J. Math. Sci. Univ. Tokyo
19 (2012), 699–716.

Lower Weight Gel'fand-Kalinin-Fuks Cohomology Groups of the Formal Hamiltonian Vector Fields on $\mathbb{R}^4$

By Kentaro Mikami* and Yasuharu Nakae

Abstract. In this paper, we investigate the relative Gel'fand-Kalinin-Fuks cohomology groups of the formal Hamiltonian vector fields on $\mathbb{R}^4$. In the case of formal Hamiltonian vector fields on $\mathbb{R}^2$, we computed the relative Gel'fand-Kalinin-Fuks cohomology groups of weight < 20 in the paper by Mikami-Nakae-Kodama. The main strategy there was decomposing the Gel'fand-Fucks cochain complex into irreducible factors and picking up the trivial representations and their concrete bases, and ours is essentially the same. By computer calculation, we determine the relative Gel'fand-Kalinin-Fuks cohomology groups of the formal Hamiltonian vector fields on $\mathbb{R}^4$ of weights 2, 4 and 6. In the case of weight 2, the Betti number of the cohomology group is equal to 1 at degree 2 and is 0 at any other degree. In weight 4, the Betti number is 2 at degree 4 and is 0 at any other degree, and in weight 6, the Betti number is 0 at any degree.

1. Introduction

Inspired by [4], we are interested in getting information about the relative Gel'fand-Kalinin-Fuks cohomology groups of the formal Hamiltonian vector fields on $\mathbb{R}^{2n}$ of a given weight.

In [6], we dealt with the case where $n = 1$ of weight < 20. In this paper, we investigate the relative Gel'fand-Kalinin-Fuks cohomology groups of the formal Hamiltonian vector fields on $\mathbb{R}^4$. Even for $n = 1$ or 2, the limitation comes from overloading of heavy computations of picking up the trivial representations and their concrete bases. Comparing the case where $n = 2$ with the case $n = 1$, we encountered more difficulty of decomposing into irreducible factors of tensor product, even though the Littlewood-Richardson formula is theoretically rather simple. So far, the information we have gotten

*Partially supported by Grant-in-Aid for Scientific Research (C) of JSPS, No.23540067 and No.20540059.

2010 *Mathematics Subject Classification.* Primary 57R32, 57R17; Secondary 17B66.

about the relative Gel'fand-Kalinin-Fuks cohomology groups of the formal Hamiltonian vector fields on $\mathbb{R}^4$ is only in the cases of weight=2,4,6, and the corresponding Euler characteristic numbers are 1,2 and 0, respectively.

2. Splitting Cochains by Weight

We are interested in the standard linear symplectic space $\mathbb{R}^{2n}$. The function space $C^\infty(\mathbb{R}^{2n})$ forms a Lie algebra with respect to the Poisson bracket $\{\cdot,\cdot\}$. For the Darboux coordinate $(x_1,\dots,x_n,y_1,\dots,y_n)$, we have $\{x_i,y_j\} = -\{y_j,x_i\} = \delta_{ij}$ and $\{x_i,x_j\} = \{y_i,y_j\} = 0$. Then the space of polynomials of $x_1,\dots,y_n$ is a subalgebra with respect to the Poisson bracket. The space $\mathfrak{ham}^0_{2n}$ of Hamiltonian vector fields which have polynomials as Hamiltonian potentials is a Lie algebra and the map $f \mapsto -H_f$ from the Hamiltonian potentials to Hamiltonian vector fields is a Lie algebra homomorphism with the kernel $\cong \mathbb{R}$.

We look at the Lie subalgebra $\mathfrak{ham}^1_{2n}$ of $\mathfrak{ham}^0_{2n}$ formed by elements which vanish at 0. $\mathfrak{ham}^1_{2n}$ corresponds to the algebra of polynomials without linear terms.

In this paper, we are interested in the Gel'fand-Kalinin-Fuks cohomology groups of $\mathfrak{ham}^1_{2n}$ when $n=2$.

We can split the polynomial functions by their homogeneity. The cochain complex is the exterior algebra of dual of polynomial functions and we introduce the "weight" on the cochain complex as follows:

DEFINITION 1. Let $\mathfrak{S}_\ell$ be the dual space of ℓ-homogeneous polynomial functions, and define the weight of each non zero element of $\mathfrak{S}_\ell$ to be $\ell-2$. For each non-zero element of $\mathfrak{S}_{\ell_1}\wedge\mathfrak{S}_{\ell_2}\wedge\cdots\wedge\mathfrak{S}_{\ell_s}$ $(\ell_1 \le \ell_2 \le \cdots \le \ell_s)$, define its weight to be $\displaystyle\sum_{i=1}^{s}(\ell_i-2)$.

PROPOSITION 1 (cf.[4],[6]). *The coboundary operator d of the Gel'fand-Kalinin-Fuks cochain complex preserves the weight, namely, if a cochain σ is of weight w, then $d\sigma$ is also of weight w.*

Hence we can decompose the total space of cochain complex by degree *and weight: namely,*

$$\mathrm{C}^m_{GF}(\mathfrak{ham}^0_{2n})_w = \textit{LinearSpan of } \{\sigma \in \Lambda^{k_1}\mathfrak{S}_1 \wedge \Lambda^{k_2}\mathfrak{S}_2 \wedge \cdots \wedge \Lambda^{k_s}\mathfrak{S}_s \mid \sum_{i=1}^{s} k_i = m \ , \ \sum_{i=1}^{s} k_i(i-2) = w \ , \ s = 1, 2, \dots\}$$

and we can define the cohomology group $\mathrm{H}^m_{GF}(\mathfrak{ham}^0_{2n})_w$.

$\mathrm{C}^\bullet_{GF}(\mathfrak{ham}^1_{2n})_w$ is the subspace of $\mathrm{C}^\bullet_{GF}(\mathfrak{ham}^0_{2n})_w$ characterized by $k_1 = 0$. If we restrict our attention to the cochain complex relative to $Sp(2n, \mathbb{R})$, then it turns out $k_2 = 0$ (cf.[6]). Thus we first look at the subcomplex $\overline{\mathrm{C}}^\bullet_{GF}(\mathfrak{ham}^1_{2n})_w$ of $\mathrm{C}^\bullet_{GF}(\mathfrak{ham}^1_{2n})_w$ spanned by cochains such that $k_1 = k_2 = 0$.

We consider finite sequences of non-negative integers $(k_3, k_4, \dots, k_s)$ satisfying

$$\sum_{i=3}^{s} k_i = m \quad \text{and} \quad \sum_{i=3}^{s} k_i(i-2) = w \ . \tag{1}$$

Shifting the indices by 2 as $\hat{k}_i = k_{i+2}$ $(i > 0)$, we see

$$\begin{aligned} w &= \hat{k}_1 + 2\hat{k}_2 + \cdots + t\hat{k}_t \\ &= (\underbrace{t + \cdots + t}_{\hat{k}_t}) + \cdots + (\underbrace{2 + \cdots + 2}_{\hat{k}_2}) + (\underbrace{1 + \cdots + 1}_{\hat{k}_1}) = \ell_1 + \ell_2 + \cdots + \ell_m \end{aligned}$$

where $t = s - 2$ and $\ell_1 \geq \ell_2 \geq \cdots \geq \ell_m \geq 1$. This is a partition of w of length m or a Young diagram of height m with w cells. Conversely, for a partition of w

$$\begin{aligned} w &= \ell_1 + \ell_2 + \cdots + \ell_m \\ \ell_1 &\geqq \ell_2 \geqq \cdots \geqq \ell_m \geqq 1 \end{aligned} \tag{2}$$

$\hat{k}_i = \#\{j \mid \ell_j = i\}$ gives a solution of (1). That means there is a one-to-one correspondence between the solutions of (1) and all the partitions of w of length m or Young diagrams of height m with w cells.

REMARK 1. Since $-$Identity $\in Sp(2n, \mathbb{R})$, we see that the relative cochain complex of odd weight must be the zero space, and hence we only deal with the complexes of even weights.

