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Multidimensional Backward Stochastic Differential Equations with Left-Lipschitz Coefficients

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Abstract. In this note, we consider multidimensional backward stochastic differential equations with coefficients which are left-Lipschitz w.r.t. y and Lipschitz w.r.t. z and without explicit constraints on the growth. An existence theorem of minimal solution is established in this framework. We also relate it to the hedging problem for interacting economic agents.

1. Introduction and Preliminaries

Backward stochastic differential equations (BSDEs) have been studied by many authors during the last two decades since the pioneering work of Pardoux and Peng [8]. An m-dimensional BSDE defined on $[0, T]$ is of the form:

$$(1)\quad {}^k y_t = {}^k\xi + \int_t^T {}^k g\left(s, {}^1 y_s, \dots, {}^k y_s, \dots, {}^m y_s, {}^1 z_s, \dots, {}^k z_s, \dots, {}^m z_s\right) ds \\ - \int_t^T {}^k z_s dB_s,\ t \in [0, T],$$

where $k = 1, \dots, m$, $(B_t)_{t\in[0,T]}$ is a standard d-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}_T, P)$ and $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented Brownian filtration. We denote by $\mathcal{M}^2_{\mathcal{F}}(0, T; \mathbf{R}^{m\times d})$ the space of all $\mathcal{F}_t$-progressively measurable $\mathbf{R}^{m\times d}$-valued processes such that $\mathbf{E}\left[\int_0^T |\psi_t|^2 dt\right] < \infty$ and $\mathcal{H}^2_{\mathcal{F}}(0, T; \mathbf{R}^m)$ the space of all continuous processes in $\mathcal{M}^2_{\mathcal{F}}(0, T; \mathbf{R}^m)$ such that $\mathbf{E}[\sup_{0\le t\le T} |\varphi_t|^2] < \infty$. The function $g : \Omega \times [0, T] \times \mathbf{R}^m \times \mathbf{R}^{m\times d} \longmapsto$

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$\mathbf{R}^m$ is called the generator of BSDE (1) and the $\mathbf{R}^m$-valued $\mathcal{F}_T$-measurable random variable ξ is the terminal condition. A BSDE is determined by its standard parameters (ξ, g, T).

The solution of BSDE (1) is a pair of processes $(Y_t, Z_t)_{t\in[0,T]} \in \mathcal{H}^2_{\mathcal{F}}(0,T;\mathbf{R}^m) \times \mathcal{M}^2_{\mathcal{F}}(0,T;\mathbf{R}^{m\times d})$. The simplest case, $g \equiv 0$, corresponds to the Martingale Representation Theorem. Multidimensional BSDEs are more complicated than the 1-dimensional one because that different dimensions are interacted in the generator g and the monotonicity of y_t with respect to (w.r.t. for short) ξ does not hold naturally. Xu [10] found some applications of multidimensional BSDEs in finance.

Comparison theorems for multidimensional BSDEs were first studied in Zhou, H. [11], extended by Zhou, S. [12], generalized by Hu and Peng [4] and Xu [10]. It says that, for two multidimensional BSDEs (ξ^i, g^i, T), $i = 1, 2$, when ${}^k g^i$ does not depend on $\left({}^j z\right)_{j\neq k}$, if for one of g^i, for each $k = 1, \dots, m$, ${}^k g^i$ is nondecreasing in $\left({}^j y\right)_{j\neq k}$ and for any $\left(y, {}^k z\right) \in \mathbf{R}^n \times \mathbf{R}^d$,

$$ {}^k g^1(t, y, {}^k z) \geq {}^k g^2(t, y, {}^k z), \tag{2} $$

and $\xi^1 \geq \xi^2$, then $Y^1_t \geq Y^2_t$, $\forall t \in [0, T]$, P-a.s. We say $x \geq y$, for $x, y \in \mathbf{R}^m$, if $x_i \geq y_i$, for all $i = 1, 2, ..., m$.

Based on the Comparison Theorem for BSDEs, we present an existence theorem for (1). Thus our results generalize Jia [5, 6] to the multidimensional case.

