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Spectral Anomalies of the Robin Laplacian in Non-Lipschitz Domains

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Abstract. We consider the spectral Laplace-Robin problem in bounded peak shaped domains of $\mathbb{R}^n$, $n \geq 2$. In case of a sufficiently sharp peak and "wrong" sign of the Robin coefficients, the spectrum becomes pathological: the residual spectrum covers the whole complex plane, while all complex numbers are eigenvalues of the adjoint problem operator. Our results solve a spectral problem posed by H. Amann and D. Daners.

1. Introduction

1.1. Preamble

For the so-called rooms-and-corridors-domain $\Omega \subset \mathbb{R}^2$ the Sobolev embedding $H^1(\Omega) \subset L^2(\Omega)$ fails, as shown in [5]; see Fig. 1.1, where squares of size $2^{-k} \times 2^{-k}$ are connected with rectangles of size $2^{-k-1} \times 2^{-2k}$. As a consequence, the spectrum of the Neumann Laplacian cannot be discrete. Many other "interesting" domains can be found, e.g., in the papers [8, 10, 33] and the books [18, 22], all having the property that the natural compact embedding is lost and the essential spectrum of the classical boundary value problem becomes nonempty. However, in all those examples the spectrum σ is contained in the closed positive semi-axis $\overline{\mathbb{R}_+}$ of the complex plane $\mathbb{C}$. The inclusion $\sigma \subset \overline{\mathbb{R}_+}$ is a consequence of the general result [4, Thm. 10.1.2], which associates a semi-bounded self-adjoint operator with a semi-bounded closed quadradic form in a proper Hilbert space. The energy quadratic form of the above mentioned boundary value problems apparently gives rise to a

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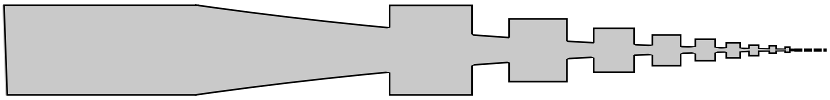

Fig. 1.1. Rooms and corridors.

positive self-adjoint operator in $L^2(\Omega)$, the spectrum of which is contained in $\overline{\mathbb{R}_+}$, and the non-discreteness of the spectrum is caused just by the non-compactness of the Sobolev embedding $H^1(\Omega) \subset L^2(\Omega)$ ([4, Thm. 10.1.5].)

In this work we give a different kind of example by considering the spectral Helmholtz equation with the Robin boundary condition in a peak-shaped domain with some geometric requirements (Fig. 1.2). We show that, first, if the Robin coefficient has the "wrong" sign (see (1.4) and (1.13) below), the above mentioned theory does not always apply, since the quadratic form may lose semi-boundedness, though it is still symmetric. Second, the spectrum of the Robin Laplacian with the natural domain becomes pathological: it covers the whole complex plane and apart from a countable set of real eigenvalues, stays *residual* with empty discrete and continuous spectra. At the same time, any point $\lambda \in \mathbb{C}$ is an eigenvalue of the adjoint operator in the Lebesgue space $L^2(\Omega)$. This is caused by the fact that the corresponding operator is symmetric but not self-adjoint. Third, we describe all possible self-adjoint extensions of the operator and prove that each of them has discrete spectrum on the real axis with two accumulation points, at $+\infty$ and $-\infty$. In this way there is no natural choice of an appropriate self-adjoint extension.

For a treatment of the Robin problem on general domains but with a "good" coefficient (see (1.4) and (1.10) below), we refer to [6] and the citations therein.

1.2. Statement of the problem; notation

Let ω be a domain in $\mathbb{R}^{n-1}$, $n \geq 2$, with Lipschitz boundary $\partial\omega$ and compact closure $\overline{\omega} = \omega \cup \partial\omega$. For technical reasons we assume that $0 \in \mathbb{R}^{n-1}$ is contained in ω and that ω is star shaped with respect to 0; this assumption may be relaxed, see the explanations just after the formula (1.8). We denote by Π_d a peak of height $d > 0$,

(1.1) $$\Pi_d = \{x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} \ : \ z \in (0, d), z^{-1-\gamma} y \in \omega\},$$

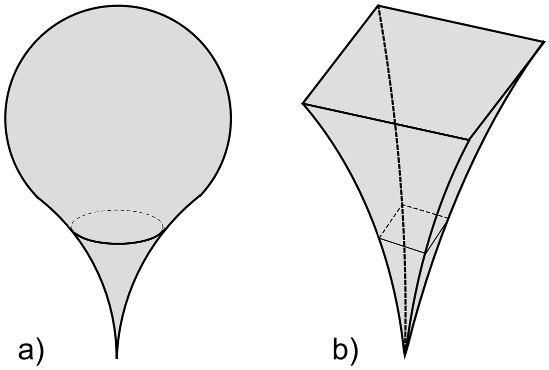

Fig. 1.2. Peak-shaped domains.

where $\gamma > 0$ is the sharpness exponent. The height d will be chosen small enough during the course of the proofs. We introduce the peak–shaped domain Ω (Fig. 1.2) which coincides with Π_1 inside the layer $\{x : y \in \mathbb{R}^{n-1}, 0 < z < 1\}$ and has the boundary $\Gamma = \partial\Omega$ and compact closure $\overline{\Omega}$. The surface Γ is assumed Lipschitz everywhere else except in the origin $\mathcal{O} = (0,0) \in \mathbb{R}^{n-1} \times \mathbb{R}$; indeed, if $\gamma > 0$, the Lipschitz property is lost at $\mathcal{O}$ due to (1.1). If γ were equal to 0, the set (1.1) would be a cone, hence, Ω would be Lipschitz. The following additional notation will be used throughout the paper:

$$\begin{aligned} \Omega(d) &:= \Omega \setminus \overline{\Pi}_d \ , \ \Gamma(d) := \Gamma \setminus \partial\Pi_d, \\ \varpi_d &:= \partial\Pi_d \cap \partial\Omega \ , \ \omega(z) := \{y : z^{-1-\gamma} y \in \omega\}, \end{aligned} \tag{1.2}$$

where $d \in (0, 1]$ and $0 < z < d$.

We consider the Helmholtz equation

$$-\Delta_x u(x) = \lambda u(x) \quad , \quad x \in \Omega, \tag{1.3}$$

where Δ_x is the Laplacian and $\lambda \in \mathbb{C}$ is a spectral parameter. Taking into account that the outward unit normal ν is defined almost everywhere on the surface Γ, we supply (1.3) with the Robin boundary condition

$$\partial_\nu u(x) + a(x)u(x) = 0 \quad \text{for a.e. } x \in \Gamma, \tag{1.4}$$

where ∂_ν is the outward normal derivative and a is a measurable function on Γ such that

$$|a(x) - a_0| \leq c|x|^\alpha \ \text{ for a.e. } x \in \Gamma, \tag{1.5}$$

and $\alpha > 0$, $c > 0$ and a_0 are some constants. In particular (1.5) means that $a \in L^\infty(\Gamma)$.

The variational formulation of the problem (1.3), (1.4) is written as the integral identity [15]

$$(\nabla_x u, \nabla_x v)_\Omega + (au, v)_\Gamma = \lambda(u, v)_\Omega \quad , \quad v \in C_c^\infty(\overline{\Omega} \setminus \mathcal{O}) \ , \tag{1.6}$$

where ∇_x is the gradient, $(\cdot, \cdot)_\Omega$ is the natural inner product of $L^2(\Omega)$ and $(\cdot, \cdot)_\Gamma$ has a similar meaning, and the linear space $C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ consists of infinitely differentiable functions with supports in $\overline{\Omega} \setminus \mathcal{O}$, i.e. functions vanishing in a neighbourhood of $\mathcal{O}$.

The main goal of the paper is to show that the loss of the Lipschitz property may lead to a pathological structure of the spectrum of the Robin problem (1.3), (1.4). The question on the structure of this spectrum was posed to the authors by H. Amann and D. Daners.

1.3. Preliminary information on the spectrum

Let us first comment on the case Ω is Lipschitz, e.g., $\gamma = 0$. Then it is known that the Sobolev space $H^1(\Omega)$ with the traditional norm $\|u; H^1(\Omega)\| = \big(\|\nabla_x u; L^2(\Omega)\|^2 + \|u; L^2(\Omega)\|^2\big)^{1/2}$ embeds compactly both into $L^2(\Omega)$ and $L^2(\Gamma)$ (cf. [15]) Hence, the spectrum σ of the problem (1.6), posed in $H^1(\Omega)$, is discrete and forms the unbounded monotone sequence

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p \leq \ldots \to +\infty \ . \tag{1.7}$$

If $a \geq 0$ almost everywhere on Γ, the first eigenvalue in (1.7) satisfies

$$\lambda_1 = \inf_{u \in H^1(\Omega) \setminus \{0\}} \frac{\|\nabla_x u; L^2(\Omega)\|^2 + (au, u)_\Gamma}{\|u; L^2(\Omega)\|^2} \geq 0. \tag{1.8}$$

Let now $\gamma > 0$. We start by remarking that in this case both $C^\infty(\overline{\Omega})$ and $C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ are dense in $H^1(\Omega)$. The first case follows from [18], Th. 2 in Section 1.1.6., since the domain Π_d is an epigraph of a continuous functions due to the star shaped assumption on ω. (This is the only point of the paper where this assumption is used. Hence, it is quite obvious that the assumption could be made weaker.) For the second, given $g \in H^1(\Omega)$ we first approximate it in the Sobolev-norm by a function $f \in C^\infty(\overline{\Omega})$. Then, for all $0 < \delta < 1/2$ we choose an infinitely smooth cut-off function $\chi_\delta : \overline{\mathbb{R}^+} \to [0,1]$ such that $\chi_\delta(z) = 0$ for $0 \le z \le \delta/2$ and $\chi_\delta(z) = 1$ for $z \ge \delta$ and such that $|\chi_\delta'(z)| \le C/\delta$ for some constant $C > 0$ and all z. For any $r > 0$, denote $A_r = \{x = (y,z) \in \Omega \ : \ z < r\}$. The quantity $\|f - \chi_\delta f; H^1(\Omega)\|^2$ can be estimated in a straightforward way by a sum of terms $\int_{A_\delta} |f|^2 dx$, $\int_{A_\delta} |\nabla_x f|^2 dx$ and $\int_{A_\delta} |f \nabla_x (1 - \chi_\delta)|^2 dx$. The first two tend to zero as $\delta \to +0$, since f and $\nabla_x f$ are bounded functions and since the measure of the integration domain tends to 0. The third is majorized by a constant times $\delta^{-2} \int_{A_\delta \setminus A_{\delta/2}} |f|^2 dx$. This also tends to zero, again since f is bounded, and the d-dimensional measure of $A_\delta \setminus A_{\delta/2}$ is at most $C\delta^{1+(1+\gamma)(d-1)}$.

The following Hardy type inequality was proven in [30, Proposition] for domains Ω with any $\gamma > 0$. The citation actually contains a proof for functions in a special function space, but the remark above generalizes it for all $u \in H^1(\Omega)$. (See also [22] for much more general results).

Lemma 1.1. *Let $\gamma > 0$. For any $u \in C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$, we have*

$$\|r^{(\gamma-1)/2} u; L^2(\Gamma)\| + \|r^{-1} u; L^2(\Omega)\| \le c_\Omega \|u; H^1(\Omega)\|, \tag{1.9}$$

where $r = |x|$ and $c_\Omega > 0$ is a constant independent of u.

From (1.9) it follows that the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact for any $\gamma > 0$. If $\gamma \in (0,1)$, the embedding operator $H^1(\Omega) \subset L^2(\Gamma)$ is also compact, because it is the sum of a compact operator (restricted to the Lipschitz domain $\Omega(d)$) and a small operator with norm $\mathcal{O}(d^{(1-\gamma)/2})$ (restricted to the peak Π_d, due to the large weight $r^{(\gamma-1)/2} \ge c d^{(\gamma-1)/2}$ in the first norm in (1.9)). Hence, the spectrum σ stays discrete for insufficiently sharp peaks with exponent $\gamma \in (0,1)$.

Let the peak be sharp, i.e. $\gamma \ge 1$. It follows from a result in [6] that in case the stabilization condition (1.5) holds with

$$a_0 > 0, \tag{1.10}$$

the spectrum is discrete as well. We give here a proof of this fact using a result [4, Ch. 10]. Let $d > 0$ be chosen such that (cf. (1.10) and (1.5))

$$a(x) > a_d > 0 \quad \text{for a.e. } x \in \varpi_d. \tag{1.11}$$

The sesquilinear form

$$q_d(u,v) = (\nabla_x u, \nabla_x v)_\Omega + (au, v)_{\varpi_d} \tag{1.12}$$

defines a scalar product in the Hilbert space $\mathcal{H} = H^1(\Omega) \cap L^2(\Gamma)$ (recall that the surface $\Gamma \setminus \varpi_d$ is Lipschitz; $\mathcal{H}$ is usually called the Maz'ya space). Hence, the theory mentioned above associates with the form (1.12) a self-adjoint positive operator A_d with domain $\mathcal{D}(A_d) \subset H^1(\Omega) \cap L^2(\Gamma)$. Since the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact, the spectrum $\sigma(A_d)$ is discrete and consists of positive eigenvalues, [4, Th.10.1.5 and 10.2.2]. The operator A of the problem (1.6) is generated by the sesquilinear form $q(u,v)$ on the left hand side of the identity (1.6), and it is a compact perturbation of A_d with $\mathcal{D}(A) = \mathcal{D}(A_d)$. Hence, by [4, Th.9.1.3], the spectrum $\sigma(A)$ is discrete, too, although the relation $\lambda_1 > 0$ may be lost.

1.4. The main goal of the paper

We are going to investigate the spectrum of the problem (1.6) in the case the Robin coefficient has "wrong" sign, that is,

$$a_0 < 0, \tag{1.13}$$

which means that for some $d > 0$ we have

$$-a(x) > a_d > 0 \quad \text{for a.e. } x \in \varpi_d\,, \tag{1.14}$$

contrary to (1.10) and (1.11). The next lemma demonstrates that if $\gamma > 1$, then the form (1.12) in the space $H^1(\Omega) \cap L^2(\Gamma)$ is not semibounded and therefore the theory in [4, Ch. 10] does not apply. The case $\gamma = 1$ is much more delicate and will be considered in Section 3.3.

LEMMA 1.2. *For any $m > 0$ there exist functions $u_\pm^m \in H^1(\Omega) \cap L^2(\Gamma)$ such that*

$$\pm q(u_\pm^m, u_\pm^m) \geq m \|u_\pm^m; L^2(\Omega)\|^2. \tag{1.15}$$

In other words, neither the form

$$q(u,v) = (\nabla_x u, \nabla_x v)_\Omega + (au, v)_\Gamma, \tag{1.16}$$

nor the form $-q$ is semibounded from below.

Proof. Clearly, it suffices to deal with large m. To define the functions u_+^m, we just pick them so as to satisfy

$$\|\nabla_x u_+^m; L^2(\Omega)\|^2 \geq m\|u_+^m; L^2(\Omega)\|^2 \tag{1.17}$$

from the subspace $H_0^1(\Omega;\Gamma)$ of functions in $H^1(\Omega)$ which vanish on Γ.

To construct the functions u_-^m, let $\chi \in C_c^\infty(1,2)$ be such that $0 \leq \chi \leq 1$ and $\chi(z) = 1$ for $z \in (5/4, 7/4)$. We set $u_-^m(z) = \chi(2^m z)$. A direct calculation shows that

$$\begin{aligned}
\|u_-^m; L^2(\Omega)\|^2 &\leq |\omega| \int\limits_{2^{-m}}^{2^{1-m}} z^{(n-1)(1+\gamma)} dz \leq c_0 2^{-m((n-1)(1+\gamma)+1)}, \\
\|\nabla_x u_-^m; L^2(\Omega)\|^2 &\leq c_\chi |\omega| \int\limits_{2^{-m}}^{2^{1-m}} z^{(n-1)(1+\gamma)} dz \leq c_1 2^{-m((n-1)(1+\gamma)-1)}, \\
\|u_-^m; L^2(\Gamma)\|^2 &\geq c_\chi |\partial\omega| \int\limits_{5\cdot 2^{-m}/4}^{7\cdot 2^{-m}/4} z^{(n-2)(1+\gamma)}(1 + c_\omega z^{2\gamma})^{1/2} dz \\
&\geq c_\Gamma 2^{-m((n-2)(1+\gamma)+1)} = c_\Gamma 2^{-m((n-1)(1+\gamma)-\gamma)}, \qquad (1.18)
\end{aligned}$$

where $|\omega| = \text{mes}_{n-1}\omega$ and $|\partial\omega| = \text{mes}_{n-2}\partial\omega$ are the $(n-1)$-dimensional volume of ω and the $(n-2)$-dimensional area of its surface, respectively. Since $\gamma > 1$ (for $\gamma = 1$ the conclusion is wrong), we infer for all sufficiently large m that

$$\begin{aligned}
-q(u_-^m, u_-^m) &\geq 2^{-m(n-1)(1+\gamma)}(c_\Gamma 2^{m\gamma} - c_1 2^m) \geq \frac{1}{2} c_\Gamma 2^{-m(n-1)(1+\gamma)} 2^{m\gamma} \\
&\geq c_0 2^{-m(n-1)(1+\gamma)} 2^{-m} m \geq m\|u_-^m; L^2(\Omega)\|^2. \;\Box \qquad (1.19)
\end{aligned}$$

1.5. Excursis to the theory of elliptic problems in peak-shaped domains

We assume for a while that $\partial\omega$ and $\partial\Omega \setminus \mathcal{O}$ are smooth. The theory of boundary value problems for elliptic systems in peak-shaped domains was developed in [20, 32, 19, 23] (see also the monographs [14, 13]). The starting point for this approach is the coordinate change

$$y \mapsto \eta = h(z)^{-1} y \ , \ z \mapsto \zeta = -\int\limits_z^d \frac{dt}{h(t)}, \tag{1.20}$$

which transforms the peak

$$\{(y,z)\,:\,z\in(0,d),\ h(z)^{-1}y\in\omega\}$$

into the semi-cylinder $\mathcal{Q}=\omega\times(-\infty,0)$. Our particular sharpness function $h(z)=z^{1+\gamma}$ leads to

$$\eta=z^{-1-\gamma}y\ ,\ \zeta=-\frac{1}{\gamma}(z^{-\gamma}-d^{-\gamma}). \tag{1.21}$$

We have

$$\nabla_\eta=h(z)^{-1}\nabla_y\ ,\ \partial_z=h(z)^{-1}(\partial_\zeta-h'(z)\eta\cdot\nabla_\eta), \tag{1.22}$$

where $h'(z)=(1+\gamma)z^\gamma\to0$ as $z\to0^+$ and $\zeta\to-\infty$. The Helmholtz equation (1.3) takes the form

$$\begin{aligned}&-\Big(\Delta_\eta+h(z)(\partial_\zeta-h'(z)\eta\cdot\nabla_\eta)h(z)^{-1}(\partial_\zeta-h'(z)\eta\cdot\nabla_\eta)\Big)u(\eta,\zeta)\\ =\ &\lambda h(z)^2u(\eta,\zeta)\ ,\quad(\eta,\zeta)\in\mathcal{Q},\end{aligned}$$

and still has the Laplace operator $\Delta_\eta+\partial_\zeta^2$ as the main part as $\zeta\to-\infty$. Moreover, the normal derivative $\partial_{\nu(\eta)}$ on $\partial\omega\times(-\infty,0)$ remains as the main part of the Robin boundary operator in (1.4). As evident, e.g., in [2, 12, 19], an appropriate scale of function spaces in cylindrical domains is the Sobolev spaces $\mathcal{W}^l_\beta(\mathcal{Q})$ with exponentially weighted norms

$$\|\exp(\beta\zeta)u;H^l(\mathcal{Q})\|. \tag{1.23}$$

It was shown in [19] and [14], [13] that the problem operator with a domain in the $\mathcal{W}^l_\beta$-class becomes Fredholm for every $\beta\in\mathbb{R}$, with the exception of a countable set of *forbidden* indices. These are produced by eigenvalues of the auxiliary spectral problem on the cross-section ω.

The inverse change $(\eta,\zeta)\mapsto(y,z)$ brings the weight

$$\exp(-\beta\gamma^{-1}z^{-\gamma})$$

from (1.23) into the Sobolev norm in Ω. In order to make (1.6) the variational formulation of the problem (1.3), (1.4) we have to accept $\beta=0$. Unfortunately, though not surprisingly, the index $\beta=0$ is forbidden, because in our case the auxiliary problem, namely the spectral Neumann problem in ω with the Laplace operator Δ_η, has the null eigenvalue. Hence, the general theory mentioned above does not suffice to answer the above posed question on the spectrum of (1.3),(1.4).

In [11] the authors studied the problem (1.3), (1.4) with $a(x) = a_0 < 0$ in a two-dimensional peak-shaped domain, that is, $\omega = (0, h) \subset \mathbb{R}$, $h > 0$, and $\gamma = 1$ in (1.1). They constructed the asymptotics of the solutions using the projection technique developed in [20, 32, 23] (see also [14] and [13]); however, estimates of the asymptotic remainders were derived in [11] in the scale of weighted Hölder spaces, which is not suitable for our purposes. Hence, to consider general peak-shaped domains in $\mathbb{R}^n$ with Lipchitz cross-sections we shall employ a different approach [25], which is partly based on some techniques in [24] and involves asymptotic ansätze from the theory of elliptic problems in thin domains (see [28], [21, Ch. 15,16], [26], [27] and others). In this connection we emphazise that the peak (1.1) is a rapidly thinning domain as $z \to 0^+$.

1.6. The case $\gamma = 1$; formal asymptotics

We proceed to construct the formal asymptotics for the solution u of the inhomogeneous problem

$$-\Delta_x u(x) = f(x)\ ,\ x \in \Omega, \tag{1.24}$$

$$\partial_\nu u(x) + a(x)u(x) = g(x)\ ,\ x \in \Gamma, \tag{1.25}$$

which is written as the integral identity

$$(\nabla_x u, \nabla_x v)_\Omega + (au, v)_\Gamma = (f, v)_\Omega + (g, v)_\Gamma,\ v \in C_c^\infty(\overline{\Omega} \setminus \mathcal{O}). \tag{1.26}$$

Notice that the term λu of (1.3) is included in f on the right hand side (see Remarks 2.4 and 2.7 for an explanation). Here, f as well as g are supposed to have "nice" properties. The asymptotic procedure applied here will be justified in Section 2.4.