Proposition 2. *When weight = 2, 4 or 6, the non-trivial cochain groups are as follows:*

$$\overline{\mathrm{C}}^1_{GF}(\mathfrak{ham}^1_{2n})_2 = \mathfrak{S}_4, \qquad \overline{\mathrm{C}}^2_{GF}(\mathfrak{ham}^1_{2n})_2 = \Lambda^2\mathfrak{S}_3$$

$$\begin{aligned}
&\overline{\mathrm{C}}^1_{GF}(\mathfrak{ham}^1_{2n})_4 = \mathfrak{S}_6,\\
&\overline{\mathrm{C}}^2_{GF}(\mathfrak{ham}^1_{2n})_4 = (\mathfrak{S}_3 \wedge \mathfrak{S}_5) \oplus \Lambda^2\mathfrak{S}_4 \cong (\mathfrak{S}_3 \otimes \mathfrak{S}_5) \oplus \Lambda^2\mathfrak{S}_4,\\
&\overline{\mathrm{C}}^3_{GF}(\mathfrak{ham}^1_{2n})_4 = \Lambda^2\mathfrak{S}_3 \wedge \mathfrak{S}_4 \cong \Lambda^2\mathfrak{S}_3 \otimes \mathfrak{S}_4, \qquad \overline{\mathrm{C}}^4_{GF}(\mathfrak{ham}^1_{2n})_4 = \Lambda^4\mathfrak{S}_3
\end{aligned}$$

In the above, we identify the exterior product $\mathfrak{S}_3 \wedge \mathfrak{S}_5$ *with the tensor product* $\mathfrak{S}_3 \otimes \mathfrak{S}_5$ *as vector spaces, and we often use this identification without comments.*

$$\begin{aligned}
&\overline{\mathrm{C}}^1_{GF}(\mathfrak{ham}^1_{2n})_6 = \mathfrak{S}_8, \qquad \overline{\mathrm{C}}^2_{GF}(\mathfrak{ham}^1_{2n})_6 = (\mathfrak{S}_3 \otimes \mathfrak{S}_7) \oplus (\mathfrak{S}_4 \otimes \mathfrak{S}_6) \oplus \Lambda^2\mathfrak{S}_5\\
&\overline{\mathrm{C}}^3_{GF}(\mathfrak{ham}^1_{2n})_6 = \left(\Lambda^2\mathfrak{S}_3 \otimes \mathfrak{S}_6\right) \oplus (\mathfrak{S}_3 \otimes \mathfrak{S}_4 \otimes \mathfrak{S}_5) \oplus \Lambda^3\mathfrak{S}_4\\
&\overline{\mathrm{C}}^4_{GF}(\mathfrak{ham}^1_{2n})_6 = \left(\Lambda^3\mathfrak{S}_3 \otimes \mathfrak{S}_5\right) \oplus \left(\Lambda^2\mathfrak{S}_3 \otimes \Lambda^2\mathfrak{S}_4\right)\\
&\overline{\mathrm{C}}^5_{GF}(\mathfrak{ham}^1_{2n})_6 = \Lambda^4\mathfrak{S}_3 \otimes \mathfrak{S}_4, \qquad \overline{\mathrm{C}}^6_{GF}(\mathfrak{ham}^1_{2n})_6 = \Lambda^6\mathfrak{S}_3
\end{aligned}$$

In general, $\Lambda^p\mathfrak{S}_q = \{\mathbf{0}\}$ *if* $p > \dim\mathfrak{S}_q = (q+2n-1)!/(q!(2n-1)!)$. *If* $n=1$, $\dim\mathfrak{S}_3 = 4$ *and we have* $\overline{\mathrm{C}}^6_{GF}(\mathfrak{ham}^1_2)_6 = \{\mathbf{0}\}$.

Proof. (1) weight= 2 case: When $m = 1$, then $\ell_1 = 2$, so we have $\hat{k}_2 = 1$ and $\hat{k}_j = 0$ $(j \neq 2)$. Thus, $k_4 = 1$. When $m = 2$, then $2 = \ell_1 + \ell_2$ ($\ell_1 \geq \ell_2 \geq 1$), $\ell_1 = \ell_2 = 1$, so we have $\hat{k}_1 = 2$ and $\hat{k}_j = 0$ $(j \neq 1)$. Thus, $k_3 = 2$.

(2) weight= 4 case: When $m = 2$, i.e., $4 = \ell_1 + \ell_2$ ($\ell_1 \geq \ell_2 \geq 1$), then $(\ell_1, \ell_2) = (3, 1)$ or $(2, 2)$, so we have $(\hat{k}_1 = 1, \hat{k}_3 = 1)$, or $\hat{k}_2 = 2$. Thus, $(k_3 = 1, k_5 = 1)$ or $(k_4 = 2)$. When $m = 3$, i.e., $4 = \ell_1 + \ell_2 + \ell_3$ ($\ell_1 \geq \ell_2 \geq \ell_3 \geq 1$), $\ell_1 = 2, \ell_2 = 1, \ell_3 = 1$. Thus $(\hat{k}_2 = 1, \hat{k}_1 = 2)$, so $(k_3 = 2, k_4 = 1)$. When $m = 4$, i.e., $4 = \ell_1 + \ell_2 + \ell_3 + \ell_4$ $(\ell_1 \geq \ell_2 \geq \ell_3 \geq \ell_4 \geq 1)$, $\ell_1 = \ell_2 = \ell_3 = \ell_4 = 1$. Thus $\hat{k}_1 = 4$, so $k_3 = 4$.

(3) weight= 6 case: Way is the same, we omit the discussion. □

3. Coboundary Operator

The relative cochain complex is defined by

$$\begin{aligned}&\mathrm{C}^m_{GF}(\mathfrak{ham}^1_{2n}, Sp(2n,\mathbb{R}))_w\\&\quad = \{\sigma \in \mathrm{C}^m_{GF}(\mathfrak{ham}^1_{2n}) \mid i_{\hat{J}(\xi)}\sigma = 0, i_{\hat{J}(\xi)}d\sigma = 0 \quad (\forall \xi \in \mathfrak{sp}(2n,\mathbb{R}))\}\ ,\end{aligned}$$

where J is the momentum mapping of $Sp(2n,\mathbb{R})$ on $\mathbb{R}^{2n}$. The first condition $i_{\hat{J}(\xi)}\sigma = 0$ $(\forall \xi \in \mathfrak{sp}(2n,\mathbb{R}))$ means that $\mathfrak{S}_2$ does never appear. The second condition $i_{\hat{J}(\xi)}d\sigma = 0$ $(\forall \xi \in \mathfrak{sp}(2n,\mathbb{R}))$ means the cochain complex consists of the trivial representation spaces.

In the rest of this paper, we use the global variables x_1, x_2, x_3, x_4 on $\mathbb{R}^4$ and the symplectic form ω on $\mathbb{R}^4$ is given by $\omega(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_4}) = -\omega(\frac{\partial}{\partial x_4}, \frac{\partial}{\partial x_1}) = 1$, $\omega(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}) = -\omega(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2}) = 1$, and the others are zero. ($x_4 = y_1$ and $x_3 = y_2$ for the Darboux coordinate x_1, x_2, y_1, y_2 we explained in the section 2).

We consider the standard basis of homogeneous polynomials of x_1, x_2, x_3, x_4 given by $\frac{x_1^i}{i!}\frac{x_2^j}{j!}\frac{x_3^k}{k!}\frac{x_4^\ell}{\ell!}$. We use the notation $z_{i,j,k,\ell}$ to express the dual basis of those. While we deal with polynomials with $0 \le i,j,k,\ell \le 9$, the 4 digit number has the unique meaning and we may denote $z_{i,j,k,\ell}$ by $z_{ijk\ell}$. Furthermore, in order to simplify expression we use the notation $z^{i+j+k+\ell}_{ijk}$ for $z_{ijk\ell}$.

Now, the Poisson bracket is given by

$$\{f,g\} = \frac{\partial f}{\partial x_1}\frac{\partial g}{\partial x_4} + \frac{\partial f}{\partial x_2}\frac{\partial g}{\partial x_3} - \frac{\partial f}{\partial x_3}\frac{\partial g}{\partial x_2} - \frac{\partial f}{\partial x_4}\frac{\partial g}{\partial x_1}.$$

If we denote $\frac{x_1^{a_1}}{a_1!}\frac{x_2^{a_2}}{a_2!}\frac{x_3^{a_3}}{a_3!}\frac{x_4^{a_4}}{a_4!}$ by $\mathbf{e}_A$, where $A = (a_1,a_2,a_3,a_4) \in \mathbb{Z}^4_{\ge 0}$, then

$$\begin{aligned}\{\mathbf{e}_A, \mathbf{e}_B\} = &+ (a_1b_4 - a_4b_1)\frac{(a_1+b_1-1)!}{a_1!b_1!}\frac{(a_2+b_2)!}{a_2!b_2!}\frac{(a_3+b_3)!}{a_3!b_3!}\frac{(a_4+b_4-1)!}{a_4!b_4!}\\&\times \mathbf{e}_{(a_1+b_1-1, a_2+b_2, a_3+b_3, a_4+b_4-1)}\\&+ (a_2b_3 - a_3b_2)\frac{(a_1+b_1)!}{a_1!b_1!}\frac{(a_2+b_2-1)!}{a_2!b_2!}\frac{(a_3+b_3-1)!}{a_3!b_3!}\frac{(a_4+b_4)!}{a_4!b_4!}\\&\times \mathbf{e}_{(a_1+b_1, a_2+b_2-1, a_3+b_3-1, a_4+b_4)}.\end{aligned}$$

From the definition of the coboundary operator d, $dz_C(\mathbf{e}_A, \mathbf{e}_B) = -\langle z_C, \{\mathbf{e}_A, \mathbf{e}_B\}\rangle$ holds good, and we see that

$$\begin{aligned} dz_C = &- \sum_{(A,B)\in \mathrm{out}(C)} (a_1 b_4 - a_4 b_1) \frac{C!}{A!B!} z_A \otimes z_B \\ &- \sum_{(A,B)\in \mathrm{inn}(C)} (a_2 b_3 - a_3 b_2) \frac{C!}{A!B!} z_A \otimes z_B \end{aligned}$$

where $C! = c_1!c_2!c_3!c_4!$, $\mathrm{out}(C)$ consists of (A, B) with $a_1 + b_1 = 1 + c_1$, $a_2 + b_2 = c_2$, $a_3 + b_3 = c_3$, $a_4 + b_4 = 1 + c_4$, $|A| > 1$ and $|B| > 1$, and also $\mathrm{inn}(C)$ consists of (A, B) with $a_1 + b_1 = c_1$, $a_2 + b_2 = 1 + c_2$, $a_3 + b_3 = 1 + c_3$, $a_4 + b_4 = c_4$, $|A| > 1$ and $|B| > 1$. Here the notation $|A|$ means $a_1 + a_2 + a_3 + a_4$. Since we are working in $\mathfrak{ham}^1_{2n}$, $|A| > 1$ and $|B| > 1$.