Due to the Comparison Theorem, we will consider generators ${}^k g$, $k = 1, \dots, m$, which do not depend on $\left({}^j z\right)_{j\neq k}$. The well known Lipschitz condition for multidimensional BSDEs is, for each $k = 1, \dots, m$, $\forall t$, $\forall (y, y')$, $\forall ({}^k z, {}^k z')$,

$$ |{}^k g\left(t, y, {}^k z\right) - {}^k g\left(t, y', {}^k z'\right)| \leq L\left(|y - y'| + |{}^k z - {}^k z'|\right), \ L \geqslant 0. \tag{3} $$

Note that if ${}^k g$ is nondecreasing in $\left({}^j y\right)_{j\neq k}$, the above Lipschitz condition is equivalent to

$$ \begin{aligned} &- L\left(\sum_{l\neq k}\left({}^l y - {}^l y'\right)^- + \left|{}^k y - {}^k y'\right| + \left|{}^k z - {}^k z'\right|\right) \\ &\leq {}^k g\left(t, y, {}^k z\right) - {}^k g\left(t, y', {}^k z'\right) \end{aligned} \tag{4} $$

$$\leq L\left(\sum_{l\neq k}\left({}^{l}y-{}^{l}y'\right)^{+}+\left|{}^{k}y-{}^{k}y'\right|+\left|{}^{k}z-{}^{k}z'\right|\right). \tag{5}$$

If $y \geq y'$, then (4) becomes

$$^{k}g\left(t,y,{}^{k}z\right)-{}^{k}g\left(t,y',{}^{k}z'\right)\geq -L\left(\left({}^{k}y-{}^{k}y'\right)+\left|{}^{k}z-{}^{k}z'\right|\right). \tag{6}$$

So we will use (6) as the left-Lipschitz condition. Left-Lipschitz condition is a kind of discontinuous condition, in which the generator g may not be continuous w.r.t. y. By the well known Feynman-Kac formula ([9]), BSDE is in fact a random version of semilinear parabolic partial differential equation (PDE), i.e., there is a one to one relationship between BSDE and semilinear parabolic PDE. Many physical phenomenons are described by PDEs with discontinuous coefficients. An important example of best response dynamics model arising in the theory of games comes from Hofbauer [2] and Hofbauer and Simon [3] in which the generator $g(y)$ satisfies:

$$\begin{cases} g(y)<0, & y\in(0,a)\,,\\ g(y)>0, & y\in(a,1)\,. \end{cases}$$

This is one of the reasons we study BSDEs with discontinuous left-Lipschitz condition. On the other hand, some financial problems are related to BSDEs with left-Lipschitz condition. See Example 2.2 at the end of the paper.

We make the following assumptions throughout the paper.

(H1) For each k, ${}^{k}g$ does not depend on $\left({}^{j}z\right)_{j\neq k}$ and is nondecreasing in $\left({}^{j}y\right)_{j\neq k}$, left-continuous[1] w.r.t. y and condition (6) holds.

(H2) There are two multidimensional BSDEs with generators ${}^{k}g^{i}:\Omega\times[0,T]\times\mathbf{R}\times\mathbf{R}^{d}\longmapsto\mathbf{R}$, $i=1,2$, $k=1,\dots,m$, such that ${}^{k}g^{1}\left(s,y,{}^{k}z\right)\leq{}^{k}g\left(s,y,{}^{k}z\right)\leq{}^{k}g^{2}\left(s,y,{}^{k}z\right)$, and for any $\xi\in L^{2}(\mathcal{F}_{T};\mathbf{R}^{m})$, they have at least a solution respectively, denoted by $\left(Y_{t}^{i},Z_{t}^{i}\right)$, $i=1,2$, with $Y_{t}^{1}\leq Y_{t}^{2}$ and $g^{i}\left(t,Y_{t}^{i},Z_{t}^{i}\right)\in\mathcal{M}_{\mathcal{F}}^{2}(0,T;\mathbf{R}^{m})$.

(H3) $g^{1}\left(t,\cdot,\cdot\right)$ is Lipschitz continuous.

Remark 1.1. Condition (6) implies that g is in fact Lipschitz-continuous w.r.t. z. Taking $y'=y$, we get that ${}^{k}g(t,y,{}^{k}z)-{}^{k}g(t,y,{}^{k}z')\geq -L|{}^{k}z-$

[1] "${}^{k}g$ is left-continuous w.r.t. y" means that for an increasing sequence in $\{y^{n}\}_{n=1}^{\infty}$ in $\mathbf{R}^{m}$ with y^{n} converging to $y\in\mathbf{R}^{m}$, ${}^{k}g(t,y,{}^{k}z)=\lim_{y^{n}\uparrow y}{}^{k}g(t,y^{n},{}^{k}z)$.

