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Long-Time Solvability of the Navier-Stokes-Boussinesq Equations with Almost Periodic Initial Large Data

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Abstract. We investigate long time existence of solutions of the Navier-Stokes-Boussinesq equations with spatially almost periodic large data when the density stratification parameter is sufficiently large. In 1996, Kimura and Herring [16] examined numerical simulations to show a stabilizing effect due to the large stratification. They observed scattered two-dimensional pancake-shaped vortex patches lying almost in the horizontal plane. Our result gives a mathematical foundation of the presence of such two-dimensional pancakes.

1. Introduction

Large-scale fluids such as atmosphere and ocean are parts of geophysical fluids, and the Coriolis force caused by the rotation of the earth plays a significant role in the large scale flows considered in meteorology and geophysics.

Mathematically, such significant role was first investigated by Poincaré [20]. Later on, the problem of strong Coriolis force was extensively studied. Babin, Mahalov and Nicolaenko (BMN) [1, 2] studied the incompressible rotating Navier-Stokes and Euler equations in the periodic case while Chemin, Desjardins, Gallagher and Grenier [8] analyzed the case of data decaying at space infinity. Recently, the second author of the present paper considered the almost periodic case [21]. Gallagher in [10] studied a more abstract parabolic system. We also refer to Paicu [19] for anisotropic viscous fluids, Benameur, Ibrahim and Majdoub [5] for rotating Magneto-Hydro-Dynamic

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system, and Gallagher and Saint-Raymond [11] for inhomogeheous rotating fluid equations.

Moreover the case when fluids are governed by both strong Coriolis force and vertical stratification effects was investigated by BMN in [3] in the periodic setting and by Charve in [6] for decaying data, respectively. However, their studies do not cover the case when fluid equations are governed by the only effect of stratification. It is now well known that a strong Coriolis force has a stabilizing effect (see [1]). However, in BMN [4, Section 9.2] they observed that for ideal fluids (i.e., with zero viscosity), the only effect of stratification leads to unbalanced dynamics. On the other hand, Kimura and Herring [16] examined numerical simulations to show a stabilizing effect due to the effect of stratification for viscous fluid. They observed scattered two-dimensional pancake-shaped vortex patches lying almost in the horizontal plane. Our result gives a mathematical foundation of the presence of such two-dimensional pancakes.

More precisely, we study long-time solvability for Navier-Stokes-Boussinesq equation with stratification effects:

$$
\begin{cases}
\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p = g\rho e_3, & x\in\mathbb{R}^3, \quad t>0\\
\partial_t \rho - \kappa\Delta\rho + (u\cdot\nabla)\rho = -\mathcal{N}^2 u_3, & x\in\mathbb{R}^3, \quad t>0\\
\nabla\cdot u = 0, & x\in\mathbb{R}^3, \quad t>0\\
u|_{t=0} = u_0, \quad \rho|_{t=0} = \rho_0, & x\in\mathbb{R}^3,
\end{cases}
\tag{1.1}
$$

where the unknown functions $u = u(x,t) = (u_1,u_2,u_3)$, $\rho = \rho(x,t)$ and $p = p(x,t)$ are the fluid velocity, the thermal disturbances and the pressure, respectively. The parameters $\nu > 0$, $\kappa > 0$ and $g > 0$ represent the viscosity, the thermal diffusivity and the gravity force, respectively. The parameter $\mathcal{N} > 0$ is Brunt-Väisälä frequency (stratification-parameter). We use the notations: $\Delta := (\partial_1^2 + \partial_2^2 + \partial_3^2)$, $\nabla := (\partial_1, \partial_2, \partial_3)$ and $e_3 := (0,0,1)$. For the physical background of (1.1), see [16].

Our method follows the ideas based on BMN. We show that the limit equations (formally obtained by letting $\mathcal{N}$ tend to infinity in equation (1.1)) is almost equivalent to the 2D-Navier-Stokes equations[1], which is known to have a unique global solution (see for example [15]). A straightforward

[1]In the sense that there is a one to one correspondence between solutions of the two equations.

application of the energy method is impossible if the initial data is almost periodic. To overcome this difficulty, we use ℓ^1-norm of amplitudes with sum closed frequency set and Fujita-Kato's method. We use the analytic functional setting (see [21]) as follows:

Definition 1.1 (Countable sum closed frequency set). A countable set Λ in $\mathbb{R}^3$ is called a sum closed frequency set if it satisfies the following properties:

$$\Lambda = \{a + b : a, b \in \Lambda\} \quad \textit{and} \quad -\Lambda = \Lambda.$$

Remark 1.2. If $\{e_j\}_{j=1}^3$ is the standard orthogonal basis in $\mathbb{R}^3$, then the sets $\mathbb{Z}^3$, $\{(m_1 + \sqrt{2}m_2)e_1 + (m_3 + \sqrt{3}m_4)e_2 + (m_5 + \sqrt{5}m_6)e_3 : m_1, \cdots, m_6 \in \mathbb{Z}\}$ and $\{m_1 e_1 + m_2(e_1 + e_2\sqrt{2}) + m_3(e_2 + e_3\sqrt{3}) : m_1, m_2, m_3 \in \mathbb{Z}\}$ are examples of such countable sum closed frequency sets. Clearly, the case $\mathbb{Z}^3$ corresponds to the periodic. Each of the other two cases is dense in $\mathbb{R}^3$ and therefore they correspond to "purely" almost periodic setting.

Definition 1.3 (An ℓ^1-type function space). Let BUC be the space of all bounded uniformly continuous functions defined in $\mathbb{R}^3$ equipped with the L^∞-norm. For a countable sum closed frequency set $\Lambda \subset \mathbb{R}^3$, let

$$\begin{aligned} X^\Lambda(\mathbb{R}^3) \quad := \quad & \Big\{ u = \sum_{n\in\Lambda} \hat{u}_n e^{in\cdot x} \in BUC(\mathbb{R}^3) \quad \text{for} \quad \{\hat{u}_n\}_{n\in\Lambda} \subset \mathbb{C} : \\ & \hat{u}_{-n} = \hat{u}_n^* \quad for \quad n \in \Lambda,\ \|u\| := \sum_{n\in\Lambda} |\hat{u}_n| < \infty \Big\}, \end{aligned}$$

where $\hat{u}_n^*$ is the complex conjugate coefficient of $\hat{u}_n$.

The condition $\hat{u}_{-n} = \hat{u}_n^*$ guarantees that functions in X^Λ are real-valued.

Remark 1.4. Note that functions in ℓ^1 do not necessarily decay as $x \to \infty$. Also, this almost periodic setting is in general, different from the periodic case since the frequency set may have accumulation points. The almost periodic setting is between the periodic and the fully non-decaying cases.

Before defining our solutions, we first recall the notion of (usual) mild solution. Note that our definition of the solution is rather unusual in geophysical field. Applying the extended Leray projection P to equations (1.1) we annihilate the gradient term ∇p and therefore deduce the integral equation of (1.1):

$$v(t) = e^{t(\tilde{\nu}\Delta - NS)}v_0 - \int_0^t e^{(t-s)(\tilde{\nu}\Delta - NS)}P(u(s)\cdot\nabla)v(s)\, ds,$$

where $v = (u, \frac{\sqrt{g}}{N}\rho)$, $\tilde{\nu} = (\nu,\nu,\nu,\kappa)$ and $S = PJP$ (for the detail, see Section 2). The pressure p can be recovered by

$$p = \sum_{1\le i,j\le 3} R_iR_ju_iu_j - g(-\Delta)^{-1}\partial_{x_3}\rho,$$

where R_j is the Riesz transform (see Section 2). As this was pointed out in [17], it is difficult to handle mild solutions in the case when $\kappa \neq \nu$. Indeed, the main difficulty is that NS and $\tilde{\nu}\Delta$ do not commute, namely $NS(\tilde{\nu}\Delta) \neq (\tilde{\nu}\Delta)NS$. Thus we cannot use this type of mild solution directly. We define a solution of (1.1) as follows

DEFINITION 1.5. Let X^Λ be a Banach space given by Definition 1.3. We call $v(t)$ a solution to (1.1) in X^Λ if v has a form

$$v = v(t,x) = \sum_{n\in\Lambda} \hat{v}_n(t)e^{in\cdot x}$$

with $\hat{v}_n \in \mathcal{C}([0,T],\mathbb{C}^4)$ solving the following ODE:

$$\begin{aligned}\partial_t\hat{v}_n(t) = &-\tilde{\nu}|n|^2\hat{v}_n(t) - S_n\hat{v}_n(t)\\ &- iP_n\sum_{\substack{k,m\in\Lambda,\\ n=k+m}} (\hat{v}_k(t)\cdot\vec{m})\hat{v}_m(t) \quad \text{with} \quad (\vec{n}\cdot\hat{v}_n(t)) = 0\end{aligned} \tag{1.2}$$

for $n\in\Lambda$, where $\vec{m} := (m,0) = (m_1,m_2,m_3,0)$ (we define $\tilde{\nu}$, P_n and S_n in the next section), and the above sum converges uniformly in $\mathcal{C}([0,T],X^\Lambda)$.