Assuming for a while that $\partial\omega$ is smooth, the normal $\nu(x)$ of the lateral peak surface ϖ_1 takes the form

$$N(\eta, z)^{1/2}\nu(y, z) = \big(\nu'(\eta), -2z\eta \cdot \nu'(\eta)\big), \tag{1.27}$$

where $\eta = z^{-2}y$ is the "rapid" variable on the cross–section of the peak such that

$$\Pi_d = \{x = (y, z)\ :\ z \in (0, d), \eta \in \omega\}$$

(cf. (1.1) and (1.20), (1.21)). Furthermore, $\nu'(\eta)$ is the unit vector of the outward normal at the boundary of $\omega \subset \mathbb{R}^{n-1}$, and the normalization factor N is given by

$$N(\eta, z) = 1 + 4z^2 |\eta \cdot \nu'(\eta)|^2, \tag{1.28}$$

where the central dot stands for the scalar product of $\mathbb{R}^{n-1}$. The same factor (1.28) appears in the relation

$$ds_x = N(\eta, z)^{1/2} z^{2(n-2)} ds_\eta dz \tag{1.29}$$

of the measures ds_x and ds_η on ϖ_1 and $\partial\omega$, respectively. Notice that $N(\eta, z) = 1 + \mathcal{O}(z^2)$ and

$$dx = dy dz = z^{2(n-1)} d\eta dz. \tag{1.30}$$

The leading term in the asymptotics of solutions in thin domains usually stays independent of the transversal variables[1], as it is demonstrated in [26] and [28], [21, Ch. 15,16] for many boundary value problems, see also Remark 1.3 with the ansatz (1.37). We accept this assumption and leave its rigorous justification for Section 2.4. Let us change $u(x)$ and $v(x)$ in (1.26) for $U(z)$ and $V(z)$, where $V \in C_c^\infty(0, d)$ and the function U, independent of y, is to be found. We use the formulas (1.29) and (1.30) and get rid of the lower order terms by replacing N and a with 1 and a_0 according to (1.28) and (1.5), respectively. After integrating over ω and $\partial\omega$, the left hand side of (1.26) becomes

$$|\omega| \int_0^d z^{2(n-1)} \partial_z U(z) \partial_z V(z) dz + a_0 |\partial\omega| \int_0^d U(z) V(z) dz. \tag{1.31}$$

Assuming the right hand sides f and g to decay sufficiently fast as $z \to +0$, we put (1.31) equal to null, and after integrating by parts arrive at the ordinary differential equation

$$-\frac{d}{dz} z^{2(n-1)} \frac{dU}{dz}(z) - A z^{2(n-2)} U(z) = 0 \ , \ z > 0, \tag{1.32}$$

[1]We emphasize that this is especially true in the theory of thin plates, where the Kirchhoff asymptotics gets a complicated structure: the leading term does indeed not depend on the transversal variable, but the important correction terms do.

where

$$A = -a_0 \frac{|\partial\omega|}{|\omega|} > 0; \tag{1.33}$$

recall that we have fixed $a_0 < 0$. The equation is of Euler type and has the general solution

$$\begin{aligned} U(z) &= c_+ U^+(z) + c_- U^-(z) \quad \text{for } A \neq (n-3/2)^2, \\ U^\pm(z) &= z^{\lambda_\pm}\ ,\ \lambda_\pm = -n + \frac{3}{2} \pm \sqrt{\Big(n - \frac{3}{2}\Big)^2 - A} \end{aligned} \tag{1.34}$$

and

$$U(z) = z^{-n+3/2}(c_0 + c_1 \ln z) \quad \text{for } A = (n-3/2)^2. \tag{1.35}$$

We emphasize that

$$\begin{aligned} &\lambda_+ > -n + 3/2 > \lambda_- \quad \text{for } A < (n-3/2)^2, \\ &\mathrm{Re}\lambda_\pm = -n + 3/2 \quad \text{for } A \geq (n-3/2)^2. \end{aligned} \tag{1.36}$$

Remark 1.3. The functions $U^\pm = z^{\lambda_\pm}$ can also be found by inserting the following asymptotic ansatz in thin domains (see, e.g., [28, Ch. 1])

$$u(x) \sim z^\lambda + z^{\lambda+2} W(z^{-2} y) \tag{1.37}$$

into the problem (1.24), (1.25) restricted to the peak and extracting terms of order $z^{\lambda-2}$ in Π_d and of order z^λ in ϖ_d. Notice that z is considered here as a small parameter. Writing these equal to zero we get the Neumann problem

$$\begin{aligned} -\Delta_\eta W(\eta) &= \lambda(\lambda - 1)\ ,\ \eta \in \omega \\ \partial_{\nu'} W(\eta) &= 2\lambda \eta \cdot \nu'(\eta) - a_0\ ,\ \eta \in \partial\omega. \end{aligned} \tag{1.38}$$

Since

$$\int_{\partial\omega} \eta \cdot \nu'(\eta) ds_\eta = \int_\omega \nabla_\eta \cdot \eta \, d\eta = (n-1)|\omega|\,,$$

the compatibility condition in problem (1.38) reads as

$$\lambda(\lambda-1)|\omega| + \lambda(n-1)|\omega| - a_0|\partial\omega| = 0$$

and yields the same quadratic equation

$$\lambda^2 + (2n-3)\lambda - a_0\frac{|\partial\omega|}{|\omega|} = 0$$

and roots $\lambda_\pm$ in (1.34) as the Euler method for the ordinary differential equation (1.32). Once the compatibility condition is satisfied by the proper choice of λ in (1.34), the problem (1.38) admits a unique solution of mean zero. We shall use this solution in Section 3.

REMARK 1.4. In the critical case $A = (n-3/2)^2$ we shall also use the particular solutions

$$U^\pm(z) = z^{-n+3/2}(1 \pm i \ln z) \tag{1.39}$$

so that the general solution

$$U(z) = c_+U^+(z) + c_-U^-(z) \quad \text{for } A = (n-3/2)^2 \tag{1.40}$$

takes the same form as in (1.34).

The structure and the properties of the solutions mentioned above establish the threshold

$$a_\ddagger = \Big(n - \frac{3}{2}\Big)^2 \frac{|\omega|}{|\partial\omega|}. \tag{1.41}$$

Namely, in the case $a_0 \in (-a_\ddagger, 0)$, i.e. above the threshold $-a_\ddagger$, the normalized solution

$$z \mapsto z^{n-3/2}U^\pm(z) \tag{1.42}$$

either grows unboundedly, or decays as $z \to +0$. In contrast, for $a < -a_\ddagger$, i.e. below the threshold, the function (1.42) oscillates in the logarithmic scale and has no limit as $z \to +0$. At the threshold $a = -a_\ddagger$ the solution (1.34), with $c_1 = 0$ and multiplied by $z^{n-3/2}$, neither grows nor decays but stays constant.

1.7. Preliminary description of the results on the spectrum

One of our aims is to give an explicit formula for the domain of the unbounded operator, which is naturally related to the problem (1.3), (1.4). To this end we shall assume in Section 3 that the boundary $\partial\omega$ and the punctured surface $\partial\Omega \setminus \mathcal{O}$ are smooth, say C^3, and that the coefficient a belongs to C^1 in a neighbourhood of $\partial\Omega$. Furthermore, we restrict ourselves to the case

$$\gamma = 1, \tag{1.43}$$

where a straightforward asymptotic analysis is available (see Section 1.6 and §2) and which provides the noncompact embedding $H^1(\Omega) \subset L^2(\Gamma)$. We discuss the case of a non-smooth $\partial\Omega$ and a in Section 3.7

If the punctured surface $\partial\Omega\setminus\mathcal{O}$ is not smooth and the normal derivative is not properly defined, we have to define the domain of the problem operator of (1.3), (1.4), which is an unbounded operator in $L^2(\Omega)$, as follows: based on the integral identity (1.6) we set

$$\begin{aligned} \mathcal{D}(\mathcal{T}) = \quad \{ \quad & u \in H^1(\Omega) \ : \ \Delta_x u \in L^2(\Omega) \text{ and} \\ & (\Delta_x u, v)_\Omega + (\nabla_x u, \nabla_x v)_\Omega + (au, v)_\Gamma = 0 \text{ for all } v \in H^1(\Omega)\}. \end{aligned} \tag{1.44}$$

Notice that owing to Lemma 1.1 and (1.43), the scalar product in $L^2(\Gamma)$ is defined correctly. However, when $\partial\Omega \setminus \mathcal{O}$ is smooth, the local elliptic estimates (cf. [1], [17]) put u into the Sobolev space $H^2(K)$ for any compact $K \subset \overline{\Omega} \setminus \mathcal{O}$, hence, $u \in H^2_{\text{loc}}(\overline{\Omega} \setminus \mathcal{O})$. Moreover, taking a test function $v \in C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ and integrating by parts turn the identity in (1.44) into

$$(\partial_\nu u, v)_\Gamma + (au, v)_\Gamma = 0 \ , \ v \in C_c^\infty(\overline{\Omega} \setminus \mathcal{O}).$$

We now see that the boundary condition (1.4) is met by functions $u \in \mathcal{D}(\mathcal{T})$ and the domain of the operator becomes

$$\begin{aligned} \mathcal{D}(\mathcal{T}) = \quad \{ \quad & u \in H^1(\Omega) \cap H^2_{\text{loc}}(\overline{\Omega} \setminus \mathcal{O}) \ : \ \Delta_x u \in L^2(\Omega) \ , \\ & \partial_\nu u(x) + a(x) u(x) = 0 \ , \ x \in \partial\Omega \setminus \mathcal{O}\}. \end{aligned} \tag{1.45}$$

Note that the boundary condition in the formula (1.45) is to be understood in the Sobolev-Slobodetskii class $H^{1/2}_{\text{loc}}(\partial\Omega \setminus \mathcal{O})$.

In Section 3 we prove in particular the following theorem which distinguishes the properties of the operator $\mathcal{T}$ above and below the threshold $-a_\ddagger$.

THEOREM 1.5. *1) In the case $a_0 > -a_\ddagger$ the operator $\mathcal{T}$ is self-adjoint with discrete spectrum.*
2) In the case $a_0 \le -a_\ddagger$ the operator is still symmetric but no longer self-adjoint. Its spectrum covers the whole complex plane $\mathbb{C}$.

Furthermore, in Section 3 we give a much more accurate description of the domain $\mathcal{D}(\mathcal{T})$ and additional information on the spectra of $\mathcal{T}$ and the adjoint $\mathcal{T}^*$ below the threshold.

The simplest formula (see Theorem 3.5)

$$\mathcal{D}(\mathcal{T}) = \{u \in H^2(\Omega) \ : \ \partial_\nu u(x) + a(x)u(x) = 0 \ , \ x \in \partial\Omega \setminus \mathcal{O}\} \tag{1.46}$$

occurs in the case[2]

$$0 > a_0 > -a_\bullet = -\Big(n^2 - 3n + \frac{5}{4}\Big)\frac{|\omega|}{|\partial\omega|} \tag{1.47}$$

while for

$$a_0 \in (-a_\ddagger, -a_\bullet) \tag{1.48}$$

the domain becomes

$$\begin{aligned} \mathcal{D}(\mathcal{T}) = \ \{ \ & u = K_+\chi_0 U^+ + \tilde{u} \ : \ \tilde{u} \in H^2(\Omega)\,,\ K_+ \in \mathbb{C}\,, \\ & \partial_\nu u(x) + a(x)u(x) = 0 \ , \ x \in \partial\Omega \setminus \mathcal{O}\}, \end{aligned} \tag{1.49}$$

where χ_0 is a smooth cut-off function with support in $\overline{\Pi_d}$ such that $\chi_0(x) = 1$ for $x \in \Pi_{d/2}$.

Notice that the function U^+ defined in (1.34) belongs to $H^2(\Pi_d)$ only under the condition (1.47), when the exponent λ_+ is bigger than $-n + 5/2$, while $U^+ \in H^1(\Pi_d)\backslash H^2(\Pi_d)$ for $a_0 \in (-a_\ddagger, -a_\bullet]$ and $U^+ \in L^2(\Pi_d)\backslash H^1(\Pi_d)$ for $a_0 \le -a_\ddagger$. At the same time the function U^- (see (1.34) and (1.35), (1.39)) always stays outside $H^1(\Pi_d)$, because the exponent λ_- is smaller than or equal to $-n + 3/2$.

This simple observation is the main distinguishing feature of the above-threshold case: none of the functions $U^\pm$ belongs to the natural domain (1.45) of $\mathcal{T}$, but both fall into the domain of the adjoint $\mathcal{T}^*$. The latter

[2] The critical case $a_0 = -a_\bullet$ leads to a complication of the formulas (1.49) and (1.46) for the domain of $\mathcal{T}$, see Remark 3.6.

makes any $\lambda \in \mathbb{C}$ an eigenvalue of $\mathcal{T}^*$ and thus a point of the residual spectrum of $\mathcal{T}$ (Section 3.5). Of course, one may try to find an intrinsic self-adjoint extension of $\mathcal{T}$ with discrete spectrum, and in Section 3.6 we describe all such extensions. However, the explicit formula (3.58) in Theorem 3.13 does not allow to select a canonical self-adjoint extension, and the discrete spectrum remains dependent on the extension parameter. We find that below the threshold, there is no possibility to construct a self-adjoint operator of the problem (1.6), which possesses the same nice properties as the operator does above the threshold.

In Section 2 we reformulate the boundary value problem (1.3), (1.4) as an integral identity and examine the solutions in weighted Sobolev spaces, namely Kondratiev spaces [12], see the norm (2.2) and also Remark 2.1. This approach with weak solutions in weighted spaces reduces the smoothness requirements on the problem data so that for example peaks and edges like in Fig. 1.2,b), can be treated. We find a necessary and sufficient condition for the problem operator to be Fredholm, and we present an asymptotic form of the solution by constructing a parametrix and making rigorous the dimension reduction, outlined in Section 1.6. As a conclusion of these results we prove Theorem 2.16 concerning the index of the problem operator in weighted spaces, see Fig. 2.4. This becomes the key point in our investigation, since it reveals the underlying reason for the pathology of the spectrum of the problem (1.3), (1.4).

Section 3 is devoted to a study of the spectrum of the problem (1.24), (1.25), that is, the spectrum of the operator $\mathcal{T}$ with the domain (1.44). Below the threshold the form (1.16) stays semibounded from below (Section 3.1) and therefore the spectrum is discrete, while the domains of $\mathcal{T}$ are described in Theorem 3.5 (and Remark 3.6). The main tool to verify the explicit formulas (1.46) and (1.49) becomes Lemma 3.2 on lifting smoothness of the weak solutions of (1.24), (1.25). Below the threshold the form (1.16) is no more semibounded (see Lemma 3.7), but the domain is still of the form (1.46). On the other hand, using Theorem 2.16 on the index and the generalized Green formula (Lemma 3.10) we find out in Section 3.4 that the domain (3.47) of the adjoint operator $\mathcal{T}^*$ is much bigger. Moreover, in Section 3.5 we describe all main properties of the spectra of $\mathcal{T}$ and $\mathcal{T}^*$, and in particular prove the second assertion of Theorem 1.5. In Section 3.6, Theorem 3.13, we investigate all possible self-adjoint extensions of the symmetric operator $\mathcal{T}$ and notice that each of them has two unbounded, positive and negative, sequences of eigenvalues. As a conclusion, the the boundary value problem (1.3), (1.4) cannot be realized above the threshold as a self-adjoint operator in $L^2(\Omega)$ in the same way as below the threshold, with similar general properties like discrete spectrum of type (1.7). Finally, in Section 3.7 we discuss the case of non-smooth problem data.

2. The Fredholm Property in Weighted Spaces and Asymptotics of the Solution

2.1. Weighted function spaces and formulation of theorems

We consider the Poisson equation (1.24) with the Robin condition (1.25) written in the variational form

$$(\nabla_x u, \nabla_x v)_\Omega + (au, v)_\Gamma = F(v) \ \ , \ v \in C_c^\infty(\overline{\Omega} \setminus \mathcal{O}); \tag{2.1}$$

cf. (1.26). Here, the functional F can determined by the right hand sides f and g in (1.24) and (1.25), but we shall also consider general (anti)linear functionals. We search for the solution of (2.1) in the function space $V_\beta^1(\Omega)$ with the weighted norm

$$\|u; V_\beta^1(\Omega)\| = \big(\|r^\beta \nabla_x u; L^2(\Omega)\|^2 + \|r^{\beta-1} u; L^2(\Omega)\|^2\big)^{1/2}, \tag{2.2}$$

where $r = |x|$ and $\beta \in \mathbb{R}$ is the weight index. Actually, $V_\beta^1(\Omega)$ is obtained as the completion of $C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ with respect to the norm (2.2) and consists of all functions in $H^1_{\rm loc}(\overline{\Omega} \setminus \mathcal{O})$ with finite norm (2.2). The last term in (2.2) defines a norm in the weighted Lebesgue space $L^2_{\beta-1}(\Omega)$.

Remark 2.1. The norm (2.2) has the same distribution of weights as in the Kondratiev spaces in conical domains (see [12] and, e.g. [29, 14]). However, the reason for this distribution in peak–shaped domains is completely different and crucially relies upon the introduced restriction (1.43), cf. weights in the inequality (1.9) and other estimates presented below in this section.

Lemma 2.2. *The following weighted trace inequality is valid:*

$$\|u; L^2_\beta(\Gamma)\| := \|r^\beta u; L^2(\Gamma)\| \le c_\beta \|u; V_\beta^1(\Omega)\|. \tag{2.3}$$

PROOF. Let $u \in C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ and insert $v = r^\beta u$ into (1.9) with $\gamma = 1$ to conclude that

$$\begin{aligned} &\|u; L^2_\beta(\Gamma)\| = \|v; L^2(\Gamma)\| \leq c\big(\|\nabla_x v; L^2(\Omega)| + \|v; L^2(\Omega)\|\big) \\ \leq\ & c\big(\|r^\beta \nabla_x u; L^2(\Omega)\| + \|r^{\beta-1}u; L^2(\Omega)\| + \|r^\beta u; L^2(\Omega)\|\big) \\ \leq\ & C\|u; V^1_\beta(\Omega)\|. \ \square \end{aligned}$$

Using a density argument and applying Lemma 2.2 to the second term on the left, we find that the integral identity (2.1) must hold for any test function $v \in V^1_{-\beta}(\Omega)$. Hence, the right hand side of (2.1) may be any continuous (anti)linear functional on $V^1_{-\beta}(\Omega)$, in other words, F belongs to the dual space $V^1_{-\beta}(\Omega)^*$. The problem (2.1) thus defines the mapping

$$T_\beta : V^1_\beta(\Omega) \to V^1_{-\beta}(\Omega)^*. \tag{2.4}$$

REMARK 2.3. The embedding $L^2_{\beta-1}(\Omega) \subset V^1_\beta(\Omega)^*$ is compact. This follows from the formula

$$(u, v)_\Omega = (u, v)_{\Pi_\varepsilon} + (u, v)_{\Omega(\varepsilon)},$$

the standard Sobolev embedding in the Lipschitz domain $\Omega(\varepsilon)$, and the estimate

$$|(u, v)_{\Pi_\varepsilon}| \leq |(r^2 r^{\beta-1}u, r^{-\beta-1}v)_{\Pi_\varepsilon}| \leq \varepsilon^2 \|u; L^2_{\beta-1}(\Pi_\varepsilon)\|\|v; V^1_{-\beta}(\Pi_\varepsilon)\| \tag{2.5}$$

with the small factor ε.

REMARK 2.4. Since $V^1_\beta(\Omega) \subset L^2_{\beta-1}(\Omega)$ by the definition (2.2), the identity mapping $I : V^1_\beta(\Omega) \to V^1_{-\beta}(\Omega)^*$ is compact, by Remark 2.3. This permits us to treat in the present section the Poisson equation instead of the Helmholtz equation.

In the sequel we prove the following two theorems.

THEOREM 2.5. *The mapping* (2.4) *is Fredholm, if and only if*

$$\beta \neq \beta_\pm := \pm \mathrm{Re}\sqrt{(n-3/2)^2 - A}. \tag{2.6}$$

In the case $\beta = \beta_+$ or $\beta = \beta_-$ the range of the operator T_β is not closed.

The proof will be presented in Section 2.3. We emphasize that Theorem 2.5 gives two forbidden indices $\beta_\pm$ for $A < (n-3/2)^2$ such that $\pm\beta_\pm > 0$. On the other hand, $\beta_\pm = 0$ for $A \geq (n-3/2)^2$, where A is computed in (1.33). The corresponding solutions (1.34) and (1.35) of the limit equation (1.32) appear in the next theorem on asymptotics.

Theorem 2.6. *Let $u \in V^1_\beta(\Omega)$ be a solution of the problem* (2.1) *with $F \in V^1_{-\theta}(\Omega)^*$ and let the weight indices β and*

$$\theta \in [\beta - \min\{\alpha, 1/2\}, \beta) \tag{2.7}$$

(α as in (1.5)*) meet the condition* (2.6)*. Then u has the asymptotic form*

$$u(x) = \chi_0(x) \sum_\pm K_\pm U^\pm(z) + \tilde{u}(x), \tag{2.8}$$

where $\tilde{u} \in V^1_\theta(\Omega)$, χ_0 is a cut–off function which equals to one in a neighbourhood of the peak tip and vanishes outside the peak Π_1, $K_\pm$ is a numerical coefficient which is null in the case

$$\mathrm{Re}\lambda_\pm \notin I(\beta,\theta) := \Big(\frac{3}{2} - n - \beta, \frac{3}{2} - n - \theta\Big), \tag{2.9}$$

where $\lambda_\pm$ is defined in (1.34), (1.35)*. The following estimate is valid:*

$$\|\tilde{u}; V^1_\theta(\Omega)\| + \sum_\pm |K_\pm| \leq c\big(\|F; V^1_{-\theta}(\Omega)^*\| + \|u; V^1_\beta(\Omega)\|\big). \tag{2.10}$$

Remark 2.7. If $u \in V^1_\beta(\Omega)$, then the functional $F_u \in V^1_{-\beta}(\Omega)^*$, given by

$$F_u(v) = (u, v)_\Omega,$$

falls into the space $V^1_{2-\beta}(\Omega)^*$ (cf. the left inequality (2.5) in Remark 2.3) and therefore into $V^1_{-\theta}(\Omega)^*$, if (2.7) holds. This is just another reason to consider in this section the Poisson equation instead of the inhomogeneous Helmholtz equation.