${}^k z'|$, then change the position of ${}^k z$, ${}^k z'$, we have ${}^k g(t, y, {}^k z) - {}^k g(t, y, {}^k z') \leq L|{}^k z - {}^k z'|$, therefore ${}^k g$ is Lipschitz-continuous w.r.t. z.

The following lemma which is a special case of Comparison Theorem 2.2 coming from [1].

Lemma 1.1. *Assume $\xi \in L^2(\mathcal{F}_T; \mathbf{R})$, the function $g(s, y, z) = ay + b|z| + \varphi_s$ for some constants $a, b \in \mathbf{R}, (\varphi_s) \in \mathcal{M}^2_{\mathcal{F}}(0, T; \mathbf{R})$. Let (Y_t, Z_t) denote the solutions of the corresponding 1-dimensional BSDE (ξ, g, T). If $\varphi_s \geq 0$, and $\xi \geq 0$, then $Y_t \geq 0$.*

2. Main Result

We now show the existence of a solution to BSDE (1) under assumptions (H1) and (H2). Let (Y^i_t, Z^i_t), $i = 1, 2$, denote solutions of the following two BSDEs:

$$Y^i_t = \xi + \int_t^T g^i\left(t, Y^i_t, Z^i_t\right) ds - \int_t^T Z^i_s dB_s,\ 0 \leq t \leq T, \tag{7}$$

where g^i satisfies (H2). Now we consider the following sequence of m-dimensional BSDEs parameterized by $n = 1, 2, \ldots$

$$\begin{aligned} {}^k y^n_t = {}^k\xi + \int_t^T &\left({}^k g\left(s, y^{n-1}_s, {}^k z^{n-1}_s\right)\right. \\ &- L(\left({}^k y^n_s - {}^k y^{n-1}_s\right) + \left|{}^k z^n_s - {}^k z^{n-1}_s\right|\Big)\, ds - \int_t^T {}^k z^n_s dB_s, \end{aligned} \tag{8}$$

where $k = 1, \ldots, m$. Define $(y^0_t, z^0_t) := \left(Y^1_t, Z^1_t\right)$. For the sequence $\{y^n_t\}$, we have

Lemma 2.1. *Under (H1) and (H2), the following properties hold true:*
(i) For any n=1,2, ..., there is a unique solution $(y^n_t, z^n_t) \in \mathcal{H}^2_{\mathcal{F}}(0, T; \mathbf{R}^m) \times \mathcal{M}^2_{\mathcal{F}}(0, T; \mathbf{R}^{m\times d})$ for BSDE (8).
(ii) For any n=1,2, ..., $Y^1_t \leq y^n_t \leq y^{n+1}_t \leq Y^2_t$, $\forall t \in [0, T]$, P-a.s.

Proof. We first prove the case $n = 1$. By $Y^2_t \geq Y^1_t$ and conditions (H1), (H2), it follows that, for each $k = 1, \ldots, m$,

$${}^k g\left(t, Y^2_t, {}^k Z^2_t\right) - {}^k g\left(t, Y^1_t, {}^k Z^1_t\right) \geq -L\left(\left(Y^2_t - Y^1_t\right) + \left|{}^k Z^2_t - {}^k Z^1_t\right|\right).$$

Hence

$$
\begin{aligned}
{}^k g^2\left(t, Y_t^2, {}^k Z_t^2\right) &+ L\left(\left(Y_t^2 - Y_t^1\right) + \left|{}^k Z_t^2 - {}^k Z_t^1\right|\right) \\
&\geq {}^k g\left(t, Y_t^2, {}^k Z_t^2\right) + L\left(\left(Y_t^2 - Y_t^1\right) + \left|{}^k Z_t^2 - {}^k Z_t^1\right|\right) \\
&\geq {}^k g\left(t, Y_t^1, {}^k Z_t^1\right) \geq {}^k g^1\left(t, Y_t^1, {}^k Z_t^1\right).
\end{aligned}
$$

Therefore ${}^k g\left(t, Y_t^1, {}^k Z_t^1\right) \in \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R})$ and by Pardoux and Peng [8], there is a unique solution $\left(y_t^1, z_t^1\right) \in \mathcal{H}_{\mathcal{F}}^2(0,T;\mathbf{R}^m) \times \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^{m\times d})$ for BSDE (8).