To construct the family $(v_n(t))_{n\in\Lambda}$, we apply Craya-Herring decomposition and use a filtering procedure enabling us to obtain this "different type" of mild solutions. See (2.8),(3.2) and (3.3) for details.

Now, we define anisotropic dilation of the frequency set as follows.

Definition 1.6. Let Λ be a countable sum closed frequency set. For $\gamma=(\gamma_1,\gamma_2)\in(0,\infty)^2$, let

$$\Lambda(\gamma):=\{(\gamma_1 n_1,\gamma_2 n_2,n_3)\in\mathbb{R}^3:(n_1,n_2,n_3)\in\Lambda\}. \tag{1.3}$$

Now, we introduce the following Quasi-Geostrophic equation which is a part of the limiting system (formally obtained from equation (1.1) when $\mathcal{N}$ tends to infinity):

$$\begin{cases}\partial_t\theta-\Delta\theta+(-\Delta_h)^{-1/2}\left[(v\cdot\nabla)\left((-\Delta_h)^{1/2}\theta)\right)\right]=0,\\ v=\left(-\partial_{x_2}(-\Delta_h)^{-1/2}\theta,\partial_{x_1}(-\Delta_h)^{-1/2}\theta\right)\\ \theta(t)|_{t=0}=\theta_0=-\partial_{x_2}(-\Delta_h)^{-1/2}u_{0,1}+\partial_{x_1}(-\Delta_h)^{-1/2}u_{0,2},\end{cases} \tag{1.4}$$

where $\theta=\theta(t)=\theta(t,x_1,x_2,x_3)$, $\Delta_h:=\partial_{x_1}^2+\partial_{x_2}^2$ and $\Delta:=\partial_{x_1}^2+\partial_{x_2}^2+\partial_{x_3}^2$. We see that $(-\Delta_h)^{-1/2}u=\sum_{n\in\Lambda}|n|_h^{-1}\hat{u}_ne^{in\cdot x}$ for $u=\sum_{n\in\Lambda}\hat{u}_ne^{in\cdot x}$. We give an explicit one-to-one correspondence between the QG and a 2D type Navier-Stokes equations:

$$\begin{cases}\partial_t v-\Delta v+(v\cdot\nabla_2)v+\nabla_2 p=0,\\ \nabla_2\cdot v=0,\ v|_{t=0}=v_0.\end{cases} \tag{1.5}$$

In this case we set $\theta=(-\Delta_h)^{-1/2}\mathrm{rot}_2 v$. For the existence of the unique global solution to 2D-Navier-Stokes equation with almost periodic initial data, see [15].

Before stating the main result, and in order to avoid resonances, we need the following setting: for any sum closed frequency set Λ, we choose a set of frequencies dilation factors $\Gamma(\Lambda)\subset(0,\infty)^2$ as (2.14). Note that its complement set Γ^c is at most countable. Let us take $\Lambda(\gamma)$ (see Definition 1.6) and fix it. Now we choose an arbitrary large initial datum $(u_0,\rho_0)\in X^{\Lambda(\gamma)}\times X^{\Lambda(\gamma)}$ and set $\theta_0:=-\partial_{x_2}(-\Delta_h)^{-1/2}u_{0,1}+\partial_{x_1}(-\Delta_h)^{-1/2}u_{0,2}$. We know that for this initial data, (1.4) has a global-in-time unique solution. We also take $\nu>0$, $\kappa>0$ and fix them. The main result is the following:

Theorem 1.7. *Set $N=\mathcal{N}\sqrt{g}$ and take arbitrarily $T>0$. If zero-mean value divergence free initial vector field $u_0\in X^{\Lambda(\gamma)}$ and initial thermal disturbance $\rho_0\in X^{\Lambda(\gamma)}$ are chosen as above, then there exists $N_0>g$*

depending only on ν, κ, u_0, ρ_0 such that if $|N| > N_0$, then there exists a unique smooth solution to the equation (1.1) (in the sense of Definition 1.5), $u(t) \in C([0,T] : X^{\Lambda(\gamma)})$ with zero-mean value and divergence free, and $\rho(t) \in C([0,T] : X^{\Lambda(\gamma)})$.

REMARK 1.8. In order to see the functional space of the pressure, it is convenient to divide it into low and high frequency parts, since we have the explicit representation of the pressure p and it has $g(-\Delta)^{-1}\partial_{x_3}\rho$. We see that the low frequency part of ∇p (or equivalently $(-\Delta)^{-1/2}p$) is in X^Λ and its high frequency part is also in X^Λ for $t \in [0,T]$.

REMARK 1.9. For the periodic case, we do not need to restrict the frequency set to Γ i.e. we can take $\Gamma(\Lambda) = (0,\infty)^2$. However, the computation in this case is more complicated and needs a"restricted convolution" type result in the spirit of [2].

2. Preliminaries

Before going any further, we first recall the following facts about the space X^Λ:

- $(X^\Lambda, \|\cdot\|)$ is a Banach space, and any almost periodic function $u \in X^\Lambda$ can be decomposed $u(x) = \Sigma_{n\in\Lambda}\hat{u}_n e^{inx}$, where each "Fourier coefficient" $\hat{u}_n$ is uniquely determined by
$$\hat{u}_n = \lim_{|B|\to\infty} \frac{1}{|B|}\int_B u(x)e^{ix\cdot n}\, dx,$$
where B stands for a ball in $\mathbb{R}^3$ (see for example [7]).
- X^Λ is a closed subspace of FM; the Fourier preimage of the space of all finite Radon measures proposed by Giga, Inui, Mahalov and Matsui in [12, 13, 14].
- Leray projection on almost periodic functions $\bar{P} = \{\bar{P}_{jk}\}_{j,k=1,2,3}$ is defined by
$$\bar{P}_{jk} := \delta_{jk} + R_j R_k \quad (1 \le j, k \le 3),$$

where δ_{jk} is Kronecker's delta and R_j is the Riesz transform defined by

$$R_j = \frac{\partial}{\partial x_j}(-\Delta)^{-\frac{1}{2}} \; for \; j = 1,2,3.$$

The symbol $\sigma(R_j)$ of R_j is $in_j/|n|$ and $\sigma((-\Delta)^{-1/2})$ is $|n|^{-1}$, where $i = \sqrt{-1}$ (see [7]). Let P denote the extended Leray projection with Fourier-multiplier $P_n = \{P_{n,ij}\}_{i,j=1,2,3,4}$ (the symbol $\sigma(P)$ is P_n) given by

$$P_{n,ij} := \begin{cases} \delta_{ij} - \frac{n_i n_j}{|n|^2} & (1 \le i, j \le 3), \\ \delta_{ij} & (otherwise). \end{cases}$$

- Helmholtz-Leray decomposition is defined on almost periodic functions in the same way as in the periodic case. Namely, any $u \in X^{\Lambda}$ is uniquely decomposed as

  $$u = w + \nabla \pi,$$

  where $\pi = -(-\Delta)^{-1}\mathrm{div}\; u \in X^{\Lambda}$ and $w = \bar{P}u \in X^{\Lambda}$.

Now we rewrite the system (1.1) in a more abstract way. Let $N := \mathcal{N}\sqrt{g}$ and $v \equiv (v_1, v_2, v_3, v_4) := (u_1, u_2, u_3, \frac{\sqrt{g}}{\mathcal{N}}\rho)$. Then (v, p) solves

$$\begin{cases} \partial_t v - \tilde{\nu}\Delta v + NJv + \nabla_3 p = -(v \cdot \nabla_3)v, \\ v|_{t=0} = v_0, \\ \nabla_3 \cdot v = 0 \end{cases} \tag{2.1}$$

with $\tilde{\nu} = \mathrm{diag}(\nu, \nu, \nu, \kappa)$, the initial data $v_0 = (u_{0,1}, u_{0,2}, u_{0,3}, \frac{\sqrt{g}}{\mathcal{N}}\rho_0)$, $\nabla_3 := (\partial_1, \partial_2, \partial_3, 0)$,

$$J := \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and $(v \cdot \nabla_3) = (v_1\partial_1 + v_2\partial_2 + v_3\partial_3)$.

Observe that under the condition $N > g$ we have $\mathcal{N} > \sqrt{g}$ and therefore $\|v_{0,4}\| = \|\frac{\sqrt{g}}{\mathcal{N}}\rho_0\| < \|\rho_0\|$. We will assume this condition throughout the paper.

