF. Consider the following Banach spaces

$$X = \{w : w(x) \in C^2([0, X_0])\}, \qquad\qquad Y = \{w : w(x) \in C([0, X_0])\}.$$

Putting $T : X \to Y$,

$$(Tw)(x) = \frac{1}{1+4\tau^2}\left[(x^{2\alpha} - 4\mu x^{\alpha})w(x) - 4x^{2\alpha}\left(D_{0+}^{2\alpha}w\right)(x)\right],$$

we rewrite equation (20) in the form $w(x) = (Tw)(x)$. Taking into account Theorems 2.2, 2.3, Lemma 2.4 and relations (15), (17), we have

$$
\begin{aligned}
&\|Tw_1 - Tw_2\|_Y \\
&= \frac{1}{|\lambda|^2} \left\| (x^{2\alpha} - 4\mu x^{\alpha})(w_1(x) - w_2(x)) - 4x^{2\alpha}(D^{2\alpha}_{0+}(w_1 - w_2))(x) \right\|_Y \\
&\leq \frac{X_0^2 + 4\mu X_0}{1 + 4|\tau|^2} \|w_1 - w_2\|_Y \\
&\quad + \frac{4X_0^2}{(1+4|\tau|^2)\ \Gamma(2(1-\alpha))\ (3-2\alpha)} \|D^2(w_1 - w_2)\|_Y \\
&\leq \frac{X_0}{1 + 4|\tau|^2} \left( X_0 + 4\mu + \frac{4X_0}{\Gamma(2(1-\alpha))\ (3-2\alpha)} \right) \|w_1 - w_2\|_X.
\end{aligned}
$$

From (22) we conclude that $T$ is a contraction. Hence, we can apply the Banach fix point theorem to complete the proof. □

## 4. Fractional Whittaker Equation

The aim of this section is to obtain particular solutions for the following fractional Whittaker equation (20), i.e.,

$$-4x^{2\alpha}(D^{2\alpha}_{0+}w)(x) + (x^{2\alpha} - 4\mu x^{\alpha} - 4\tau^2 - 1)w(x) = 0, \qquad \alpha \in ]0, 1[,$$

where $\mu, \tau \in \mathbb{C}$, $x \in [x_0, X_0]$, $x_0, X_0 \in \mathbb{R}_+$ and $D^{\alpha}_{0+}$ is the operator of the Riemann fractional derivative (10).

### 4.1. Recurrence relation for the coefficients of the series solution

Here, we derive a recurrence relation for coefficients of the series solution associated to (20). We seek a solution of (20) in the form of a generalized power series in increasing powers of argument $x$

$$w(x) = \sum_{n=0}^{+\infty} a_n x^{\alpha n}, \tag{23}$$

such that the following condition holds

$$\sum_{n=0}^{\infty} |a_n| n^2 X_0^n < \infty. \tag{24}$$

In fact, by Stirling's asymptotic formula for Gamma function we have (see [13], Ch. 1)

$$\frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n-2)+1)} = O(n^{2\alpha}), \qquad n \to \infty.$$

Therefore, from relation (11), condition (24) and Lemma 2.4 we can guarantee the absolute and uniform convergence, on $[x_0, X_0]$, of the corresponding series in Lemma 2.6. Hence, we get

$$(D^{2\alpha}_{0+} w)(x) = \sum_{n=0}^{+\infty} a_n \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha n+1-2\alpha)}\, x^{\alpha n-2\alpha}. \tag{25}$$

Substituting expressions (23) and (25) in (20), and collecting the terms containing equal powers of $x$, we derive

$$\begin{aligned} & -4\sum_{n=0}^{+\infty} a_n \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha n+1-2\alpha)}\, x^{\alpha n} + \sum_{n=2}^{+\infty} a_{n-2}\, x^{\alpha n} - 4\mu \sum_{n=1}^{+\infty} a_{n-1}\, x^{\alpha n} \\ & \quad - (1+4\tau^2)\sum_{n=0}^{+\infty} a_n\, x^{\alpha n} = 0. \end{aligned} \tag{26}$$

Evidently all coefficients of $x^{\alpha n}$ should be equal to zero. Hence

$$\begin{cases} a_0\left(-\dfrac{4}{\Gamma(1-2\alpha)} - 1 - 4\tau^2\right) = 0 \\ a_1\left(-\dfrac{4\Gamma(1+\alpha)}{\Gamma(1-\alpha)} - 1 - 4\tau^2\right) - 4\mu a_0 = 0 \\ a_n - 4\mu a_{n+1} - \left(\dfrac{4\Gamma(\alpha n+1)}{\Gamma(\alpha(n-2)+1)} + 1 + 4\tau^2\right) a_{n+2} = 0, \\ \qquad n \in \mathbb{N}_0. \end{cases} \tag{27}$$

The analysis of the previous system will be split into two different cases.