By BSDE (7) and BSDE (8), we have

$$
\begin{aligned}
{}^k y_t^1 - {}^k Y_t^1 = &\int_t^T \left(-L(\left({}^k y_s^1 - {}^k Y_s^1\right) + \left|{}^k z_s^1 - {}^k Z_s^1\right|) + {}^k\varphi_s\right) ds \\
&- \int_t^T \left({}^k z_s^1 - {}^k Z_s^1\right) dB_s,
\end{aligned}
$$

where ${}^k\varphi_s = {}^k g\left(t, Y_t^1, {}^k Z_t^1\right) - {}^k g^1\left(t, Y_t^1, {}^k Z_t^1\right) \geq 0$ and $\varphi_s \in \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^m)$. Then by Lemma 1.1, we deduce that ${}^k y_t^1 \geq {}^k Y_t^1$, for each $k = 1, \dots, m$.

By (7) and (8) again, we get that

$$
\begin{aligned}
{}^k Y_t^2 - {}^k y_t^1 = &\int_t^T \left(-L(\left({}^k Y_s^2 - {}^k y_s^1\right) + \left|{}^k Z_s^2 - {}^k z_s^1\right|) + {}^k\psi_s\right) ds \\
&- \int_t^T \left({}^k Z_s^2 - {}^k z_s^1\right) dB_s,
\end{aligned}
$$

where

$$
\begin{aligned}
{}^k\psi_s &= {}^k g^2\left(t, Y_t^2, {}^k Z_t^2\right) - {}^k g\left(t, Y_t^1, {}^k Z_t^1\right) \\
&\qquad + L\left(\left({}^k Y_s^2 - {}^k y_s^1\right) + \left|{}^k Z_s^2 - {}^k z_s^1\right|\right) + L(\left({}^k y_s^1 - {}^k Y_s^1\right) + \left|{}^k z_s^1 - {}^k Z_s^1\right|) \\
&\geq {}^k g\left(t, Y_t^2, {}^k Z_t^2\right) - {}^k g\left(t, Y_t^1, {}^k Z_t^1\right) + L((Y_t^2 - Y_t^1) + \left|{}^k Z_t^2 - {}^k Z_t^1\right|) \\
&\geq 0.
\end{aligned}
$$

Obviously $\psi_s \in \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^m)$. Then by Lemma 1.1, we deduce that ${}^k Y_t^2 \geq {}^k y_t^1$, for each $k = 1, \dots, m$.

Similarly to the above procedure,

$$
\begin{aligned}
{}^k g^2\left(t, Y_t^2, {}^k Z_t^2\right) - {}^k g\left(t, y_t^1, {}^k z_t^1\right) &\geq {}^k g\left(t, Y_t^2, {}^k Z_t^2\right) - {}^k g\left(t, y_t^1, {}^k z_t^1\right) \\
&\geq -L(\left({}^k Y_t^2 - {}^k y_t^1\right) + \left|{}^k Z_t^2 - {}^k z_t^1\right|).
\end{aligned}
$$

Hence ${}^k g^2\left(t, Y_t^2, {}^k Z_t^2\right) + L(\left({}^k Y_t^2 - {}^k y_t^1\right) + \left|{}^k Z_t^2 - {}^k z_t^1\right|) \geq {}^k g\left(t, y_t^1, {}^k z_t^1\right)$. On the other hand, ${}^k g\left(t, y_t^1, {}^k z_t^1\right) \geq {}^k g^1\left(t, y_t^1, {}^k z_t^1\right) - L(\left({}^k y_s^1 - {}^k Y_s^1\right) + \left|{}^k z_s^1 - {}^k Z_s^1\right|)$, which implies that ${}^k g\left(t, y_t^1, {}^k z_t^1\right) \in \mathcal{M}_{\mathcal{F}}^2(0, T; \mathbf{R})$ and BSDE (8) has a unique solution when $n = 2$. By an analogous proof, we can obtain that

$$
y_t^2 \geq y_t^1 \text{ and } y_t^2 \leq Y_t^2.
$$

Now we use the induction method to prove this lemma. Assume that

$$
Y_t^1 \leq y_t^{n-1} \leq y_t^n \leq Y_t^2 \text{ and } g\left(t, y_t^{n-1}, z_t^{n-1}\right) \in \mathcal{M}_{\mathcal{F}}^2(0, T; \mathbf{R}^m).
$$