Applying the extended Leray projection P to (2.1) yields the following system of equations that we will solve

$$\begin{cases} dv/dt + (-\tilde{\nu}\Delta + NS)v = -P(v \cdot \nabla_3)v, \\ v|_{t=0} = Pv_0 = v_0, \end{cases} \tag{2.2}$$

with $S := PJP$. Recall that for $|n|_h \neq 0$, the matrix $S_n := P_n J P_n$ (the symbol $\sigma(S)$ is S_n) has the following Craya-Herring orthonormal eigenvectors (for fixed n) $\{q_n^1, q_n^{-1}, q_n^0, q_n^{div}\}$ (see [3, 9]) associated to the eigenvalues $\{i\omega_n, -i\omega_n, 0, 0\}$, respectively. Here,

$$\omega_n = \frac{|n|_h}{|n|},\ \ |n|_h = \sqrt{n_1^2 + n_2^2}$$

and

$$\begin{aligned}
q_n^1 &:= (q_{1,n}^1, q_{2,n}^1, q_{3,n}^1, q_{4,n}^1) := \frac{1}{\sqrt{2}|n|_h^2}(i\omega_n n_1 n_3, i\omega_n n_2 n_3, -i|n|_h^2\omega_n, |n|_h^2),\\
q_n^{-1} &:= (q_{1,n}^{-1}, q_{2,n}^{-1}, q_{3,n}^{-1}, q_{4,n}^{-1}) := \frac{1}{\sqrt{2}|n|_h^2}(-i\omega_n n_1 n_3, -i\omega_n n_2 n_3, i|n|_h^2\omega_n, |n|_h^2),\\
q_n^0 &:= (q_{1,n}^0, q_{2,n}^0, q_{3,n}^0, q_{4,n}^0) := \frac{1}{|n|_h}(-n_2, n_1, 0, 0),\\
q_n^{div} &:= (q_{1,n}^{div}, q_{2,n}^{div}, q_{3,n}^{div}, q_{4,n}^{div}) := \frac{1}{|n|}(n_1, n_2, n_3, 0).
\end{aligned}$$

Note that $q_n^1 = q_n^{-1*}$ and $q_n^0 = q_n^{0*}$, where the $*$ notation stands for the complex conjugate. We have $S_n q_n^1 = i\omega_n q_n^1$, $S_n q_n^{-1} = -i\omega_n q_n^{-1}$, and $S_n q_n^0 = S_n q_n^{div} = 0$. In the case when $|n|_h = 0$ and $n_3 \neq 0$, we define

$$\begin{aligned}
q_n^1 :=& (1/2, 1/2, 0, 1/\sqrt{2})\\
q_n^{-1} :=& (-1/2, -1/2, 0, 1/\sqrt{2})\\
q_n^0 :=& (-1/\sqrt{2}, 1/\sqrt{2}, 0, 0)\\
q_n^{div} :=& (0, 0, 1, 0).
\end{aligned}$$

In fact, for $|n|_h = 0$ and $n_3 \neq 0$, we have $S_n = P_n J P_n = 0$. We point out that, the above choice of the basis is uniquely determined by the conditions $(\tilde{\nu} q_n^1 \cdot q_n^{1*}) = \left(\frac{\nu+\kappa}{2}\right)$, $(\tilde{\nu} q_n^{-1} \cdot q_n^{-1*}) = \left(\frac{\nu+\kappa}{2}\right)$ and $(\tilde{\nu} q_n^0 \cdot q_n^{0*}) = \nu$. Moreover, the divergence-free condition requires that $(\hat{v}_n(t) \cdot q_n^{div}) = 0$, giving $q_n^{div} := (0, 0, 1, 0)$.

Now we explain our strategy in solving (2.2). Recall that S and $\tilde{\nu}\Delta$ do not necessarily commute i.e. $S(\tilde{\nu}\Delta) \neq (\tilde{\nu}\Delta)S$. Hence, we see that

$$e^{t(\tilde{\nu}\Delta - NS)} \neq e^{t\tilde{\nu}\Delta} e^{tNS}.$$

In [17], the authors considered the case $\kappa = \nu$ and therefore were able to use the semigroup $e^{t(\tilde{\nu}\Delta - NS)}$ that enjoys the commutation property in this case. It is not clear how to effectively use the semigroup $e^{t(\tilde{\nu}\Delta - NS)}$ under the assumption $\kappa \neq \nu$. Our idea is to "filter out" solutions of (2.2) using the semigroup e^{tNS}. After doing so, we prove a key observation showing that the operator $e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}$ keeps diffusivity and is independent of time t. To be more precise, first recall that the standard basis of $[L^2(\mathbb{T}^3)]^4$ is given by

$$[(e^{in\cdot x}, 0, 0, 0)]_{n\in\mathbb{Z}^3}\ [(0, e^{in\cdot x}, 0, 0)]_{n\in\mathbb{Z}^3}, [(0, 0, e^{in\cdot x}, 0)]_{n\in\mathbb{Z}^3} \quad \text{and}$$
$$[(0, 0, 0, e^{in\cdot x})]_{n\in\mathbb{Z}^3}.$$

Clearly, these functions are not eigenfunctions of the operator $NS = N(PJP)$. However, introducing the functions $[\Phi_n^j(x)]_{n\in\mathbb{Z}^3}$ defined, using Craya-Herring eigenvectors, by

$$\begin{aligned}
[\Phi_n^1(x)]_{n\in\mathbb{Z}^3} &:= [q_n^1 e^{in\cdot x}]_{n\in\mathbb{Z}^3} \\
&= [(q_{1,n}^1 e^{in\cdot x}, q_{2,n}^1 e^{in\cdot x}, q_{3,n}^1 e^{in\cdot x}, q_{4,n}^1 e^{in\cdot x})]_{n\in\mathbb{Z}^3}, \\
[\Phi_n^2(x)]_{n\in\mathbb{Z}^3} &:= [q_n^{-1} e^{in\cdot x}]_{n\in\mathbb{Z}^3} \\
&= [(q_{1,n}^{-1} e^{in\cdot x}, q_{2,n}^{-1} e^{in\cdot x}, q_{3,n}^{-1} e^{in\cdot x}, q_{4,n}^{-1} e^{in\cdot x})]_{n\in\mathbb{Z}^3}, \\
[\Phi_n^3(x)]_{n\in\mathbb{Z}^3} &:= [q_n^0 e^{in\cdot x}]_{n\in\mathbb{Z}^3} \\
&= [(q_{1,n}^0 e^{in\cdot x}, q_{2,n}^0 e^{in\cdot x}, q_{3,n}^0 e^{in\cdot x}, q_{4,n}^0 e^{in\cdot x})]_{n\in\mathbb{Z}^3}, \\
[\Phi_n^4(x)]_{n\in\mathbb{Z}^3} &:= [q_n^{div} e^{in\cdot x}]_{n\in\mathbb{Z}^3} \\
&= [(q_{1,n}^{div} e^{in\cdot x}, q_{2,n}^{div} e^{in\cdot x}, q_{3,n}^{div} e^{in\cdot x}, q_{4,n}^{div} e^{in\cdot x})]_{n\in\mathbb{Z}^3},
\end{aligned}$$

we easily see that $[\Phi_n^j(x)]_{n\in\mathbb{Z}^3}$ are eigenfunctions of NS and constitue a complete orthogonal basis in $[L^2(\mathbb{T}^3)]^4$. The eigenvalues are

$$\{i\omega_n\}_{n\in\mathbb{Z}^3},\ \{-i\omega_n\}_{n\in\mathbb{Z}^3},\ \{0\}_{n\in\mathbb{Z}^3} \quad \text{and} \quad \{0\}_{n\in\mathbb{Z}^3}$$

respectively. Second, set $V(t) := e^{tNS}v(t)$ and $V_0 = V(0) = v(0)$. From (2.2), we see that the function V satisfies the following equation (we say "filtering procedure"):

$$\begin{cases} dV/dt + e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}V = -e^{tNS}P(e^{-tNS}V\cdot\nabla_3)e^{-tNS}V, \\ V|_{t=0} = PV_0 = V_0. \end{cases} \tag{2.3}$$

Finally, we show that the operator $e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}$ is diffusive and time independent. Indeed, a direct calculation enables us to see that

$$\begin{aligned}
e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}\Phi_n^1(x) &= (\tilde{\nu}q_n^1\cdot q_n^{1*})|n|^2\Phi_n^1(x) = \left(\frac{\nu+\kappa}{2}\right)|n|^2\Phi_n^1(x),\\
e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}\Phi_n^2(x) &= (\tilde{\nu}q_n^{-1}\cdot q_n^{-1*})|n|^2\Phi_n^2(x) = \left(\frac{\nu+\kappa}{2}\right)|n|^2\Phi_n^2(x),\\
e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}\Phi_n^3(x) &= (\tilde{\nu}q_n^0\cdot q_n^{0*})|n|^2\Phi_n^3(x) = \nu|n|^2\Phi_n^3(x),\\
e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}\Phi_n^4(x) &= (\tilde{\nu}q_n^{div}\cdot q_n^{div*})|n|^2\Phi_n^4(x) = \nu|n|^2\Phi_n^4(x)
\end{aligned}$$

which can be rewritten as

$$e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}\Phi_n^j(x) = |n|^2(\tilde{\nu}q_n^j\cdot q_n^{j*})\Phi_n^j(x), \quad j=1,2,3,4. \tag{2.4}$$

Thus, $e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}$ acts like a diffusion operator. In this way we can therefore handle even the case $\kappa\neq\nu$. However, the price to pay is to consider the three wave interaction in the almost periodic case. Up to now, the analysis of such interactions is difficult and not known how one can handle it. To avoid this problem, we use anisotropic dilation of the frequency set (see Definition 2.7) so that these interactions cancel (see Lemma 2.7).

