J. Math. Sci. Univ. Tokyo
19 (2012), 613–659.

Log Néron Models over Surfaces

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Abstract. We prove that admissible normal functions over surfaces extend to sections of log Néron models.

Contents

Introduction

To study admissible normal functions, various analytic Néron models have been introduced by several authors (Green-Griffiths-Kerr [4], Brosnan-Pearlstein-Saito [2], Schnell [14],...). Log Néron models ([10], [12]) are among them. They have geometric structures and, via the work of Hayama ([6]), in case of the 1-dimensional base, they are homeomorphic to the ones of Green-Griffiths-Kerr.

In this paper, we study the problem to construct a log Néron model for each admissible normal function ν, that is, a log Néron model J_Σ which "graphs" ν in the sense that ν extends to a section of J_Σ (cf. the paragraph before 1.8). Over the 1-dimensional base, it is relatively easy to see that there is *the* log Néron model which graphs *any* admissible normal function (cf. [10] §7 and [12] 6.1.8). But, over a general base of any dimension, we

2010 *Mathematics Subject Classification*. Primary 14C30; Secondary 14D07, 32G20.

cannot expect that there is such a nice model which graphs any ν simultaneously. Instead, we hope that there is a nice model for each ν. We call such a model *a log Néron model for* ν. This model depends on ν.

More precisely, as is explained in [10] and in [12] §5 respectively, there are two ways to formulate log Néron models, i.e., the absolute formulation and the relative formulation. Roughly speaking, in the former, we use cones in the Lie algebra, while, in the latter, we use cones in the fiber product of the cone of the log structure of the base and the Lie algebra. The absolute one is more understandable and is studied earlier from the pioneer work [10], whereas the relative one has some advantages, and, in [12], we adopted the relative formulation to define some log Néron models (see ibid. 5.5 for a comparison of both methods).

As is said in the above, over the 1-dimensional base, both formulations (the absolute one [10] and the relative one [12]) works well and yield the best model. Over the higher dimensional base, there are some results in the relative formulation so far. First, the log Néron model in the relative formulation ([12]) graphs the admissible normal functions with torsion singularities, i.e., the admissible normal functions whose associated local systems are $\mathbf{Q}$-split. Further, it is not very hard to see that, for each admissible normal function ν with any singularity (not necessarily with torsion singularity), there is a model in the relative formulation which graphs it. On the other hand, we have studied the problem little in the absolute setting.

In this paper, we return to the absolute setting and find it an interesting problem to construct a log Néron model in the absolute formulation for each admissible normal function ν over the base of any dimension. The main result in this paper is to carry it out in the surface base case. Since the machinery is already established in [12] §2 to associate a nice model to a weak fan, the problem is in essence to construct an appropriate weak fan for ν, whose proof we will outline soon. (Here "weak fan" is a relaxed concept of fan, which admits some overlappings of cones.)

Another problem is to construct a model which graphs given two or more admissible normal functions simultaneously, which has not yet been studied well for neither context (absolute nor relative). We will investigate this problem in a forthcoming paper.

In Section 1, we formulate the problem, state the main results, and deduce some corollaries.

The proofs start in Section 2. Roughly speaking, the idea is as follows. (See 1.17 for a more precise outline.) Let σ be the admissible nilpotent cone associated to ν. The problem is to subdivide σ into a finite set of cones such that each cone of this set generates a weak fan, that is, we have to prove that, after replacing σ by each member of a finite subdivision of σ, all the translations $g(\sigma)$ ($g \in G_{\mathbf{Z}}$) of σ make a weak fan, where $G_{\mathbf{Z}}$ is the group of automorphisms of the lattice.

Generally, the given σ and its translation $g(\sigma)$ are overlapped, that is, the intersection $g(\sigma) \cap \sigma$ is not necessarily a face of σ. Sometimes, we see that, after a finite subdivision, the intersection $g(\sigma) \cap \sigma$ becomes a face of σ. Sometimes, it is not the case, but still we can prove that after a finite subdivision, σ generates a weak fan. In this introduction, we temporarily call the former case (A) and the latter case (B).

In case (A), we subdivide σ in a careful way: First, in Section 2, we prove some lemmas in an abstract setting which provide several methods to subdivide cones. In Section 3, we prove some properties of polarized nilpotent orbits, which are necessary to apply the methods in Section 2 to our situation.