Consider the $(n+1)$'th BSDE with the form:

$$
\begin{aligned}
(9) \quad {}^k y_t^{n+1} = {}^k \xi + \int_t^T &\left({}^k g\left(s, y_s^n, {}^k z_s^n\right)\right. \\
&\left. - L(\left({}^k y_s^{n+1} - {}^k y_s^n\right) + \left|{}^k z_s^{n+1} - {}^k z_s^n\right|)\right) ds - \int_t^T {}^k z_s^{n+1} dB_s.
\end{aligned}
$$

By direct calculation,

$$
\begin{aligned}
{}^k g^2\left(t, Y_t^2, {}^k Z_t^2\right) - {}^k g\left(t, y_t^n, {}^k z_t^n\right) &\geq {}^k g\left(t, Y_t^2, {}^k Z_t^2\right) - {}^k g\left(t, y_t^n, {}^k z_t^n\right) \\
&\geq -L(\left({}^k Y_t^2 - {}^k y_t^n\right) + \left|{}^k Z_t^2 - {}^k z_t^n\right|).
\end{aligned}
$$

Hence ${}^k g^2\left(t, Y_t^2, {}^k Z_t^2\right) + L(\left({}^k Y_t^2 - {}^k y_t^n\right) + \left|{}^k Z_t^2 - {}^k z_t^n\right|) \geq {}^k g\left(t, y_t^n, {}^k z_t^n\right)$. On the other hand, ${}^k g\left(t, y_t^n, {}^k z_t^n\right) \geq {}^k g\left(t, Y_t^1, {}^k Z_t^1\right) - L(\left({}^k y_s^n - {}^k Y_s^1\right) + \left|{}^k z_s^n - {}^k Z_s^1\right|) \geq {}^k g^1\left(t, Y_t^1, {}^k Z_t^1\right) - L(\left({}^k y_s^n - {}^k Y_s^1\right) + \left|{}^k z_s^n - {}^k Z_s^1\right|)$, which implies that ${}^k g\left(t, y_t^n, {}^k z_t^n\right) \in \mathcal{M}_{\mathcal{F}}^2(0, T; \mathbf{R})$ and BSDE (9) has a unique solution $\left(y_t^{n+1}, z_t^{n+1}\right) \in \mathcal{H}_{\mathcal{F}}^2(0, T; \mathbf{R}^m) \times \mathcal{M}_{\mathcal{F}}^2(0, T; \mathbf{R}^{m \times d})$. Similarly we can get that $y_t^n \leq y_t^{n+1} \leq Y_t^2$. The proof is complete. $\square$

Now we present an existence theorem for BSDE (1).

THEOREM 2.1. *Let (H1) and (H2) hold for g and $\xi \in L^2(\mathcal{F}_T; \mathbf{R}^m)$. Then the sequence (y_t^n, z_t^n) converges in $\mathcal{H}_{\mathcal{F}}^2(0,T;\mathbf{R}^m) \times \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^{m\times d})$ to $(\underline{y}_t, \underline{z}_t)$ and $(\underline{y}_t, \underline{z}_t)$ is a solution of BSDE* (1). *Assume further that (H3) holds for g^1, then $(\underline{y}_t, \underline{z}_t)$ is the minimal solution of BSDE* (1), *i.e., for any solution (y_t) of* (1), *we have $\underline{y}_t \leq y_t$.*