*4.1.1 Case of $\mu = 0$*

It is immediate from the previous system that the case $a_0 = a_1 = 0$ drives to the trivial solution of the equation (20). Moreover, in order to obtain non-trivial solutions, we will additionally assume that

$$\tau^2 \neq -\frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n-2)+1)} - \frac{1}{4}, \quad \alpha \in ]0,1[, \quad n \in \mathbb{N}_0.$$

From the first and second equation of system (27) we obtain the following equation

$$\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} - \frac{1}{\Gamma(1-2\alpha)} = 0. \tag{28}$$

Numerical simulations indicate that equation (28) is impossible for every $\alpha \in ]0,1[$ and $n \in \mathbb{N}_0$. Hence, $a_0$ and $a_1$ cannot be simultaneously different from zero because it will lead to an impossible system. So, consider, for instance, $a_0 \neq 0$. Consequently,

$$\tau^2 = -\frac{1}{\Gamma(1-2\alpha)} - \frac{1}{4}. \tag{29}$$

Hence the third equation of (27) becomes

$$a_n - 4\left(\frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n-2)+1)} - \frac{1}{\Gamma(1-2\alpha)}\right) a_{n+2} = 0, \qquad n \in \mathbb{N}_0.$$

Furthermore, numerical simulations indicate

$$\frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n-2)+1)} - \frac{1}{\Gamma(1-2\alpha)} \neq 0$$

for all $\alpha \in ]0,1[$ and $n \in \mathbb{N}_0$. Therefore one can express coefficients $a_n$ with even indices by the relation

$$a_{2n} = \frac{a_0}{4}\prod_{k=1}^{n}\left(\frac{\Gamma(2(k-1)+1)}{\Gamma(2\alpha(k-2)+1)} - \frac{1}{\Gamma(1-2\alpha)}\right)^{-1}, \qquad n \in \mathbb{N}_0. \tag{30}$$

Since in this case $a_1 = 0$, we have, from the third equation of (27), that all odd coefficients are zero.

The above discussion can be summarized in the following result:

THEOREM 4.1. *Let $\mu = 0$, $a_0 \neq 0$, $\alpha \in ]0,1[$, $x \in [x_0, X_0]$, and*

$$\tau^2 = -\frac{1}{\Gamma(1-2\alpha)} - \frac{1}{4}.$$

*Then fractional Whittaker equation (20) admits a particular solution in terms of series as solution the power series (23), with even coefficients satisfying condition (25) and given by formula (30).*

In the odd case, we presume $a_1 \neq 0$ and the determining equation becomes

$$\tau^2 = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} - \frac{1}{4}. \tag{31}$$

As above, numerical simulations indicate that

$$\frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n-2)+1)} - \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} \neq 0,$$

for every $\alpha \in ]0,1[$ and $n \in \mathbb{N}_0$. Hence in the same manner, we express the odd coefficients by the relation

$$a_{2n+1} = \frac{a_1}{4} \prod_{k=1}^{n} \left( \frac{\Gamma(\alpha(2k-1)+1)}{\Gamma(\alpha(2k-3)+1)} - \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} \right)^{-1}, \quad n \in \mathbb{N}_0. \tag{32}$$

Since in this case $a_0 = 0$, we have, from the third equation of (27), that all even coefficients are zero. We summarize the above discussion in the next theorem:

THEOREM 4.2. *Let $\mu = 0$, $a_1 \neq 0$, $\alpha \in ]0,1[$, $x \in [x_0, X_0]$, and*

$$\tau^2 = -\frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} - \frac{1}{4}.$$

*Then fractional Whittaker equation (20) admits a particular solution in terms of series as solution the power series (23), with odd coefficients satisfying condition (25) and given by formula (32).*

*4.1.2 Case of $\mu \neq 0$*

The case $a_0 = a_1 = 0$ leads to the trivial solution of equation (20) (similar to what happened when $\mu = 0$). If we consider $a_0 = 0$ and $a_1 \neq 0$, we are in the odd case when $\mu = 0$ which was studied in the previous subsection. When $a_1 = 0$ and $a_0 \neq 0$ the second equation of system (27) become impossible.