2.2. Weighted inequalities

The next lemma is formulated for the peak (1.1) with any sharpness exponent $\gamma > 0$, although it will be used with $\gamma = 1$ only. We assume here that $v \in V^1_\beta(\Pi_d)$ satisfies the orthogonality conditions

$$\int_{\omega(z)} v(y,z)dy = 0 \quad \text{for a.e. } z \in (0,d). \tag{2.11}$$

Lemma 2.8. *The following inequality is valid:*

$$\begin{aligned} &\|z^{\beta-1-\gamma}v; L^2(\Pi_d)\|^2 + \|z^{\beta-(1+\gamma)/2}v; L^2(\varpi_d)\|^2 \\ \leq\ & c\|z^\beta \nabla_y v; L^2(\Pi_d)\|^2. \end{aligned} \tag{2.12}$$

Proof. From (2.11) we obtain for the function $\omega \times (0,d) \ni (\eta, z) \mapsto V(\eta,z) = v(z^{1+\gamma}\eta, z)$ the relation

$$\int_\omega V(\eta,z)d\eta = 0 \quad \text{for a.e. } z \in (0,d).$$

Applying now the Poincare inequality in ω and then the standard trace inequality in $\partial\omega$ (see e.g. [15]), we get

$$\|V; L^2(\omega)\|^2 + \|V; L^2(\partial\omega)\|^2 \leq c_\omega \|\nabla_\eta V; L^2(\omega)\|^2. \tag{2.13}$$

Performing the change $\eta \mapsto y = z^{1+\gamma}\eta$ and multiplying (2.13) by $z^{\beta-2(1+\gamma)+(n-1)(1+\gamma)}$, we integrate the result over $(0,d) \ni z$ and recall the relation (1.29) to arrive at (2.12). Note that to make the integrals on the left of (2.12) a priori converge, one may integrate over (ε, d) and then send ε to $+0$. □

Weighted trace inequalities can be found for example in [22], but we shall need the exact constant, which is not available in the cited standard formulation. However, it is given in the next lemma, taken from [30]. For the convenience of the reader we give here an abbreviated proof.

LEMMA 2.9. *Let $\gamma = 1$. For any $u \in H^1(\Omega)$ and any $\varepsilon > 0$, there holds the inequality*

$$\|u; L^2(\Gamma)\|^2 \leq \Big(\Big(n - \frac{3}{2}\Big)^2 \frac{|\omega|}{|\partial\omega|} - \varepsilon\Big)^{-1} \|\nabla_x u; L^2(\Omega)\|^2 + C_\varepsilon \|u; L^2(\Omega)\|^2\Big), \tag{2.14}$$

where $C_\varepsilon \to +\infty$ as $\varepsilon \to +0$.

PROOF. In the well-known inequality (see, e.g., [15])

$$\|u; L^2(\Gamma(\varepsilon))\|^2 \leq \varepsilon \|\nabla_x u; L^2(\Omega(\varepsilon))\|^2 + C_\varepsilon \|u; L^2(\Omega(\varepsilon))\|^2, \tag{2.15}$$

where $\varepsilon > 0$ is arbitrary and $C_\varepsilon \to \infty$ as $\varepsilon \to +0$, we take $\varepsilon = d/2$, and thus we may restrict the consideration to functions $u \in C_c^\infty(\overline{\Pi}_d \setminus \mathcal{O})$ which also vanish at $z = d$. Notice that d will eventually be chosen small enough.

Recalling the notation in (1.2), we have

$$z^{2(n-1)}|\omega| = |\omega(z)| \quad , \quad z^{2(n-2)}|\partial\omega| = |\partial\omega(z)|. \tag{2.16}$$

Let us make the decomposition

$$u(y,z) = u_0(z) + u_\perp(y,z) \quad \text{with} \quad u_0(z) = \frac{1}{z^{2(n-1)}|\omega|} \int\limits_{\omega(z)} u(y,z)\, dy. \tag{2.17}$$

Notice that then $u_\perp$ satisfies the orthogonality condition (2.11).

We apply Lemma 2.8 with $\gamma = 1$, $\beta = 0$ and $d > 0$ so small that

$$d \leq \varepsilon c^{-1/2} \tag{2.18}$$

(c is as in (2.12)) to get the bound

$$\|u_\perp; L^2(\varpi_d)\|^2 \leq \varepsilon^2 \int\limits_{\Pi_d} |\nabla_y u_\perp(x)|^2\, dx = \varepsilon^2 \int\limits_{\Pi_d} |\nabla_y u(x)|^2\, dx. \tag{2.19}$$

Next we derive a bound for the component $\overline{u}$ on Π_d. First,

$$\int\limits_{\Pi_d} |\nabla_x u(x)|^2\, dx = \int\limits_{\Pi_d} |\nabla_y u_\perp(y,z)|^2\, dx + \int\limits_{\Pi_d} \Big|\frac{\partial u_\perp}{\partial z}(y,z)\Big|^2\, dx$$

$$+ \int\limits_{\Pi_d} \left| \frac{du_0}{dz}(z) \right|^2 dx + 2 \int\limits_{\Pi_d} \frac{\partial u_0}{dz}(z) \frac{\partial u_\perp}{\partial z}(y,z)\, dx$$

$$=: \quad I_1 + I_2 + I_3 + 2I_4 \geq I_3 + 2I_4. \tag{2.20}$$

The differentiation rule for integrals with varying limits and definition (1.1) with $\gamma = 1$ establish the estimate

$$\left| \frac{d}{dz} \int\limits_{\omega(z)} u_\perp(y,z)\, dy - \int\limits_{\omega(z)} \frac{\partial u_\perp}{\partial z}(y,z)\, dy \right|$$

$$\leq \quad 2zc_\omega \int\limits_{\partial\omega(z)} |u_\perp(y,z)|\, ds_y \quad , \; z \in (0,d), \tag{2.21}$$

while the first integral on the left vanishes due to (2.17). Using (2.16) we thus get

$$\begin{aligned} |I_4| \quad &\leq \quad c \int\limits_0^d \left| \frac{du_0}{dz}(z) \right| \int\limits_{\omega(z)} \left| \frac{\partial u_\perp}{\partial z}(y,z) \right| dydz \\ &\leq \quad c \int\limits_0^d \left| \frac{du_0}{dz}(z) \right| z \int\limits_{\partial\omega(z)} |u_\perp(y,z)|\, ds_y dz \\ &\leq \quad c \Big(\int\limits_0^d z^{2(n-1)} \left| \frac{du_0}{dz}(z) \right|^2 dz \Big)^{1/2} \\ &\times \quad \Big(\int\limits_0^d z^{2-2(n-1)} \Big(\int\limits_{\partial\omega(z)} |u_\perp(y,z)|\, ds_y \Big)^2 dz \Big)^{1/2}. \end{aligned} \tag{2.22}$$

Taking into account (2.16), the Cauchy-Bunyakovskii-Schwartz inequality implies

$$\Big(\int_{\partial\omega(z)} |u_\perp| ds_y \Big)^2 \leq C z^{2(n-2)} \int_{\partial\omega(z)} |u_\perp|^2 ds_y,$$

and hence (2.22) can be bounded by

$$c\, \|\partial_z u_0; L_2(\Pi_d)\|\, \|u_\perp; L_2(\varpi_d)\| = c\, I_3^{1/2}\, \|u_\perp; L_2(\varpi_d)\|$$

$$\leq \quad c\,\varepsilon I_3 + c\varepsilon^{-1} \, \|u_\perp; L_2\,(\varpi_d)\,\|^2. \tag{2.23}$$

To estimate I_3 we need the one-dimensional Hardy inequality

$$\int_0^d z^{2\kappa-1}|U(z)|^2 dz \leq \frac{1}{\kappa^2}\int_0^d z^{2\kappa+1}\Big|\frac{dU}{dz}(z)\Big|^2 dz, \tag{2.24}$$

where $\kappa = (2(n-1)-1)/2 = n - 3/2 > 0$ and $U(d) = 0$ is assumed, so that extending U as null for $z > d$ reduces (2.24) to the case $d = \infty$. Taking also into account (2.16) we obtain

$$\begin{aligned} I_3 &= |\omega| \int_0^d z^{2\kappa+1} \left|\frac{du_0}{dz}(z)\right|^2 dz \geq \kappa^2 |\partial\omega| \int_0^d z^{2\kappa-1}|u_0(z)|^2\,dz \\ &= \kappa^2 \frac{|\omega|}{|\partial\omega|} \int_0^d \int_{\partial\omega(z)} |u_0(z)|^2\,ds_y dz \\ &= \Big(n - \frac{3}{2}\Big)^2 \frac{|\omega|}{|\partial\omega|} \|\overline{u}; L^2(\varpi_d)\|. \end{aligned} \tag{2.25}$$

The estimate

$$(1 + c\varepsilon)\|\nabla_x u; L^2(\Pi_d)\| \geq \Big(n - \frac{3}{2}\Big)^2 \frac{|\omega|}{|\partial\omega|}(1 - c'\varepsilon)\|u_0; L^2(\varpi_d)\| \tag{2.26}$$

for some positive constants c and c', follows by combining (2.20) with (2.25) and (2.23); in (2.23) apply (2.19) to the second term. Moreover, using again (2.19), the triangle inequality $\|u_0; L^2(\varpi_d)\| + \|u_\perp; L^2(\varpi_d)\| \geq \|u; L^2(\varpi_d)\|$ and increasing the constant c on the left of (2.26), we obtain the inequality (2.26) with u replacing u_0. The result (2.14) follows from this together with the inequality (2.15). $\square$

2.3. The parametrix and the proof of Theorem 2.5

Let us construct a right parametrix of the operator (2.4), i.e. a mapping

$$R_\beta : V^1_{-\beta}(\Omega)^* \to V^1_\beta(\Omega) \tag{2.27}$$

such that the operator

$$I - T_\beta R_\beta : V^1_{-\beta}(\Omega)^* \to V^1_{-\beta}(\Omega)^* \tag{2.28}$$

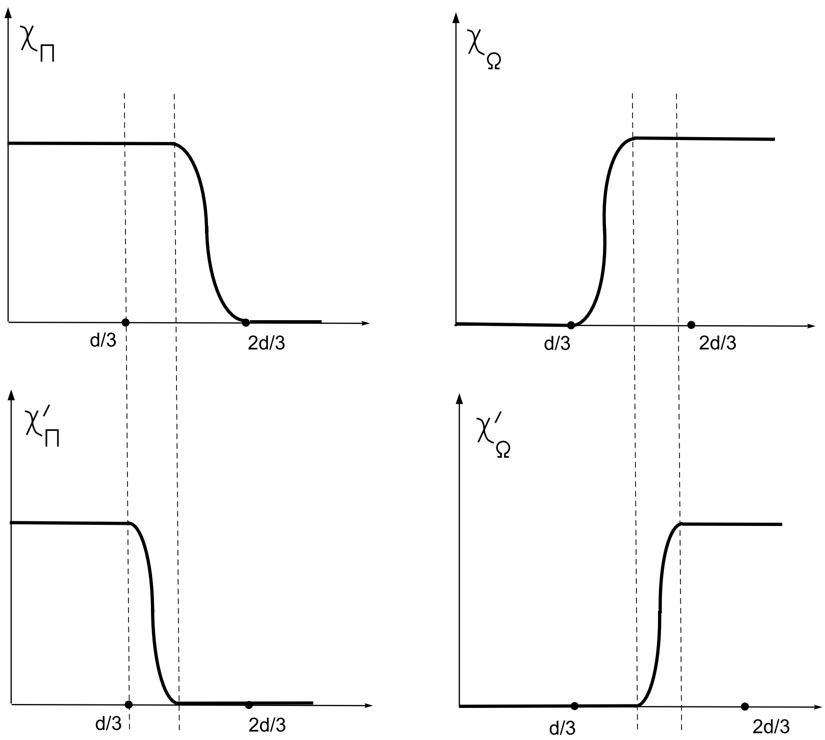

Fig. 2.1. Cut-off functions.

is compact. Since $T_{-\beta}$ is the adjoint of the operator T_β and the requirement (2.6) holds true for both weight indices $\pm\beta$ simultaneously, the adjoint operator $R^*_{-\beta}$ serves as a left parametrix for T_β, namely

$$I - R^*_{-\beta} T_\beta : V^1_\beta(\Omega) \to V^1_\beta(\Omega) \tag{2.29}$$

is compact as well. It is known (see, e.g., [3]) and can be readily verified, that the statements (2.28) and (2.29) imply the Fredholm property for the operator T_β. The loss of this property at $\beta = \beta_\pm$, see (2.6), will be shown in Remark 2.13.

We introduce the smooth cut–off functions χ_Ω, χ'_Ω, χ_Π, and χ'_Π such that

$$\chi_\Omega \chi'_\Omega = \chi'_\Omega \ , \ \chi_\Omega(z) = 0 \text{ for } z < d/3 \ , \ \chi'_\Omega(z) = 1 \text{ for } z > 2d/3;$$

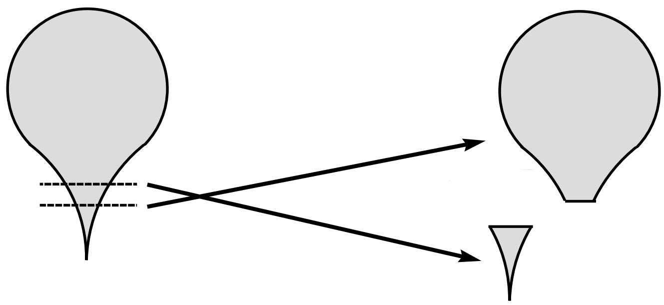

Fig. 2.2. Blunted and truncated peaks.

$$\chi_\Pi \chi'_\Pi = \chi'_\Pi \ , \ \chi_\Pi(z) = 0 \text{ for } z > 2d/3 \ , \ \chi'_\Pi(z) = 1 \text{ for } z < d/3; \tag{2.30}$$

and $\{\chi'_\Omega, \chi'_\Pi\}$ is a partition of unity in Ω. Possible graphs of these functions are drawn and compared in Fig. 2.1.

Let us consider auxiliary problems in the domain $\Omega(d/3)$ with the blunted peak and truncated peak $\Pi_{2d/3}$ (see Fig. 2.2), namely,

$$\begin{aligned} & (\nabla_x u^\Omega, \nabla_x v^\Omega)_{\Omega(d/3)} + (au^\Omega, v^\Omega)_{\Gamma(d/3)} + M(u^\Omega, v^\Omega)_{\Omega(d/3)} \\ = \ & F^\Omega(v^\Omega), \ v^\Omega \in H^1(\Omega(d/3)), \qquad (2.31) \\ & (\nabla_x u^\Pi, \nabla_x v^\Pi)_{\Pi_{2d/3}} + a_0(u^\Pi, v^\Pi)_{\varpi_{2d/3}} \\ = \ & F^\Pi(v^\Pi), \ v^\Pi \in V^1_{-\beta}(\Pi_{2d/3}), \qquad (2.32) \end{aligned}$$

where $M > 0$ is a number to be fixed later on. Owing to the properties of the cut–off functions in (2.30), the right–hand sides are defined as follows:

$$\begin{aligned} & v^\Omega \in H^1(\Omega(d/3)) \Rightarrow \chi_\Omega v^\Omega \in V^1_{-\beta}(\Omega) \text{ and } F^\Omega(v^\Omega) = F(\chi_\Omega v^\Omega), \\ & v^\Pi \in V^1_{-\beta}(\Pi_{2d/3}) \Rightarrow \chi_\Pi v^\Pi \in V^1_{-\beta}(\Omega) \text{ and } F^\Pi(v^\Pi) = F(\chi_\Pi v^\Pi). \qquad (2.33) \end{aligned}$$

Furthermore,

$$\begin{aligned} & v \in V^1_{-\beta}(\Omega(d/3)) \Rightarrow \chi'_\Omega v \in H^1(\Omega(d/3)) \ , \\ & \chi'_\Pi v \in V^1_{-\beta}(\Pi_{2d/3}) \text{ and} \\ & F^\Omega(\chi'_\Omega v) + F^\Pi(\chi'_\Pi v) = F(\chi_\Omega \chi'_\Omega v) + F(\chi_\Pi \chi'_\Pi v) \\ = \ & F(\chi'_\Omega v) + F(\chi'_\Pi v) = F(v). \qquad (2.34) \end{aligned}$$

Then, after solving the problems (2.31) and (2.32), we may determine the approximate solution of the problem (2.1) by the formula

$$R_\beta F = \chi'_\Omega u^\Omega + \chi'_\Pi u^\Gamma. \tag{2.35}$$

The inequality (2.15), with $d/3$ replacing d, exhibits the compactness of the embedding $H^1(\Omega(d/3)) \subset L^2(\Gamma(d/3))$, and with its help we find the number $M = M_d > 0$ such that the problem (2.31) is uniquely solvable and its solution $u^\Omega \in H^1(\Omega(d/3))$ obeys the estimate

$$\|u^\Omega; H^1(\Omega(d/3))\| \le c_d \|F^\Omega; H^1(\Omega(d/3))^*\| \le c_d \|F; V^1_{-\beta}(\Omega)^*\|. \tag{2.36}$$

Let us turn to the second auxiliary problem (2.32). We set

$$u^\Pi(y,z) = u_0^\Pi(z) + u_\perp^\Pi(y,z), \tag{2.37}$$

$$u_0^\Pi(z) = |\omega(z)|^{-1} \int\limits_{\omega(z)} u^\Pi(y,z)dy = |\omega|^{-1} \int\limits_{\omega} u^\Pi(z^2\eta, z)d\eta,$$

$$0 = \int\limits_{\omega(z)} u_\perp^\Pi(y,z)dy = z^{2(n-1)} \int\limits_{\omega} u_\perp^\Pi(z^2\eta, z)d\eta. \tag{2.38}$$

Clearly, $u_0^\Pi \in V^1_\beta(\Pi_{2d/3})$ in the case $u^\Pi \in V^1_\beta(\Pi_{2d/3})$, but also

$$\begin{aligned}
&\|u_0^\Pi; V^1_{\beta+n-1}(0, 2d/3)\|^2 \\
= &\int\limits_0^{2d/3} z^{2(\beta+n-1)}(|\partial_z u_0^\Pi(z)|^2 + z^{-2}|u_0^\Pi(z)|^2)dz \\
\le &\; c \int\limits_0^{2d/3} z^{2(\beta+n-1)}\Big(\int\limits_\omega \big(|\partial_z u^\Pi(z^2\eta, z)|^2 + 4z^2|\eta\cdot\nabla_y u^\Pi(z^2\eta,z)|^2\big)d\eta \\
+ &\; |\omega(z)|^{-2} z^{-2} \Big| \int\limits_{\omega(z)} u^\Pi(y,z)dy\Big|^2\Big)dz \\
\le &\; c \int\limits_0^{2d/3} z^{2(\beta+n-1)}
\end{aligned}$$

$$\times \Big(z^{-2(n-1)} \int\limits_{\omega(z)} \big(|\partial_z u^{\Pi}(y,z)|^2 + z^2 c_\omega |\nabla_y u^{\Pi}(y,z)|^2\big) dy$$

$$+ \ c_\omega z^{-2(n-1)} z^2 \int\limits_{\omega(z)} |u^{\Pi}(y,z)|^2 dy\Big) dz \leq C \| u^{\Pi}; V^1_\beta(\Pi_{2d/3})\|^2. \tag{2.39}$$

The same decomposition (2.37) applies to the test function v in (2.32). We plug both the decompositions into the integral identity (2.32) and write the system

$$(\partial_z u_0^{\Pi}, \partial_z v_0^{\Pi})_{\Pi_{2d/3}} + a_0(u_0^{\Pi}, v_0^{\Pi})_{\varpi_{2d/3}} = F^{\Pi}(v_0^{\Pi}) - S_0^{\Pi}(v_0^{\Pi}), \tag{2.40}$$

$$(\nabla_x u_\perp^{\Pi}, \nabla_x v_\perp^{\Pi})_{\Pi_{2d/3}} + a_0(u_0^{\Pi}, v_\perp^{\Pi})_{\varpi_{2d/3}} = F^{\Pi}(v_\perp^{\Pi}) - S_\perp^{\Pi}(v_\perp^{\Pi}), \tag{2.41}$$

where v_0^{Π} and $v_\perp^{\Pi}$ are arbitrary elements of the appropriate function spaces and the cross terms are given by

$$\begin{aligned} S_0^{\Pi}(v_0^{\Pi}) &= (\partial_z u_\perp^{\Pi}, \partial_z v_0^{\Pi})_{\Pi_{2d/3}} + a_0(u_\perp^{\Pi}, v_0^{\Pi})_{\varpi_{2d/3}}, \\ S_\perp^{\Pi}(v_\perp^{\Pi}) &= (\partial_z u_0^{\Pi}, \partial_z v_\perp^{\Pi})_{\Pi_{2d/3}} + a_0(u_0^{\Pi}, v_\perp^{\Pi})_{\varpi_{2d/3}}. \end{aligned} \tag{2.42}$$

First of all, we investigate the solvability of the problems (2.40) with $S_0^{\Pi} = 0$ and (2.41) with $S_\perp^{\Pi} = 0$; these are denoted by $(2.40)'$ and $(2.32)'$, respectively.

PROPOSITION 2.10. *If $d > 0$ is small and $\beta \neq \beta_\pm$, then the problem $(2.40)'$ has a unique solution $u_0^{\Pi} \in V^1_{\beta+n-1}(0, 2d/3)$ which satisfies the estimate*

$$\begin{aligned} &\| u_0^n; V^1_\beta(\Pi_{2d/3})\| \leq c_0 \| u_0^n; V^1_{\beta+n-1}(0, 2d/3)\| \\ \leq \ & C_0 \| F^{\Pi}; V^1_{-\beta}(\Pi_{2d/3})^* \| \leq c_d \| F; V^1_{-\beta}(\Omega)^*\|, \end{aligned} \tag{2.43}$$

where the constants c_0 and C_0 are independent of d.