Since we look for an almost periodic solution to (2.2), we formally write the solution v as

$$v(t,x) = \sum_{n\in\Lambda}\hat{v}_n(t)e^{in\cdot x},$$

where for $n\in\Lambda$, $\hat{v}_n := (\hat{v}_{n,1},\hat{v}_{n,2},\hat{v}_{n,3},\hat{v}_{n,4})$, $\hat{v}_n\cdot\vec{n} = 0$ and $\vec{n} := (n,0) = (n_1,n_2,n_3,0)$. Then we decompose $\hat{v}_n$ using Craya-Herring eigenvectors,

$$\hat{v}_n = \sum_{\sigma_0\in\{-1,0,1\}} a_n^{\sigma_0}q_n^{\sigma_0} \quad \text{with} \quad a_n^{\sigma_0} := (\hat{v}_n\cdot q_n^{\sigma_0 *}).$$

Hence the filtered solution $V(t) := e^{tNS}v(t)$ can be written as

$$V(t) = \sum_{n\in\Lambda} \sum_{\sigma_0\in\{-1,0,1\}} c_n^{\sigma_0}(t)\Phi_n^{\sigma_0}.$$

Using the important property of diffusivity of $e^{tNS}(-\tilde{\nu}\Delta)e^{-tNS}$, equation (2.3) yields

$$\begin{aligned} \partial_t c_n^{\sigma_0}(t) &= -c_n^{\sigma_0}(t)|n|^2(\tilde{\nu} q_n^{\sigma_0} \cdot q_n^{\sigma_0 *}) \\ &- i \sum_{n=k+m,\ \sigma_1,\sigma_2\in\{-1,0,1\}} e^{iNt\omega_{nkm}^{\sigma}} \\ &\times c_k^{\sigma_1} c_m^{\sigma_2}(q_k^{\sigma_1}\cdot m)(q_m^{\sigma_2}\cdot q_n^{\sigma_0 *}), \end{aligned} \tag{2.5}$$

where we set

$$\omega_{nkm}^{\sigma} := (-\sigma_0\omega_n + \sigma_1\omega_k + \sigma_2\omega_m). \tag{2.6}$$

Now we split the nonlinear part into the resonant (independent of N) and non-resonant two parts defined by

$$\bar{B}_n^{\sigma_0}(g^{\sigma_1}, h^{\sigma_2}) := -i \sum_{n=k+m,\ \omega_{nkm}^{\sigma}=0} (q_k^{\sigma_1}\cdot m)(q_m^{\sigma_2}\cdot q_n^{\sigma_0 *}) g_k^{\sigma_1} h_m^{\sigma_2}$$

and

$$\begin{aligned} &\tilde{B}_n^{\sigma_0}(Nt, g^{\sigma_1}, h^{\sigma_2}) \\ &\quad := -i \sum_{n=k+m,\ \omega_{nkm}^{\sigma}\neq 0} (q_k^{\sigma_1}\cdot m)(q_m^{\sigma_2}\cdot q_n^{\sigma_0 *}) g_k^{\sigma_1} h_m^{\sigma_2} \exp(i\omega_{nkm}^{\sigma} Nt), \end{aligned}$$

respectively. In addition, thanks to the standard smoothing estimates of the heat kernal we have the following estimates:

$$\begin{cases} \|\{e^{-\nu|n|^2 t}\tilde{B}_n^{\sigma_0}(Nt, g^{\sigma_1}, h^{\sigma_2})\}_{n\in\Lambda}\| \leq \frac{C_\nu}{t^{1/2}}\|g^{\sigma_1}\|\|h^{\sigma_2}\| \\ \\ \|\{e^{-\nu|n|^2 t}\bar{B}_n^{\sigma_0}(g^{\sigma_1}, h^{\sigma_2})\}_{n\in\Lambda}\| \leq \frac{C_\nu}{t^{1/2}}\|g^{\sigma_1}\|\|h^{\sigma_2}\| \end{cases} \tag{2.7}$$

(for $\sigma_0 = -1, 0, 1$) obtained by estimating the first derivative of the heat kernel as follows

$$\sup_{n\in\Lambda} \left| |n| e^{-\nu|n|^2 t} \right| \leq \frac{C_\nu}{t^{1/2}}.$$

The constant $C_\nu > 0$ is independent of N.

Then we have the following equations:

$$(2.8)\quad \begin{cases} \partial_t c_n^0(t) = -\nu|n|^2 c_n^0(t) \\ \qquad\qquad + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \left(\bar{B}_n^0(c^{\sigma_1}, c^{\sigma_2}) + \tilde{B}_n^0(Nt, c^{\sigma_1}, c^{\sigma_2})\right) \\ \partial_t c_n^{\sigma_0}(t) = -\left(\frac{\nu+\kappa}{2}\right)|n|^2 c_n^{\sigma_0}(t) \\ \qquad\qquad + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \left(\bar{B}_n^{\sigma_0}(c^{\sigma_1}, c^{\sigma_2}) + \tilde{B}_n^{\sigma_0}(Nt, c^{\sigma_1}, c^{\sigma_2})\right), \\ c_n^0(t)|_{t=0} = c_n^0(0), \quad c_n^{\sigma_0}(t)|_{t=0} = c_n^{\sigma_0}(0) \end{cases}$$

for $\sigma_0 = \pm 1$. From the condition $\omega_{nkm}^\sigma = 0$, we easily see that the terms $\bar{B}_n^0(c^1, c^1)$, $\bar{B}_n^0(c^{-1}, c^{-1})$, $\bar{B}_n^0(c^0, c^{\pm 1})$, $\bar{B}_n^0(c^{\pm 1}, c^0)$, $\bar{B}_n^{\pm 1}(c^{\mp 1}, c^0)$, $\bar{B}_n^{\pm 1}(c^0, c^{\mp 1})$ and $\bar{B}_n^{\pm 1}(c^0, c^0)$ disappear. Now, we define the "limit equations" by

$$(2.9)\quad \begin{cases} \partial_t b_n^0(t) = -\nu|n|^2 b_n^0(t) + \bar{B}_n^0(b^0, b^0) + \bar{B}_n^0(b^1, b^{-1}) + \bar{B}_n^0(b^{-1}, b^1), \\ \partial_t b_n^{\sigma_0}(t) = -\left(\frac{\nu+\kappa}{2}\right)|n|^2 b_n^{\sigma_0}(t) \\ \qquad\qquad + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2\setminus D} \bar{B}_n^{\sigma_0}(b^{\sigma_1}, b^{\sigma_2}), \quad \sigma_0 = \pm 1, \\ b_n^0(t)|_{t=0} = c_n^0(0), \quad b_n^{\sigma_0}(t)|_{t=0} = c_n^{\sigma_0}(0), \end{cases}$$

where $D := \{(0,0), (-1,0), (0,-1)\}$ for $\sigma_0 = 1$ and $D := \{(0,0), (1,0), (0,1)\}$ for $\sigma_0 = -1$. Formally, we can get (2.9) from (2.8) when $N \to \infty$. We will justify this convergence in Lemma 3.2. Now we show that there is more non trivial cancellation in the limit equations. More precisely,

LEMMA 2.1. *We have*

$$\bar{B}_n^0(c^1, c^{-1}) + \bar{B}_n^0(c^{-1}, c^1) = 0.$$

PROOF. To prove the lemma, it suffices to show

$$(2.10) \quad (q_k^1 \cdot m)(q_m^{-1} \cdot q_n^{0*}) + (q_m^{-1} \cdot k)(q_k^1 \cdot q_n^{0*}) = 0 \quad \text{for any} \quad n = k + m \quad \text{with} \quad \omega_k = \omega_m.$$

First we show that $\omega_k = \omega_m$ if and only if

$$k, m \in \{n \in \mathbb{Z}^3 : |n|_h^2 = \lambda n_3^2\} \quad \text{for some} \quad \lambda > 0.$$

($\Leftarrow$): This direction is clear. Thus we omit it.