PROOF. By Lemma 2.1, the sequence $\{y_t^n\}$ converges to a process $\left(\underline{y}_t\right) \in \mathcal{H}_{\mathcal{F}}^2(0,T;\mathbf{R}^m)$ and $\sup\limits_n \mathbf{E}[\sup\limits_{0\leq t\leq T} |y_t^n|^2] \leq \mathbf{E}[\sup\limits_{0\leq t\leq T} |Y_t^1|^2] + \mathbf{E}[\sup\limits_{0\leq t\leq T} |Y_t^2|^2] < \infty$. By the last paragraph in the proof of Lemma 2.1, we have

$$\begin{aligned}\left|{}^k g\left(t, y_t^n, {}^k z_t^n\right)\right| &\leq \left|{}^k g^2\left(t, Y_t^2, {}^k Z_t^2\right) + L(\left({}^k Y_t^2 - {}^k y_t^n\right) + \left|{}^k Z_t^2 - {}^k z_t^n\right|)\right| \\ &\quad + \left|{}^k g^1\left(t, Y_t^1, {}^k Z_t^1\right) - L(\left({}^k y_s^n - {}^k Y_s^1\right) + \left|{}^k z_s^n - {}^k Z_s^1\right|)\right| \\ &\leq \sum_{i=1}^2 \left({}^k g^i\left(t, Y_t^i, {}^k Z_t^i\right)\right. \\ &\quad \left. + L(\left|{}^k Y_t^i\right| + \left|{}^k Z_t^i\right|) + 2L(\left|{}^k y_t^n\right| + \left|{}^k z_t^n\right|)\right).\end{aligned}$$

It follows from an application of Itô's formula to $\left|{}^k y_t^{n+1}\right|^2$ that

$$\begin{aligned}\mathbf{E}\left[\int_0^T |{}^k z_t^{n+1}|^2 dt\right] &= 2\mathbf{E}[\int_0^T {}^k y_t^{n+1}({}^k g\left(t, y_t^n, {}^k z_t^n\right) \\ &\quad - L(\left({}^k y_s^{n+1} - {}^k y_s^n\right) + \left|{}^k z_s^{n+1} - {}^k z_s^n\right|))dt \\ &\quad + \mathbf{E}|\xi|^2 - \left|{}^k y_0^{n+1}\right|^2] \\ &\leq C + \frac{1}{16}\mathbf{E}[\int_0^T \left(|{}^k z_t^{n+1}|^2 + |{}^k z_t^n|^2\right) dt],\end{aligned}$$

where C is a constant independent of n. Consequently we have

$$\mathbf{E}\left[\int_0^T |{}^k z_t^{n+1}|^2 dt\right] \leq \frac{16C}{15} + \frac{1}{15}\mathbf{E}[\int_0^T |{}^k z_t^n|^2 dt],$$

which yields that $\sup\limits_n \mathbf{E}[\int_0^T |{}^k z_t^n|^2 dt] < \infty$ and ${}^k\Phi_s^{n+1} = {}^k g\left(t, y_t^n, {}^k z_t^n\right) - L(\left({}^k y_s^{n+1} - {}^k y_s^n\right) + \left|{}^k z_s^{n+1} - {}^k z_s^n\right|)$ are uniformly bounded in $\mathcal{M}_{\mathcal{F}}^2$ w.r.t. n.

Set $\tilde{C} = \sup_n \mathbf{E}[\int_0^T |\Phi_s^n|^2 ds]$. Then by Itô's formula, for any two positive integers $n, n\prime$,

$$\mathbf{E}\left[|{}^k y_t^n - {}^k y_t^{n'}|^2\right] + \mathbf{E}\left[\int_0^T |{}^k z_t^n - {}^k z_t^{n'}|^2 dt\right]$$
$$= 2\mathbf{E}\left[\int_0^T \left({}^k y_s^n - {}^k y_s^{n'}\right)({}^k\Phi_s^n - {}^k\Phi_s^{n'})ds\right]$$
$$\leq 4\tilde{C}^{\frac{1}{2}}\left(\mathbf{E}\int_0^T \left|{}^k y_s^n - {}^k y_s^{n'}\right|^2 ds\right)^{\frac{1}{2}}.$$

Thus $\{z_t^n\}$ is a Cauchy sequence and converges to some process $\underline{z}_t \in \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^{m\times d})$. Now by passing to the limit on both sides of BSDE (8), we obtain that $(\underline{y}_t, \underline{z}_t)$ satisfy BSDE (1).