PROOF. Performing integration with respect to y, we rewrite $(2.40)'$ as follows:

$$\begin{aligned} &|\omega| (z^{2(n-1)} \partial_z u_0^{\Pi}, \partial_z v_0^{\Pi})_{(0,2d/3)} + a_0 |\partial\omega| (z^{2(n-2)} u_0^{\Pi}, v_0^{\Pi})_{(0,2d/3)} \\ + \ & (s(z) z^{2(n-2)} u_0^{\Pi}, v_0^{\Pi})_{(0,2d/3)} = F^{\Pi}(v_0^{\Pi}) \ , \ v_0^{\Pi} \in V_{-\beta+n-1}(0, 2d/3). \end{aligned} \tag{2.44}$$

The term with the multiplier

$$s(z) = \int_{\partial\omega} \Big((1 + 4z^2|\eta \cdot \nu'(\eta)|^2 - 1\Big) ds_\eta = \mathcal{O}(z^2) \tag{2.45}$$

arises from (1.28), (1.29), and it is small when $d \ll 1$. The integral identity (2.44) with $s = 0$ generates the Cauchy problem

$$\begin{aligned} -\frac{d}{dz} z^{2(n-1)} \frac{du_0^\Pi}{dz}(z) - A z^{2(n-2)} u_0^\Pi &= \mathcal{F}(z), \ z \in (0, 2d/3), \\ &\partial_z u_0^\Pi(2d/3) = 0. \end{aligned} \tag{2.46}$$

We recall the notation in Section 2.1, the Euler change of variables $z \mapsto t = -\ln z$ which reduces (1.32) to an ordinary differential equation with constant coefficients, and the standard variation of constants method. Putting together theses pieces of information provides the assertion. □

Let $V_\beta^1(\Pi_{2d/3})_\perp$ be the subspace of functions $u_\perp^\Pi \in V_\beta^1(\Pi_{2d/3})$ satisfying the orthogonality conditions in (2.38) for almost all $z \in (0, 2d/3)$. Note that $\mathrm{codim} V_\beta^1(\Pi_{2d/3})_\perp = \infty$.

Proposition 2.11. *For a small $d > 0$, the problem $(2.41)'$ has a unique solution $u_\perp^\Pi \in V_\beta^1(\Pi_{2d/3})_\perp$ such that*

$$\begin{aligned} &\|z^\beta \nabla_x u_\perp^\Pi; L^2(\Pi_{2d/3})\| + \|z^{\beta-2} u_\perp^\Pi; L^2(\Pi_{2d/3})\| \\ \leq\ & c_0 \|F^\Pi; V_{-\beta}^1(\Pi_{3d/2})^*\| \leq c_d \|F; V_{-\beta}^1(\Omega)^*\|, \end{aligned} \tag{2.47}$$

where c_0 does not depend on d.

Proof. If $w_\perp^\Pi \in V_\beta^1(\Pi_{2d/3})_\perp$, then $v_\perp^\Pi = z^{2\beta} w_\perp^\Pi \in V_{-\beta}^1(\Pi_{2d/3})_\perp$. Hence, we can write the problem $(2.41)'$ in the form

$$\begin{aligned} &(z^\beta \nabla_x u_\perp^\Pi, z^\beta \nabla_x w_\perp^\Pi)_{\Pi_{2d/3}} + \beta (z^\beta \nabla_x u_\perp^\Pi, z^{\beta-1} w_\perp^\Pi)_{\Pi_{2d/3}} \\ +\ & a_0 (z^\beta u_\perp^\Pi, z^\beta w_\perp^\Pi) = F^\Pi(z^{2\beta} w_\perp^\Pi)\ ,\ w_\perp^\Pi \in V_\beta^1(\Pi_{2d/3})_\perp. \end{aligned} \tag{2.48}$$

By the inequality (2.12) with $\gamma = 1$ and the formula $z^\beta \nabla_x u_\perp^\Pi = \nabla_x(z^\beta u_\perp^\Pi) - \beta z^{\beta-1} u_\perp^\Pi$, we have

$$\|z^\beta \nabla_x u_\perp^\Pi; L^2(\Pi_{2d/3})\|^2 \geq \frac{1}{2} \|\nabla_x (z^\beta u_\perp^\Pi); L^2(\Pi_{2d/3}\|^2$$

$$
\begin{aligned}
- &\quad \beta^2 \| z^{\beta-1} u_\perp^\Pi ; L^2(\Pi_{2d/3}) \|^2 \geq \frac{1}{4} \| \nabla (z^\beta u_\perp^\Pi); L^2(\Pi_{2d/3}) \|^2 \\
+ &\quad \frac{c}{4} \| z^{\beta-2} u_\perp^\Pi ; L^2(\Pi_{2d/3}) \|^2 - \beta^2 \| z^{\beta-1} u_\perp^\Pi ; L^2(\Pi_{2d/3}) \|^2 .
\end{aligned}
$$

Choosing a small $d > 0$, we can make the last term smaller than the previous one, since the exponent $\beta - 1$ is larger than $\beta - 2$. Thus, the first term on the left of (2.48) serves for a scalar product in $V_\beta^1(\Pi_{2d/3})_\perp$, and the left–hand side of (2.47) does not exceed $c \| z^\beta \nabla_x u_\perp^\Pi (\Pi_{2d/3}) \|^2$ with $c > 0$. Furthermore, using Lemma 2.8 again, we see that

$$
\begin{aligned}
&\| z^{\beta-1} w_\perp^\Pi ; L^2(\Pi_{2d/3}) \|^2 \leq c d \| z^\beta \nabla_x w_\perp^\Pi ; L^2(\Pi_{2d/3}) \|^2 , \\
&| (z^\beta u_\perp^\Pi , z^\beta w_\perp^\Pi)_{\varpi_{2d/3}} | \leq c d^2 | (z^{\beta-1} u_\perp^\Pi , z^{\beta-1} w_\perp^\Pi)_{\varpi_{2d/3}} | \\
\leq &\quad c d^2 \| z^\beta \nabla_x u_\perp^\Pi ; L^2(\Pi_{2d/3}) \| \| z^\beta \nabla_x w_\perp^\Pi ; L^2(\Pi_{2d/3}) \| .
\end{aligned}
$$

These mean that the second and third terms on the left of (2.48) give rise to small operators in the Hilbert space $V_\beta^1(\Pi_{2d/3})_\perp$ so that our assertion is established by the Riesz representation theorem. □

Differentiating the orthogonality condition (2.38) with respect to z and using (2.16) we observe that

$$
0 = 2 \int_{\omega(z)} z^{-1} y \cdot \nabla_y u_\perp^\Pi (y,z) dy + \int_{\omega(z)} \partial_z u_\perp^\Pi (y,z) dy. \tag{2.49}
$$

Hence, replacing the last integral in (2.49) by the first one and taking into account that $|y| \leq c_\omega z^2$ on $\omega(z)$, we infer

$$
\begin{aligned}
&| (\partial_z u_\perp^\Pi , \partial_z v_0^\Pi)_{\Pi_{2d/3}} | = \Big| \int_0^{2d/3} \partial_z v_0^\Pi (z) \int_{\omega(z)} \partial_z u_\perp^\Pi (y,z) dy dz \Big| \\
\leq &\quad c \int_0^{2d/3} z^{-\beta} | \partial_z v_0^\Pi (z) | \int_{\omega(z)} z^{\beta+1} | \nabla_y u_\perp^\Pi (y,z) | dy dz \\
\leq &\quad c d \| z^\beta \nabla_y u_\perp^\Pi ; L^2(\Pi_{2d/3}) \| \| z^{-\beta} \partial_z v_0^\Pi ; L^2(\Pi_{2d/3}) \|
\end{aligned}
$$

and

$$
| (\partial_z u_0^\Pi , \partial_z v_\perp^\Pi)_{\Pi_{2d/3}} |
$$

$$\leq \quad cd\|z^{\beta}\partial_z u_0^{\Pi}; L^2(\Pi_{2d/3})\|\|z^{-\beta}\nabla_y v_{\perp}^{\Pi}; L^2(\Pi_{2d/3})\|.$$

These, together with the estimates

$$\begin{aligned} & |(u_{\perp}^{\Pi}, v_0^{\Pi})_{\varpi_{2d/3}}| \leq c\|z^{\beta}\nabla_y u_{\perp}^{\Pi}; L^2(\Pi_{2d/3})\|\|z^{-\beta+1}v_0^{\Pi}; L^2(\varpi_{2d/3})\| \\ \leq \quad & c\|z^{\beta}\nabla_y u_{\perp}^{\Pi}; L^2(\Pi_{2d/3})\|\|z^{-\beta}v_0^{\Pi}; L^2(\Pi_{2d/3})\| \\ \leq \quad & cd\|z^{\beta}\nabla_y u_{\perp}^{\Pi}; L^2(\Pi_{2d/3})\|\|z^{-\beta-1}v_0^{\Pi}; L^2(\Pi_{2d/3})\|, \end{aligned}$$

$$|(u_0^{\Pi}, v_{\perp}^{\Pi})_{\varpi_{2d/3}}| \leq cd\|z^{\beta-1}u_0^{\Pi}; L^2(\Pi_{2d/3})\|\|z^{-\beta}\nabla_y v_{\perp}^{\Pi}; L^2(\Pi_{2d/3})\|, \tag{2.50}$$

taken from Lemma 2.8 with $\gamma = 1$, ensure that the cross terms (2.42) generate in (2.40) and (2.41) operators with norms of order d. Thus, Propositions 2.10 and 2.11 establish the unique solvability of the full problems (2.40), (2.41) in case $d > 0$ is fixed small. As a corollary, the same holds for the second auxiliary problem (2.32).

Recalling the notation (2.35), we take any $v \in V_{-\beta}^1(\Omega)$ and put the test functions $v^{\Omega} = \chi'_{\Omega}v \in H^1(\Omega(d/3))$ and $v^{\Pi} = \chi'_{\Pi}v \in V_{-\beta}^1(\Pi_{2d/3})$ into the integral identities (2.31) and (2.32). After commuting twice the cut–off functions with the gradient operator, we have

$$\begin{aligned} & (\nabla_x(\chi'_{\Omega}u^{\Omega}), \nabla_x v)_{\Omega} + (\nabla_x u^{\Omega}, v\nabla_x\chi'_{\Omega})_{\Omega} - (u^{\Omega}\nabla_x\chi'_{\Omega}, \nabla_x v)_{\Omega} \\ + \quad & a(\chi'_{\Omega}u^{\Omega}, v) + M(\chi'_{\Omega}u^{\Omega}, v)_{\Omega} = F(\chi'_{\Omega}v), \end{aligned} \tag{2.51}$$

and

$$\begin{aligned} & (\nabla_x(\chi'_{\Pi}u^{\Pi}), \nabla_x v)_{\Omega} + (\nabla_x u^{\Pi}), v\nabla_x\chi'_{\Pi})_{\Omega} - (u^{\Pi}\nabla_x\chi'_{\Pi}, \nabla_x v)_{\Omega} \\ + \quad & a_0(\chi'_{\Pi}u^{\Pi}, v) = F(\chi'_{\Pi}v). \end{aligned}$$

The integrals can be extended over Ω and Π using the disposition of the supports of the cut–off functions. In view of (2.34) and (2.35), the summation of the above relations gives

$$(\nabla_z R_{\beta}F, \nabla_x v)_{\Omega} + (aR_{\beta}F, v)_{\Gamma} = F(v) - (S_{\beta}F)(v) \ , \quad v \in V_{-\beta}^1(\Omega), \tag{2.52}$$

where S_{β} is an operator in $V_{-\beta}^1(\Omega)^*$ defined by

$$\begin{aligned} & (S_{\beta}F)(v) = M(\chi' u^{\Omega}, v)_{\Omega} + ((a_0 - a)\chi'_{\Pi}u^{\Pi}, v)_{\Gamma} \\ + \quad & (\nabla_x u^{\Omega}, v\nabla_x\chi'_{\Omega})_{\Omega} \end{aligned}$$

$$(2.53) \qquad - \quad (u^\Omega \nabla_x \chi'_\Omega, \nabla_x v)_\Omega + (\nabla_x u^\Gamma, v \nabla_x \chi'_\Pi)_\Omega - (u^\Gamma \nabla_x \chi'_\Pi, \nabla_x v)_\Omega.$$

By estimates (2.43) and (2.47), the operators R_β and S_β are continuous. Moreover, S_β is compact. Indeed, the first and second terms on the right of (2.53) give rise to compact operators. The reasons are that the embedding $H^1(\Omega) \subset L^2(\Omega)$ is compact and that

$$|((a - a^0)\chi'_\Pi u^\Pi, v)_{\partial\varpi_\varepsilon}| \le c\varepsilon^\alpha \|u^\Pi; V^1_\beta(\Pi_{2d/3})\| \|v; V^1_{-\beta}(\Omega)\|$$

for any ε (due to the stabilization condition (1.5)), and that the embedding $V^1_\beta(\Pi_{2d/3} \setminus \Pi_\varepsilon) \subset L^2(\varpi_{2d/3} \setminus \varpi_\varepsilon)$ is compact (the peak top is cut off).

The last four terms in (2.53) do not contain products of derivatives of u^Ω, u^Γ and v, therefore, they produce compact operators as well. In other words, the property (2.28) of the operator $S_\beta = I - T_\beta R_\beta$ holds true, and hence the desired parametrix R_β is constructed and Theorem 2.5 is proven. □

Corollary 2.12. *If β meets the restriction* (2.6)*, any function $u \in V^1_\beta(\Omega)$ satisfies the relation*

$$(2.54) \qquad \|u; V^1_\beta(\Omega)\| \le c_d(\beta)\big(\|T_\beta u; V^1_{-\beta}(\Omega)^*\| + \|u; L^2(\Omega(d))\|\big),$$

where the factor $c_d(\beta)$ is independent of u.

Proof. Theorem 2.5 ensures that T_β is Fredholm, so, the dimension of the kernel $\ker T_\beta$ is finite, and the estimate

$$\|u; V^1_\beta(\Omega)\| \le c_f\big(\|T_\beta u; V^1_{-\beta}(\Omega)^*\| + |f(u)|\big)$$

holds. Here, f is any nonlinear weakly continuous functional in $V^1_\beta(\Omega)$ such that

$$\begin{aligned} f(tu) &= tf(u), \ t \in [0, +\infty) \ , \ u \in V^1_\beta(\Omega) \ ; \\ f(u) &= 0 \ , \ u \in \ker T_\beta \Leftrightarrow u = 0. \end{aligned}$$

Since a nontrivial harmonic function cannot vanish everywhere in $\Omega(d)$, we take $f(u) = \|u; L^2(\Omega(d))\|$ and finish the proof. □

Remark 2.13. To verify that the range $T_{\beta_\pm}(V^1_{\beta_\pm}(\Omega))$ is not closed in $V^1_{-\beta_\pm}(\Omega)^*$, we introduce a family of test functions

$$(2.55) \qquad u^m(x) = \chi_m(-\ln z) U^\mp(z)$$

Fig. 2.3. A cut-off function.

with small supports located at the peak $\overline{\Pi}_1$. Here, $U^{\pm}$ are given in (1.34),

$$\chi_m(t) = \chi_0(t - 2^{m+1} + 1)\chi_0(2^m + t + 1), \tag{2.56}$$

and $\chi_0 \in C^\infty(\mathbb{R})$, $0 \le \chi \le 1$, $\chi_0(t) = 1$ for $t \le 0$ and $\chi_0(t) = 0$ for $t \ge 1$. The graph of (2.56) is depicted in Fig. 2.3. By (1.34), (2.6) and (2.16), we have

$$|u^m(z)| = z^{-\beta_\pm - n + 3/2} \quad \text{for} \quad z \in [\exp(-2^{m+1} + 1), \exp(-2^m - 1)],$$

hence,

$$\begin{aligned} &\|u^m; V^1_{\beta_\pm}(\Omega)\|^2 \ge \|u^m; L^2_{\beta_\pm - 1}(\Pi_1)\|^2 \ge \\ \ge &\int\limits_{\exp(-2^{m+1}+1)}^{\exp(-2^m-1)} r^{2(\beta_\pm - 1)} |\omega(z)| z^{-2\beta_\pm - 2n + 3} dz \ge \int\limits_{\exp(-2^{m+1}+1)}^{\exp(-2^m-1)} \frac{dz}{z} \\ \ge &\ c(2^{m+1} - 2^m - 2) = c(2^m - 2)\ ,\ c > 0. \end{aligned} \tag{2.57}$$

Differentiating the function (2.55) in the formula

$$(T_{\beta_\pm} u^m, v)_\Omega = -(\Delta u^m, v)_\Omega + (\partial_\nu u^m, v)_\Gamma\ ,\ v \in V^1_{-\beta_\pm}(\Omega)^* \tag{2.58}$$

and using (1.27) for the normal ν, we deduce that

$$\Delta u^m = \chi_m f_0^m + f_\chi^m\ ,\ f_0^m(z) = \lambda_\mp(\lambda_\mp - 1) z^{\lambda_\mp - 2}$$

and

$$\partial_\nu u^m + a u^m = \chi_m g_0^m + \chi_m \widetilde{g}^m + g_\chi^m\ ,\ g_0^m(y,z) = \eta \cdot \nu'(\eta)\lambda_\pm - a_0\,,$$

where

$$|f_\chi^m(z)| \le c z^{\mathrm{Re}\lambda_\mp - 2}\ ,\ |g_\chi^m(y,z)| \le c z^{\mathrm{Re}\lambda_\mp}\ ,\ |\widetilde{g}^m(y,z)| \le c z^{\mathrm{Re}\lambda_\mp \min\{\alpha,1\}}\,,$$

$$f_\chi(z)=0\ ,\ g_\chi(y,z)=0 \text{ for } z\notin[\exp(-2^{m+1}),\exp(-2^{m+1}+1)] \cup[\exp(-2^m-1),\exp(-2^m)].$$

In view of the disposition of $\operatorname{supp} f_\chi^m$ and $\operatorname{supp} g_\chi^m$, we have

$$\begin{aligned} &\|f_\chi^m; L^2_{\beta_\pm+1}(\Omega)\|^2 \\ \le\ & c\Big(\int\limits_{\exp(-2^{m+1})}^{\exp(-2^{m+1}+1)} + \int\limits_{\exp(-2^m-1)}^{\exp(-2^m)}\Big) z^{2(\beta_\pm+1)} z^{2(n-1)} z^{2(\mathrm{Re}\lambda_\mp-2)}dz \le C, \\ &\|g_\chi^m; L^2_{\beta_\pm}(\Gamma)\|^2 \\ \le\ & c\Big(\int\limits_{\exp(-2^{m+1})}^{\exp(-2^{m+1}+1)} + \int\limits_{\exp(-2^m-1)}^{\exp(-2^m)}\Big) z^{2\beta_\pm} z^{2(n-2)} z^{2\mathrm{Re}\lambda_\mp}dz \le C. \end{aligned}$$

Moreover,

$$\|\chi_m\widetilde{g}^m; L^2_{\beta_\pm}(\Gamma)\| \le C\exp(-\min\{\alpha,1\}2^m).$$

Recalling a calculation in Remark 1.3 we also observe that

$$I_m(v) := (\chi_m f_0^m, v)_\Omega + (\chi_m g_0^m, v)_\Gamma = (\chi_m f_0^m, v_\perp)_{\Pi_d} + (\chi_m g_0^m, v_\perp)_{\varpi_d},$$

where $v_\perp$ is the component in the decomposition (2.17). The inequality (2.12) with $\gamma=1$ yields

$$|I_m(v)| \le \exp(-2^m).$$

The above formulas provide the estimate

$$\|T_{\beta_\pm}u^m; V^1_{-\beta_\pm}(\Omega)^*\| \le C,$$

which contradicts with (2.57), if we assume that the range $T_{\beta_\pm}(V^1_{\beta_\pm}(\Omega))$ is a closed subspace; the reason is that $\ker T_{\beta_\pm}\subset\ker T_\beta$ for any index $\beta>\beta_\pm$ which is not forbidden, and hence $\dim\ker T_\beta<+\infty$, owing to the already exposed part of the proof of Theorem 2.5.

2.4. Dimension reduction and the proof of Theorem 2.6

First of all, the result on asymptotics is local, because Theorem 2.6 does not improve properties of the solution $u \in V^1_\beta(\Omega)$ outside a neighbourhood of the peak top $\mathcal{O}$. Indeed, by the definition (2.2), a function $v \in V^1_\beta(\Omega)$ with support in $\overline{\Omega} \setminus \mathcal{O}$ belongs to the Sobolev space $H^1(\Omega)$ and moreover to the Kondratiev space $V^1_\gamma(\Omega)$ with any weight exponent. Therefore, we may work with a function $u \in V^1_\beta(\Omega)$ which vanishes outside the peak (1.1) with any fixed $d > 0$.

We again employ the decomposition (2.37) for $u(y, z)$ with the terms $u_0(z)$ and $u_\perp(y, z)$. We have the estimate

$$\begin{aligned} &\|u_0; V^1_{\beta+n-1}(0, d)\| + \|z^\beta \nabla_x u_\perp; L^2(\Pi_d)\| \\ + \quad &\|z^{\beta-2} u_\perp; L^2(\Pi_d)\| \leq c\|u; V^1_\beta(\Omega)\| \end{aligned} \tag{2.59}$$

(cf. (2.39) and Lemma 2.8). The following integral identities are derived from (2.1) similarly to (2.40), (2.44) and (2.41), but without using the decomposition for the test function:

$$\begin{aligned} &(z^{2(n-1)}\partial_z u_0, \partial_z v_0)_{(0,d)} + a_0(z^{2(n-2)} u_0, v_0)_{(0,d)} \\ = \quad &F_0(v_0) := F(v_0) - \mathcal{F}_0(v_0) \ , \quad v_0 \in C^\infty_c(0, d), \end{aligned} \tag{2.60}$$

$$\begin{aligned} &(\nabla_x u_\perp, \nabla_x v)_{\Pi_d} + (a u_\perp, v)_{\omega_d} \\ = \quad &F_\perp(v) := F(v) - \mathcal{F}_\perp(v) \ , \quad v \in C^\infty_c(\overline{\Pi}_d \setminus \mathcal{O}). \end{aligned} \tag{2.61}$$

Here, the perturbation terms

$$\begin{aligned} \mathcal{F}_0(v_0) \quad &= \quad (\partial_z u_\perp, \partial_z v_0)_{\Pi_d} + (a u_\perp, v_0)_{\varpi_d} \\ &- \quad \big((a_0 - a)u_0, v_0\big)_{\varpi_d} - (s z^{2(n-2)} u_0, v_0)_{(0,d)}, \end{aligned} \tag{2.62}$$

$$\mathcal{F}_\perp(v) \quad = \quad (\partial_z u_0, \partial_z v)_{\Pi_d} + (a u_0, v)_{\varpi_d} \tag{2.63}$$

are analogous to (2.42) and (2.45). Note that (2.60) is obtained just by setting $v = v_0$ in (2.1).