4. Irreducible Decomposition of Cochain Complex

Our purpose here is to find the multiplicity of the trivial representations for a given cochain complex. We try getting a complete irreducible decomposition by finding the maximal weight vectors. A way of finding maximal weight vectors is to find the invariant vectors by the maximal unipotent subgroup of $Sp(4, \mathbb{R})$.

4.1. weight = 2

Since $\overline{\mathrm{C}}^1_{GF}(\mathfrak{ham}^1_4)_2 = \mathfrak{S}_4$, and $\mathfrak{S}_4$ is a non-trivial irreducible representation of $Sp(4, \mathbb{R})$, $\mathrm{C}^1_{GF}(\mathfrak{ham}^1_4, Sp(4, \mathbb{R}))_2 = \{\mathbf{0}\}$. Concerning $\overline{\mathrm{C}}^2_{GF}(\mathfrak{ham}^1_4)_2 = \Lambda^2\mathfrak{S}_3$, by finding maximal weight vectors, we get an irreducible decomposition

$$\Lambda^2\mathfrak{S}_3 = V_{\langle 0\rangle} \oplus V_{\langle 1,1\rangle} \oplus V_{\langle 2,2\rangle} \oplus V_{\langle 3,3\rangle} \oplus V_{\langle 4\rangle} \oplus V_{\langle 5,1\rangle} \ ,$$

where $V_{\langle p,q\rangle}$ is the irreducible representation of $Sp(4, \mathbb{R})$ corresponding to a Young diagram of length not greater than 2. We often denote $V_{\langle p,0\rangle}$ by $V_{\langle p\rangle}$, and $V_{\langle p\rangle}$ is identical with $\mathfrak{S}_p$, and $V_{\langle 0,0\rangle} = V_{\langle 0\rangle} = \mathfrak{S}_0$ is the the trivial representation. Hence, $\mathrm{C}^2_{GF}(\mathfrak{ham}^1_4, Sp(4, \mathbb{R}))_2 \cong \mathbb{R}$.

The following is the reason why we try getting a complete irreducible decomposition. By Weyl's dimension formula, we can calculate the dimension of $V_{\langle p,q\rangle}$ for each $p \geq q \geq 0$ and also $\dim \Lambda^r V_{\langle p,q\rangle}$, $\dim \left(\Lambda^r V_{\langle p,q\rangle} \otimes \Lambda^{r'} V_{\langle p',q'\rangle}\right)$

and so on. When we get an irreducible decomposition of $\Lambda^r V_{\langle p,q \rangle}$, we can calculate the dimension of the left hand side of the decomposition by using the dimension formula, and also the dimensions of all terms which appears in the right hand side. This gives us an evidence which supports that our decomposition given by a computer program would be correct.

PROPOSITION 3. *When weight* $= 2$, *then we have the following table.*

degree	0	1	2
dim	0	0	1

Thus, we see the Euler characteristic number of weight 2 *is* $(-1)^0 0 + (-1)^1 0 + (-1)^2 1 = 1$.

Note that we can also see that in the case of weight 2, Betti number h^2 of the cohomology is equal to 1 and $h^0 = h^1 = 0$ by the observation of Proposition 3.

4.2. weight = 4 and relative

The Littlewood-Richardson rule is used to decompose the tensor product of two irreducible representations into irreducible components in general. In many places below, by the help of Littlewood-Richardson rule, we decompose the tensor product $V_{\langle p,q \rangle} \otimes V_{\langle p',q' \rangle}$ of two irreducible representations (of length at most 2) into irreducible components. We write down the results only, but by the symbol $\overset{\text{LR}}{\cong}$, we suggest using of the Littlewood-Richardson rule.

Here we review the Littlewood-Richardson rule and the specialization algorithm briefly. The product of Schur functions is given by

$$s_\mu s_\nu = \sum_\lambda LR^\lambda_{\mu\nu} s_\lambda \ ,$$

where λ, μ, ν are general partitions and $LR^\lambda_{\mu\nu}$ is called the Littlewood-Richardson coefficient. $LR^\lambda_{\mu\nu}$ is obtained by

$$LR^\lambda_{\mu\nu} = \#\{T \in SSTab(\lambda/\mu : \nu) \mid w(T) \text{ is a lattice permutation}\}$$

(this is called the Littlewood-Richardson rule). $SSTab(\lambda/\mu, \nu)$ is the set of semistandard tableaux of shape λ/μ and weight ν. When T_i denotes the

i-th row of T and reverse(T_i) the word obtained by reading T_i from right to left, the $w(T)$ is the concatenated word (reverse(T_1), ..., reverse(T_m)) whose length equals $|\nu| = |\lambda| - |\mu|$, where $m = \ell(T)$, the length of T.

In order to obtain the character corresponding to each irreducible representation of $Sp(2n, \mathbb{R})$, the virtual character $S_{\langle\lambda\rangle}$ and the universal character $s_{\langle\lambda\rangle}$ are defined in the completely same form as follows.

$$S_{\langle\lambda\rangle} = \frac{1}{2}\det(H_{\lambda_i-i+j} + H_{\lambda_i-i-j})_{1\le i,j\le \ell(\lambda)}$$
$$s_{\langle\lambda\rangle} = \frac{1}{2}\det(h_{\lambda_i-i+j} + h_{\lambda_i-i-j})_{1\le i,j\le \ell(\lambda)}$$

where H_k is the character of the k-th symmetric tensor product of the natural representation $\mathbb{R}^{2n}$ of $Sp(2n, \mathbb{R})$, $H_k = 0$ if $k < 0$, and h_k is the k-th complete symmetric function.

As the Schur functions $\{s_\lambda\}$ is a basis of the space of symmetric functions Sym_∞, $\{s_{\langle\lambda\rangle}\}$ is also a basis of Sym_∞. It is known that the product of two universal characters is given by

$$s_{\langle\mu\rangle}s_{\langle\nu\rangle} = \sum_{\lambda} LR^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle} s_{\langle\lambda\rangle} \ , \tag{3}$$

where

$$LR^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle} = \sum_{\alpha,\beta,\gamma} LR^{\mu}_{\alpha\beta} LR^{\nu}_{\alpha\gamma} LR^{\lambda}_{\beta\gamma} \ .$$

There is a homomorphism

$$\pi : \mathrm{Sym}_\infty \to Rep(Sp(2n, \mathbb{R})) : h_k \mapsto H_k \quad (k = 1, 2, \dots) \ ,$$

which is surjective and whose kernel is generated by $e_k - e_{2n-k}$ $(0 \le k \le n)$ and e_k $(k > 2n)$, where e_k is the k-th elementary symmetric function. π is called the specialization homomorphism and satisfies

$$\pi(s_{\langle\lambda\rangle}) = S_{\langle\lambda\rangle}.$$

In the $Sp(2n, \mathbb{R})$-representation theory, equivalence classes of the irreducible representations are parametrized by the set of partitions λ whose length $\ell(\lambda) \le n$. We use the notation $V_{\langle\lambda\rangle}$ for that representation. Although the character $\pi(s_{\langle\lambda\rangle}) = S_{\langle\lambda\rangle}$ is defined for each general partition

λ, if $\ell(\lambda) \le n$ then $\pi(s_{\langle\lambda\rangle}) = S_{\langle\lambda\rangle}$ is the irreducible character of $V_{\langle\lambda\rangle}$. If $\ell(\lambda) > n$, it is known that $S_{\langle\lambda\rangle} = 0$ or $\pm S_{\langle\lambda'\rangle}$ with $\ell(\lambda') \le n$ by some rule, which is called the specialization algorithm. Applying the specialization homomorphism π to the formula (3) , we see that

$$\begin{aligned} S_{\langle\mu\rangle} \otimes S_{\langle\nu\rangle} &= \sum_{\lambda} LR^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle} S_{\langle\lambda\rangle} = \sum_{\ell(\lambda)\le n} LR^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle} S_{\langle\lambda\rangle} + \sum_{\ell(\lambda)>n} LR^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle} S_{\langle\lambda\rangle} \\ &= \sum_{\ell(\lambda)\le n} \overline{LR}^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle} S_{\langle\lambda\rangle}, \end{aligned}$$

where $\overline{LR}^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle}$ is the slight modification of $LR^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle}$ caused by applying the specialization algorithm. In our context,

$$V_{\langle\mu\rangle} \otimes V_{\langle\nu\rangle} = \sum_{\ell(\lambda)\le n} \overline{LR}^{\langle\lambda\rangle}_{\langle\mu\rangle\langle\nu\rangle} V_{\langle\lambda\rangle}.$$

About the Littlewood-Richardson rule for $Sp(2n,\mathbb{R})$, we refer to Soichi Okada's book: *Representation theory of classical groups and Combinatorics, Baifukan, 2006 (in Japanase)* Volume2, p.258 and also Volume 2, p.253 for specialization algorithm.