Assume further that (H3) hold for g^1, for any solution y_t of (1), by the multidimensional comparison theorem (Zhou, S. [12] Theorem 2.1), we can obtain that $Y_t^1 \leq y_t$. Then it is easy to prove that $y_t^1 \leq y_t$ and $y_t^n \leq y_t$, $n > 1$. Thus the limit $\underline{y}_t \leq y_t$. The proof is complete. □

Remark 2.1. If we set $(\bar{y}_t^0, \bar{z}_t^0) := \left(Y_t^2, Z_t^2\right)$ and assume that $g(t,\cdot,z)$ is right-continuous in condition (H1) and $g^2(t,\cdot,\cdot)$ is Lipschitz continuous, consider the following sequence

$$(10) \qquad {}^k y_t^n = {}^k\xi + \int_t^T \left({}^k g\left(s, y_s^{n-1}, {}^k z_s^{n-1}\right)\right.$$
$$\left. + L(\left({}^k y_s^{n-1} - {}^k y_s^n\right) + \left|{}^k z_s^{n-1} - {}^k z_s^n\right|)\right) ds - \int_t^T {}^k z_s^n dB_s,$$

then we can prove that

(i) For any n, $Y_t^1 \leq y_t^{n+1} \leq y_t^n \leq Y_t^2$, $\forall t \in [0,T]$, P-a.s.

(ii) The sequence (y_t^n, z_t^n) converges in $\mathcal{H}_{\mathcal{F}}^2(0,T;\mathbf{R}^m) \times \mathcal{M}_{\mathcal{F}}^2(0,T;\mathbf{R}^{m\times d})$ to $(\overline{y}_t, \overline{z}_t)$ and $(\overline{y}_t, \overline{z}_t)$ is the maximal solution of BSDE (1), i.e., for any solution (y_t) of (1), we have $\overline{y}_t \geq y_t$.

Recall that Lepeltier and San Martin [7] established an existence theorem of minimal and maximal for 1-dimensional BSDE with continuous coefficients. Here we have considered multidimensional BSDEs with coefficients which may be discontinuous w.r.t. y.

REMARK 2.2. The following condition, for $y \geq y' \in \mathbf{R}^m$, ${}^k z$, ${}^k z' \in \mathbf{R}^d$,

$$(11) \quad {}^k g\left(t, y, {}^k z\right) - {}^k g\left(t, y', {}^k z'\right) \geq -L\left(\sum_{j=1}^{m}\left({}^j y - {}^j y'\right) + \left|{}^k z - {}^k z'\right|\right)$$

does not work, because the multidimensional comparison theorem could not be applied to prove that the following sequence

$$(12) \quad \begin{aligned} {}^k y_t^n = {}^k \xi + \int_t^T &\Bigg({}^k g\left(s, y_s^{n-1}, {}^k z_s^{n-1}\right) \\ &- L\left(\sum_{j=1}^{m}\left({}^j y_s^n - {}^j y_s^{n-1}\right) + \left|{}^k z_s^n - {}^k z_s^{n-1}\right|\right)\Bigg) ds \\ &- \int_t^T {}^k z_s^n dB_s, \end{aligned}$$

is increasing in n, since its generator

$${}^k g^n\left(t, y, {}^k z\right) = {}^k g\left(t, y_t^{n-1}, {}^k z_t^{n-1}\right) - L\left(\sum_{j=1}^{m}\left({}^j y - {}^j y_t^{n-1}\right) + \left|{}^k z - {}^k z_t^{n-1}\right|\right)$$

is not nondecreasing in $\left({}^j y\right)_{j \neq k}$.

Example 2.1. Consider the following m-dimensional BSDE

$$(13) \quad {}^k y_t = \int_t^T \frac{k}{m} \sum_{j=1}^{m} \mathbf{1}_{\{{}^j y_s > 0\}}\left(y_s\right) ds - \int_t^T {}^k z_s dB_s, \ 0 \leq t \leq T.$$

Since (H1) and (H2) hold for ${}^k g = \frac{k}{m} \sum_{j=1}^{m} \mathbf{1}_{\{{}^j y_s > 0\}}\left(y_s\right)$, and ${}^k g$ is **left-continuous** w.r.t. y, so there is a minimal solution to (13). In fact, the minimal solution is $\left(\underline{y}_t, \underline{z}_t\right) = (\mathbf{0}, \mathbf{0})$. Since $0 \leq {}^k g \leq k$, by the Multidimensional Comparison Theorem, for any solution of (13), we have, for each $k = 1, \ldots, m$, $0 \leq {}^k y_t \leq k\,(T - t)$.