Hence, it remains to study the case $a_0 \neq 0$ and $a_1 \neq 0$. For this situation, we obtain from the first and second equations of (27) the following equation

$$-\frac{1}{\Gamma(1-2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)} = -\frac{\mu a_0}{a_1}. \tag{33}$$

Numerical simulations indicate that the previous equation is only possible when $\mu$, $a_0$ and $a_1$ are such that $\frac{\mu a_0}{a_1} \in ]-M, 0[$, where $M$ is the maximum of

$$-\frac{1}{\Gamma(1-2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)},$$

with $\alpha \in ]0,1[$. For this case the coefficients of the series will be given by

$$a_{n+2} = \left( \frac{\Gamma(\alpha n+1)}{\Gamma(\alpha(n-2)+1)} - \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-1)} - \frac{\mu a_0}{a_1} \right)^{-1} \times \frac{a_n - 4\mu a_{n+1}}{4}, \quad n \in \mathbb{N}_0. \tag{34}$$

The above discussion can be summarized in the following theorem:

Theorem 4.3. *Let $x \in [x_0, X_0]$, $\mu \neq 0$, $a_0 \neq 0$, $a_1 \neq 0$ such that $\frac{\mu a_0}{a_1} \in ]-M, 0[$, where $M$ is the maximum of*

$$-\frac{1}{\Gamma(1-2\alpha)} + \frac{\Gamma(1+\alpha)}{\Gamma(1-\alpha)},$$

*with $\alpha \in ]0,1[$. Then fractional Whittaker equation (20) admits a particular solution in terms of series as solution the power series (23), with coefficients satisfying condition (25) and given by formula (34).*

### 4.2. The Mellin transform method for solving fractional Whittaker's equation

The aim of this section is to obtain an approximate solution of fractional Whittaker equation by using the direct and inverse Mellin transforms $M$ and $M^{-1}$, respectively (1) and (2). Applying Mellin's transform to (20) and taking into account properties (7), (8) and (9), we have

$$-4\frac{\Gamma(1-s)}{\Gamma(1-s-2\alpha)} W(s) - W(s) + W(s+2\alpha) - 4\mu W(s+\alpha) - 4\tau^2 W(s) = 0. \tag{35}$$

Denoting $H(s) = \Gamma(1-s)W(s)$ and

$$h(x) = \int_0^\infty e^{-xt} w(t)\, dt, \tag{36}$$

we get owing to Theorem 2.1 that $h(x)$ is the inverse Mellin transform of $H(s)$. Thus, (35) became

$$-4\ H(s) - \frac{\Gamma(1-s-2\alpha)}{\Gamma(1-s)}\ H(s) +\ H(s+2\alpha) - 4\mu\frac{\Gamma(1-s-2\alpha)}{\Gamma(1-s-\alpha)}\ H(s+\alpha) - 4\tau^2\frac{\Gamma(1-s-2\alpha)}{\Gamma(1-s)}\ H(s) = 0. \tag{37}$$

Taking the inverse Mellin transform we obtain, correspondingly, the equality

$$(x^{2\alpha}-4)\ h(x) - 4\mu\int_0^\infty \left(\frac{x}{t}\right)^\alpha h\left(\frac{x}{t}\right) k_2(t)\frac{dt}{t} - (1+4\tau^2)\int_0^\infty h\left(\frac{x}{t}\right) k_1(t)\frac{dt}{t} = 0, \tag{38}$$

where

$$k_1(t) = M^{-1}\left\{\frac{\Gamma(1-s-2\alpha)}{\Gamma(1-s)}\right\} = \frac{1}{\Gamma(2\alpha)}\ (t-1)_+^{2\alpha-1},$$

$$k_2(t) = M^{-1}\left\{\frac{\Gamma(1-s-2\alpha)}{\Gamma(1-s-\alpha)}\right\} = \frac{t^\alpha}{\Gamma(\alpha)}\ (t-1)_+^{\alpha-1}. \tag{39}$$