We stress that all the $\overset{\mathrm{LR}}{\cong}$ in the paper are done by ourselves by following the Littlewood-Richardson rule and the specialization algorithm faithfully.

The decomposition for weight = 4 is as follows.

$$\begin{aligned} \overline{\mathrm{C}}^1_{GF}(\mathfrak{ham}^1_4)_4 =& \mathfrak{S}_6 \\ \overline{\mathrm{C}}^2_{GF}(\mathfrak{ham}^1_4)_4 \cong& (\mathfrak{S}_3 \otimes \mathfrak{S}_5) \oplus \Lambda^2\mathfrak{S}_4 \\ \overset{\mathrm{LR}}{\cong}& (V_{\langle 2\rangle} \oplus V_{\langle 4\rangle} \oplus V_{\langle 6\rangle} \oplus V_{\langle 8\rangle} \oplus V_{\langle 3,1\rangle} \oplus V_{\langle 4,2\rangle} \\ & \oplus V_{\langle 5,1\rangle} \oplus V_{\langle 5,3\rangle} \oplus V_{\langle 6,2\rangle} \oplus V_{\langle 7,1\rangle}) \oplus \Lambda^2\mathfrak{S}_4 \\ =& (V_{\langle 2\rangle} \oplus V_{\langle 4\rangle} \oplus V_{\langle 6\rangle} \oplus V_{\langle 8\rangle} \oplus V_{\langle 3,1\rangle} \oplus V_{\langle 4,2\rangle} \\ & \oplus V_{\langle 5,1\rangle} \oplus V_{\langle 5,3\rangle} \oplus V_{\langle 6,2\rangle} \oplus V_{\langle 7,1\rangle}) \\ & \quad \oplus (V_{\langle 2\rangle} + V_{\langle 3,1\rangle} + V_{\langle 4,2\rangle} + V_{\langle 5,3\rangle} + V_{\langle 6\rangle} + V_{\langle 7,1\rangle}) \\ =& 2V_{\langle 2\rangle} + V_{\langle 4\rangle} + 2V_{\langle 6\rangle} + V_{\langle 8\rangle} + 2V_{\langle 3,1\rangle} + 2V_{\langle 4,2\rangle} \\ & + V_{\langle 5,1\rangle} + 2V_{\langle 5,3\rangle} + V_{\langle 6,2\rangle} + 2V_{\langle 7,1\rangle} \end{aligned}$$

$$\begin{aligned}
\overline{\mathrm{C}}^3_{GF}(\mathfrak{ham}^1_4)_4 \cong& \Lambda^2\mathfrak{S}_3 \otimes \mathfrak{S}_4 \\
\cong& (V_{\langle 0\rangle} + V_{\langle 4\rangle} + V_{\langle 1,1\rangle} + V_{\langle 2,2\rangle} + V_{\langle 3,3\rangle} + V_{\langle 5,1\rangle}) \otimes V_{\langle 4\rangle} \\
\overset{\mathrm{LR}}{\cong}& + V_{\langle 4\rangle} \\
& + (V_{\langle 0\rangle} + V_{\langle 2\rangle} + V_{\langle 4\rangle} + V_{\langle 6\rangle} + V_{\langle 8\rangle} + V_{\langle 1,1\rangle} + V_{\langle 2,2\rangle} + V_{\langle 3,1\rangle} \\
& \quad + V_{\langle 3,3\rangle} + V_{\langle 4,2\rangle} + V_{\langle 4,4\rangle} + V_{\langle 5,1\rangle} + V_{\langle 5,3\rangle} + V_{\langle 6,2\rangle} + V_{\langle 7,1\rangle}) \\
& + (V_{\langle 4\rangle} + V_{\langle 3,1\rangle} + V_{\langle 5,1\rangle}) \\
& + (V_{\langle 4\rangle} + V_{\langle 2,2\rangle} + V_{\langle 3,1\rangle} + V_{\langle 4,2\rangle} + V_{\langle 5,1\rangle} + V_{\langle 6,2\rangle}) \\
& + (V_{\langle 4\rangle} + V_{\langle 3,1\rangle} + V_{\langle 3,3\rangle} + V_{\langle 4,2\rangle} + V_{\langle 5,1\rangle} + V_{\langle 5,3\rangle} \\
& \quad + V_{\langle 6,2\rangle} + V_{\langle 7,3\rangle}) \\
& + (V_{\langle 2\rangle} + V_{\langle 4\rangle} + V_{\langle 6\rangle} + V_{\langle 8\rangle} + V_{\langle 1,1\rangle} + V_{\langle 2,2\rangle} + 2V_{\langle 3,1\rangle} + V_{\langle 3,3\rangle} \\
& \quad + 2V_{\langle 4,2\rangle} + V_{\langle 4,4\rangle} + 2V_{\langle 5,1\rangle} + 2V_{\langle 5,3\rangle} + V_{\langle 5,5\rangle} + 2V_{\langle 6,2\rangle} \\
& \quad + V_{\langle 6,4\rangle} + 2V_{\langle 7,1\rangle} + V_{\langle 7,3\rangle} + V_{\langle 8,2\rangle} + V_{\langle 9,1\rangle}) \\
=& V_{\langle 0\rangle} + 2V_{\langle 2\rangle} + 6V_{\langle 4\rangle} + 2V_{\langle 6\rangle} + 2V_{\langle 8\rangle} + 2V_{\langle 1,1\rangle} + 3V_{\langle 2,2\rangle} \\
& + 6V_{\langle 3,1\rangle} + 3V_{\langle 3,3\rangle} + 5V_{\langle 4,2\rangle} + 2V_{\langle 4,4\rangle} + 6V_{\langle 5,1\rangle} + 4V_{\langle 5,3\rangle} \\
& + V_{\langle 5,5\rangle} + 5V_{\langle 6,2\rangle} + V_{\langle 6,4\rangle} + 3V_{\langle 7,1\rangle} + 2V_{\langle 7,3\rangle} + V_{\langle 8,2\rangle} + V_{\langle 9,1\rangle} \\
\overline{\mathrm{C}}^4_{GF}(\mathfrak{ham}^1_4)_4 =& \Lambda^4\mathfrak{S}_3 \\
=& 3V_{\langle 0\rangle} + 4V_{\langle 4\rangle} + 2V_{\langle 6\rangle} + V_{\langle 8\rangle} + 2V_{\langle 1,1\rangle} + 4V_{\langle 2,2\rangle} + 3V_{\langle 3,1\rangle} \\
& + 4V_{\langle 3,3\rangle} + 3V_{\langle 4,2\rangle} + 3V_{\langle 4,4\rangle} + 5V_{\langle 5,1\rangle} + 2V_{\langle 5,3\rangle} + V_{\langle 5,5\rangle} \\
& + 4V_{\langle 6,2\rangle} + V_{\langle 6,4\rangle} + V_{\langle 6,6\rangle} + V_{\langle 7,1\rangle} + 2V_{\langle 7,3\rangle} + V_{\langle 8,2\rangle}.
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
&\mathrm{C}^1_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_4 = \{\mathbf{0}\} && \mathrm{C}^2_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_4 = \{\mathbf{0}\} \\
&\mathrm{C}^3_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_4 \cong \mathbb{R} && \mathrm{C}^4_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_4 \cong \mathbb{R}^3.
\end{aligned}$$

PROPOSITION 4. *When the weight* $= 4$, *we have*

degree	0	1	2	3	4
dim	0	0	0	1	3

Thus, the Euler characteristic number of weight 4 *is* $(-1)^0 0 + (-1)^3 1 + (-1)^4 3 = 2$.