The solutions of (13) are not unique. One can check that for any $c \in [0, T]$, $\left({}^k y_t, {}^k z_t\right) = (k \max\{c - t, 0\}, 0)$ is a solution of (13). The maximal solution of (13) also exists and $\left({}^k\overline{y}_t, {}^k\overline{z}_t\right) = \left(k\left(T-t\right),0\right)$, though ${}^kg$ is not right-continuous w.r.t. y.

Now we consider the following

$$(14)\qquad {}^ky_t = \int_t^T \frac{k}{m}\sum_{j=1}^{m} \mathbf{1}_{\{{}^jy_s\geq 0\}}\left(y_s\right)ds - \int_t^T {}^kz_s dB_s,\ 0\leq t\leq T$$

with a little difference from (13). Conditions (H1) and (H2) still hold for ${}^kf = \frac{k}{m}\sum_{j=1}^{m}\mathbf{1}_{\{{}^jy_s\geq 0\}}\left(y_s\right)$, but ${}^kf$ is **right** -continuous w.r.t. y. So the maximal solution of (14) exists and $\left({}^k\overline{y}_t, {}^k\overline{z}_t\right) = \left(k\left(T-t\right),0\right)$. Since solutions of (14) can not be negative and ${}^kf$ satisfies the right-Lipschitz condition (5) on $\{\ y\in\mathbf{R}^m|y\geq 0\}$, therefore the solution of (14) is unique.

Correlation is a huge issue in finance. It comes into play in the products (multi-assets, hybrids), in the models (price/volatility correlation), in asset allocation (diversification) and in credit (between names). Much effort is now being expended on the study of market microstructure to try to better understand price formation. Multidimensional BSDEs provide an alternative way to study complicated financial issues. Correlation could be reflected in the generators. See the following example.

Example 2.2. Consider a firm with two subsidiaries hedging their possible risk positions ξ^1 and ξ^2 at time T respectively. Let (r_t^1, r_t^2) and (θ_t^1,θ_t^2) be their risk exposure to the spot interest and risky investment. Let (Y_t^1, Y_t^2) and (Z_t^1, Z_t^2) be their hedging processes and portfolios of risk investment. Assume that if the whole wealth of the company is below certain scale $(\phi_t)\in M_{\mathcal{F}}^2(0,T;\mathbf{R})$, i.e., $Y_t^1+Y_t^2\leq\phi_t$, then the two subsidiaries will receive sectoral subsidies (h_t^1),$(h_t^2)\in M_{\mathcal{F}}^2(0,T;\mathbf{R}^+)$ respectively. Therefore in a complete financial market the hedging processes should satisfy:

$$(15)\qquad \begin{cases} Y_t^1 = \xi^1 - \int_t^T \left(r_s^1Y_s^1 + h_s^1\cdot\mathbf{1}_{\{Y_s^1+Y_s^2\leq\phi_s\}}\left(Y_s^1,Y_s^2\right) + \theta_s^1Z_s^1\right)dr \\ \qquad\quad + \int_t^T Z_s^1 dB_t, \qquad t\in[0,T], \\ Y_t^2 = \xi^2 - \int_t^T \left(r_s^2Y_s^2 + h_s^2\cdot\mathbf{1}_{\{Y_s^1+Y_s^2\leq\phi_s\}}\left(Y_s^1,Y_s^2\right) + \theta_s^2Z_s^2\right)dr \\ \qquad\quad + \int_t^T Z_s^2 dB_t, \qquad t\in[0,T]. \end{cases}$$

It is easy to check that the generator $g^i(s,y^1,y^2,z^i) = -(r_s^iy^i + h_s^i\cdot\mathbf{1}_{\{y^1+y^2\leq\phi_s\}}\left(y^1,y^2\right) + \theta_s^iz^i)$, $i = 1,2$, satisfies (H1)$\sim$(H3) with $\overline{g}^i =$

$-\left(r_s^i y^i + \theta_s^i z^i\right)$, $\underline{g}^i = -\left(r_s^i y^i + h_s^i + \theta_s^i z^i\right)$. Hence by Theorem 2.1 there is a minimal solution for BSDE (15). The minimal solution is the least endowment to hedge their risk positions.

In the above example, different dimensions represent different subsidiaries. See Xu [10] for more applications of multidimensional BSDEs on risk measuring for interacted subsidiaries. Multidimensional BSDEs can be also applied to differential games with interacted players.

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