Consider that $h(x)$ admits formal series representation

$$h(x) \sim \sum_{n=1}^{\infty} b_n x^{-\alpha n}, \tag{40}$$

i.e.,

$$h(x) = \sum_{n=1}^{N} b_n x^{-\alpha n} + O(x^{-\alpha N}), \qquad x\to\infty,\ N\in\mathbb{N}.$$

Substituting this into (38) and using Lemma 2.5 with (39), we come out with asymptotic equality

$$(x^{2\alpha}-4)\sum_{n=1}^{N} b_n x^{-\alpha n} - \frac{4\mu}{\Gamma(\alpha)}x^\alpha\sum_{n=1}^{N} b_n x^{-\alpha n}\int_1^\infty t^{\alpha n-1}(t-1)^{\alpha-1}dt - \frac{(1+4\tau^2)}{\Gamma(2\alpha)}\sum_{n=1}^{N} b_n x^{-\alpha n}\int_1^\infty t^{\alpha n-1}(t-1)^{2\alpha-1}dt = O(x^{-\alpha N}). \tag{41}$$

At the meantime, the elementary Beta-integrals that appear in (41) are calculated explicitly under condition $\alpha < \frac{1}{N+2}$.

$$\begin{aligned} &\int_1^\infty t^{\alpha n-1}(t-1)^{2\alpha-1}dt \\ &\quad = \frac{\Gamma(2\alpha)\,\Gamma(-\alpha(n+2)+1)}{\Gamma(-\alpha n+1)}, \qquad n=0,\dots,N, \\ &\int_1^\infty t^{\alpha n-1}(t-1)^{\alpha-1}dt \\ &\quad = \frac{\Gamma(\alpha)\,\Gamma(-\alpha(n+1)+1))}{\Gamma(-\alpha n+1)}, \qquad n=0,\dots,N. \end{aligned} \tag{42}$$

Therefore, substituting (42) into (41), we arrive at the following truncated equation

$$\begin{aligned} (x^{2\alpha}-4)\sum_{n=1}^N b_n x^{-\alpha n} - 4\mu x^\alpha \sum_{n=1}^N b_n \frac{\Gamma(-\alpha(n+1)+1)}{\Gamma(-\alpha n+1)} x^{-\alpha n} \\ -(1+4\tau^2)\sum_{n=1}^N b_n \frac{\Gamma(-\alpha(n+2)+1)}{\Gamma(-\alpha n+1)}\, x^{-\alpha n} = 0. \end{aligned}$$

Collecting the terms which contain equal powers of $x$ and equating them to zero, we find

$$\begin{aligned} b_{n+2} - 4\mu\frac{\Gamma(1-\alpha(n+2))}{\Gamma(1-\alpha(n+1))} b_{n+1} \\ -\left[4+(1+4\tau^2)\,\frac{\Gamma(1-\alpha(n+2))}{\Gamma(1-\alpha n)}\right] b_n = 0. \end{aligned} \tag{43}$$

Analysis of the equation (43) is made considering several cases, depending on the values assumed by some of the parameters, in particular,$\tau$, $\mu$, $b_1$, and $b_2$.

*4.2.1 Case 1*

Suppose that

$$\tau^2 \neq -\frac{\Gamma(1-\alpha n)}{\Gamma(1-\alpha(n+2))} - \frac{1}{4},$$

with $\alpha \in \left]0, \frac{1}{N+2}\right]$, $n \in \mathbb{N}$, $n \leq N$. This means, from the previous equation, that the case $b_1 = b_2 = 0$ leads to the trivial solution of the equation (20).

In order to obtain non-trivial solutions, we need to split our study in sub-cases.

*Sub-case 1.1*: $\mu = 0$ and $b_1$, $b_2$ non simultaneously zero.