($\Rightarrow$): Rewrite the identity $\omega_k = \omega_m$ as $F(X) = F(Y)$, where $X := |k|_h^2/k_3^2$, $Y := |m|_h^2/m_2^3$ and $F(X) := X/(X+1)$. Since the function F is monotone increasing, we see $X = Y$. This means that

$$k_3 = \pm\frac{|k|_h}{\sqrt{\lambda}} \quad \text{and} \quad m_3 = \pm\frac{|m|_h}{\sqrt{\lambda}}.$$

We only consider the case $k_3 = \frac{|k|_h}{\sqrt{\lambda}}$ and $m_3 = \frac{|m|_h}{\sqrt{\lambda}}$, since the other cases are similar. A direct calculation shows that

$$\begin{aligned}
&(q_k^1 \cdot m)(q_m^{-1} \cdot q_n^{0*}) \\
&\quad = \frac{1}{\sqrt{2}\lambda|m||k||n|_h}\left(\frac{k_h \cdot m_h}{\lambda} - \frac{m_3|k|}{\sqrt{1+\lambda^2}}\right)(-m_2k_1 + m_1k_2), \\
&(q_m^{-1} \cdot k)(q_k^1 \cdot q_n^{0*}) \\
&\quad = \frac{1}{\sqrt{2}\lambda|m||k||n|_h}\left(\frac{k_h \cdot m_h}{\lambda} - \frac{k_3|m|}{\sqrt{1+\lambda^2}}\right)(-k_2m_1 + k_1m_2).
\end{aligned}$$

From $k_3 = \frac{|k|_h}{\sqrt{\lambda}}$ and $m_3 = \frac{|m|_h}{\sqrt{\lambda}}$, we have (2.10) as desired. □

Now we show that the function c^0 in the limit equations satisfies a quasi geostrophic (QG) equation type and that this QG equation is equivalent to the 2D type Navier-Stokes equation. By the following lemma, we can see that the function c^0 satisfies the QG equation (1.4).

LEMMA 2.2. *Let $|n|_h = \sqrt{n_1^2 + n_2^2}$. The resonant part $\bar{B}_n^0(c^0, c^0)$ can be expressed as follows:*

$$\bar{B}_n^0(c^0, c^0) = -\sum_{n=k+m} \frac{i(k \times m)|m|_h}{|k|_h|n|_h} c_k^0 c_m^0.$$

PROOF. Since $q_n^0 = \frac{1}{|n|_h}(-n_2, n_1, 0, 0)$ and $q_n^0 = \frac{1}{|n|_h}(|k|_h q_k^0 + |m|_h q_m^0)$ for $n = k + m$, we have

$$\begin{aligned}
\bar{B}_n^0(c^0, c^0) &= -\sum_{n=k+m} c_k^0 c_m^0 (q_k^0 \cdot im)(q_m^0 \cdot q_n^{0*}) \\
&= -\sum_{n=k+m} \frac{i}{|k|_h |n|_h}(k_2 m_1 - k_1 m_2) \\
&\quad \times \left(q_m^0 \cdot (|k|_h q_k^{0*} + |m|_h q_m^{0*})\right) c_k^0 c_m^0 \\
&= \sum_{n=k+m} -\frac{i|m|_h}{|k|_h |n|_h}(k_2 m_1 - k_1 m_2) c_k^0 c_m^0 \\
&\quad - \sum_{n=k+m} \frac{i}{|n|_h}(k_2 m_1 - k_1 m_2)(q_m^0 \cdot q_k^{0*}) c_k^0 c_m^0 .
\end{aligned}$$

Since $k \times m = -(m \times k)$, we see that

$$\sum_{n=k+m} \frac{i}{|n|_h}(k_2 m_1 - k_1 m_2)(q_m^0 \cdot q_k^{0*}) c_k^0 c_m^0 = 0,$$

which leads to the desired formula. □

Now we show that there is a one-to-one correspondence between the QG and a 2D type Navier-Stokes equations.

LEMMA 2.3. *Recall $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ and let*

$$\begin{aligned}
w &:= (w_1(x_1, x_2, x_3, t), w_2(x_1, x_2, x_3, t)) \\
&:= \left(\sum_{n \in \Lambda} \hat{w}_{1,n}(t) e^{in \cdot x}, \sum_{n \in \Lambda} \hat{w}_{2,n}(t) e^{in \cdot x} \right).
\end{aligned}$$

Define $\theta = \theta(t, x_1, x_2, x_3) := (-\Delta_h)^{-1/2} rot_2 w$, with rot_2 is the 2 dimensional curl given by

$$rot_2 w = \partial_1 w_2 - \partial_2 w_1.$$

Then, w solves the following 2D type Navier-Stokes equation:

$$\begin{cases} \partial_t w - \Delta w + (w \cdot \nabla_2) w + \nabla_2 p = 0, \\ \nabla_2 \cdot w = 0, \ w|_{t=0} = w_0 \end{cases} \tag{2.11}$$

if and only if θ solves (1.4), where $\nabla_2 = (\partial_{x_1}, \partial_{x_2})$ and p is the pressure (a scalar function).

PROOF. First, we point the following claim:

$$\begin{cases} \theta = (-\Delta_h)^{-\frac{1}{2}}\mathrm{rot}_2 w, \\ \nabla_2 \cdot w = 0 \end{cases}$$

if and only if $w = \partial_2(-\Delta_h)^{-\frac{1}{2}}\theta, -\partial_1(-\Delta_h)^{-\frac{1}{2}}\theta)$. The indirect implication ($\Leftarrow$) is clear by a straightforward computation of $\mathrm{rot}_2 w$ given $\nabla_2 \cdot w = 0$. To prove ($\Rightarrow$), set $\tilde{\theta} := (-\Delta_h)^{\frac{1}{2}}\theta$. Then we want to solve for a divergence free w satisfying $\mathrm{rot}_2 w = \tilde{\theta}$. Applying ∂_2 to both sides, and using the divergence free condition, we get $-\Delta_h w_1 = \partial_2\tilde{\theta}$, which as a consequence gives w_1. The component w_2 can be derived in a similar way.

Second, observe that for $\theta = (-\Delta_h)^{-1/2}\mathrm{rot}_2 w = \sum_{n\in\Lambda}\hat{\theta}_n(t)e^{in\cdot x}$, we have by $\nabla_2 \cdot w = 0$,

$$\begin{aligned} w &= (i\sum_{n\in\Lambda}\frac{n_2}{|n|_h}\hat{\theta}_n(t)e^{in\cdot x}, -i\sum_{n\in\Lambda}\frac{n_1}{|n|_h}\hat{\theta}_n(t)e^{in\cdot x}) \\ &= (\partial_2(-\Delta_h)^{-\frac{1}{2}}\theta, -\partial_1(-\Delta_h)^{-\frac{1}{2}}\theta). \end{aligned}$$

Then, applying rot_2 to (2.11), we get

$$\partial_t \mathrm{rot}_2 w - \Delta \mathrm{rot}_2 w + (w \cdot \nabla_2)\mathrm{rot}_2 w = 0. \tag{2.12}$$

Here, we used the fact that

$$(w \cdot \nabla_2)\mathrm{rot}_2 w = \mathrm{rot}_2[(w \cdot \nabla_2)w]. \tag{2.13}$$

Finally, applying $(-\Delta_h)^{-1/2}$ to both sides of (2.12), we see that $\theta = (-\Delta_h)^{-1/2}\mathrm{rot}_2 w$ satisfies the desired QG equation (1.4). Conversely, applying $L := (-(-\Delta_h)^{-1}\partial_{x_2}, (-\Delta_h)^{-1}\partial_{x_1})$ (which commutes with Δ) to (2.12), and by (2.13) we can see that $L\mathrm{rot}_2$ is nothing but the two dimensional Leray projection. Therefore, this implies (2.11) as desired. □

REMARK 2.4. We refer to [15] for the existence of a unique global solution to 2D type Navier-Stokes equation (2.11) with almost periodic initial data.