To investigate the problems (2.60) and (2.61), we need the next auxiliary assertions.

Proposition 2.14. *Assume that $u_\perp \in V^1_\beta(\Pi_d)$ satisfies the integral identity* (2.61), *where the functional $F_\perp$ meets the bound*

$$|F_\perp(v)| \quad \leq \quad N_F\Big(\|r^{-\theta}\nabla_x v; L^2(\Pi_d)\| + \|r^{-\theta-2} v; L^2(\Pi_d)\|\Big)$$

$$\leq \quad cN_F\|v; V^1_{-\theta}(\Pi_d)\| \tag{2.64}$$

for some $\theta \in (-\infty, \beta)$ *and any* $v \in V^1_{-\theta}(\Pi_d)$ *with zero mean as in* (2.11). *Then* $u_\perp$ *belongs to the space* $V^1_\theta(\Pi_d)$*, and there holds the inequality*

$$\|r^\theta \nabla_x u_\perp; L^2(\Pi_d)\| + \|r^{\theta-2}u_\perp; L^2(\Pi_d)\| \leq cN_F \tag{2.65}$$

PROOF. By a completion argument, we extend (2.61) for test functions in $V^1_{-\beta}(\Pi_d)$ and then introduce the weight function

$$\mathcal{R}_\zeta(z) = \begin{cases} z^\theta & \text{for } z \geq \zeta, \\ z^\beta \zeta^{-\beta+\theta} & \text{for } z \leq \zeta, \end{cases} \tag{2.66}$$

where $\zeta \in (0, d)$ is a parameter which will eventually be sent to $+0$. Since $\mathcal{R}_\zeta(z) \leq c(\zeta)z^\beta$ and $|\partial_z \mathcal{R}(z)| \leq c(\zeta)z^{\beta-1}$, we find that $v = \mathcal{R}^2_\zeta u_\perp$ belongs to $V^1_{-\beta}(\Pi_d)$ and hence it can be inserted as a test function into (2.61). We denote $U_\perp = \mathcal{R}_\zeta u_\perp \in V^1_0(\Pi_d)$, and after simple transformations, namely commuting $\mathcal{R}_\zeta$ and ∇_x several times, we take the real part of (2.61) and convert it into

$$\begin{aligned} \|\nabla_x U_\perp; L^2(\Pi_d)\|^2 &= \mathrm{Re} F_\perp(\mathcal{R}_\zeta U_\perp) + \|U_\perp \mathcal{R}^{-1}_\zeta \nabla_x \mathcal{R}_\zeta; L^2(\Pi_d)\|^2 \\ &- (aU_\perp, U_\perp)_{\varpi_d}. \end{aligned} \tag{2.67}$$

We emphasize once more that all integrals here converge absolutely. The weight function (2.66) does not depend on y and meets the estimates

$$|\mathcal{R}_\zeta(z)| \leq z^\theta \ , \ |\partial_z \mathcal{R}_\zeta(z)| \leq cz^{\theta-1} \ , \ |\mathcal{R}_\zeta(z)\partial_z \mathcal{R}_\zeta(z)| \leq cz^{-1} \tag{2.68}$$

with a constant independent of ζ. Note that $U_\perp$ still satisfies the orthogonality condition (2.11) and, by Lemma 2.8 with $\gamma = 1$,

$$\|z^{-2}U_\perp; L^2(\Pi_d)\| + \|z^{-1}U_\perp; L^2(\varpi_d)\| \leq c\|\nabla_x U_\perp; L^2(\varpi_d)\|. \tag{2.69}$$

Therefore,

$$\|U_\perp R^{-1}_\zeta \nabla R_\zeta; L^2(\Pi_d)\|^2 + |(aU_\perp, U_\perp)_{\varpi_d}| \leq cd^2\|\nabla_x U_\perp; L^2(\Pi_d)\|^2. \tag{2.70}$$

Moreover, owing to the first couple of formulas in (2.68), the inequality (2.64) gives

$$\begin{aligned} |F_\perp(\mathcal{R}_\zeta U_\perp)| &\leq N_F\big(\|r^{-\theta}\nabla_x(\mathcal{R}_\zeta U_\perp);L^2(\Pi_d)\| + \|r^{-\theta-2}\mathcal{R}_\zeta U_\perp;L^2(\Pi_d)\|\big) \\ &\leq cN_F\big(\|\nabla_x U_\perp;L^2(\Pi_d)\| + \|z^{-2}U_\perp;L^2(\Pi_d)\|\big). \end{aligned} \tag{2.71}$$

Collecting together these calculations, we obtain for small $d > 0$ that

$$\begin{aligned} &\|\mathcal{R}_\zeta\nabla_x u_\perp;L^2(\Pi_d)\|^2 + \|z^{-2}\mathcal{R}_\zeta u_\perp;L^2(\Pi_d)\|^2 \\ \leq\ & c\big(\|\nabla_x U_\perp;L^2(\Pi_d)\|^2 + \|z^{-2}U_\perp;L^2(\Pi_d)\|^2\big) \\ \leq\ & cN_F\|\nabla_x U_\perp;L^2(\Pi_d)\|. \end{aligned} \tag{2.72}$$

Thus, the left hand side of (2.72) remains uniformly bounded when $\zeta \to +0$. It increases monotonely, cf. (2.66), and thus has a limit which readily becomes (2.65). □

To conclude the estimate

$$\|u_\perp;V^1_\theta(\Pi_d)| \leq c\big(\|u;V^1_\beta(\Omega)\| + \|F;V^1_{-\theta}(\Omega)^*\|\big) \tag{2.73}$$

for θ as in (2.7) of Theorem 2.6, it suffices to verify the condition (2.64) for the functional (2.63), because this property is obvious for the functional $F \in V^1_{-\theta}(\Omega)^*$, due to Lemma 2.8. For the last term in (2.63), the inequalities (2.3), (2.59) for u_0 and (2.12) for v (with $\gamma = 1$, $\beta \mapsto -\beta + 1$ and orthogonality condition (2.11)) give

$$\begin{aligned} |(au_0,v)_{\varpi_d}| &\leq c\|z^\beta u_0;L^2(\varpi_d)\|\,\|z^{-\beta}v;L^2(\varpi_d)\| \\ &\leq c\|u_0;V^1_\beta(\Pi_d)\|\,\|z^{-\beta+1}\nabla_x v;L^2(\Pi_d)\| \\ &\leq c\|u_0;V^1_{\beta+n-1}(0,d)\|\,\|z^{-\theta}\nabla_x v;L^2(\Pi_d)\| \\ &\leq C_{u_0}\|r^{-\theta}\nabla_x v;L^2(\Pi_d)\|, \end{aligned}$$

where the inequality $-\beta+1 \leq -\theta$ for exponents follows from the restriction (2.7). The same argument and (2.49) for v (recall that (2.11) is assumed) is used in the calculation

$$\begin{aligned} &|(\partial_z u_0,\partial_z v)_{\Pi_d}| = 2\big|(\partial_z u_0, z^{-1}y\cdot\nabla_y v)_{\Pi_d}\big| \\ \leq\ & c\|z^\beta\partial_z u_0;L^2(\Pi_d)\|\,\|z^{-\beta-1}y\cdot\nabla_y v;L^2(\Pi_d)\| \end{aligned}$$

$$\begin{aligned} &\leq c\|u_0; V^1_{\beta+n-1}(0,d)\| \, \|z^{-\beta+1}\nabla_y v; L^2(\Pi_d)\| \\ &\leq C_{u_0}\|r^{-\theta}\nabla_x v; L^2(\Pi_d)\|. \end{aligned}$$

The desired property of the functional (2.63) is thus verified and the proof of (2.73) is complete. □

The problem (2.60) leads to the variational formulation of the ordinary differential equation derived in Section 2.1, although in a perturbed form. Thus, the next assertion on the asymptotics of its solution can be obtained by standard methods. However, we first need to give appropriate estimates for the perturbation terms.

Using the stabilization condition (1.5) and the relation (2.45) for the coefficients generated by the Jacobian in (1.29), we easily derive the estimate

$$\begin{aligned} &|((a_0-a)u_0, v_0)_{\varpi_d}| + |(sz^{n-2}u_0, v_0)_{(0,d)}| \\ \leq\;& c\|z^{\beta+n-2}u_0; L^2(0,d)\|\Big(\|z^{-\beta+\alpha+n-2}v_0; L^2(0,d)\| \\ +\;& \|z^{2-\beta+n-2}v_0; L^2(0,d)\|\Big) \\ \leq\;& c\|u_0; V^1_{\beta+n-1}(0,d)\| \, \|v_0; V^1_{-\theta+n-1}(0,d)\|. \end{aligned} \tag{2.74}$$

Notice that by (2.7), $\alpha - \beta \geq -\theta$. Based on the result which has been concluded in Proposition 2.14, we now make use of the inequality (2.12) to get the estimates of the two other terms in (2.62):

$$\begin{aligned} &|(\partial_z u_\perp, \partial_z v_0)_{\Pi_d}| \leq \|z^{\theta}\partial_z u_\perp; L^2(\Pi_d)\| \, \|z^{-\theta}\partial_z v_0; L^2(\Pi_d)\| \\ \leq\;& c\|u_\perp; V^1_\theta(\Pi_d)\| \, \|v_0; V^1_{-\theta+n-1}(0,d)\|, \end{aligned} \tag{2.75}$$

and

$$\begin{aligned} &|(au_\perp, v_0)_{\varpi_d}| \leq c\|z^{\theta}u_\perp; L^2(\varpi_d)\| \, \|z^{-\theta}v_0; L^2(\varpi_d)\| \\ \leq\;& c\|u_\perp; V^1_\theta(\Pi_d)\| \, \|v_0; V^1_{-\theta+n-1}(0,d)\|. \end{aligned} \tag{2.76}$$

Formulas (2.74) and (2.75) mean that $\mathcal{F}_0$ is a continuous functional on $V^1_{-\theta+n-1}(0,d)$ and, by virtue of (2.59) and (2.73), the norm of $\mathcal{F}_0$ does not exceed $c\|u; V^1_\beta(\Omega)\|$.

PROPOSITION 2.15. *The asymptotic representation*

$$v_0(z) = \sum_{\pm} K_\pm U_\pm(z) + \tilde{v}_0(z) \quad , \quad z \in (0,d), \tag{2.77}$$

holds true; the notation is as in Theorem 2.6 and in particular the coefficients $K_\pm$ *satisfy the same requirement as in* (2.8)–(2.9). *Here,* $\tilde v \in V^1_{\theta+n-1}(0,d)$ *and moreover*

$$\|\tilde v_0; V^1_{\theta+n-1}(0,d)\| + \sum_\pm |K_\pm| \le c\big(\|F; V^1_{-\theta}(\Omega)\| + \|u; V^1_\beta(\Omega)\|\big). \tag{2.78}$$

The representation (2.8) and the inequality (2.10) in Theorem 2.6 follow by applying Propositions 2.14 and 2.15 to the components in the formula $u = u_0 + u_\perp$ (cf. (2.37)).

2.5. Calculation of the index

Let us assume the condition (2.6) and calculate the index

$$\operatorname{Ind} T_\beta = \dim \ker T_\beta - \dim \operatorname{coker} T_\beta, \tag{2.79}$$

where $\ker T_\beta$ and $\operatorname{coker} T_\beta$ stand for the kernel and cokernel of the operator (2.4).

Above the threshold, i.e., for $a_0 > -a_\ddagger$, the interval $I(\beta_-, \beta_+)$, see (2.6), includes $\beta_0 = 0$ while, by the definition, the Fredholm operator T_0 is self-adjoint, hence, for $\beta = 0$ we have

$$\operatorname{Ind} T_\beta = 0. \tag{2.80}$$

By Theorem 2.6 on asymptotics, ker $T_\beta = \ker T_\theta$, if the interval $I(\beta,\theta)$ does not include any of the indices $\beta_\pm$. Noting that $I(\beta,\theta)$ and $I(-\theta,-\beta)$ are free of the forbidden indices simultaneously and thus $\operatorname{coker} T_\beta = \operatorname{coker} T_\theta$, we extend the formula (2.80) for

$$|\beta| < \sqrt{\Big(n - \frac{3}{2}\Big)^2 + a_0 \frac{|\partial\omega|}{|\omega|}}. \tag{2.81}$$

Assume that $\beta > \beta_- > \theta$ and that the hypotheses of Theorem 2.6 are fulfilled. We introduce the space $\mathfrak{V}^1_\theta(\Omega)$ consisting of functions u of the form

$$u(x) = \chi_0(x) K_- U^-(x) + \tilde u(x)$$

with the norm

$$\|u; \mathfrak{V}^1_\theta(\Omega)\| = \big(|K_-|^2 + \|\tilde u; V^1_\theta(\Omega)\|^2\big)^{1/2}.$$

Theorem 2.6 passes all the properties of the operator T_β (under our conditions) to the operator

$$\mathfrak{T}_\theta : \mathfrak{V}^1_\theta(\Omega) \to V^1_\theta(\Omega)^*.$$

In particular, $\ker \mathfrak{T}_\theta = \ker T_\beta$ and $\operatorname{coker} \mathfrak{T}^1_\theta = \operatorname{coker} T_\beta$. Since the dimension of the quotient space $\mathfrak{V}^1_\theta(\Omega)/V^1_\theta(\Omega)$ is equal to 1, we have

$$\operatorname{Ind} T_\theta = \operatorname{Ind} \mathfrak{T}_\theta - 1 = \operatorname{Ind} T_\beta - 1 = -1. \tag{2.82}$$

Again Theorem 2.6 extends the equality (2.82) for all $\theta < \beta_-$ while the relation $A_{-\theta} = A^*_\theta$ implies

$$\operatorname{Ind} T_\theta = 1 \quad \text{for} \quad \theta > \beta_+ = -\beta_-. \tag{2.83}$$

THEOREM 2.16. *1) Assume a_0 is above the threshold, i.e. $a_0 > -a_\ddagger$. We have*

$$\operatorname{Ind} T_\beta = 0 \text{ for } |\beta| < \sqrt{\Big(n - \frac{3}{2}\Big)^2 + a_0 \frac{|\partial\omega|}{|\omega|}}$$

and

$$\operatorname{Ind} T_\beta = \pm 1 \text{ for } \pm\beta \geq \pm\operatorname{Re}\sqrt{\Big(n - \frac{3}{2}\Big)^2 + a_0 \frac{|\partial\omega|}{|\omega|}}. \tag{2.84}$$

2) Below the threshold, $a_0 \leq -a_\ddagger$, the formula (2.84) *holds for all $\pm\beta > 0 = \operatorname{Re}\big((n-3/2)^2 + a_0|\partial\omega|/|\omega|\big)$.*

PROOF. It suffices to confirm the second assertion only. Let $\beta > 0$ and $\theta = -\beta$ satisfy the hypotheses of Theorem 2.6. By $\mathfrak{V}^1_{-\beta}(\Omega)$ we understand the space of functions of the form (2.8), endowed with the norm

$$\|u; \mathfrak{V}^1_{-\beta}(\Omega)\| = \big(|K_-|^2 + |K_+|^2 + \|\tilde{u}; V^1_{-\beta}(\Omega)\|^2\big)^{1/2}.$$

Again we have $\ker \mathfrak{T}_{-\beta} = \ker T_\beta$, $\operatorname{coker} \mathfrak{T}_{-\beta} = \operatorname{coker} T_\beta$, but $\dim\big(\mathfrak{V}^1_{-\beta}(\Omega)/V^1_{-\beta}(\Omega)\big) = 2$ so that

$$\operatorname{Ind} T_\beta = \operatorname{Ind} \mathfrak{T}_{-\beta} = \operatorname{Ind} T_{-\beta} + 2 = -\operatorname{Ind} T_\beta + 2.$$

This formula, Theorem 2.6 and the inequality $T^*_\beta = T_{-\beta}$ yield (2.84) for any $\beta \neq 0$. $\square$

The index areas for T_β are drawn in Fig. 5.

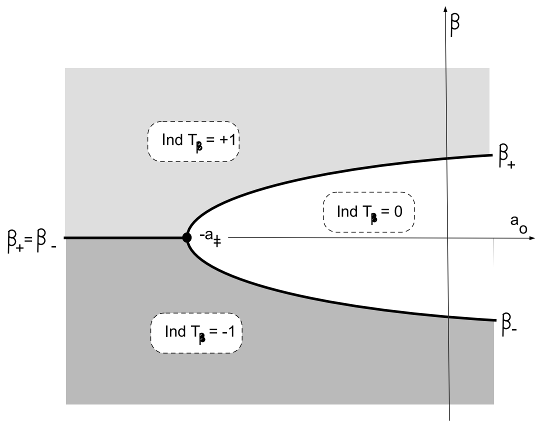

Fig. 2.4. The index areas.

3. Problem Operator in $L^2(\Omega)$

3.1. The discrete spectrum above the threshold

In the case $a_0 > -a_\ddagger$ the sesquilinear form

$$q(u,v) = (\nabla_x u, \nabla_x v)_\Omega + (au, v)_\Gamma \tag{3.1}$$

is semi–bounded from below. To verify this property we use Lemma 2.9 with the exact constant in the embedding $H^1(\Omega) \subset L^2(\Gamma)$. Namely, we put $\varepsilon = (2(1+a_\ddagger))^{-1}(a_\ddagger + a_0)$ in the inequality (2.14) including the number C_ε and obtain

$$\begin{aligned} & q(u,u) + M\|u; L^2(\Omega)\|^2 \\ = \; & \varepsilon\|\nabla_x u; L^2(\Omega)\|^2 + (au,u)_\Gamma + (1-\varepsilon)\Big(\|\nabla_x u; L^2(\Omega)\|^2 \\ + \; & (a_\ddagger - \varepsilon)C_\varepsilon\|u; L^2(\Omega)\|^2\Big) + \big(M - (1-\varepsilon)(a_\ddagger - \varepsilon)C_\varepsilon\big)\|u; L^2(\Omega)\|^2 \\ \geq \; & \varepsilon\|\nabla_x u; L^2(\Omega)\|^2 + (au,u)_\Gamma + (1-\varepsilon)(a_\ddagger - \varepsilon)\|u; L^2(\Gamma)\|^2 \\ + \; & \big(M - (1-\varepsilon)(a_\ddagger - \varepsilon)C_\varepsilon\big)\|u; L^2(\Omega)\|^2. \end{aligned}$$

Owing to (1.4) we have

$$a(x) + (1-\varepsilon)(a_\ddagger - \varepsilon) = a(x) + a_\ddagger - (1+a_\ddagger)\varepsilon + \varepsilon^2$$

$$\geq \;\; a(x) + a_\ddagger - \frac{1}{2}(a_\ddagger - a_0) \to \frac{1}{2}(a_0 + a_\ddagger) > 0 \;\; \text{as } x \to \mathcal{O}.$$

We thus find a small $d > 0$ such that $a(x) + (1-\varepsilon)(a_\ddagger - \varepsilon) > \frac{1}{4}(a_0 + a_\ddagger)$ for $z \in [0, d]$, and therefore

$$(au, u)_{\varpi(d)} + (1-\varepsilon)(a_\ddagger - \varepsilon)\|u; L^2(\varpi(d))\|^2 \geq \frac{1}{4}(a_0 + a_\ddagger)\|u; L^2(\varpi(d))\|^2.$$

We now use the inequality (2.15) to get an estimate for $(au, u)_{\Gamma(d)}$ and then choose a small $t > 0$ and a large M such that

$$t \max_{x \in \Gamma} |a(x)| \leq \frac{\varepsilon}{2} \;\; , \;\; M > (1-\varepsilon)(a_\ddagger - \varepsilon)C_\varepsilon + C_d(t), \tag{3.2}$$

where $C_d(t)$ is the coefficient in (2.15). Gathering the above calculations yields

$$q(u, u) + M\|u; L^2(\Omega)\|^2 \geq 0 \Rightarrow q(u, u) \geq -M\|L^2(\Omega)\|^2. \tag{3.3}$$

Hence, q is semi–bounded from below and, evidently, it is also closed. According to [4, Th. 10.1.2], this form generates a semi-bounded self-adjoint operator $\mathcal{T}$ in $L^2(\Omega)$ associated with the problem (1.6) (or (1.3), (1.4) in differential form). Finally, [4, Th. 10.1.5] establishes the following assertion.

THEOREM 3.1. *In the case*

$$\lim_{x \to \mathcal{O}} a(x) = a_0 > -a_\ddagger = -\Big(n - \frac{3}{2}\Big)^2 \frac{|\omega|}{|\partial\omega|}, \tag{3.4}$$

i.e. above the threshold, the spectrum of $\mathcal{T}$ *and, hence, of the problem* (1.6), *are discrete and form the eigenvalue sequence* (1.7), *where* $\lambda_1 = -M$ *and* M *is the lowest constant in the estimate* (3.3).

3.2. The explicit description of the operator domain above the threshold

Theorem 3.1, which is based on the general results in [4, Ch. 10], does not yield the explicit form of the domain $\mathcal{D}(\mathcal{T})$. Moreover, the Lipschitz property does not prevent the surfaces $\partial\omega$ and $\partial\Omega \setminus \mathcal{O}$ from having irregularities like conical points, edges and so on, while a solution of the problem (1.3), (1.4) may possess singularities and thus live outside $H^2_{\rm loc}(\overline{\Omega}\setminus\mathcal{O})$ (cf. the introductory chapters in [29] and [14]). In order to describe $\mathcal{D}(\mathcal{T})$ explicitely we assume from now on that the surfaces $\partial\omega$ and $\partial\Omega\setminus\mathcal{O}$ are C^3-smooth and that the Robin coefficient a is continuously differentiable in a neighbourhood of $\overline{\partial\Omega}$ (see Section 3.7 for the case of Lipschitz surfaces).