Suppose that $b_1 \neq 0$ and $b_2 = 0$. Hence (43) becomes

$$b_{2n+1} = \left[4 + (1+4\tau^2)\,\frac{\Gamma(1-\alpha(2n+1))}{\Gamma(1-\alpha(2n-1))}\right] b_{2n-1}, \qquad n \in \mathbb{N}, \quad n \leq N.$$

Therefore one can express coefficients $b_n$ with odd indices by the relation

$$b_{2n+1} = b_1 \prod_{k=1}^{n} \left[4 + (1+4\tau^2)\,\frac{\Gamma(1-\alpha(2k+1))}{\Gamma(1-\alpha(2k-1))}\right], \qquad n \in \mathbb{N}. \tag{44}$$

Since in this case $b_2 = 0$, we have, from the equation (43), that all even coefficients are zero. Therefore, we get an approximate solution of equation (41) in the form

$$h(x) \sim \sum_{n=1}^{\infty} b_{2n+1}\; x^{-\alpha(2n+1)}.$$

We obtain the corresponding expression for the solution $w(t)$ employing operational relation for the Laplace transform. Namely, equality (36) gives $w(t)$ as a formal series

$$w(t) \sim \sum_{n=1}^{\infty} \frac{b_{2n+1}}{\Gamma(\alpha(2n+1))}\; t^{\alpha(2n+1)-1}. \tag{45}$$

The above discussion can be summarized in the following result:

THEOREM 4.4. *Let $b_1 \neq 0$, $\mu = b_2 = 0$, $\alpha \in \left]0, \frac{1}{N+2}\right]$, $N \in \mathbb{N}_0$, $x \in [x_0, X_0]$, and*

$$\tau^2 \neq -\frac{\Gamma(1-\alpha n)}{\Gamma(1-\alpha(n+2))} - \frac{1}{4}.$$

*Then fractional Whittaker equation (20) admits a particular solution in terms of series as solution the power series (45), with odd coefficients given by formula (44).*

Now we turn to the even case. Assuming that $b_2 \neq 0$, equation (43) becomes

$$b_{2n+2} = \left[4 + (1+4\tau^2)\, \frac{\Gamma(1-2\alpha(n+1))}{\Gamma(1-2\alpha n)}\right] b_{2n}, \qquad n \in \mathbb{N}, \quad n \leq N.$$

for all $\alpha \in \left]0, \frac{1}{N+2}\right]$, $N, n \in \mathbb{N}$ with $n \leq N$. In the same manner, we have the following relation for the even coefficients

$$b_{2(n+1)} = b_2 \prod_{k=1}^{n} \left[4 + (1+4\tau^2)\, \frac{\Gamma(1-\alpha(2k+2))}{\Gamma(1-\alpha 2k)}\right], \qquad n \in \mathbb{N}. \tag{46}$$

Since in this case $b_1 = 0$, we have, from the equation of (43), that all odd coefficients are zero. Hence, we get an approximate solution of equation (41) in the form

$$h(x) \sim \sum_{n=1}^{\infty} b_{2n} x^{-2\alpha n}.$$

Similarly as we had done for the odd case, equality (36) gives $w(t)$ as a formal series

$$w(t) \sim \sum_{n=1}^{\infty} \frac{b_{2n}}{\Gamma(2n\alpha)}\, t^{2\alpha n - 1}. \tag{47}$$

The above discussion can be summarized in the following theorem:

THEOREM 4.5. *Let $b_2 \neq 0, \mu = b_1 = 0$, $\alpha \in \left]0, \frac{1}{N+2}\right]$, $N \in \mathbb{N}_0$, $x \in [x_0, X_0]$, and*

$$\tau^2 \neq -\frac{\Gamma(1-\alpha n)}{\Gamma(1-\alpha(n+2))} - \frac{1}{4}.$$

*Then fractional Whittaker equation (20) admits a particular solution in terms of series as solution the power series (47), with even coefficients given by formula (46).*

Finally, consider that $b_1 \neq 0$ and $b_2 \neq 0$. In this case equation (43) becomes

$$b_{n+2} = \left[4 + (1+4\tau^2)\, \frac{\Gamma(1-\alpha(n+2))}{\Gamma(1-\alpha n)}\right] b_n, \qquad n \in \mathbb{N}, \quad n \leq N, \tag{48}$$

and we get an approximate solution of equation (41) in the form

$$h(x) \sim \sum_{n=1}^{\infty} b_n x^{-\alpha n}.$$

Then, equality (36) gives $w(t)$ as a formal series

$$w(t) \sim \sum_{n=1}^{\infty} \frac{b_n}{\Gamma(n\alpha)} t^{\alpha n-1}. \tag{49}$$

The above discussion can be summarized in the following result:

THEOREM 4.6. *Let* $b_1, b_2 \neq 0, \mu = 0$, $\alpha \in \left]0, \frac{1}{N+2}\right]$, $N \in \mathbb{N}_0$, $x \in [x_0, X_0]$, *and*

$$\tau^2 \neq -\frac{\Gamma(1-\alpha n)}{\Gamma(1-\alpha(n+2))} - \frac{1}{4}.$$

*Then fractional Whittaker equation (20) admits a particular solution in terms of series as solution the power series (49), with coefficients given by formulas (48).*

*Sub-case 1.2*: $\mu \neq 0$ and $b_1$, $b_2$ non simultaneously zero.