4.3. weight = 6 and relative

The decomposition into irreducible representations for degrees 1,2 and 3 are as follows.

$$\overline{\mathrm{C}}^1_{GF}(\mathfrak{ham}^1_4)_6 = \mathfrak{S}_8$$

$$\begin{aligned}
\overline{\mathrm{C}}^2_{GF}(\mathfrak{ham}^1_4)_6 \cong & (\mathfrak{S}_3 \otimes \mathfrak{S}_7) \oplus (\mathfrak{S}_4 \otimes \mathfrak{S}_6) \oplus \Lambda^2\mathfrak{S}_5 \\
\overset{\mathrm{LR}}{\cong} & (V_{\langle 4\rangle} + V_{\langle 6\rangle} + V_{\langle 8\rangle} + V_{\langle 10\rangle} + V_{\langle 5,1\rangle} + V_{\langle 6,2\rangle} + V_{\langle 7,1\rangle} + V_{\langle 7,3\rangle} \\
& \quad + V_{\langle 8,2\rangle} + V_{\langle 9,1\rangle}) \\
& + (V_{\langle 2\rangle} + V_{\langle 4\rangle} + V_{\langle 6\rangle} + V_{\langle 8\rangle} + V_{\langle 10\rangle} + V_{\langle 3,1\rangle} + V_{\langle 4,2\rangle} + V_{\langle 5,1\rangle} \\
& \quad + V_{\langle 5,3\rangle} + V_{\langle 6,2\rangle} + V_{\langle 6,4\rangle} + V_{\langle 7,1\rangle} + V_{\langle 7,3\rangle} + V_{\langle 8,2\rangle} + V_{\langle 9,1\rangle}) \\
& + (V_{\langle 0\rangle} + V_{\langle 1,1\rangle} + V_{\langle 2,2\rangle} + V_{\langle 3,3\rangle} + V_{\langle 4\rangle} + V_{\langle 4,4\rangle} + V_{\langle 5,1\rangle} \\
& \quad + V_{\langle 5,5\rangle} + V_{\langle 6,2\rangle} + V_{\langle 7,3\rangle} + V_{\langle 8\rangle} + V_{\langle 9,1\rangle}) \\
= & V_{\langle 0\rangle} + V_{\langle 2\rangle} + 3V_{\langle 4\rangle} + 2V_{\langle 6\rangle} + 3V_{\langle 8\rangle} + 2V_{\langle 10\rangle} + V_{\langle 1,1\rangle} + V_{\langle 2,2\rangle} \\
& + V_{\langle 3,1\rangle} + V_{\langle 3,3\rangle} + V_{\langle 4,2\rangle} + V_{\langle 4,4\rangle} + 3V_{\langle 5,1\rangle} + V_{\langle 5,3\rangle} + V_{\langle 5,5\rangle} \\
& + 3V_{\langle 6,2\rangle} + V_{\langle 6,4\rangle} + 2V_{\langle 7,1\rangle} + 3V_{\langle 7,3\rangle} + 2V_{\langle 8,2\rangle} + 3V_{\langle 9,1\rangle}
\end{aligned}$$

$$\begin{aligned}
\overline{\mathrm{C}}^3_{GF}(\mathfrak{ham}^1_4)_6 \cong & (\Lambda^2\mathfrak{S}_3 \otimes \mathfrak{S}_6) \oplus (\mathfrak{S}_3 \otimes \mathfrak{S}_4 \otimes \mathfrak{S}_5) \oplus \Lambda^3\mathfrak{S}_4 \\
\overset{\mathrm{LR}}{\cong} & (2V_{\langle 2\rangle} + 2V_{\langle 4\rangle} + 6V_{\langle 6\rangle} + 2V_{\langle 8\rangle} + 2V_{\langle 10\rangle} + V_{\langle 1,1\rangle} + V_{\langle 2,2\rangle} \\
& \quad + 3V_{\langle 3,1\rangle} + 2V_{\langle 3,3\rangle} + 5V_{\langle 4,2\rangle} + V_{\langle 4,4\rangle} + 6V_{\langle 5,1\rangle} + 4V_{\langle 5,3\rangle} \\
& \quad + V_{\langle 5,5\rangle} + 5V_{\langle 6,2\rangle} + 3V_{\langle 6,4\rangle} + 6V_{\langle 7,1\rangle} + 4V_{\langle 7,3\rangle} + V_{\langle 7,5\rangle} \\
& \quad + 5V_{\langle 8,2\rangle} + V_{\langle 8,4\rangle} + 3V_{\langle 9,1\rangle} + 2V_{\langle 9,3\rangle} + V_{\langle 10,2\rangle} + V_{\langle 11,1\rangle}) \\
& + (V_{\langle 0\rangle} + 6V_{\langle 2\rangle} + 10V_{\langle 4\rangle} + 10V_{\langle 6\rangle} + 6V_{\langle 8\rangle} + 3V_{\langle 10\rangle} + V_{\langle 12\rangle} \\
& \quad + 3V_{\langle 1,1\rangle} + 5V_{\langle 2,2\rangle} + 12V_{\langle 3,1\rangle} + 6V_{\langle 3,3\rangle} + 15V_{\langle 4,2\rangle} + 5V_{\langle 4,4\rangle} \\
& \quad + 16V_{\langle 5,1\rangle} + 13V_{\langle 5,3\rangle} + 3V_{\langle 5,5\rangle} + 15V_{\langle 6,2\rangle} + 8V_{\langle 6,4\rangle} + V_{\langle 6,6\rangle} \\
& \quad + 12V_{\langle 7,1\rangle} + 10V_{\langle 7,3\rangle} + 3V_{\langle 7,5\rangle} + 9V_{\langle 8,2\rangle} + 4V_{\langle 8,4\rangle} \\
& \quad + 6V_{\langle 9,1\rangle} + 4V_{\langle 9,3\rangle} + 3V_{\langle 10,2\rangle} + 2V_{\langle 11,1\rangle}) \\
& + (2V_{\langle 2\rangle} + 3V_{\langle 3,1\rangle} + V_{\langle 3,3\rangle} + V_{\langle 4\rangle} + 4V_{\langle 4,2\rangle} + 3V_{\langle 5,1\rangle} + 3V_{\langle 5,3\rangle} \\
& \quad + 3V_{\langle 6,0\rangle} + 2V_{\langle 6,2\rangle} + 2V_{\langle 6,4\rangle} + 2V_{\langle 7,1\rangle} + 2V_{\langle 7,3\rangle} + V_{\langle 7,5\rangle} \\
& \quad + 2V_{\langle 8,2\rangle} + V_{\langle 9,1\rangle} + V_{\langle 9,3\rangle} + V_{\langle 10\rangle}) \\
= & V_{\langle 0\rangle} + 8V_{\langle 2\rangle} + 12V_{\langle 4\rangle} + 16V_{\langle 6\rangle} + 8V_{\langle 8\rangle} + 5V_{\langle 10\rangle} + V_{\langle 12\rangle}
\end{aligned}$$

$$\begin{aligned}&+ 4V_{\langle 1,1\rangle} + 2V_{\langle 2\rangle} + 6V_{\langle 2,2\rangle} + 18V_{\langle 3,1\rangle} + 9V_{\langle 3,3\rangle} + V_{\langle 4\rangle}\\ &+ 24V_{\langle 4,2\rangle} + 6V_{\langle 4,4\rangle} + 25V_{\langle 5,1\rangle} + 20V_{\langle 5,3\rangle} + 4V_{\langle 5,5\rangle} + 3V_{\langle 6\rangle}\\ &+ 22V_{\langle 6,2\rangle} + 13V_{\langle 6,4\rangle} + V_{\langle 6,6\rangle} + 20V_{\langle 7,1\rangle} + 16V_{\langle 7,3\rangle} + 5V_{\langle 7,5\rangle}\\ &+ 16V_{\langle 8,2\rangle} + 5V_{\langle 8,4\rangle} + 10V_{\langle 9,1\rangle} + 7V_{\langle 9,3\rangle} + V_{\langle 10\rangle} + 4V_{\langle 10,2\rangle}\\ &+ 3V_{\langle 11,1\rangle}.\end{aligned}$$

For degree 4, since

$$\overline{\mathrm{C}}^4_{GF}(\mathfrak{ham}^1_4)_6 \cong (\Lambda^3\mathfrak{S}_3 \otimes \mathfrak{S}_5) \oplus (\Lambda^2\mathfrak{S}_3 \otimes \Lambda^2\mathfrak{S}_4),$$

we shall decompose two kinds of tensor products: The first one is

$$\begin{aligned}\Lambda^3\mathfrak{S}_3 \otimes \mathfrak{S}_5 =&(V_{\langle 2,1\rangle} + 3V_{\langle 3,0\rangle} + V_{\langle 3,2\rangle} + 2V_{\langle 4,1\rangle} + V_{\langle 4,3\rangle} + 2V_{\langle 5,2\rangle}\\ &\quad + V_{\langle 6,1\rangle} + V_{\langle 6,3\rangle} + V_{\langle 7,0\rangle}) \otimes V_{\langle 5\rangle}\\ \overset{\mathrm{LR}}{\cong}& 9V_{\langle 2\rangle} + 13V_{\langle 4\rangle} + 13V_{\langle 6\rangle} + 10V_{\langle 8\rangle} + 2V_{\langle 10\rangle} + V_{\langle 12\rangle} + 3V_{\langle 1,1\rangle}\\ &+ 6V_{\langle 2,2\rangle} + 18V_{\langle 3,1\rangle} + 8V_{\langle 3,3\rangle} + 23V_{\langle 4,2\rangle} + 7V_{\langle 4,4\rangle} + 22V_{\langle 5,1\rangle}\\ &+ 22V_{\langle 5,3\rangle} + 4V_{\langle 5,5\rangle} + 24V_{\langle 6,2\rangle} + 13V_{\langle 6,4\rangle} + 2V_{\langle 6,6\rangle} + 19V_{\langle 7,1\rangle}\\ &+ 15V_{\langle 7,3\rangle} + 6V_{\langle 7,5\rangle} + 13V_{\langle 8,2\rangle} + 7V_{\langle 8,4\rangle} + V_{\langle 8,6\rangle} + 8V_{\langle 9,1\rangle}\\ &+ 7V_{\langle 9,3\rangle} + V_{\langle 9,5\rangle} + 5V_{\langle 10,2\rangle} + V_{\langle 10,4\rangle} + 2V_{\langle 11,1\rangle} + V_{\langle 11,3\rangle}.\end{aligned}$$