In what follows and in order to show the main theorem, we need the following lemma (which is needed only for the almost periodic case) on the dilation of the frequency set (1.3). This kind of restrictions is technical. However we do not know whether or not such constraints are removable. This means that the general almost periodic setting seems to remain open. Let

$$\Gamma := \Gamma(\Lambda) := \bigcap_{\substack{n,k,m\in\Lambda\setminus\{0\}\\ |n|_h,|k|_h,|m|_h\neq 0}} \{\gamma\in(0,\infty)^2 : P_{nkm}(\gamma)\neq 0\}, \tag{2.14}$$

where

$$P_{nkm}(\gamma) := |\tilde{n}|^8|\tilde{k}|^8|\tilde{m}|^8 \prod_{\sigma\in\{-1,1\}^3} \omega^{\sigma}_{\tilde{n}\tilde{k}\tilde{m}}$$

with $\tilde{n} = (\gamma_1 n_1, \gamma_2 n_2, n_3)$, $\tilde{k} = (\gamma_1 k_1, \gamma_2 k_2, k_3)$ and $\tilde{m} = (\gamma_1 m_1, \gamma_2 m_2, m_3)$, and $\sigma = (\sigma_0, \sigma_1, \sigma_2)$ and $\omega^{\sigma}_{\tilde{n}\tilde{k}\tilde{m}}$ is given by (2.6). (We put $|\tilde{n}|^8|\tilde{k}|^8|\tilde{m}|^8$ in front of $\prod_{\sigma\in\{-1,1\}^3}\omega^{\sigma}_{\tilde{n}\tilde{k}\tilde{m}}$, since we can have a polynomial $P_{nkm}(\gamma)$).

LEMMA 2.5. *Γ is not empty and $|\Gamma| = \infty$.*

REMARK 2.6. To extract γ from Γ, we need to admit "Axiom of choice". In other words, "Every set includes a countable set". This seems to be a crucial point in order to consider the almost periodic setting. Once we extract γ from Γ (it seems difficult to figure out the concrete value of γ), then we can see that $\omega^{\sigma}_{\tilde{n}\tilde{k}\tilde{m}} \neq 0$ for any $n, k, m \in \Lambda(\gamma)\setminus\{0\}$ with $|n|_h, |k|_h, |m|_h \neq 0$.

PROOF. We show that Γ cannot be empty. By a direct calculation, we have

$$\begin{aligned}
P_{nkm}(\gamma) &= |\tilde{n}|^8|\tilde{k}|^8|\tilde{m}|^8 \\
&\quad \Big((\omega_{\tilde{n}} + \omega_{\tilde{k}} + \omega_{\tilde{m}})(-\omega_{\tilde{n}} + \omega_{\tilde{k}} + \omega_{\tilde{m}}) \\
&\quad (\omega_{\tilde{n}} - \omega_{\tilde{k}} + \omega_{\tilde{m}})(\omega_{\tilde{n}} + \omega_{\tilde{k}} - \omega_{\tilde{m}})\Big)^2 \\
&= |\tilde{n}|^8|\tilde{k}|^8|\tilde{m}|^8 \left(\omega_{\tilde{n}}^2 - (\omega_{\tilde{k}} + \omega_{\tilde{m}})^2\right)^2 \left((\omega_{\tilde{n}}^2 - (\omega_{\tilde{k}} - \omega_{\tilde{m}})^2\right)^2 \\
&= |\tilde{n}|^8|\tilde{k}|^8|\tilde{m}|^8 \left((\omega_{\tilde{n}}^2 - \omega_{\tilde{k}}^2 - \omega_{\tilde{m}}^2)^2 - 4\omega_{\tilde{k}}^2\omega_{\tilde{m}}^2\right)^2 \\
&= |\tilde{n}|^8|\tilde{k}|^8|\tilde{m}|^8 \left(\omega_{\tilde{n}}^4 + \omega_{\tilde{k}}^4 + \omega_{\tilde{m}}^4 - 2\omega_{\tilde{k}}^2\omega_{\tilde{m}}^2 - 2\omega_{\tilde{m}}^2\omega_{\tilde{n}}^2 - 2\omega_{\tilde{n}}^2\omega_{\tilde{k}}^2\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \Big(|\tilde n|_h^4 |\tilde k|^4 |\tilde m|^4 + |\tilde n|^4 |\tilde k|_h^4 |\tilde m|^4 + |\tilde n|^4 |\tilde k|^4 |\tilde m|_h^4 \\
&\quad -2|\tilde n|^2 |\tilde n|_h^2 |\tilde k|^2 |\tilde k|_h^2 |\tilde m|^4 - 2|\tilde n|^2 |\tilde n|_h^2 |\tilde k|^4 |\tilde m|^2 |\tilde m|_h^2 \\
&\quad -2|\tilde n|^4 |\tilde k|^2 |\tilde k|_h^2 |\tilde m|^2 |\tilde m|_h^2 \Big)^2 \\
&= \Big(- 3n_1^2 k_1^2 m_1^2 \gamma_1^6 - 3n_2^2 k_2^2 m_2^2 \gamma_2^6 \\
&\quad -3n_1^2 k_1^2 m_2^2 \gamma_1^4 \gamma_2^2 - 3n_1^2 k_2^2 m_1^2 \gamma_1^4 \gamma_2^2 - 3n_2^2 k_1^2 m_1^2 \gamma_1^4 \gamma_2^2 \\
&\quad -3n_2^2 k_2^2 m_1^2 \gamma_1^2 \gamma_2^4 - 3n_2^2 k_1^2 m_2^2 \gamma_1^2 \gamma_2^4 - 3n_1^2 k_2^2 m_2^2 \gamma_1^2 \gamma_2^4 + l.o.t \Big)^2,
\end{aligned}
$$

where $l.o.t$ stands for lower order terms. Since $|n|_h, |k|_h, |m|_h \neq 0$, then the highest order terms never disappear. This means that $P_{nkm}(\gamma)$ is a polynomial and coefficients of highest order terms in both γ_1 and γ_2 are not zero. Thus

$$
|\{\gamma \in \Gamma : P_{nkm}(\gamma) = 0\}| = 0 \quad \text{for fixed} \quad n, k \quad \text{and} \quad m. \tag{2.15}
$$

Since

$$
(0,\infty)^2 = \{\gamma \in (0,\infty)^2 : P_{nkm}(\gamma) \neq 0\} \cup \{\gamma \in (0,\infty)^2 : P_{nkm}(\gamma) = 0\},
$$

then we see

$$
\begin{aligned}
(0,\infty)^2 = &\Big(\cap_{\substack{n,k,m \in \Lambda\setminus\{0\} \\ |n|_h, |k|_h, |m|_h \neq 0}} \{\gamma : P_{nkm}(\gamma) \neq 0\} \Big) \\
&\bigcup \Big(\cup_{\substack{n,k,m \in \Lambda\setminus\{0\} \\ |n|_h, |k|_h, |m|_h \neq 0}} \{\gamma : P_{nkm}(\gamma) = 0\} \Big).
\end{aligned}
$$

By (2.15) we have

$$
\Big| \cap_{\substack{n,k,m \in \Lambda\setminus\{0\} \\ |n|_h, |k|_h, |m|_h \neq 0}} \{\gamma \in (0,\infty)^2 : P_{nkm}(\gamma) \neq 0\} \Big| = |(0,\infty)^2| = \infty. \ \Box
$$

Now we show that the limit equations have a global solution. In the almost periodic case, the non-resonant part $\bar B^{\pm 1}(c^{\pm 1}, c^{\pm 1})$ disappears just by restricting the frequencies set to $\Lambda(\gamma)$.

LEMMA 2.7. *Let Λ be a sum closed frequency set. We extract γ from $\Gamma(\Lambda)$ and fix it. Then for $\sigma_0 = \pm 1$ there exists a global-in-time unique solution $c^{\sigma_0}(t)$ to equations (2.9) such that $c^{\sigma_0}(t) \in C([0,\infty) : \ell^1(\Lambda(\gamma)))$ with $(c_n^{\sigma_0}(t) \cdot n) = 0$ for all $n \in \Lambda(\gamma)$ and $c_0^{\sigma_0}(t) = 0$.*

PROOF. Recall that $\tilde{n} = (\gamma_1 n_1, \gamma_2 n_2, n_3)$, $\tilde{k} = (\gamma_1 k_1, \gamma_2 k_2, k_3)$ and $\tilde{m} = (\gamma_1 m_1, \gamma_2 m_2, m_3)$. By restricting $\gamma \in \Gamma$, we can eliminate the worst nonlinear term. More precisely, for all $n \in \Lambda$ and $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \{-1, 1\}^3$, the term

$$\bar{B}_{\tilde{n}}^{\sigma_0}(c^{\sigma_1}, c^{\sigma_2}) = \sum_{\substack{\tilde{n} = \tilde{k} + \tilde{m} \\ \omega^{\sigma}_{\tilde{n}\tilde{k}\tilde{m}} = 0}} (q_{\tilde{k}}^{\sigma_1} \cdot i\tilde{m})(q_{\tilde{m}}^{\sigma_2} \cdot q_{\tilde{n}}^{\sigma_0 *}) c_{\tilde{k}}^{\sigma_1} c_{\tilde{m}}^{\sigma_2} \quad \text{disappears.}$$

Then we have two coupled linear equations for $\{c_n^{-1}\}_n$ and $\{c_n^1\}_n$. In this case, the global existence will immediately follow from estimates (2.7). □

3. Proof of the Main Theorem

Before proving the main theorem, we first mention the local existence result. Using estimate (2.7), we obtain a local-in-time unique solution to (2.8) in $C([0, T] : \ell^1(\Lambda))$ as stated in the following lemma.