Next, in Section 4, we add more lemmas in an abstract setting to subdivide cones suitably in the case (B). After reviewing some basic consequences of admissibility in Section 5, we prove the main results in the final section 6 by combining the propositions in the preceding sections.

Acknowledgments. The author is thankful to K. Kato and S. Usui for collaboration for log intermediate Jacobians, from which this subject arose. The author thanks J. C. for suggesting this work. He also thanks the referee for the careful reading and pointing out some unclear points in 6.2 and in 6.5.

Notation and Terminology. All combinatorial notions are the rational ones, i.e., are considered over $\mathbf{Q}$, unless explicitly stated otherwise. For example, a polyhedral cone is a finitely generated and integral (i.e., cancellative) $\mathbf{Q}_{\geq 0}$-monoid. A fan in a $\mathbf{Q}$-vector space V is a set Σ of strictly convex polyhedral cones in V satisfying: (1) A face of a member of Σ also belongs to Σ; (2) For $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of σ. A finite subdivision of a polyhedral cone σ is a finite fan Σ whose support coincides with σ.

Let $N\colon V \to V'$ be a map of sets. For a subset A of V and a subset A' of V', we write NA for $N(A)$ and $N^{-1}A'$ for $N^{-1}(A')$. For example, for maps $N_1, N_2\colon V \to V'$, the symbol $(N_2N_1^{-1})^2A'$ means $N_2(N_1^{-1}(N_2(N_1^{-1}(A'))))$.

1. Main Results

1.1. First we review the definition of weak fans. The weak fan is the relaxed concept of fan introduced in [11]. We follow the formulation in [12]. For a slight (inessential) difference between the formulations [11] and [12], see [12] 2.2.5 Remark 2.

As is explained in [12] §2 and in ibid. §5 respectively, there are the absolute formulation and the relative formulation of weak fans. In this paper, we study weak fans in the absolute setting, i.e., the one in [12] §2.

Thus the following definition is the same as that in [12] §2 except that we work over $\mathbf{Q}$, which yields no difference in essence.

1.2. In this section, we fix a free $\mathbf{Z}$-module $H'_{\mathbf{Z}}$ of finite rank and define $H_{\mathbf{Z}} := H'_{\mathbf{Z}} \oplus \mathbf{Z}$. Let W be the increasing filtration on $H_{\mathbf{Q}} := H_{\mathbf{Z}} \otimes \mathbf{Q}$ characterized by $\mathrm{gr}^W_{-1}(H_{\mathbf{Q}}) = H'_{\mathbf{Q}}$ and $\mathrm{gr}^W_0(H_{\mathbf{Q}}) = \mathbf{Q}$. Let $\langle\ ,\ \rangle_0$ be the pairing $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Q}; (a,b) \mapsto ab$. Let $\langle\ ,\ \rangle_{-1}\colon H'_{\mathbf{Z}} \times H'_{\mathbf{Z}} \to \mathbf{Q}$ be a non-degenerate anti-symmetric pairing. Let $(h^{p,q})_{p,q}$ are non-negative integers given for each integers p, q satisfying the following conditions (1)–(4).

(1) $h^{p,q} = 0$ unless $p + q = -1$ or $p = q = 0$.

(2) $h^{0,0} = 1$.

(3) $\sum_{p+q=-1} h^{p,q} = \mathrm{rank}_{\mathbf{Z}} H'_{\mathbf{Z}}$.

(4) $h^{p,q} = h^{q,p}$ for any p, q.

1.3. Let $D' := D(H'_{\mathbf{Z}}, (h^{p,q})_{p+q=-1}, \langle\ ,\ \rangle_{-1})$ be the classifying space of polarized Hodge structures of weight -1 defined by P. A. Griffiths in [5]. Let $D := D(H_{\mathbf{Z}}, W, (h^{p,q})_{p,q}, \langle\ ,\ \rangle_{-1}, \langle\ ,\ \rangle_0)$ be the classifying space of gradedly polarized mixed Hodge structures introduced by S. Usui in [16].

For $A = \mathbf{Z}, \mathbf{Q}$, let G'_A be the group of the A-automorphisms of $(H'_A, \langle\ ,\ \rangle_{-1})$. Let G_A be the group of the A-automorphisms of $(H_A, W \cap H_A, \langle\ ,\ \rangle_{-1}, \langle\ ,\ \rangle_0)$.

Let $\mathfrak{g}'_{\mathbf{Q}}$ be the Lie algebra associated to $G'_{\mathbf{Q}}$. Let $\mathfrak{g}_{\mathbf{Q}}$ be the Lie algebra associated to $G_{\mathbf{Q}}$.