Concerning $\Lambda^2\mathfrak{S}_3\otimes\Lambda^2\mathfrak{S}_4$, by using the Littlewood-Richardson rule as usual, we have a decomposition into terms which include a term of partition length 4. By an easy observation, $V_{\langle i,j,1,1\rangle} = -V_{\langle i,j\rangle}$, and we see

$$\begin{aligned}\Lambda^2\mathfrak{S}_3 \otimes \Lambda^2\mathfrak{S}_4 \overset{\mathrm{LR}}{\cong}& 18V_{\langle 2\rangle} + 22V_{\langle 4\rangle} + 26V_{\langle 6\rangle} + 12V_{\langle 8\rangle} + 5V_{\langle 10\rangle} + V_{\langle 12\rangle} + 6V_{\langle 1,1\rangle}\\ &+ 12V_{\langle 2,2\rangle} + 34V_{\langle 3,1\rangle} + 15V_{\langle 3,3\rangle} + 45V_{\langle 4,2\rangle} + 13V_{\langle 4,4\rangle}\\ &+ 41V_{\langle 5,1\rangle} + 39V_{\langle 5,3\rangle} + 7V_{\langle 5,5\rangle} + 40V_{\langle 6,2\rangle} + 25V_{\langle 6,4\rangle}\\ &+ 3V_{\langle 6,6\rangle} + 32V_{\langle 7,1\rangle} + 26V_{\langle 7,3\rangle} + 11V_{\langle 7,5\rangle} + 24V_{\langle 8,2\rangle}\\ &+ 12V_{\langle 8,4\rangle} + 3V_{\langle 8,6\rangle} + 12V_{\langle 9,1\rangle} + 12V_{\langle 9,3\rangle} + 2V_{\langle 9,5\rangle}\\ &+ 7V_{\langle 10,2\rangle} + 3V_{\langle 10,4\rangle} + 4V_{\langle 11,1\rangle} + V_{\langle 11,3\rangle} + V_{\langle 12,2\rangle}\\ &+ 8V_{\langle 3,2,2,1\rangle} + V_{\langle 3,3,2,2\rangle} + 4V_{\langle 3,3,3,1\rangle} + 3V_{\langle 4,2,2,2\rangle}\\ &+ 12V_{\langle 4,3,2,1\rangle} + 2V_{\langle 4,3,3,2\rangle} + V_{\langle 4,4,2,2\rangle} + 3V_{\langle 4,4,3,1\rangle}\end{aligned}$$

$$+ 8V_{\langle 5,2,2,1\rangle} + 4V_{\langle 5,3,2,2\rangle} + 4V_{\langle 5,3,3,1\rangle} + V_{\langle 5,3,3,3\rangle} + 8V_{\langle 5,4,2,1\rangle} + V_{\langle 5,4,3,2\rangle} + 2V_{\langle 5,5,3,1\rangle} + V_{\langle 6,2,2,2\rangle} + 9V_{\langle 6,3,2,1\rangle} + V_{\langle 6,3,3,2\rangle} + V_{\langle 6,4,2,2\rangle} + 2V_{\langle 6,4,3,1\rangle} + 2V_{\langle 6,5,2,1\rangle} + 5V_{\langle 7,2,2,1\rangle} + 3V_{\langle 7,3,3,1\rangle} + 2V_{\langle 7,4,2,1\rangle} + 2V_{\langle 8,3,2,1\rangle}.$$

Next, we use $V_{\langle i,j,k,1\rangle} = 0$ and $V_{\langle i,j,k,2\rangle} = 0$ where $i \geq j \geq k > 1$, and have

$$\begin{aligned}\Lambda^2\mathfrak{S}_3 \otimes \Lambda^2\mathfrak{S}_4 \cong & 18V_{\langle 2\rangle} + 22V_{\langle 4\rangle} + 26V_{\langle 6\rangle} + 12V_{\langle 8\rangle} + 5V_{\langle 10\rangle} + V_{\langle 12\rangle} + 6V_{\langle 1,1\rangle} \\ & + 12V_{\langle 2,2\rangle} + 34V_{\langle 3,1\rangle} + 15V_{\langle 3,3\rangle} + 45V_{\langle 4,2\rangle} + 13V_{\langle 4,4\rangle} \\ & + 41V_{\langle 5,1\rangle} + 39V_{\langle 5,3\rangle} + 7V_{\langle 5,5\rangle} + 40V_{\langle 6,2\rangle} + 25V_{\langle 6,4\rangle} \\ & + 3V_{\langle 6,6\rangle} + 32V_{\langle 7,1\rangle} + 26V_{\langle 7,3\rangle} + 11V_{\langle 7,5\rangle} + 24V_{\langle 8,2\rangle} \\ & + 12V_{\langle 8,4\rangle} + 3V_{\langle 8,6\rangle} + 12V_{\langle 9,1\rangle} + 12V_{\langle 9,3\rangle} + 2V_{\langle 9,5\rangle} \\ & + 7V_{\langle 10,2\rangle} + 3V_{\langle 10,4\rangle} + 4V_{\langle 11,1\rangle} + V_{\langle 11,3\rangle} + V_{\langle 12,2\rangle} \\ & + V_{\langle 5,3,3,3\rangle}.\end{aligned}$$

Since $V_{\langle 5,3,3,3\rangle} = 0$, we finally have

$$\begin{aligned}\Lambda^2\mathfrak{S}_3 \otimes \Lambda^2\mathfrak{S}_4 \cong & 18V_{\langle 2\rangle} + 22V_{\langle 4\rangle} + 26V_{\langle 6\rangle} + 12V_{\langle 8\rangle} + 5V_{\langle 10\rangle} + V_{\langle 12\rangle} + 6V_{\langle 1,1\rangle} \\ & + 12V_{\langle 2,2\rangle} + 34V_{\langle 3,1\rangle} + 15V_{\langle 3,3\rangle} + 45V_{\langle 4,2\rangle} + 13V_{\langle 4,4\rangle} \\ & + 41V_{\langle 5,1\rangle} + 39V_{\langle 5,3\rangle} + 7V_{\langle 5,5\rangle} + 40V_{\langle 6,2\rangle} + 25V_{\langle 6,4\rangle} \\ & + 3V_{\langle 6,6\rangle} + 32V_{\langle 7,1\rangle} + 26V_{\langle 7,3\rangle} + 11V_{\langle 7,5\rangle} + 24V_{\langle 8,2\rangle} \\ & + 12V_{\langle 8,4\rangle} + 3V_{\langle 8,6\rangle} + 12V_{\langle 9,1\rangle} + 12V_{\langle 9,3\rangle} + 2V_{\langle 9,5\rangle} \\ & + 7V_{\langle 10,2\rangle} + 3V_{\langle 10,4\rangle} + 4V_{\langle 11,1\rangle} + V_{\langle 11,3\rangle} + V_{\langle 12,2\rangle}.\end{aligned}$$

The decomposition is obtained by a computer program. Validity of our computation is supported by the following fact: The sum of dimensions of each components above is 113050. On the other hand, $\dim(\Lambda^2\mathfrak{S}_3 \otimes \Lambda^2\mathfrak{S}_4) = 190 \times 595$. These numbers are equal.

Combining above decompositions, we have

$$\begin{aligned}\overline{\mathrm{C}}^4_{GF}(\mathfrak{ham}^1_4)_6 \cong & (\Lambda^3\mathfrak{S}_3 \otimes \mathfrak{S}_5) \oplus (\Lambda^2\mathfrak{S}_3 \otimes \Lambda^2\mathfrak{S}_4) \\ = & 27V_{\langle 2\rangle} + 35V_{\langle 4\rangle} + 39V_{\langle 6\rangle} + 22V_{\langle 8\rangle} + 7V_{\langle 10\rangle} + 2V_{\langle 12\rangle} + 9V_{\langle 1,1\rangle} \\ & + 18V_{\langle 2,2\rangle} + 52V_{\langle 3,1\rangle} + 23V_{\langle 3,3\rangle} + 68V_{\langle 4,2\rangle} + 20V_{\langle 4,4\rangle}\end{aligned}$$

$$
\begin{aligned}
&+ 63V_{\langle 5,1\rangle} + 61V_{\langle 5,3\rangle} + 11V_{\langle 5,5\rangle} + 64V_{\langle 6,2\rangle} + 38V_{\langle 6,4\rangle} \\
&+ 5V_{\langle 6,6\rangle} + 51V_{\langle 7,1\rangle} + 41V_{\langle 7,3\rangle} + 17V_{\langle 7,5\rangle} + 37V_{\langle 8,2\rangle} \\
&+ 19V_{\langle 8,4\rangle} + 4V_{\langle 8,6\rangle} + 20V_{\langle 9,1\rangle} + 19V_{\langle 9,3\rangle} + 3V_{\langle 9,5\rangle} \\
&+ 12V_{\langle 10,2\rangle} + 4V_{\langle 10,4\rangle} + 6V_{\langle 11,1\rangle} + 2V_{\langle 11,3\rangle} + V_{\langle 12,2\rangle}.
\end{aligned}
$$