LEMMA 3.1. *Assume that $c(0) := \{c_n^{\sigma_0}(0)\}_{n \in \Lambda, \sigma_0 \in \{-1,0,1\}} \in \ell^1(\Lambda)$ and $c_0^{\sigma_0}(0) = 0$ for $\sigma_0 \in \{-1, 0, 1\}$. Then there is a local-in-time unique solution $c(t) \in C([0, T_L] : \ell^1(\Lambda))$ and $c_0^{\sigma_0}(t) = 0$ ($\sigma_0 \in \{-1, 0, 1\}$) to* (2.8) *satisfying*

$$T_L \geq \frac{C}{\|c(0)\|^2}, \quad \sup_{0<t<T_L} \|c(t)\| \leq 10 \|c(0)\|, \tag{3.1}$$

where C is a positive constant independent of N.

PROOF. First we consider a mild formulation of (2.8):

$$\begin{aligned} c_n^0(t) = {} & e^{-\nu|n|^2 t} c_n^0(0) \\ & + \sum_{(\sigma_1, \sigma_2) \in \{-1,0,1\}^2} \\ & \times \int_0^t e^{-\nu(t-s)|n|^2} \left(\bar{B}_n^0(c^{\sigma_1}, c^{\sigma_2}) + \tilde{B}_n^0(Nt, c^{\sigma_1}, c^{\sigma_2}) \right) ds \end{aligned} \tag{3.2}$$

and

$$c_n^{\sigma_0}(t) = e^{-\frac{\nu+\kappa}{2}|n|^2 t} c_n^{\sigma_0}(0) + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \times \int_0^t e^{-\frac{\nu+\kappa}{2}(t-s)|n|^2} \left(\bar{B}_n^{\sigma_0}(c^{\sigma_1}, c^{\sigma_2}) + \tilde{B}_n^{\sigma_0}(Nt, c^{\sigma_1}, c^{\sigma_2})\right) ds. \tag{3.3}$$

By (2.7), we have the estimates

$$\|c_n^0(t)\| \leq \|c_n^0(0)\| + C_\nu t^{1/2} \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \left(\sup_{0\leq s<t} \|c^{\sigma_1}(s)\| \sup_{0\leq s<t} \|c^{\sigma_2}(s)\|\right)$$

and

$$\|c_n^{\sigma_0}(t)\| \leq \|c_n^{\sigma_0}(0)\| + C_{(\frac{\nu+\kappa}{2})} t^{1/2} \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \left(\sup_{0\leq s<t} \|c^{\sigma_1}(s)\| \sup_{0\leq s<t} \|c^{\sigma_2}(s)\|\right).$$

These *à-priori* estimates of $\sup_t \|c^0(t)\|$ and $\sup_t \|c^{\sigma_0}(t)\|$ give us through a standard fixed point argument the existence of a local-in-time unique solution (for the detailed computation, see [12] for example). We can obtain inequalities (3.1) from the above inequalities (in this case, we take sufficiently small T_L depending on C_ν and $C_{(\frac{\nu+\kappa}{2})}$, and then use an absorbing argument). $\square$

Let $b^{\sigma_0}(t)$ be the solution of the limit equations (2.9) and $c^{\sigma_0}(t)$ be the solution to the original equation (2.8). The main point is to control the ℓ^1-norm of the remainder term $r_n^{\sigma_0}(t) := c_n^{\sigma_0}(t) - b_n^{\sigma_0}(t)$ $(\sigma_0 = -1, 0, 1)$ by the large parameter N. Note that the initial data $r_n^0(0)$ and $r_n^{\sigma_0}(0)$ are zeros. More precisely, r^0 and r^{σ_0} satisfy

$$\partial_t r_n^0(t) = -\nu|n|^2 r_n^0(t) + \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \left(\bar{B}_n^0(r^{\sigma_1}, c^{\sigma_2}) + \bar{B}_n^0(b^{\sigma_1}, r^{\sigma_2}) + \tilde{B}_n^0(Nt, c^{\sigma_1}, c^{\sigma_2})\right)$$

and

$$\begin{aligned}\partial_t r_n^{\sigma_0}(t) = &-\left(\frac{\nu+\kappa}{2}\right)|n|^2 r_n^{\sigma_0}(t)\\ &+\sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\Big(\bar{B}_n^{\sigma_0}(r^{\sigma_1},c^{\sigma_2})+\bar{B}_n^{\sigma_0}(b^{\sigma_1},r^{\sigma_2})\\ &\qquad\qquad+\tilde{B}_n^{\sigma_0}(Nt,c^{\sigma_1},c^{\sigma_2})\Big),\end{aligned}$$

respectively. Once we control the remainder term in the ℓ^1-norm, we easily have the main result by a standard bootstrapping argument (see [21] for example). Now we show the following lemma concerning the smallness of the reminder term. Let $b(t) := \{b_n^{\sigma_0}(t)\}_{n\in\Lambda,\sigma_0\in\{-1,0,1\}}$ and $r(t) := \{r_n^{\sigma_0}(t)\}_{n\in\Lambda,\sigma_0\in\{-1,0,1\}}$.

LEMMA 3.2. *For all $\epsilon > 0$, there is $N_0 > 0$ such that $\|r(t)\| \le \epsilon$ for $|N| > N_0$ and $0 < t < T_L$ with T_L is the local existence time (see Lemma 3.1).*

PROOF. To simplify the remainder equation, we introduce the following notation. Let

$$\begin{aligned}\bar{R}_n^{\sigma_0}(r,c,b) : &= \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\left(\bar{B}_n^{\sigma_0}(r^{\sigma_1},c^{\sigma_2})+\bar{B}_n^{\sigma_0}(b^{\sigma_1},r^{\sigma_2})\right),\\ \tilde{R}_n^{\sigma_0}(Nt,c) : &= \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2}\tilde{B}_n^{\sigma_0}(Nt,c^{\sigma_1},c^{\sigma_2}).\end{aligned}$$

Rewrite the remainder equations as follows:

$$\begin{cases}\partial_t r_n^0(t) = -\nu|n|^2 r_n^0(t) + \bar{R}_n^0(r,c,b) + \tilde{R}_n^0(Nt,c)\\ \partial_t r_n^{\sigma_0}(t) = -\left(\frac{\nu+\kappa}{2}\right)|n|^2 r_n^{\sigma_0}(t) + \bar{R}_n^{\sigma_0}(r,c,b) + \tilde{R}_n^{\sigma_0}(Nt,c)\\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad \sigma_0 = -1, 1.\end{cases} \tag{3.4}$$

In order to control r, the key is to estimate $\tilde{R}_n^0(Nt,c)$ and $\tilde{R}_n^{\sigma_0}(Nt,c)$ in (3.4). To do so, we need to analyze the following oscillatory integral of the non-resonant part as follows:

$$\tilde{\mathcal{B}}_n^{\sigma_0}(Nt, g^{\sigma_1}, h^{\sigma_2}) \tag{3.5}$$

$$:= \sum_{n=k+m,\omega^{\sigma}_{nkm}\neq 0} \frac{1}{iN\omega^{\sigma}_{nkm}} e^{iNt\omega^{\sigma}_{nkm}} (q^{\sigma_1}_k \cdot im)(q^{\sigma_2}_m \cdot q^{\sigma_0 *}_n) g^{\sigma_1}_k h^{\sigma_2}_m$$

and

$$\tilde{\mathcal{R}}^{\sigma_0}_n(Nt, c) := \sum_{(\sigma_1,\sigma_2)\in\{-1,0,1\}^2} \tilde{\mathcal{B}}^{\sigma_0}_n(c^{\sigma_1}, c^{\sigma_2}). \tag{3.6}$$