Since the problem data is smooth, standard local elliptic estimates (see, e.g., [1, 17]) guarantee that a solution $u\in H^1(\Omega)$ of (1.26), or (1.24), (1.25) with

$$f\in L^2(\Omega)\ , g=0 \tag{3.5}$$

belongs to $u\in H^2_{\rm loc}(\overline{\Omega}\setminus\mathcal{O})$ and satisfies the estimate

$$\|u;H^2(\Omega(2\varepsilon))\|\le c_\varepsilon\big(\|f;L^2(\Omega(\varepsilon))\|+\|u;L^2(\Omega(\varepsilon))\|\big) \tag{3.6}$$

with any $\varepsilon>0$ and coefficient c_ε which may grow unboundedly for $\varepsilon\to+0$.

The next auxiliary lemma yields weighted estimates of the second derivatives of solutions to the problem (1.24), (1.25) with

$$f\in L^2_{\beta+1}(\Omega)\ ,\ g=G\big|_\Gamma\ ,\ G\in L^2_{\beta-1}(\Omega)\ ,\ \nabla_x G\in L^2_{\beta+1}(\Omega). \tag{3.7}$$

The right hand side of (1.25) is defined as the trace of the function G which belongs to $H^1_{\rm loc}(\overline{\Omega}\setminus\mathcal{O})$, hence, $g\in H^{1/2}_{\rm loc}(\Gamma\setminus\mathcal{O})$. The solution u meets the estimate (3.6) but with an additional term, either $\|G;H^1(\Omega(\varepsilon))\|$ or $\|g;H^{1/2}(\Gamma(\varepsilon))\|$ on the left. In what follows we still refer to this modified estimate as (3.6).

LEMMA 3.2. *Let $u\in V^1_\beta(\Omega)$ be a solution to the problem* (1.26) *with data* (3.7). *Then $\nabla^2_x u\in L^2_{\beta+1}(\Omega)$ and the following estimate is valid:*

$$\|r^{\beta+1}\nabla^2_x u;L^2(\Omega)\|\le c\big(\|f;L^2_{\beta+1}(\Omega)\|+\|u;V^1_\beta(\Omega)\|\big). \tag{3.8}$$

PROOF. Put $u\in H^2_{\rm loc}(\overline{\Omega}\setminus\mathcal{O})$, $v\in C^\infty_c(\overline{\Omega}\setminus\mathcal{O})$ into the integral identity (1.26). Integrating it by parts shows that

$$(-\Delta_x u,v)_\Omega+(au+\partial_\nu u,v)_\Gamma=(f,v)_\Omega+(g,v)_\Gamma. \tag{3.9}$$

Taking here test functions $v\in C^\infty_c(\Omega)$ yields the equation (1.24) and test functions $v\in C^\infty_c(\overline{\Omega}\setminus\mathcal{O})$ give the boundary condition (1.25). In view of

(3.6), where $\varepsilon = d/4$, it suffices to consider only the peak $\Pi_{d/2}$. We introduce the sets

$$\begin{aligned} \Xi_j &= \Big\{x : z \in \Big(\frac{d}{2}(1+j)^{-1}, \frac{d}{2}j^{-1}\Big) , \ y \in \omega(z)\Big\}, \\ \Xi'_j &= \Big\{x : z \in \Big(\frac{d}{2}\big(\frac{3}{2}+j\big)^{-1}, \frac{d}{2}\big(j-\frac{1}{2}\big)^{-1}\Big) , \ y \in \omega(z)\Big\} \end{aligned} \tag{3.10}$$

and notice that

$$\Xi_j \subset \Xi'_j \ , \ \overline{\Pi_{d/2}} = \bigcup_{j\in\mathbb{N}} \overline{\Xi_j} \ , \ \Xi'_j \subset \Pi_d \ , \ \Xi'_j \cap \Xi'_{j+2} = \emptyset. \tag{3.11}$$

Since $\frac{1}{j} - \frac{1}{1+j} = \frac{1}{j^2} + O(\frac{1}{j^3})$ and the sets (3.10) have diameters $O(\frac{1}{j^2})$, we make the coordinate dilatation

$$x \mapsto \widehat{x}^j = \big(j^2 y, j^2(z - \frac{d}{2}j^{-1})\big), \tag{3.12}$$

which transforms (3.10) into

$$\begin{aligned} \widehat{\Xi}_j &= \Big\{\widehat{x} : -\frac{d}{2}\frac{j}{j+1} < \widehat{z} < 0 \ , \Big(\frac{d}{2} + \frac{1}{j}\widehat{z}\Big)^{-2}\widehat{y} \in \omega\Big\}, \\ \widehat{\Xi}'_j &= \Big\{\widehat{x} : -\frac{3d}{4}\frac{j}{j+3/2} < \widehat{z} < \frac{d}{4}\frac{j}{j-1/2} < 0 \ , \Big(\frac{d}{2} + \frac{1}{j}\widehat{z}\Big)^{-2}\widehat{y} \in \omega\Big\}. \end{aligned}$$

We emphasize that

$$\begin{aligned} \widehat{\Xi}_j \subset \widetilde{\Xi}_j &= \Big\{\widehat{x} : -\frac{d}{4} < \widehat{z} < 0 \ , \Big(\frac{d}{2} + \frac{1}{j}\widehat{z}\Big)^{-2}\widehat{y} \in \omega\Big\}, \\ \widehat{\Xi}'_j \supset \widetilde{\Xi}'_j &= \Big\{\widehat{x} : -\frac{3d}{10} < \widehat{z} < \frac{d}{2} \ , \Big(\frac{d}{2} + \frac{1}{j}\widehat{z}\Big)^{-2}\widehat{y} \in \omega\Big\} \supset \widetilde{\Xi}_j. \end{aligned} \tag{3.13}$$

The change of variables (3.12) converts the problem (1.24), (1.25) into

$$\begin{aligned} -\Delta_{\widehat{x}}\widehat{u}_j(\widehat{x}) &= j^{-4}\widehat{f}_j(\widehat{x}) \ , \ \widehat{x} \in \widetilde{\Xi}'_j, \\ -\partial_{\widehat{\nu}}\widehat{u}_j(\widehat{x}) &= -j^{-2}\widehat{a}_j(\widehat{x})\widehat{u}_j(\widehat{x}) + j^{-2}\widehat{g}_j(\widehat{x}) \ , \ \widehat{x} \in \widetilde{\Gamma}'_j, \end{aligned} \tag{3.14}$$

where $\widetilde{\Gamma}'_j = \{\widehat{x} \in \partial\widetilde{\Xi}'_j : -\frac{3d}{10} < \widehat{z} < \frac{d}{2}\}$ is the lateral side of the "cylinder" $\widetilde{\Xi}'_j$ in (3.13) and $\widehat{u}_j(\widehat{x}) = u(x)$ for $\widehat{x} \in \widetilde{\Xi}'_j$, and $\widehat{f}_j$, $\widehat{g}_j$ and $\widehat{a}_j$ are defined similarly. We apply to (3.14) the local estimates for the Poisson equation with Neumann condition (cf. [1, 17]) and write

$$\begin{aligned} \|\nabla^2_{\widehat{x}}\widehat{u}_j; L^2(\widetilde{\Xi}_j)\|^2 \quad &\leq \quad c\big(j^{-8}\|\widehat{f}_j; L^2(\widetilde{\Xi}'_j)\|^2 + j^{-4}\|\widehat{a}_j\widehat{f}_j; H^{1/2}(\widetilde{\Gamma}'_j)\|^2 \\ &+ \quad j^{-2}\|\widehat{g}_j; H^{1/2}(\widetilde{\Gamma}'_j)\|^2 + \|\nabla_{\widehat{x}}\widehat{u}_j; L^2(\widetilde{\Xi}'_j)\|^2\big), \end{aligned} \tag{3.15}$$

where $H^{1/2}(\widetilde{\Gamma}'_j)$ is the Sobolev-Slobodetskii space. Moreover, the bound

$$\|\widehat{a}_j\widehat{f}_j; H^{1/2}(\widetilde{\Gamma}'_j)\| \leq c\|\widehat{u}_j; H^1(\widetilde{\Xi}'_j)\| \tag{3.16}$$

holds by virtue of the assumed smoothness property of the Robin coefficient.

Remark 3.3. Local estimates of type (3.15) are usually written with the "weak" term $\|\widehat{u}_j; L^2(\widetilde{\Xi}'_j)\|^2$ at the end, however, the norm $\|\nabla_{\widehat{x}}\widehat{u}_j; L^2(\widetilde{\Xi}'_j)\|$ is much more convenient here. This substitution is due to the following simple observation: the estimate can initially be written for $\widehat{u}_j - \overline{u}_j$, where $\overline{u}_j$ is the mean value of $\widehat{u}_j$ over $\widetilde{\Xi}'_j$. The difference satisfies the same Neumann problem in $\widetilde{\Xi}'_j$ and meets the relations $\nabla_{\widehat{x}}(\widehat{u}_j - \overline{u}_j) = \nabla_{\widehat{x}}\widehat{u}_j$ and $\|\widehat{u}_j - \overline{u}_j, L^2(\widetilde{\Xi}'_j)\| \leq c\|\nabla_{\widehat{x}}\widehat{u}_j, L^2(\widetilde{\Xi}'_j)\|$.

For big indices j, the domain $\widetilde{\Xi}'_j$ in (3.13) is a small regular perturbation of the cylinder

$$\Big\{\widehat{x} \,:\, -\frac{3d}{10} < \widehat{z} < \frac{d}{2}\ ,\ \frac{4}{d^2}\widehat{y} \in \omega\Big\},$$

and hence the constant c in (3.15) can be chosen independent of $j \in \mathbb{N}$ and, of course, $\widehat{u}_j$. Relations (3.15) and (3.16) give

$$\begin{aligned} \|\nabla^2_{\widehat{x}}\widehat{u}_j; L^2(\widehat{\Xi}_j)\|^2 \quad &\leq \quad c\big(j^{-8}\|\widehat{f}_j; L^2(\widehat{\Xi}'_j)\|^2 + j^{-4}\|\widehat{G}_j; H^1(\widehat{\Xi}'_j)\|^2 \\ &+ \quad \|\nabla_{\widehat{x}}\widehat{u}_j; L^2(\widehat{\Xi}'_j)\|^2 + j^{-4}\|\widehat{u}_j; L^2(\widehat{\Xi}'_j)\|^2\big), \end{aligned}$$

and after the inverse change $\widehat{x}_j \mapsto x$, we obtain

$$\begin{aligned} j^{-8}\|\nabla^2_x u; L^2(\Xi_j)\|^2 \quad &\leq \quad c\big(j^{-8}\|f; L^2(\Xi'_j)\|^2 + j^{-8}\|\nabla_x G; L^2(\Xi'_j)\|^2 \\ &+ \quad j^{-4}\|G; L^2(\Xi'_j)\|^2 + j^{-4}\|\nabla_x u; L^2(\Xi'_j)\|^2 \\ &+ \quad j^{-4}\|u; L^2(\Xi'_j)\|^2\big). \end{aligned} \tag{3.17}$$

We multiply (3.17) with $j^{-2\beta+4}$ and make use of the relation

$$0 < cj^{-1} < r < Cj^{-1} \ \text{ for } x \in \Xi_j', \tag{3.18}$$

see (3.10), in order to turn the factor j^{-1} into the weight r inside the norm. In this way we have

$$\begin{aligned} \|r^{\beta+2}\nabla_x^2 u; L^2(\Xi_j)\|^2 \quad &\leq \quad c\big(\|r^{\beta+2} f; L^2(\Xi_j')\|^2 + \|r^{\beta+2}\nabla_x G; L^2(\Xi_j')\|^2 \\ &+ \quad \|r^{\beta} G; L^2(\Xi_j')\|^2 + \|r^{\beta}\nabla_x u; L^2(\Xi_j')\|^2 \\ &+ \quad \|r^{\beta} u; L^2(\Xi_j')\|^2\big). \end{aligned} \tag{3.19}$$

Summing up the inequalities (3.19) with respect to $j \in \mathbb{N}$ and recalling the formulas (3.11) lead to the estimate

$$\begin{aligned} \|\nabla_x^2 u; L^2_{\beta+2}(\Pi_{d/2})\|^2 \quad &\leq \quad 2c\big(\|f; L^2_{\beta+2}(\Pi_d)\|^2 + \|\nabla_x G; L^2_{\beta+2}(\Pi_d)\|^2 \\ &+ \quad \|G; L^2_{\beta}(\Pi_d)\|^2 + \|\nabla_x u; L^2_{\beta}(\Pi_d)\|^2 \\ &+ \quad \|u; L^2_{\beta}(\Pi_d)\|^2\big); \end{aligned} \tag{3.20}$$

the reason for the factor 2 is that the family $\{\Xi_j'\}_{j\in\mathbb{N}}$ covers the set $\Pi_{d/2}$ twice, see the last formula in (3.11). Note that all norms on the right of (3.20) are bounded.

The inequality (3.20) together with (3.6) at $\varepsilon = d/4$ provide the inclusion $\nabla_x^2 u \in L^2_{\beta+2}(\Omega)$ and the estimate

$$\|r^{\beta+2}\nabla_x^2 u; L^2(\Omega)\| \leq c\big(\|f; L^2_{\beta+1}(\Omega)\| + \|u; V^1_{\beta}(\Omega)\|\big), \tag{3.21}$$

which however is not satisfactory because of the "wrong" exponent $\beta+2$ on the left. However, it is straightforward to improve the exponent and turn (3.21) into (3.8) by means of our previous calculations.

Multiplying the solution with an appropriate cut-off function χ and recalling the estimate (3.6), we may again assume that the support of u is contained in $\overline{\Pi}_d$. Applying the decomposition (2.17), we write down the integral identities of type (2.60) and (2.61) for the components $u_0(z)$ and $u_\perp(y,z)$, respectively. The first identity, namely

$$|\omega|(z^{2(n-1)}\partial_z u_0, \partial_z v_0)_{(0,d)} = (\mathbf{F}_0, v_0)_{(0,d)} \ , \ v \in C_c^\infty(0,d)$$

turns into the ordinary differential equation

$$-|\omega|\partial_z z^{2(n-1)}\partial_z u_0 = \mathbf{F}_0(z) \ , \ z \in (0,d), \tag{3.22}$$

where

$$\begin{aligned} \mathbf{F}_0(z) &= \int\limits_{\omega(z)} f(y,z)dz + \partial_z \int\limits_{\omega(z)} \partial_z u_\perp(y,z)dy \\ &- \int\limits_{\partial\omega(z)} N(y,z)^{1/2} a(y,z) u(y,z) ds_y \\ &=: \mathbf{F}_0^f(z) + \mathbf{F}_0^u(z) - \mathbf{F}_0^a(z). \end{aligned} \tag{3.23}$$

Clearly, by virtue of (2.16),

$$\int\limits_0^d z^{2(\beta+2-n)} |\mathbf{F}_0^f(z)|^2 dz \leq \int\limits_0^d z^{2(\beta+2-n)} |\omega(z)|^2 \int\limits_{\omega(z)} |f(y,z)|^2 dydz$$
$$\leq c\|f; L^2_{\beta+1}(\Pi_d)\|^2.$$

To process $\mathbf{F}_0^u$ we take into account the relation (2.49) following from the orthogonality condition (2.11) for $u_\perp$:

$$\int\limits_{\omega(z)} \partial_z u_\perp(y,z)dy = -2 \int\limits_{\omega(z)} z^{-1} y \cdot \nabla_y u_\perp(y,z)dy$$
$$= -2 \int\limits_{\omega(z)} z^{-1} y \cdot \nabla_y u(y,z)dy.$$

Together with the formula $|y| \leq c_\omega z^2$ for $y \in \omega(z)$, this yields

$$\begin{aligned} &\int\limits_0^d z^{2(\beta+2-n)} |\mathbf{F}_0^u(z)|^2 dz \\ \leq\ & c \int\limits_0^d z^{2(\beta+2-n)} \Big| \int\limits_{\omega(z)} \big(|\nabla_y u(y,z)| + z|\partial_z \nabla_y u(y,z)|\big) dy \Big|^2 dz \\ \leq\ & c \int\limits_0^d z^{2(\beta+1)} \int\limits_{\omega(z)} \big(|\nabla_y u(y,z)|^2 + z^2 |\partial_z \nabla_y u(y,z)|^2\big) dydz \end{aligned}$$

$$\leq \quad c\big(\|r^{\beta+1}\nabla_x u; L^2(\Pi_d)\|^2 + \|r^{\beta+2}\nabla_x^2 u; L^2(\Pi_d)\|^2\big),$$

where both of the norms on the right are finite due to our assumption and the estimate (3.21). Finally, we apply the trace inequality to obtain

$$\int_0^d z^{2(\beta+2-n)}|\mathbf{F}_0^a(z)|^2 dz \leq c\int_0^d z^{2(\beta+2-n)}|\partial\omega(z)| \int_{\partial\omega(z)} |u(y,z)|^2 ds_y dz$$
$$\leq \quad c\int_{\varpi_d} r^{2\beta}|u(y,z)|^2 ds_x \leq c\|u; V_\beta^1(\Pi_d)\|^2.$$

It remains to mention that the proof of Lemma 2.8 yields

$$(3.24) \qquad \|z^{\beta-2+n}u_0; L^2(0,d)\| + \|z^{\beta-1+n}\partial_z u_0; L^2(0,d)\| \leq c\|u; V_\beta^1(\Pi_d)\|,$$

and that the following estimate for the solution of (3.22) is evident:

$$\|r^{\beta+1}\partial_z^2 u_0; L^2(\Pi_d)\| \leq c\|r^{\beta+n}\partial_z^2 u_0; L^2(0,d)\|$$
$$(3.25) \qquad \leq \quad c\big(\|z^{\beta+1-n}\mathbf{F}_0; L^2(0,d)\| + \|z^{\beta-1+n}\partial_z u_0; L^2(0,d)\|\big).$$

The second integral identity is nothing but (2.61) with the functionals $F(v) = (f,v)_{\Pi_d}$ and (2.63). Both satisfy the condition (2.64) with $\theta = \beta - 1$. Indeed, in view of Lemma 2.8 with $\gamma = 1$ we have

$$\begin{aligned} |(f,v)_{\Pi_d}| &\leq \|z^{\beta+1}f; L^2(\Pi_d)\|\|z^{-\theta-2}v; L^2(\Pi_d)\| \\ &\leq c\|f; L^2_{\beta+1}(\Pi_d)\|\|\nabla_x v; L^2_{-\theta}(\Pi_d)\|, \\ |(au_0,v)_{\varpi_d}| &\leq c\|z^\beta u_0; L^2(\varpi_d)\|\|z^{-\theta-1}v; L^2(\varpi_d)\| \\ &\leq c\|z^{\beta+n-1}u_0; L^2(0,d)\|\|\nabla_x v; L^2_{-\theta}(\Pi_d)\|. \end{aligned}$$

Moreover, integrating by parts and recalling the formula (1.27) for the normal $\nu(y,z)$ we derive similarly

$$|(\partial_z u_0, \partial_z v)_{\Pi_d}| = |(\partial_z^2 u_0, \partial_z v)_{\Pi_d} + 2(z^{-1}y\cdot\nu'\partial_z u_0, v)_{\varpi_d}|$$
$$\leq \quad c\big(\|z^{\beta+n-2}\partial_z^2 u_0; L^2(0,d)\| + \|z^{\beta+n-1}\partial_z u_0; L^2(0,d)\|\big)\|\nabla_x v; L^2_{-\theta}(\Pi_d)\|,$$

where both norms of u_0 also appear in (3.24) and (3.25). In other words, Proposition 2.14 furnishes the estimate

$$\|r^{\beta-1}\nabla_x u_\perp; L^2(\Pi_d)\| + \|r^{\beta-3}u_\perp; L^2(\Pi_d)\|$$

$$\leq \quad c\big(\|r^{\beta+1}f;L^2(\Pi_d)\| + \|u;V^1_\beta(\Pi_d)\|\big). \tag{3.26}$$

We emphasize that the increasing of weights from r^β on $\nabla_x u$ to $r^{\beta-1}$ on $\nabla_x u_\perp$ and from $r^{\beta-1}$ on u to $r^{\beta-3}$ on $u_\perp$ allows us to derive the necessary estimate

$$\|r^{\beta+1}\nabla_x^2 u_\perp;L^2(\Pi_d)\|^2 \leq c\big(\|r^{\beta+1}f;L^2(\Pi_d)\|^2 + \|u;V^1_\beta(\Pi_d)\|^2\big);$$

here we also use the local estimates for the solution $u_\perp$ of the problem

$$\begin{aligned} -\Delta_x u_\perp(x) &= f_\perp(x) := f(x) + \partial_z^2 u_0(x)\ ,\ x\in\Pi_d,\\ \partial_\nu u_\perp(x) + a(x)u_\perp(x) &= g_\perp(x) := -\partial_\nu u_0(z) - a(x)u_0(x)\ ,\ x\in\varpi_d. \end{aligned} \tag{3.27}$$

To this end, we rewrite the estimate (3.17) as follows:

$$\begin{aligned} j^{-8}\|\nabla_x^2 u_\perp;L^2(\Xi_j)\|^2 &\leq c\big(j^{-8}\|f_\perp;L^2(\Xi_j')\|^2\\ &+\ j^{-4}\|\nabla_x u_\perp;L^2(\Xi_j')\|^2 + j^{-4}\|u_\perp;L^2(\Xi_j')\|^2\big). \end{aligned}$$

Multiplying with $j^{-2\beta+6}$ and using (3.18), one can again introduce weighted norms and obtain

$$\begin{aligned} \|r^{\beta+1}\nabla_x^2 u_\perp;L^2(\Xi_j)\|^2 &\leq c\big(\|r^{\beta+1}f_\perp;L^2(\Xi_j')\|^2\\ &+\ \|r^{\beta-1}\nabla_x u_\perp;L^2(\Xi_j')\|^2 + \|r^{\beta-1}u_\perp;L^2(\Xi_j')\|^2\big). \end{aligned} \tag{3.28}$$

The right hand sides possess the necesssary properties because of the estimates (3.24) and (3.25) for the component u_0. For example, the extension $G_\perp$ of $g_\perp$ in (3.27) is given by

$$G_\perp(y,z) = -a(y,z)u_0(z) - 2z^{-1}y\cdot N'(z^{-2}y)\partial_z u_0(z), \tag{3.29}$$

where $N' = (N_1', N_2')$ is a vector function in $\overline{\omega}$ coinciding with $N(z^{-2}y)^{-1/2}\nu'(z^{-2}y)$ at $\partial\omega$, see the formulas (1.27) and (1.28) for the exterior normal $\nu(y,z)$ on $\partial\varpi_d$. The inclusions for $G_\perp$ in (3.7) are verified by a direct calculation.