For degree 5, applying the Littlewood-Richardson rule to the following decomposition

$$
\begin{aligned}
\overline{\mathrm{C}}^5_{GF}(\mathfrak{ham}^1_4)_6 \cong \Lambda^4\mathfrak{S}_3 \otimes \mathfrak{S}_4 \cong (&3V_{\langle 0\rangle} + 2V_{\langle 1,1\rangle} + 4V_{\langle 2,2\rangle} + 3V_{\langle 3,1\rangle} + 4V_{\langle 3,3\rangle} \\
&+ 4V_{\langle 4\rangle} + 3V_{\langle 4,2\rangle} + 3V_{\langle 4,4\rangle} + 5V_{\langle 5,1\rangle} + 2V_{\langle 5,3\rangle} + V_{\langle 5,5\rangle} + 2V_{\langle 6\rangle} + 4V_{\langle 6,2\rangle} \\
&+ V_{\langle 6,4\rangle} + V_{\langle 6,6\rangle} + V_{\langle 7,1\rangle} + 2V_{\langle 7,3\rangle} + V_{\langle 8\rangle} + V_{\langle 8,2\rangle}) \otimes V_{\langle 4\rangle}
\end{aligned}
$$

and using the Littlewood-Richardson rule many times, we have

$$
\begin{aligned}
V_{\langle 0\rangle} \otimes V_{\langle 4\rangle} &\cong V_{\langle 4\rangle}, \qquad V_{\langle 1,1\rangle} \otimes V_{\langle 4\rangle} \overset{\mathrm{LR}}{\cong} V_{\langle 4\rangle} + V_{\langle 3,1\rangle} + V_{\langle 5,1\rangle}, \\
V_{\langle 2,2\rangle} \otimes V_{\langle 4\rangle} &\overset{\mathrm{LR}}{\cong} V_{\langle 4\rangle} + V_{\langle 2,2\rangle} + V_{\langle 3,1\rangle} + V_{\langle 4,2\rangle} + V_{\langle 5,1\rangle} + V_{\langle 6,2\rangle} \\
V_{\langle 3,1\rangle} \otimes V_{\langle 4\rangle} &\overset{\mathrm{LR}}{\cong} V_{\langle 2\rangle} + V_{\langle 4\rangle} + V_{\langle 6\rangle} + V_{\langle 1,1\rangle} + V_{\langle 2,2\rangle} + 2V_{\langle 3,1\rangle} + V_{\langle 3,3\rangle} + 2V_{\langle 4,2\rangle} \\
&\quad + 2V_{\langle 5,1\rangle} + V_{\langle 5,3\rangle} + V_{\langle 6,2\rangle} + V_{\langle 7,1\rangle} \\
V_{\langle 3,3\rangle} \otimes V_{\langle 4\rangle} &\overset{\mathrm{LR}}{\cong} V_{\langle 4\rangle} + V_{\langle 3,1\rangle} + V_{\langle 3,3\rangle} + V_{\langle 4,2\rangle} + V_{\langle 5,1\rangle} + V_{\langle 5,3\rangle} + V_{\langle 6,2\rangle} + V_{\langle 7,3\rangle} \\
V_{\langle 4\rangle} \otimes V_{\langle 4\rangle} &\overset{\mathrm{LR}}{\cong} V_{\langle 0\rangle} + V_{\langle 2\rangle} + V_{\langle 4\rangle} + V_{\langle 6\rangle} + V_{\langle 8\rangle} + V_{\langle 1,1\rangle} + V_{\langle 2,2\rangle} + V_{\langle 3,1\rangle} \\
&\quad + V_{\langle 3,3\rangle} + V_{\langle 4,2\rangle} + V_{\langle 4,4\rangle} + V_{\langle 5,1\rangle} + V_{\langle 5,3\rangle} + V_{\langle 6,2\rangle} + V_{\langle 7,1\rangle} \\
&\vdots \quad \text{(decompositions of 12 tensor products are omitted)} \\
V_{\langle 8,2\rangle} \otimes V_{\langle 4\rangle} &\overset{\mathrm{LR}}{\cong} V_{\langle 6\rangle} + V_{\langle 8\rangle} + V_{\langle 10\rangle} + V_{\langle 4,2\rangle} + V_{\langle 5,1\rangle} + V_{\langle 5,3\rangle} + 2V_{\langle 6,2\rangle} + V_{\langle 6,4\rangle} \\
&\quad + 2V_{\langle 7,1\rangle} + 2V_{\langle 7,3\rangle} + V_{\langle 7,5\rangle} + 3V_{\langle 8,2\rangle} + 2V_{\langle 8,4\rangle} + V_{\langle 8,6\rangle} \\
&\quad + 2V_{\langle 9,1\rangle} + 2V_{\langle 9,3\rangle} + V_{\langle 9,5\rangle} + 2V_{\langle 10,2\rangle} + V_{\langle 10,4\rangle} \\
&\quad + V_{\langle 11,1\rangle} + V_{\langle 11,3\rangle} + V_{\langle 12,2\rangle}.
\end{aligned}
$$

The complete list of decompositions which we need is available in [7].

Gathering above decompositions, we see that

$$
\overline{\mathrm{C}}^5_{GF}(\mathfrak{ham}^1_4)_6 \cong 4V_{\langle 0\rangle} + 17V_{\langle 2\rangle} + 41V_{\langle 4\rangle} + 29V_{\langle 6\rangle} + 20V_{\langle 8\rangle} + 5V_{\langle 10\rangle} + V_{\langle 12\rangle}
$$

$$
\begin{aligned}
&+ 12V_{\langle 1,1\rangle} + 23V_{\langle 2,2\rangle} + 45V_{\langle 3,1\rangle} + 27V_{\langle 3,3\rangle} + 59V_{\langle 4,2\rangle}\\
&+ 24V_{\langle 4,4\rangle} + 61V_{\langle 5,1\rangle} + 55V_{\langle 5,3\rangle} + 15V_{\langle 5,5\rangle} + 65V_{\langle 6,2\rangle}\\
&+ 36V_{\langle 6,4\rangle} + 8V_{\langle 6,6\rangle} + 42V_{\langle 7,1\rangle} + 44V_{\langle 7,3\rangle} + 16V_{\langle 7,5\rangle} + 2V_{\langle 7,7\rangle}\\
&+ 31V_{\langle 8,2\rangle} + 21V_{\langle 8,4\rangle} + 5V_{\langle 8,6\rangle} + 18V_{\langle 9,1\rangle} + 15V_{\langle 9,3\rangle} + 6V_{\langle 9,5\rangle}\\
&+ 10V_{\langle 10,2\rangle} + 4V_{\langle 10,4\rangle} + V_{\langle 10,6\rangle} + 3V_{\langle 11,1\rangle} + 3V_{\langle 11,3\rangle} + V_{\langle 12,2\rangle}.
\end{aligned}
$$

For degree 6, direct computation of maximal weight vectors shows that

$$
\begin{aligned}
\overline{\mathrm{C}}^{6}_{GF}(\mathfrak{ham}^1_4)_6 \cong{}& \Lambda^6\mathfrak{S}_3\\
\cong{}& 4V_{\langle 0\rangle} + 6V_{\langle 1,1\rangle} + 2V_{\langle 2,0\rangle} + 10V_{\langle 2,2\rangle} + 10V_{\langle 3,1\rangle} + 12V_{\langle 3,3\rangle}\\
&+ 13V_{\langle 4,0\rangle} + 14V_{\langle 4,2\rangle} + 9V_{\langle 4,4\rangle} + 19V_{\langle 5,1\rangle} + 14V_{\langle 5,3\rangle} + 7V_{\langle 5,5\rangle}\\
&+ 7V_{\langle 6,0\rangle} + 18V_{\langle 6,2\rangle} + 9V_{\langle 6,4\rangle} + 4V_{\langle 6,6\rangle} + 10V_{\langle 7,1\rangle} + 13V_{\langle 7,3\rangle}\\
&+ 4V_{\langle 7,5\rangle} + 4V_{\langle 8,0\rangle} + 7V_{\langle 8,2\rangle} + 5V_{\langle 8,4\rangle} + V_{\langle 8,6\rangle} + 3V_{\langle 9,1\rangle}\\
&+ 3V_{\langle 9,3\rangle} + 2V_{\langle 9,5\rangle} + V_{\langle 10,0\rangle} + V_{\langle 10,2\rangle}.
\end{aligned}
$$

From the observation above, we have the following:

PROPOSITION 5. *When the weight* $= 6$, *we have*

degree	0	1	2	3	4	5	6
dim	0	0	1	1	0	4	4

The Euler characteristic number for weight 6 *is* $(-1)^0 0+(-1)^2 1+(-1)^3 1+(-1)^5 4+(-1)^6 4=0$.