In this case, there is no possible simplification of relation (43) for the coeficientes. Hence, from operation techniques previously used for the Laplace's transform, we obtain $w(t)$ as a formal series

$$w(t) \sim \sum_{n=1}^{\infty} \frac{b_n}{\Gamma(n\alpha)} t^{\alpha n-1}. \tag{50}$$

The above discussion can be summarized in the following theorem:

THEOREM 4.7. *Let* $b_1, b_2$ *non simultaneously zero,* $\mu = 0$, $\alpha \in \left]0, \frac{1}{N+2}\right]$, $N \in \mathbb{N}_0$, $x \in [x_0, X_0]$, *and*

$$\tau^2 \neq -\frac{\Gamma(1-\alpha n)}{\Gamma(1-\alpha(n+2))} - \frac{1}{4}.$$

*Then fractional Whittaker equation (20) admits a particular solution in terms of series as solution the power series (50), with coefficients given by formulas (43).*

*4.2.2 Case 2*

Suppose that

$$\tau^2 = -\frac{\Gamma(1-\alpha n)}{\Gamma(1-\alpha(n+2))} - \frac{1}{4},$$

with $\alpha \in \left]0, \frac{1}{N+2}\right]$, $n \in \mathbb{N}$, $n \leq N$. It is immediate that our conclusions will be independent from the value of $b_1$. However, we need to consider two sub-cases.

*Sub-case 2.1*: $\mu = 0$.

From equation (43), if $b_1 = b_2 = 0$ we get the trivial solution of the equation (20). If $b_1 = 0$ and $b_2 \neq 0$, or $b_1 \neq 0$ and $b_2 = 0$ then $b_n = b_2$, $b_n = b_1$, $n \in \mathbb{N}$, respectively. Finally, $b_n = b_1 + b_2$, with $n \in \mathbb{N}$, in the case that $b_1$ and $b_2$ are both different form zero.

*Sub-case 2.2*: $\mu \neq 0$.

In this case we only need to take into account the value of $b_2$ because the conclusions will be the same for all $b_1$. From (43), if $b_2 = 0$ we obtain the trivial solution of the equation (20). So, let, for instance, $b_2 \neq 0$. Hence, (43) becomes

$$b_{n+2} = 4\mu \, \frac{\Gamma(1-\alpha(n+2))}{\Gamma(1-\alpha(n+1))} \, b_{n+1}, \qquad n \in \mathbb{N}, \quad n \leq N.$$

Therefore one can express coefficients $b_n$ by the relation

$$b_{n+2} = b_2 \prod_{k=1}^{n} 4\mu \, \frac{\Gamma(1-\alpha(k+2))}{\Gamma(1-\alpha(k+1))}, \qquad n \in \mathbb{N}. \tag{51}$$

Since in this case $b_2 \neq 0$, we get, from (43), an approximate solution of equation (41) in the form

$$h(x) \sim \sum_{n=1}^{\infty} b_n x^{-\alpha n}.$$

Since $h(x)$ is approximated by the previous series, we use Laplace operational calculus to conclude that equality (36) gives $w(t)$ as a formal series

$$w(t) \sim \sum_{n=1}^{\infty} \frac{b_n}{\Gamma(n\alpha)} \, t^{\alpha n - 1}. \tag{52}$$

The above discussion can be summarized in the following result:

THEOREM 4.8. *Let* $b_2 \neq 0$, $\mu \neq 0$, $\alpha \in \left]0, \frac{1}{N+2}\right]$, $N \in \mathbb{N}_0$, $x \in [x_0, X_0]$, *and*

$$\tau^2 = -\frac{\Gamma(1-\alpha n)}{\Gamma(1-\alpha(n+2))} - \frac{1}{4}.$$

*Then fractional Whittaker equation (20) admits a particular solution in terms of series as solution the power series (52), with coefficients given by formula (51).*

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