Summing (3.28) up with respect to the index $j\in\mathbb{N}$ leads to the estimate (3.26), which together with (3.25) completes the proof. □

Remark 3.4. Introducing the weighted Kondratiev space $V^2_\theta(\Omega)$ with the norm

$$\|u;V^2_\theta(\Omega)\| \quad = \quad \big(\|r^\theta\nabla_x^2 u;L^2(\Omega)\|^2$$

$$+ \quad \|r^{\theta-1}\nabla_x u; L^2(\Omega)\|^2 + \|r^{\theta-2}u; L^2(\Omega)\|^2\big)^{1/2}, \tag{3.30}$$

we may rewrite the inequality (3.8) in the same way as in (2.2):

$$\|u; V^2_{\beta+1}(\Omega)\| \le c(\|f; L^2_{\beta+1}(\Omega)\| + \|u; V^1_\beta(\Omega)\|). \tag{3.31}$$

We have assumed here that $g = 0$ in the boundary condition (1.25) so that u satisfies (1.4). Furthermore, the evident multiplicative inequality

$$\|r^\beta \nabla_x u; L^2(\Omega)\| \le c\|\nabla_x u; V^1_{\beta+1}(\Omega)\| \, \|u; L^2_{\beta-1}(\Omega)\|$$

allows us to write (3.31) in the new form

$$\|u; V^2_{\beta+1}(\Omega)\| \le c(\|f; L^2_{\beta+1}(\Omega)\| + \|u; L^2_{\beta-1}(\Omega)\|), \tag{3.32}$$

and a completion argument proves that a solution $u \in H^2_{\rm loc}(\overline{\Omega}\backslash\mathcal{O}) \cap L^2_{\beta-1}(\Omega)$ of the problem (1.24), (1.4) with $f \in L^2_{\beta+1}(\Omega)$ falls into $V^2_{\beta+1}(\Omega)$. There are crucial differences between the estimates (3.31) (or (3.32)) and (2.54): in both (3.31) and (3.32) the forbidden indices $\beta = \beta_\pm$ of (2.6) are accepted, but (2.54) does not permit $\beta = \beta_\pm$. On the other hand, (2.54) involves the $L^2(\Omega(d))$-norm of u and $V^1_\beta(\Omega)$ is embedded compactly into this space, while none of the embeddings $V^2_{\beta+1}(\Omega) \subset V^1_\beta(\Omega) \subset L^2_{\beta-1}(\Omega)$ is compact.

We are now in a position to verify the given formulas for $\mathcal{D}(\mathcal{T})$.

THEOREM 3.5. *In the case* $a_0 \in (-a_\bullet, 0)$ *the domain* $\mathcal{D}(\mathcal{T})$ *of the operator* $\mathcal{T}$ *takes the form* (1.46), *while for* $a_0 \in (-a_\ddagger, -a_\bullet)$ *it equals* (1.49).

PROOF. If u belongs to the linear space (1.45), then $u \in H^1(\Omega) = V^1_0(\Omega)$ is a solution of the problem (1.24) with the right-hand sides $f \in L^2(\Omega) = L^2_0(\Omega)$ and $g = 0$. At the same time it becomes a solution of the problem (2.1) with the functional

$$F(v) = (f, v)_\Omega + \lambda(u, v)_\Omega \ , \ F \in V^1_{-\theta}(\Omega)^* \text{ with } \theta = -1.$$

In the case (1.47) the exponents $\lambda_\pm$ in (1.34) live outside the interval $I(0, -1)$, see (2.9). Hence, recalling the restriction (2.7), a recursive application of Theorem 2.6 shows that $u \in V^1_{-1}(\Omega)$ and

$$\|u; V^1_{-1}(\Omega)\| \le c\big(\|F; V^1_1(\Omega)^*\| + \|u; V^1_0(\Omega)\|\big) \le c\|u; H^1(\Omega)\|.$$

Furthermore, Lemma 3.2 with $\beta = -1$ proves that $\nabla_x^2 u \in L_0^2(\Omega) = L^2(\Omega)$ and

$$\|\nabla_x^2 u; L^2(\Omega)\| \le c\big(\|f; L^2(\Omega)\| + \|u; V_1^{-1}(\Omega)\|\big).$$

This yields (1.46).

In the case (1.48) the exponent λ_+, (1.34), falls into the interval $I(0,-1)$, (2.9), and so Theorem 2.6 assures that

$$\begin{aligned} &u(x) = \chi_0(x)K_+U^+(z) + \widetilde{u}(z)\,, \\ &|K_+| + \|\widetilde{u}(z); V_{-1}^1(\Omega)\| \le c\big(\|F; V_1^1(\Omega)^*\| + \|u; V_0^1(\Omega)\|\big). \end{aligned} \tag{3.33}$$

We rewrite the representation (3.33) as follows:

$$u(x) = \chi_0(x)K_+\big(z^{\lambda_+} + z^{\lambda_+ + 2}W^+(z^{-2}y)\big) + \breve{u}(x)\,, \tag{3.34}$$

where W^+ is a solution of the Neumann problem (1.38) described in Remark 1.3. Noting that

$$\chi_0 K_+ z^{\lambda_+ + 2} W^+ \in H^2(\Omega) \subset V_{-1}^1(\Omega) \tag{3.35}$$

we find that the new remainder $\breve{u} = \widetilde{u} - \chi_0 K_+ z^{\lambda_+ + 2}W^+ \in V_{-1}^1(\Omega)$ satisfies the problem (1.24) with the right-hand sides

$$\begin{aligned} \breve{f} &= f + K_+\Delta_x\chi_0(U^+ + z^{\lambda_+ + 2}W^+)\,, \\ \breve{g} &= -K_+(\partial_\nu + a)\chi_0(U^+ + z^{\lambda_+ + 2}W^+)\,. \end{aligned}$$

A direct calculation repeating the arguments in Section 1.6 shows that

$$\breve{f} \in L_0^2(\Omega)\ ,\ \breve{g} = \breve{G}\big|_\Gamma\ , \breve{G} \in L_{-2}^2(\Omega)\ ,\ \nabla_x\breve{G} \in L_0^2(\Omega), \tag{3.36}$$

while the corresponding norms do not exceed $c(\|f; L^2(\Omega)\| + |K_+|)$. Thus, Lemma 3.2 yields the following:

$$\nabla_x^2\breve{u} \in L^2(\Omega),\ \text{i.e.,}\ \breve{u} \in V_0^2(\Omega)\ \text{and}\ \|\nabla_x^2\breve{u}; L^2(\Omega)\| \le c\big(\|f; L^2(\Omega)\| + |K_+|\big).$$

This and (3.33) show that the representation (3.34) readily converts into the one in (1.49). □

REMARK 3.6. In the case $a_0 = -a_\bullet$ we have $\beta_\pm = \pm 1$ and

$$U^\pm(z) = z^{\pm 1 - n + 3/2}, \tag{3.37}$$

cf. (2.6), (1.34) and (1.47), so that we cannot derive the formulas (3.33) from Theorem 2.6, since the weight index $\theta = -1$ is forbidden. This means that we need to repeat the dimension reduction procedure to yield the differential equation (3.22) with $\mathbf{F}_0 \in L^2_{1-n}(0,d)$. Taking into account the last weight exponent $1-n$ and the form (3.37) of the solution U^+, we get the representation

$$u(x) = \chi_0(x)K_+(z)U^+(z) + \widetilde{u}(x) \tag{3.38}$$

by using the variation of constants-method found in standard textbooks of differential equations. However, the function K_+ has logarithmic growth as $z \to +0$, instead of the constant K_+ in (3.33). The estimate for the remainder $\widetilde{u}$ also differs from the one in (3.33). The present particular case $a_0 = -a_\bullet$ is above the threshold and thus has discrete spectrum, however, it requires cumbersome calculations which lay a bit outside the scope of the paper. Hence, we only formulate an easily accessible result: for any $\delta > 0$,

$$\begin{aligned} \mathcal{D}(\mathcal{T}) \;=\; &\{u \in V^2_\delta(\Omega) \,:\, \Delta u \in L^2(\Omega), \\ &\partial_\nu u(x) + a(x)u(x) = 0,\ x \in \partial\Omega \setminus \mathcal{O}\}. \end{aligned} \tag{3.39}$$

We also refer to papers [20], [32], [23] and the book [13], which contain general methods for describing in detail the properties of the terms in (3.38). Finally, we mention that the Sobolev space $H^2(\Omega)$ can be replaced in (3.33) and (3.35) by the Kondratiev space $V^2_0(\Omega)$, but (3.39) does not hold for the index $\delta = 0$.

3.3. The operator below the threshold

We proceed with the following assertion which demonstrates that under the condition

$$\lim_{x\to\mathcal{O}} a(x) = a_0 < -a_\ddagger = -\Big(n - \frac{3}{2}\Big)^2 \frac{|\omega|}{|\partial\omega|} \tag{3.40}$$

the sesquilinear form (3.1) is not semibounded. Hence, the results in [4, Ch. 10] do not apply, and we have to investigate the operator of the problem (1.6) by special methods.

Lemma 3.7. *If (3.40) holds, one can find for any m a function $u^m_\pm \in H^1(\Omega) \subset L^2(\partial\Omega)$ such that the inequality (1.15) is valid.*

PROOF. As in the proof of Lemma 1.2, we again define the functions u_+^m by choosing for every large m one from $C_c^\infty(\Omega)$ satisfying (1.17)

Let us define the functions $u_-^m \in H^1(\Omega)$ by

$$u_-^m(z) = \chi_m(-\ln z) z^{-n+3/2},$$

where χ_m is the cut-off function (2.56), see Fig.2.3, so the support of u_-^m is contained in the interval $[\exp(-2^{m+1}), \exp(-2^m)]$. Recalling (2.16) we can calculate

$$\begin{aligned} \|u_-^m; L^2(\Omega)\|^2 &\leq |\omega| \int\limits_{\exp(-2^{m+1})}^{\exp(-2^m)} z^{-2n+3} z^{2(n-1)} dz \\ &\leq \frac{1}{2}|\omega|\big(\exp(-2^{m+1}) - \exp(-2^{m+2})\big), \end{aligned}$$

$$\begin{aligned} &\|\nabla_x u_-^m; L^2(\Omega)\|^2 = \|\partial_z u_-^m; L^2(\Omega)\|^2 \\ \leq\ & |\omega| \int\limits_{\exp(-2^{m+1}+1)}^{\exp(-2^m-1)} \Big(\Big(-n+\frac{3}{2}\Big)^2 z^{2(-n+1/2)} z^{2(n-1)} dz \\ +\ & \Big(\int\limits_{\exp(-2^{m+1})}^{\exp(-2^{m+1}+1)} + \int\limits_{\exp(-2^m-1)}^{\exp(-2^m)} \Big)\Big(c_\chi z^{-n+3/2} + \Big(n-\frac{3}{2}\Big) z^{-n+\frac{1}{2}}\Big)^2 z^{2n-1} dz\Big) \\ \leq\ & |\omega|\Big(\Big(n-\frac{3}{2}\Big)^2 (2^{m+1} - 2^m - 2) + c_{\chi,n}\exp(-2^m)\Big), \end{aligned}$$

$$\begin{aligned} \|u_-^m; L^2(\partial\Omega)\|^2 &\geq |\partial\omega| \int\limits_{\exp(-2^{m+1}+1)}^{\exp(-2^m-1)} (1 + c_N z^2) z^{-2(n-3/2)} z^{2(n-2)} dz \\ &\geq |\partial\omega|(2^{m+1} - 2^m - 2)). \end{aligned} \tag{3.41}$$

In the last formula we have used (1.28) and (1.29). The relation (1.15) follows now from the threshold condition (3.40): we can find a $\delta > 0$ such that, for large enough m,

$$a(x)|\partial\omega| < -(n-3/2)^2|\omega| - \delta \tag{3.42}$$

for $x = (y, z)$ with $z < \exp(-2^m)$. Then (3.42) holds in the support of u_-^m, we can deduce using (3.42)

$$\begin{aligned} -q(u_-^m, u_-^m) &\geq -\inf_{z<e^{-2^m}} a(x)\|u_-^m; L^2(\partial\Omega)\|^2 - \|\nabla_x u_-^m; L^2(\Omega)\|^2 \\ &\geq -\inf_{z<e^{-2^m}} a(x)|\partial\omega|(2^{m+1} - 2^m - 2)) \\ &- |\omega|(n-3/2)^2(2^{m+1} - 2^m - 2) - c\exp(-2^m) \\ &\geq \delta(2^{m+1} - 2^m - 2) - c\exp(-2^m) \\ &\geq m\|u_-^m; L^2(\Omega)\|^2. \quad \Box \end{aligned} \tag{3.43}$$

Remark 3.8. In the proof of Lemma 3.7 we did not care for the Robin boundary condition (1.4), which however will become important in Section 3.6. For smooth data it is not difficult to modify the test functions (3.62) so as to satisfy (1.4). Similarly to (3.34) we set

$$\mathbf{u}_-^m(x) = \chi_m(-\ln z)\big(z^{\lambda_+} + z^{\lambda_+ + 2} W_+(z^{-2}y)\big) + \mathcal{W}^m(y, z)$$

with two correction terms. The first one, defined according to Remark 1.3, satisfies

$$\begin{aligned} \|\chi_m z^{\lambda_+ + 2} W_+; L^2(\Omega)\|^2 &\leq c \int\limits_{\exp(-2^{m+1})}^{\exp(-2^m)} z^{2(n-1)} z^{2(-n+7/2)} dz \\ &\leq c\exp(-6 \cdot 2^m), \\ \|\nabla_x(\chi_m z^{\lambda_+ + 2} W_+); L^2(\Omega)\|^2 &\leq c \int\limits_{\exp(-2^{m+1})}^{\exp(-2^m)} z^{2(n-1)} z^{2(-n+3/2)} dz \\ &\leq c\exp(-2 \cdot 2^m), \\ \|\chi_m z^{\lambda_+ + 2} W_+; L^2(\partial\Omega)\|^2 &\leq c \int\limits_{\exp(-2^{m+1})}^{\exp(-2^m)} z^{2(n-2)} z^{2(2-n+3/2)} dz \\ &\leq c\exp(-4 \cdot 2^m), \end{aligned} \tag{3.44}$$

and thus cannot spoil the calculations (3.41) and (3.43). The second term $\mathcal{W}^m$ is intended to compensate for a discrepancy left in (1.4), which can be put to the following form using the formula (1.27) for the normal on ϖ_d:

$$\begin{aligned}&\partial_\nu\big(\chi_m(-\ln z)(z^{\lambda_+}+z^{\lambda_++2}W_+(z^{-2}y))\big)\\ =\;&\chi_m(-\ln z)z^{\lambda_+}\big(\partial_{\nu'}W_+(\eta)-2\lambda_+\eta\cdot\nu'(\eta)+a_0\big)+g^m(y,z).\end{aligned}$$

The first term on the right vanishes in view of the boundary condition in the problem (1.38) for W_+. The remainder has support in $\{(y,z)\in\varpi_d : z\in[\exp(-2^{m+1}),\exp(-2^m)]\}$ and meets the estimates

$$\begin{aligned}&|g^m(y,z)|\le cz^{-n+5/2}\ ,|\partial_\nu g^m(y,z)|\le cz^{-n+3/2}\ ,\\ &|\nabla_y g^m(y,z)|\le cz^{-n+1/2}.\end{aligned}$$

Going over to the coordinates (η,ζ), see (1.21), we can readily find a function $\mathcal{W}^m$ without solving a differential equation, such that the following holds: its support is contained in $\{(y,z)\in\varpi_d : z\in[2^{-1}\exp(-2^{m+1}),2\exp(-2^m)]\}$, and moreover $\partial_\nu\mathcal{W}^n+a\mathcal{W}^n=0$ on $\partial\Omega$, and

$$\begin{aligned}&|\mathcal{W}^m(y,z)|\le cz^{-n+7/2}\ ,\ |\partial_z\mathcal{W}^m(y,z)|\le cz^{-n+5/2},\\ &|\nabla_y\mathcal{W}^m(y,z)|\le cz^{-n+3/2}.\end{aligned}$$

Hence, the relations (3.44) are valid for the second correction term as well and therefore the family $\{u^m_-\}$ satisfies (1.17). Notice that there is no need to modify the family $\{u^m_+\}$ because the functions u^m_+ vanish near the boundary.

In the next assertion we show that below the threshold the domain (1.45) again takes the form (1.46). However, later we discover that the operator $\mathcal{T}$ loses all nice properties.

THEOREM 3.9. *The formula* (1.46) *holds for* $a_0<-a_\ddagger$.

PROOF. We have to study once more the solution $u\in H^1(\Omega)=V^1_0(\Omega)$ of the problem (1.24), (1.25) with right-hand sides $f\in L^2(\Omega)$ and $g=0$. However, we cannot directly apply Theorem 2.6, because the index $\beta=0$ becomes forbidden due to (2.6). So, we fix a small positive β', and using $u\in V^1_0(\Omega)\subset V^1_{\beta'}(\Omega)$ and Theorem 2.6 with $\beta=\beta'$, $\theta=\beta'-1/2$ we obtain the representation (2.8), where $\widetilde{u} \in V^1_\theta(\Omega) \subset H^1(\Omega)$. None of the functions $\chi_0 z^{\lambda_\pm}$, (1.34), lives in $H^1(\Omega)$, since the integral

$$\int\limits_{\Pi_d} |\partial_z z^{\lambda_\pm}|^2 dx = |\lambda_\pm|^2 |\omega| \int\limits_0^d z^{2(-n+1/2)} z^{2(n-1)} dz$$

diverges. Notice that if $a_0 = -a_\ddagger$, none of the functions (1.35) and (1.39) belongs to $H^1(\Omega)$ either. Hence, the coefficients $K_\pm$ in the decomposition (2.8) of $u \in H^1(\Omega)$ must vanish, and we conlude that $u = \widetilde{u} \in V^1_{\beta'-1/2}(\Omega)$. Now an application of Theorem 2.6 with $\beta = \beta' - 1/2 \in (0,1)$ and $\theta = -1$ (this index is no more forbidden here) shows that $u \in V^1_{-1}(\Omega)$, and Lemma 3.2 completes the proof. □

3.4. The generalized Green formula and the adjoint operator above the threshold

If both $u \in V^2_0(\Omega)$ and $v \in V^2_0(\Omega) \subset L^2(\Omega)$ satisfy the two conditions imposed in (1.46), the traditional Green formula is valid, by a completion argument, and as a consequence, the symplectic (sesquilinear and anti-Hermitian) form

$$s(u,v) = (-\Delta_x u, v)_\Omega - (u, -\Delta_x v)_\Omega \tag{3.45}$$

vanishes. Note here in particular that the form (3.45) can be defined by continuity in $V^2_\beta(\Omega) \times V^2_{-\beta}(\Omega)$ for any $\beta \in \mathbb{R}$.

Let $\delta \in (0, 1/2)$ be fixed and let $\mathfrak{V}^1_{-1}(\Omega)$ denote the space of functions of the form (2.8) with the norm

$$\|u; \mathfrak{V}^1_{-1}(\Omega)\| = \big(|K_-|^2 + |K_+|^2 + \|\tilde{u}; V^1_{-1}(\Omega)\|^2\big)^{1/2}. \tag{3.46}$$

Clearly $\mathfrak{V}^1_{-1}(\Omega) \subset V^1_1(\Omega) \subset L^2(\Omega)$. Notice that $U^\pm$ are taken from (1.34) in the case $A > (n-3/2)^2$ and from (1.39), if $A = (n-3/2)^2$.

Remark. 1.) If the Fredholm operator T_1 is an epimorhism and T_{-1} is thus a monomorphism, then $\mathfrak{V}^1_{-1}(\Omega)$ is but the preimage $(T_1)^{-1} V^1_{-1}(\Omega)$ with the induced topology.

2.) The space $\mathfrak{V}^1_{-1}(\Omega)$ inherits the Hilbert space structure from $V^1_{-1}(\Omega)$, although this fact will not be used later. The superscripts 2 and 1/2 can be omitted simultaneously in (3.46).

Let $\mathcal{T}^*$ denote an unbounded operator in $L^2(\Omega)$ with the differential expression $-\Delta_x$ and the domain

$$\mathcal{D}(\mathcal{T}^*) = \{ \quad u \in \mathfrak{V}^1_{-1}(\Omega) \, : \, \Delta_x u \in L^2(\Omega)\,, \quad \partial_\nu u(z) + a(x)u(x) = 0, \; x \in \partial\Omega \setminus \mathcal{O}\}. \tag{3.47}$$

We emphasize that the boundary condition can be understood in $H^{1/2}(\partial\Omega \setminus \mathcal{O})$. As the chosen notation suggests, we are going to verify that $\mathcal{T}^*$ is the adjoint for $\mathcal{T}$.

The form (3.45) is defined for functions in (3.47), but it does not vanish any more.

LEMMA 3.10. *For functions $u, v \in \mathcal{D}(\mathcal{T}^*)$ with the attributes $K_\pm$, $\tilde{u}$, and $L_\pm$, $\tilde{v}$, respectively, the following generalized Green formula is valid:*

$$s(u,v) = \mu i \sum_{\pm} \pm K_\pm \overline{L}_\pm, \tag{3.48}$$

where

$$\mu = 2|\omega| \begin{cases} \left(A - (n-3/2)^2\right)^{1/2} & \text{for } A > (n-3/2)^2, \\ 1 & \text{for } A = (n-3/2)^2. \end{cases} \tag{3.49}$$

PROOF. Since $C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$ is dense in $V^1_{-1}(\Omega)$, it suffices to consider $\breve{u}, \breve{v} \in C_c^\infty(\overline{\Omega} \setminus \mathcal{O})$. For a small $d > 0$ we thus have

$$\begin{aligned} & (-\Delta_x u, v)_{\Omega(d)} - (u, \Delta_x v)_{\Omega(d)} \\ = & \int\limits_{\omega(d)} \left(\overline{v(y,d)}\partial_z u(y,d) - u(y,d)\overline{\partial_z v(y,d)}\right) dy \\ = & \; |\omega(d)| \sum_{\varphi=\pm} \sum_{\psi=\pm} K_\varphi \overline{L}_\psi \Big(\big(\overline{U^\psi(d)} + O(d^{-n+7/2})\big) \\ \times & \; \big(\partial_z U^\varphi(d) + O(d^{-n+3/2})\big) \\ - & \; \big(U^\varphi(d) + O(d^{-n+7/2})\big)\big(\overline{\partial_z U^\psi(d)} + O(d^{-n+3/2})\big)\Big). \end{aligned} \tag{3.50}$$

In the case $A > (n-3/2)^2$ the formulas (1.34), (3.49) and (2.16) show that the expression (3.50) indeed tends to the right-hand side of (3.48) as

$d \to 0^+$. The case $A = (n-3/2)^2$ is treated similary, using the functions $U^\pm$ in (1.39). It suffices to note that $s(u,v)$ is the limit of (3.50) as $d \to 0^+$. □

Proposition 3.11. *Let $\mathcal{T}^*$ be the adjoint of the operator $\mathcal{T}$ with domain* (1.45) = (1.46). *The domain $\mathcal{D}(\mathcal{T}^*)$ is given by* (3.47).