First, note that we have the following relation between $\tilde{B}$ and $\tilde{\mathcal{B}}$, and $\tilde{R}$ and $\tilde{\mathcal{R}}$, respectively:

$$\begin{aligned} \partial_t \left(\tilde{\mathcal{B}}^{\sigma_0}_n(Nt, g^{\sigma_1}, h^{\sigma_2}) \right) &= \tilde{B}^{\sigma_0}_n(Nt, g^{\sigma_1}, h^{\sigma_2}) + \tilde{\mathcal{B}}^{\sigma_0}_n(Nt, \partial_t g^{\sigma_1}, h^{\sigma_2}) \\ &\quad + \tilde{\mathcal{B}}^{\sigma_0}_n(Nt, g^{\sigma_1}, \partial_t h^{\sigma_2}) \end{aligned}$$

and

$$\partial_t \left(\tilde{\mathcal{R}}^{\sigma_0}_n(Nt, c) \right) = \tilde{R}^{\sigma_0}_n(Nt, c) + \tilde{\mathcal{R}}^{\sigma_0}_n(Nt, \partial_t c).$$

To control r, we split (3.4) into two parts: finitely many terms and small (in $\ell^1(\Lambda)$) remainder terms, respectively (cf. [1, Theorem 6.3]). For $r \in \ell^1(\Lambda)$ and $\eta = 1, 2, \cdots$, we choose $\{s_j\}_{j=1}^{\infty} \subset \mathbb{N}$ ($s_1 < s_2 < \cdots$) in order to satisfy $\|(I - \mathcal{P}_\eta) r\| \to 0$, as ($\eta \to \infty$), where

$$\begin{aligned} \mathcal{P}_\eta r := \Big\{ & r_{n_1}, r_{n_2}, \cdots, r_{n_{s_\eta}} : \\ & n_1, \cdots, n_{s_\eta} \in \Lambda : n_k \neq n_\ell \ (k \neq \ell), |n_j| \leq \eta \quad \text{for all} \quad j = 1, \cdots, s_\eta \Big\}. \end{aligned}$$

The choice of $n_1 \cdots n_{s_\eta}$ is not uniquely determined, however this does not matter. Then we can divide r into two parts: finitely many terms $r_{n_1}, \cdots, r_{n_{s_\eta}}$ and small remainder terms $\{(I - \mathcal{P}_\eta) r_n\}_{n\in\Lambda}$.

Remark 3.3. By (2.7) and (3.5), we have the following estimates (we omit the time variable t):

$$\|\mathcal{P}_\eta \tilde{\mathcal{B}}^{\sigma_0}(\mathcal{P}_\eta c, \mathcal{P}_\eta c)\| \leq \frac{\beta(\eta)}{N} (1 + \eta^2)^{1/2} \|\mathcal{P}_\eta c\|^2,$$

$$\|\mathcal{P}_\eta \bar{R}^{\sigma_0}(\mathcal{P}_\eta y, c, b)\| \leq (1 + \eta^2)^{1/2} \|\mathcal{P}_\eta y\| (\|c\| + \|b\|)$$

for $0 < t < T_L$ (T_L is a local existence time, see (3.1)), where

$$\beta(\eta) := \max\{|\omega^\sigma_{nkm}|^{-1} : k = k_1, \cdots k_{s_\eta},\ n = n_1, \cdots, n_{s_\eta},\ m = n - k\}.$$

Let $y \in \ell^1(\Lambda)$. Since $\mathcal{P}y$ only have finitely many terms, we see

$$\||n|^2 \mathcal{P}_\eta y\| \leq (1 + \eta^2)\|\mathcal{P}_\eta y\|.$$

Note that $\beta(\eta)$ is always finite, since it only have finite combinations for the choice of n, k and m. We can also have the same type estimate for $\|\partial_t \mathcal{P}_\eta c\|$ by using (2.9).

Second, we use a change of variables to control $\tilde{R}^0$ and $\tilde{R}^{\sigma_0}$. Let us set y as

$$y^0_n(t) := r^0_n(t) - \tilde{\mathcal{R}}^0_n(Nt, \mathcal{P}_\eta c) \quad \text{and} \quad y^{\sigma_0}_n(t) := r^{\sigma_0}_n(t) - \tilde{\mathcal{R}}^{\sigma_0}_n(Nt, \mathcal{P}_\eta c).$$

From (3.4), we see that

$$\begin{aligned}
\partial_t \left(y^0_n + \tilde{\mathcal{R}}^0_n \right) &= -\nu|n|^2 (y^0_n + \tilde{\mathcal{R}}^0_n) + \bar{R}^0_n(y^0 + \tilde{\mathcal{R}}^0, c, b) \\
&\quad + \tilde{R}^0_n(Nt, \mathcal{P}_\eta c) + \tilde{R}^0_n(Nt, (I - \mathcal{P}_\eta)c), \\
\partial_t \left(y^{\sigma_0}_n + \tilde{\mathcal{R}}^{\sigma_0}_n \right) &= -\left(\frac{\nu + \kappa}{2} \right) |n|^2 (y^{\sigma_0}_n + \tilde{\mathcal{R}}^{\sigma_0}_n) + \bar{R}^{\sigma_0}_n(y^{\sigma_0} + \tilde{\mathcal{R}}^{\sigma_0}, c, b) \\
&\quad + \tilde{R}^{\sigma_0}_n(Nt, \mathcal{P}_\eta c) + \tilde{R}^{\sigma_0}_n(Nt, (I - \mathcal{P}_\eta)c).
\end{aligned}$$

Next, we control $\mathcal{P}_\eta y^0$ and $\mathcal{P}_\eta y^{\sigma_0}$ for fixed η. If y^0_n is one of the element of $\mathcal{P}_\eta y$, then y^0_n satisfies the following equation. By (3.6),

$$\begin{aligned}
\partial_t y^0_n(t) &= -\nu|n|^2 y^0_n + \bar{R}^0_n(\mathcal{P}_\eta y, c, b) + E^0_n, \\
\partial_t y^{\sigma_0}_n(t) &= -\left(\frac{\nu + \kappa}{2} \right) |n|^2 y^{\sigma_0}_n + \bar{R}^{\sigma_0}_n(\mathcal{P}_\eta y, c, b) + E^{\sigma_0}_n,
\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}
E^0_n : &= -\tilde{\mathcal{R}}^0_n(Nt, \mathcal{P}_\eta \partial_t c) + \bar{R}^0_n(\mathcal{P}_\eta \tilde{\mathcal{R}}^0_n(Nt, \mathcal{P}_\eta c), c, b) \\
&\quad -\nu|n|^2 \tilde{\mathcal{R}}^0_n(Nt, \mathcal{P}_\eta c) + \tilde{R}^0_n(Nt, (I - \mathcal{P}_\eta)c)
\end{aligned}$$

and

$$\begin{aligned} E_n^{\sigma_0} : &= -\tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta \partial_t c) + \bar{R}_n^{\sigma_0}(\mathcal{P}_\eta \tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta c), c, b) \\ &\quad - \left(\frac{\nu + \kappa}{2}\right) |n|^2 \tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta c) + \tilde{R}_n^{\sigma_0}(Nt, (I - \mathcal{P}_\eta)c). \end{aligned}$$

Note that (3.7) are linear heat type equations with external force E^0 and E^{σ_0}. Thus the point is to control E^0 and E^{σ_0}. By Remark 3.3, we can see that for any $\epsilon > 0$, there is η_0 and N_0 (depending on η_0) such that if $N > N_0$ and $\eta > \eta_0$, then $\|E^{\sigma_0}\| < \epsilon$ and $\|E^{\sigma_0}\| < \epsilon$. Thus we have from (3.7),

$$\|\mathcal{P}_\eta y_n^0(t)\| \leq \int_0^t \left(\frac{C_\nu}{(t-s)^{1/2}} \|\mathcal{P}_\eta y^0(s)\| (\|c(s)\| + \|b(s)\|) + \epsilon \right) ds$$

and

$$\|\mathcal{P}_\eta y_n^{\sigma_0}(t)\| \leq \int_0^t \left(\frac{C_{\nu,\kappa}}{(t-s)^{1/2}} \|\mathcal{P}_\eta y^{\sigma_0}(s)\| (\|c(s)\| + \|b(s)\|) + \epsilon \right) ds.$$

Finally, the rest of the proof is rather standard manner, thus we only give an outline of the proof. By Gronwall's inequality, we conclude that for any $\epsilon > 0$, there is η_0 (independent of N_0) and N_0 (depending on η_0) such that if $\eta > \eta_0$ and $N > N_0$, then $\|\mathcal{P}_\eta y^0\| < \epsilon$ and $\|\mathcal{P}_\eta y^{\sigma_0}\| < \epsilon$ for $0 < t < T_L$ (T_L is dependeng only on ℓ^1-norm of the initial datum, independent of η_0 and N_0). Clearly, we can also control $(I - \mathcal{P}_\eta)y$ with sufficiently large η (independent of N_0), and $\mathcal{P}_\eta \tilde{\mathcal{R}}_n^{\sigma_0}(Nt, \mathcal{P}_\eta c)$ with sufficiently large N (depending on η_0). Thus we can control r for sufficiently large η and N. □

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