5. Betti Numbers

In order to get each Betti number, we have to know the image and the kernel of d itself. For that purpose, we have to fix some bases of cochain complexes and matrix representation of d by those concrete bases.

5.1. weight = 2

We already see that the Betti numbers of the cohomology group in the case of weight 2 are the following which is an immediate consequence of Proposition 3.

degree	0	1	2
dim	0	0	1
Betti	0	0	1

5.2. weight = 4

In order to know properties of $d : \mathrm{C}^3_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_4 \to \mathrm{C}^4_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_4$, we prepare a basis, say A of $\mathrm{C}^3_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_4$:

$$\begin{aligned} A = &- 4z^3_{100} \wedge z^3_{210} \wedge z^4_{101} + 2z^3_{001} \wedge z^3_{100} \wedge z^4_{310} + 4z^3_{100} \wedge z^3_{201} \wedge z^4_{110} \\ &- 2z^3_{001} \wedge z^3_{021} \wedge z^4_{211} + z^3_{001} \wedge z^3_{030} \wedge z^4_{202} - 2z^3_{100} \wedge z^3_{300} \wedge z^4_{100} \\ &- 4z^3_{001} \wedge z^3_{201} \wedge z^4_{120} + 4z^3_{001} \wedge z^3_{210} \wedge z^4_{111} \\ &\pm \mathbf{164\ terms} \\ &- 2z^3_{000} \wedge z^3_{201} \wedge z^4_{210} + 2z^3_{000} \wedge z^3_{210} \wedge z^4_{201} + z^3_{000} \wedge z^3_{300} \wedge z^4_{200} \\ &+ z^3_{001} \wedge z^3_{010} \wedge z^4_{400} + 2z^3_{001} \wedge z^3_{011} \wedge z^4_{310} + z^3_{001} \wedge z^3_{012} \wedge z^4_{220} \end{aligned}$$

The complete expression of all terms in A is available in [7].

We prepare a basis $\{B_1, B_2, B_3\}$ of $\mathrm{C}^4_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_4$ and we get $dA = -32B_1 + 32B_2 - 28B_3$ by using computer programs. The complete form of its basis is also available in [7]. But we can conclude $dA \neq 0$ directly by an observation of the calculation of the d-image of A. Thus, $h^3 = 0$ and $h^4 = 2$, that is, $h^0 = h^1 = h^2 = h^3 = 0$. Consequently, $h^4 = 2$ and the others are zero. Therefore, the alternating sum of the Betti numbers, which is another definition of the Euler characteristic number, is 2.

THEOREM 1. *When the weight = 4, we have*

degree	0	1	2	3	4
dim	0	0	0	1	3
Betti	0	0	0	0	2

5.3. weight = 6

As a basis of $\mathrm{C}^2_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_6$, we can select

$$\begin{aligned} A = &6z^5_{112} \wedge z^5_{121} + \frac{1}{10} z^5_{000} \wedge z^5_{500} + \frac{1}{2} z^5_{004} \wedge z^5_{140} + \frac{1}{2} z^5_{001} \wedge z^5_{410} + z^5_{002} \wedge z^5_{320} \\ &+ z^5_{003} \wedge z^5_{230} - 2z^5_{011} \wedge z^5_{311} + \frac{1}{10} z^5_{005} \wedge z^5_{050} - \frac{1}{2} z^5_{010} \wedge z^5_{401} \end{aligned}$$

$$
\begin{aligned}
&- 3z^5_{012} \wedge z^5_{221} - 2z^5_{013} \wedge z^5_{131} - \frac{1}{2} z^5_{014} \wedge z^5_{041} + z^5_{020} \wedge z^5_{302} + 3z^5_{021} \wedge z^5_{212} \\
&+ 3z^5_{022} \wedge z^5_{122} + z^5_{023} \wedge z^5_{032} - z^5_{030} \wedge z^5_{203} - 2z^5_{031} \wedge z^5_{113} + \frac{1}{2} z^5_{040} \wedge z^5_{104} \\
&- \frac{1}{2} z^5_{100} \wedge z^5_{400} - 2z^5_{101} \wedge z^5_{310} - 3z^5_{102} \wedge z^5_{220} - 2z^5_{103} \wedge z^5_{130} \\
&+ 2z^5_{110} \wedge z^5_{301} + 6z^5_{111} \wedge z^5_{211} - 3z^5_{120} \wedge z^5_{202} + z^5_{200} \wedge z^5_{300} + 3z^5_{201} \wedge z^5_{210}.
\end{aligned}
$$

We can also see $dA \neq 0$ by an observation for A. Thus, $h^2 = 0$ and $h^3 = 0$. We also prepared a basis B of $\mathrm{C}^3_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_6$ and we got $dA = 12B$, the complete expression of B is available in [7].

Concerning the coboundary operator $d : \mathrm{C}^5_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_6 \to \mathrm{C}^6_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_6$, we have concrete bases P_1, P_2, P_3, P_4 of $\mathrm{C}^5_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_6$ and Q_1, Q_2, Q_3, Q_4 of $\mathrm{C}^6_{GF}(\mathfrak{ham}^1_4, Sp(4,\mathbb{R}))_6$. Those have very long expressions as P_1 is a sum of 3696 terms, P_2 of 3358 terms, P_3 of 1406 terms, P_4 of 2960 terms, and Q_1 of 120 terms, Q_2 of 466 terms, Q_3 of 756 terms, Q_4 of 866 terms. Each P_i is constructed by the terms of the form $\left(\wedge^4_{a=1} z^3_{i_a j_a k_a}\right) \wedge z^4_{i_5 j_5 k_5}$, and each Q_i is by the terms of $\wedge^6_{a=1} z^3_{i_a j_a k_a}$. The complete forms of each P_i and Q_i are also available in [7].

Using those bases, we have a matrix representation of d:

$$
\begin{aligned}
d(P_1) &= -90Q_1 - 22Q_2 + 5Q_3 - 2Q_4 \\
d(P_2) &= -\frac{25}{2}Q_1 - \frac{7}{6}Q_2 + \frac{55}{12}Q_3 + 3Q_4 \\
d(P_3) &= -\frac{17}{2}Q_1 - \frac{4}{3}Q_2 + \frac{7}{6}Q_3 \\
d(P_4) &= 6Q_1 + \frac{3}{2}Q_2 - \frac{9}{2}Q_3 - 4Q_4.
\end{aligned}
$$

It is non-singular and so $h^5 = 0$ and $h^6 = 0$. Namely, $h^j = 0$ for $j = 0, \dots, 6$.

THEOREM 2. *When the weight* $= 6$, *we have*

degree	0	1	2	3	4	5	6
dim	0	0	1	1	0	4	4
Betti	0	0	0	0	0	0	0

Acknowledgement. The authors would like to express their gratitude to the referee for many useful comments and suggestions.

References

[1] Fulton, W. and J. Harris, *Representation Theory*, volume 129 of *Graduate Texts in Math.* Springer Verlag, 1991.

[2] Gel′fand, I. M. and D. B. Fuks, Cohomologies of the Lie algebra of formal vector fields, Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 322–337.

[3] Goodman, R. and N. R. Wallach, *Symmetry, Representations, and Invariants*, volume 255 of *Graduate Texts in Math.* Springer Verlag, 2009.

[4] Kotschick, D. and S. Morita, The Gel'fand-Kalinin-Fuks class and characteristic classes of transversely symplectic foliations, *arXiv:0910.3414*, October 2009.

[5] Metoki, S., *Non-trivial cohomology classes of Lie algebras of volume preserving formal vector fields*, PhD thesis, Univ. of Tokyo, 2000.

[6] Mikami, K., Nakae, Y. and H. Kodama, Higher weight Gel'fand-Kalinin-Fuks classes of formal Hamiltonian vector fields of symplectic $\mathbb{R}^2$, *arXiv:1210.1662*, October 2012.

[7] Mikami, K. and Y. Nakae, Lower weight Gel'fand-Kalinin-Fuks cohomology groups of the formal Hamiltonian vector fields on $\mathbb{R}^4$, *arXiv:1211.0185*, November 2012.

[8] Morita, S., *Characteristic classes and Geometry*, Iwanami, 2008 (in Japanase).

[9] Morita, S., *Geometry of characteristic classes*, AMS Translations of Math. Monographs, Vol. 199, 2001.

[10] Okada, S., *Representation theory of classical groups and Combinatorics*, Baifukan, 2006 (in Japanase).

[11] Perchik, J., Cohomology of Hamiltonian and related formal vector field Lie algebras, Topology **15**(4) (1976), 395–404.

[12] Takamura, M., The relative cohomology of formal contact vector fields with respect to formal Poisson vector fields, J. Math. Soc. Japan, **60**(1) (2008), 117–125.

(Received September 19, 2012)
(Revised October 15, 2012)

Department of Computer Science and Engineering
Akita University
1-1 Tegata Gakuen-machi
Akita City, 010-8502 Japan
E-mail: mikami@math.akita-u.ac.jp
nakae@math.akita-u.ac.jp