Proof. We need to verify the following: If u, $f \in L^2(\Omega)$ and

$$(u,f)_\Omega = (-\Delta_x u, v)_\Omega \text{ for any } u \in \mathcal{D}(\mathcal{T}), \tag{3.51}$$

then $v \in \mathcal{D}(\mathcal{T}^*)$ and $-\Delta_x v = f$. First, we can conlude that $u \in H^2_{\mathrm{loc}}(\overline{\Omega} \setminus \mathcal{O})$ using the general results on lifting the regularity of solutions of elliptic problems in domains with smooth boundaries, see [17]. Integrating by parts in (3.51) yields

$$\begin{aligned} 0 &= (u,f)_\Omega + (\Delta_x u, v)_\Omega = (u,f)_\Omega + (u, \Delta_x v)_\Omega + (\partial_\nu u, v)_\Gamma - (u, \partial_\nu v)_\Gamma \\ &= (u, \Delta_x v + f)_\Omega - (u, \partial_\nu v + av)_\Gamma, \end{aligned}$$

where the boundary condition in (1.45) was used. Hence, v satisfies the problem (1.24), (1.4) with $f \in L^2(\Omega)$. Next, since $f \in L^2_0(\Omega) \subset L^2_2(\Omega)$ and $v \in L^2_0(\Omega)$, Remark 3.4 with $\beta = 1$ imply $\in V^2_2(\Omega) \subset V^1_1(\Omega)$. Applying Theorem 2.6 with $\beta = 1$ and $\theta = -\delta$ (recursively, because $\beta - \theta > 1/2$) and Lemma 3.2 we conclude that v falls into the linear space (3.47).

Finally, we observe that by Lemma 3.10,

$$s(u,v) = 0$$

for all $u \in \mathcal{D}(\mathcal{T})$ and $v \in \mathfrak{V}^1_{-1}(\Omega)$ satisfying the condition (1.4). This means that, indeed, any element of the space (3.47) belongs to the domain of the adjoint operator $\mathcal{T}^*$. □

3.5. The spectra of $\mathcal{T}$ and $\mathcal{T}^*$

We proceed with the following simple observation.

Lemma 3.12. *If $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the kernel of the operator $\mathcal{T} - \lambda$ is trivial.*

Proof. Assuming $u \in \ker(\mathcal{T} - \lambda) \subset \mathcal{D}(\mathcal{T}) \subset H^1(\Omega)$, we deduce from the integral identity in (1.44) with $-\Delta_x u = \lambda u \in L^2(\Omega)$ that

$$(\nabla_x u, \nabla_x u)_\Omega + (au, u)_\Gamma = \lambda (u,u)_\Omega. \tag{3.52}$$

Let $\operatorname{Im}\lambda \neq 0$. Since the left hand side of (3.52) is real, we conclude that $\|u; L^2(\Omega)\| = 0$ and $u = 0$. □

Let us now consider the Fredholm operator

$$T_{-1} : V^1_{-1}(\Omega) \to V^1_1(\Omega)^* \tag{3.53}$$

(cf. (2.4)). Since evidently

$$V^1_{-1}(\Omega) \subset L^2_{-2}(\Omega) \subset L^2_0(\Omega) \subset V^1_1(\Omega)^*, \tag{3.54}$$

the embedding $V^1_{-1}(\Omega) \subset V^1_1(\Omega)^*$ is compact. We can thus use the general result [9, Th.1.5.1] to conclude that the spectrum of the operator (3.53) consists of normal eigenvalues and has no finite accumulation point. Since $\ker(T_{-1} - \lambda) = \{0\}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ (cf. Lemma 3.12), the eigenvalues fall into the real axis of the complex plane and form a countable set Λ of separated points. The set Λ thus belongs to the point spectrum $\sigma_p(\mathcal{T})$ of the operator $\mathcal{T}$. Moreover the formula

$$\sigma_p(\mathcal{T}) = \Lambda \tag{3.55}$$

holds, because $\mathcal{T}$ is the restriction of T^1_{-1} onto the linear set (1.46), which is included in $V^2_0(\Omega)$, while all eigenfunctions of T^1_{-1} also belong to $V^2_0(\Omega)$, by Lemma 3.2.

Since $\ker(T^1_{-1} - \lambda) = \{0\}$ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\ker(T^1_{-1} - \lambda) = \operatorname{coker}(T^1_{-1} - \overline{\lambda}) = \{0\} \quad , \quad \lambda \in \mathbb{C} \setminus \Lambda, \tag{3.56}$$

and, by the index formula (2.84),

$$\dim \operatorname{coker}(T^1_{-1} - \lambda) = \dim \ker(T^1_{-1} - \overline{\lambda}) = 1. \tag{3.57}$$

From the above information, we conclude for $\mathcal{T}$ that the regularity field $\hat{\varrho}(\mathcal{T})$ equals $\mathbb{C} \setminus \Lambda$ and the resolvent set $\varrho(\mathcal{T})$ is empty. So, the spectrum $\sigma(\mathcal{T}) = \mathbb{C} \setminus \varrho(\mathcal{T})$ fills the whole plane while Λ coincides with the spectral kernel $\hat{\sigma} = \mathbb{C} \setminus \hat{\varrho}(\mathcal{T})$. Its complement is the residual spectrum: $\sigma_r(\mathcal{T}) = \sigma(\mathcal{T}) \setminus \hat{\sigma}(\mathcal{T}) = \mathbb{C} \setminus \Lambda$.

For the adjoint operator $\mathcal{T}^*$ we have

$$\varrho(\mathcal{T}^*) = \hat{\varrho}(\mathcal{T}^*) = \emptyset \ , \ \sigma(\mathcal{T}^*) = \sigma_p(\mathcal{T}^*) = \mathbb{C}$$

(cf. the second inequality (3.57)). Furthermore, any point in $\mathbb{C} \setminus \Lambda$ is a simple eigenvalue, while eigenvalues in Λ have finite multiplicities bigger than 1.

For both $\mathcal{T}$ and $\mathcal{T}^*$, the continuous and essential spectra are empty (the latter because of the finite multiplicities).

3.6. Self-adjoint extensions

Formulas (3.56) and (3.57) show in particular that defect number $d(\lambda)$ of $\mathcal{T} - \lambda$ is equal to 1 everywhere in $\mathbb{C} \setminus \Lambda \supset \{\lambda \in \mathbb{C} \ : \ \operatorname{Im} \lambda > 0\}$. Hence, any symmetric extension of $\mathcal{T}$ is self-adjoint (see, e.g., [4, § 4.4]). The famous Neumann formulas give a parametrization of all self-adjoint extensions. This can also be determined via the generalized Green formula (Lemma 3.10). Indeed, if $\mathcal{T}^\vartheta$ is a symmetric extension of $\mathcal{T}^*$, the identity

$$s(u, v) = 0 \quad , \quad u, v \in \mathcal{D}(\mathcal{T}^\vartheta),$$

is valid due to the symmetry. The null spaces of the symplectic form s have a direct description (see [16]), namely

$$\mathcal{D}(\mathcal{T}^\vartheta) = \{u \in \mathcal{D}(\mathcal{T}^*) \ : \ K_+ = e^{i\vartheta} K_-\} \tag{3.58}$$

with the parameter $\vartheta \in [0, 2\pi)$. Notice that by (3.48) we have

$$s(u, v) = i\mu^2 \big(K_+^u \overline{K_-^v} - K_-^u \overline{K_-^u}\big) = i\mu^2 K_+^u K_+^v (1 - e^{i\vartheta} e^{-i\vartheta}) = 0$$

for any u, $v \in \mathcal{D}(\mathcal{T}^\vartheta)$ with the attributes $K_\pm^u$, $K_\pm^v$.

Theorem 3.13. *Any self-adjoint extension of $\mathcal{T}$ has the domain* (3.58), *and the restriction $\mathcal{T}^\vartheta$ of $\mathcal{T}^*$ to* (3.58) *is self-adjoint.*

The embedding $\mathcal{D}(\mathcal{T}^\vartheta) \subset L^2(\Omega)$ is still compact because $\mathcal{D}(\mathcal{T}^\vartheta)$ differs from $\mathcal{D}(\mathcal{T}) \subset H^2(\Omega) \subset V_{-1}^1(\Omega)$ by a one-dimensional subspace. Thus, by [4, Thm. 9.2.1], the spectrum $\sigma^\vartheta = \sigma(\mathcal{T}^\vartheta)$ is discrete. We divide it into two sets, $\sigma_+^\vartheta = \{\lambda \in \sigma^\vartheta \ : \ \lambda \geq 0\}$ and $\sigma_-^\vartheta = \{\lambda \in \sigma^\vartheta \ : \ \lambda < 0\}$. If σ_-^ϑ contains only a finite number $\sharp(\sigma_-^\vartheta)$ of points, then the operator $\mathcal{T}^\vartheta$ happens to be semibounded from below, and therefore the smallest eigenvalue can be computed from the minimum principle

$$\min_{u \in \mathcal{D}(\mathcal{T}^\vartheta) \setminus \{0\}} \frac{(\mathcal{T}^\vartheta u, u)_\Omega}{(u, u)_\Omega}. \tag{3.59}$$

On the other hand, a calculation in Remark 3.8 shows that the minimum (3.59) does not exist, i.e., it equals $-\infty$. The same argument applied to the operator $-\mathcal{T}^\vartheta$ (notice the minus-sign) proves that $\#(\sigma_+^\vartheta) = \infty$. Hence, the spectrum $\sigma_+^\vartheta \cup \sigma_-^\vartheta$ of $\mathcal{T}$ forms two unbounded sequences

$$\begin{aligned} &0 \le \lambda_1^\vartheta \le \lambda_2^\vartheta \le \ldots \le \lambda_n^\vartheta \ldots \to +\infty, \\ &0 > \lambda_{-1}^\vartheta \ge \lambda_{-2}^\vartheta \ge \ldots \ge \lambda_{-n}^\vartheta \ldots \to -\infty. \end{aligned} \tag{3.60}$$

In this way, none of the self-adjoint operators associated to the problem (1.3), (1.4) above the threshold possesses the properties of $\mathcal{T}$ below the threshold, such as the monotone sequence (1.7) of eigenvalues. We emphasize that in this situation, a much more physically relevant tool would be to impose a radiation condition at the peak tip (see [31] for a different geometrical setting), but we do not discuss this in the present paper. Let us only mention that a radiation condition leads to an anti-symmetric extension of the operator onto the subspace of waves $\mathcal{D}(\mathcal{T}^*)/\mathcal{D}(\mathcal{T})$ (cf. [29, Ch. 6]).

3.7. Returning to the case of Lipschitz domains

For Lipschitz surfaces $\partial\omega$, $\partial\Omega \setminus \mathcal{O}$ and a Robin coefficient $a \in L^\infty(\Gamma)$, the formulas (1.46) and (1.49) are of course no longer true. However, using Theorem 2.6 we readily conclude that below the threshold, in case (1.47), the space $H^1(\Omega)$ in (1.44) can be replaced by the smaller space $V^1_{-1}(\Omega)$. Furthermore, assuming the condition (1.48) the functions $u \in \mathcal{D}(\mathcal{T})$ have the representation

$$u = K_+\chi_0 U^+ + \widetilde{u}\ ,\ K_+ \in \mathbb{C}\ ,\ \widetilde{u} \in V^1_\theta(\Omega)\ ,\ \theta = \beta_- - \min\{\alpha, 1/2\}, \tag{3.61}$$

see (2.7) and (2.8).

At the same time, all main conclusions about spectra and self-adjoint extensions in Sections 3.5 and 3.6 remain valid above the threshold. This is due to the following formulas for the domains of $\mathcal{T}$ and $\mathcal{T}^*$:

$$\begin{aligned} \mathcal{D}(\mathcal{T}) &= \{u \in V^1_{-1}(\Omega) : u \text{ satisfies the conditions in (1.44)}\} \\ \mathcal{D}(\mathcal{T}^*) &= \{u \in V^1_1(\Omega) : u \text{ takes the form (3.61) and satisfies} \\ &\qquad \text{the conditions in (1.44)}\}. \end{aligned}$$

To derive these formulas one may argue in the same way as in Section 3.4, using only Theorem 2.6 and the generalized Green formula (3.48); they do not require smooth data, since the form (3.45) is defined properly for u, $v \in H^1(\Omega)$ such that $\Delta_x u$, $\Delta_x v \in L^2(\Omega)$.

Let $\mathfrak{V}^1_{-1}(\Omega)$ denote the space of functions of the form

$$u(x) = \chi_0(x) \sum_{\pm} K_{\pm}\big(z^{\lambda_\pm} + z^{\lambda_\pm+2} W^{\pm}(z^{-2}y)\big) + \breve{u}(x), \tag{3.62}$$

where $K_\pm \in \mathbb{C}$, $\breve{u} \in V^2_0(\Omega)$ and $W^\pm$ are solutions of the Neumann problem in ω described in Remark 1.3. This space is endowed with the norm

$$\|u; \mathfrak{V}^1_{-1}(\Omega)\| = \big(|K_-|^2 + |K_+|^2 + \|\breve{u}; V^2_0(\Omega)\|^2\big)^{1/2}.$$

Clearly $\mathfrak{V}^1_{-1}(\Omega) \subset V^1_1(\Omega) \subset L^2(\Omega)$. Notice that $U^\pm$ are taken from (1.34) in the case $A > (n-3/2)^2$ and from (1.39), if $A = (n-3/2)^2$.

The representation (3.62) looks like (3.34) and it is a bit different from (2.8). It is obtained in the following way. First, we apply Theorem 2.6 iteratively and obtain the representation (2.8) with the remainder $\widetilde{u} \in V^1_{-1+\delta}(\Omega)$ for any $\delta > 0$. It is impossible to take $\delta = 0$, for example due to the observation that

$$\widetilde{f} = f + \sum_{\pm} K_\pm \Delta_x \chi_0 U^\pm \in L^2_\delta(\Omega) \ \text{ but } \widetilde{f} \notin L^2_0(\Omega).$$

However, similarly to the proof of Theorem 3.5 we find that the right hand sides

$$\begin{aligned} \breve{f} &= f + \sum_{\pm} K_\pm \Delta_x \chi_0 (U^\pm + z^{\lambda_\pm+2} W^\pm), \\ \breve{g} &= -\sum_{\pm} K_\pm (\partial_\nu + a) \chi_0 (U^\pm + z^{\lambda_\pm+2} W^\pm) \end{aligned} \tag{3.63}$$

of the problem (1.24), (1.25) for $\widetilde{u}$ satisfy (3.36), a fact caused by the correction terms $z^{\lambda_\pm+2} W^\pm(z^{-2}y)$ of (3.62). We emphasize that $|U^+(z)| = |U^-(z)|$ and the inclusion (3.35) fails above the threshold. Nevertheless, the functions (3.63) generate a functional $\breve{F} \in V^1_1(\Omega)^*$ on the right-hand side of the integral identity (1.26) for $\breve{u}$. Thus, Theorem 2.6 proves that $\breve{u} \in V^1_{-1}(\Omega)$ while $\breve{u} \in V^2_0(\Omega)$ due to Lemma 3.2 (and Remark 3.4).

References

[1] Agmon, S., Douglis, A. and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I. Comm. Pure Appl. Math. **12** (1959), 626–727.

[2] Agmon, S. and L. Nirenberg, Properties of solutions of ordinary differential equations in Banach space, Comm. Pure Appl. Math. **16** (1963), 121–239.

[3] Atkinson, F. V., The normal solubility of linear equations in normed spaces, (Russian) Mat. Sbornik **28**(70) (1951), 3–14.

[4] Birman, M. S. and M. Z. Solomyak, Spectral theory of self-adjoint operators in Hilber space, Reidel Publ. Company, Dordrecht (1986).

[5] Courant, R. and D. Hilbert, Methods of mathematical physics, Vol. I, Interscience Publishers, Inc., New York, N.Y., (1953).

[6] Daners, D., Robin boundary problems on arbitrary domains, Trans. Amer. Math. Soc. **352**(9) (2000), 4207–4236.

[7] Daners, D., A Faber-Krahn inequality for Robin problems in any space dimension, Math. Ann. **335**(4) (2006), 767–785.

[8] Davies, E. B. and B. Simon, Spectral properties of Neumann Laplacian of horns, Geom. Funct. Anal. **2**(1) (1992), 105–117.

[9] Gohberg, I. M. and M. G. Krein, Introduction to the theory of linear nonselfadjoint operators, Am. Math. Soc. Providence, RI (1969).

[10] Hempel, R., Seco, L. A. and B. Simon, The essential spectrum of Neumann Laplacians on some bounded singular domains, J. Funct. Anal. **102**(2) (1991), 448–483.

[11] Kamotskii, I. and V. G. Maz'ya, On the third boundary value problem in domains with cusps, J. Math. Sci. **173**(5) (2011), 609–631.

[12] Kondratiev, V. A., Boundary problems for elliptic equations in domains with conical or angular points, Trudy Moskov. Mat. Obshch. **16** (1967), 209–292. (Engl. transl. in Trans. Moscow Math. Soc. **16** (1967), 227–313).

[13] Kozlov, V. A. and V. G. Maz'ya, Differential equations with operator coefficients (with applications to boundary value problems for partial differential equations), Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1999.

[14] Kozlov, V. A., Maz'ya, V. G. and J. Rossmann, Elliptic boundary value problems in domains with point singularities, Amer. Math. Soc., Providence (1997).

[15] Ladyzhenskaya, O. A., Boundary value problems of mathematical physics, Springer Verlag, New York (1985).

[16] Lang, S., Linear algebra, 3rd Edition, Springer Verlag, New York (1999).

[17] Lions, J. L. and E. Magenes, Non–homogeneus boundary value problems and applications (French), Dunod, Paris (1968). English transl.: Springer-Verlag, Berlin–Heidelberg–New York (1972).

[18] Maz'ja, V. G., Sobolev spaces, 2nd edition, Grundlehren der mathematischen Wissenschaften 342, Springer-Verlag, Berlin (2011).

[19] Maz'ja, V. G. and B. A. Plamenevskii, Estimates in L_p and Hölder classes and the Miranda-Agmon maximum principle for solutions of elliptic boundary value problems in domains with singular points on the boundary, Math. Nachr. **81** (1978), 25–82. (Engl. Transl. in: Amer. Math. Soc. Transl. (Ser. 2) **123** (1984), 1–56)

[20] Maz'ya, V. G. and B. A. Plamenevskii, The asymptotic behavior of solutions of differential equations in Hilbert space, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 1080–1133; erratum, ibid. **37** (1973), 709–710 (English transl.: Math. USSR-Izv. **6** (1972), 1067–1116 (1973)).

[21] Maz'ya, V. G., Nazarow, S. A. and B. A. Plamenevskii, Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten. 1, Akademie-Verlag, Berlin (1991). (English transl.: Maz'ya, V., Nazarov, S., Plamenevskij, B., Asymptotic theory of elliptic boundary value problems in singularly perturbed domains, Vol. 1, Birkhäuser Verlag, Basel (2000)).

[22] Maz'ya, V. G. and S. V. Poborchi, Differentiable functions on bad domains, World Scientific Publishing Co., River Edge, NJ, (1997).

[23] Maz'ya, V. G. and M. Sulimov, Asymptotics of solutions of difference equations with variable coefficients, Math. Nachr. **161** (1993), 155–170.

[24] Verzhbinskii, G. M. and V. G. Maz'ya, Asymptotic behavior of solutions of elliptic equations of the second order close to a boundary, I. Sib. Math. J. **12** (1971), 874–899, II. Sib. Math. J. **13** (1972), 858–885.

[25] Nazarov, S. A., Asymptotics of the solution of the Neumann problem at a point of tangency of smooth components of the boundary of the domain, Izv. Ross. Akad. Nauk. Ser. Mat. **58**(1) (1994), 92–120 (English transl. in Math. Izvestiya **44**(1) (1995), 91–118).

[26] Nazarov, S. A., A general scheme for averaging self-adjoint elliptic systems in multidimensional domains, including thin domains, Algebra Analiz. **7**(5) (1995), 1–92 (English transl. in St. Petersburg Math. J. **7**(5) (1996), 681–748).

[27] Nazarov, S. A., The polynomial property of self-adjoint elliptic boundary-value problems and the algebraic description of their attributes, Uspehi Mat. Nauk. **54**(5) (1999), 77–142. (English transl.: Russ. Math. Surveys. **54**(5) (1999), 947–1014).

[28] Nazarov, S. A., Asymptotic theory of thin plates and rods, Dimension reduction and integral estimates, Nauchnaya Kniga, Novosibirsk (2001).

[29] Nazarov, S. A. and B. A. Plamenevskii, Elliptic problems in domains with piecewise smooth boundaries, Walter be Gruyter, Berlin, New York (1994).

[30] Nazarov, S. A. and J. Taskinen, On the spectrum of the Steklov problem in a domain with a peak, Vestnik St. Petersburg Univ. Math. **41**(1) (2008), 45–52.

[31] Nazarov, S. A. and J. Taskinen, Radiation conditions at the top of a rotational cusp in the theory of water-waves, Math. Model. Numer. Anal. **45**(4) (2011), 947–979.

[32] Plamenevskii, B. A., The asymptotic behavior of the solutions of quasielliptic differential equations with operator coefficients, (Russian) Izv. Akad. Nauk SSSR Ser. Mat. **37** (1973), 1332–1375.

[33] Simon, B., The Neumann Laplacian of a jelly roll, Proc. Amer. Math. Soc. **114**(3) (1992), 783–785.

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