1.4. A *nilpotent cone* is a polyhedral cone σ in the $\mathbf{Q}$-vector space $\mathfrak{g}_{\mathbf{Q}}$ whose elements are all nilpotent and mutually commutative, i.e., $NN' = N'N$ for any $N, N' \in \sigma$.

A nilpotent cone σ is said to be *sharp* if it is strictly convex, i.e., $\sigma \cap (-\sigma) = \{0\}$.

Let σ be a nilpotent cone and let F be an element of the "compact dual" $\check{D}$ of D. We say that (σ, F) *generates a nilpotent orbit* if the following three conditions (1)–(3) are satisfied.

(1) The adjoint action of σ on $H_{\mathbf{Q}}$ is admissible. (For the definition of the admissibility, see below.)

(2) $NF^p \subset F^{p-1}$ for any $N \in \sigma$ and $p \in \mathbf{Z}$.

(3) For a set $N_1, \ldots, N_n$ of generators of σ, we have $\exp(\sum_{j=1}^{n} iy_j N_j)F$ belongs to D for any $y_j \gg 0$.

The important concept of admissibility, which appears in the above (1), was introduced and studied in [15] and [8]. Its definition was reviewed in [12] 1.2.2, where the formulation is over $\mathbf{R}$. To define the admissibility over $\mathbf{Q}$, read $\mathbf{R}$ there as $\mathbf{Q}$, or, equivalently, define the admissibility of the action of σ on $H_{\mathbf{Q}}$ by that of $\sigma \otimes_{\mathbf{Q}_{\geq 0}} \mathbf{R}_{\geq 0}$ on $H_{\mathbf{R}}$. Because of its importance, we repeat the definition here: We say that the action of σ on $H_{\mathbf{Q}}$ is *admissible* if there exists a family $(M(\tau, W))_{\tau}$ of finite rational increasing filtrations $M(\tau, W)$ on $H_{\mathbf{Q}}$ given for each face τ of σ satisfying the following conditions (1)–(4).

(1) $M(\sigma \cap (-\sigma), W) = W$.

(2) For any face τ of σ, any $N \in \sigma$ and any $w \in \mathbf{Z}$, we have $N(M(\tau, W)_w) \subset M(\tau, W)_w$.

(3) For any face τ of σ, any $N \in \tau$ and any $w \in \mathbf{Z}$, we have

$$N(M(\tau, W)_w) \subset M(\tau, W)_{w-2}.$$

(4) For any faces τ, τ' of σ and for any $N \in \sigma$ such that τ' is the smallest face of σ containing τ and N, $M(\tau', W)$ is the relative monodromy filtration of N with respect to $M(\tau, W)$.

1.5. A *weak fan* Σ in $\mathfrak{g}_{\mathbf{Q}}$ is a non-empty set of sharp nilpotent cones satisfying the following conditions (1) and (2).

(1) Any face of an element of Σ also belongs to Σ.

(2) Let $\sigma_1, \sigma_2 \in \Sigma$. Assume that they have a common interior point. Assume also that there is an $F \in \check{D}$ such that (σ_1, F) and (σ_2, F) generate nilpotent orbits. Then $\sigma_1 = \sigma_2$.

A *fan* in $\mathfrak{g}_{\mathbf{Q}}$ is defined, as usual, by replacing (2) with the condition: If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2$ is a face of σ_1.

Any fan is a weak fan ([12] 2.2.4, cf. [11] 1.7), but the converse does not hold, that is, $\sigma_1 \cap \sigma_2$ is not necessarily a face of σ_1 in a weak fan. See [11] 4.13 and [12] 7.2 for examples and the necessity of weak fans.

1.6. Next we review log Néron models and their variants.

Let $\sigma' \subset \mathfrak{g}'_{\mathbf{Q}}$ be a sharp nilpotent cone.

Let $\Gamma' := \exp(\sigma'^{\mathrm{gp}}) \cap G'_{\mathbf{Z}}$, where $\sigma'^{\mathrm{gp}} = \sigma' + (-\sigma')$. Let Γ be the subgroup of $G_{\mathbf{Z}}$ consisting of all the elements whose restrictions to $H'_{\mathbf{Z}}$ belong to Γ' and which induce 1 on $\mathrm{gr}^W_0(H_{\mathbf{Z}}) = \mathbf{Z}$.

Let Σ' be the fan consisting of all faces of σ'. Let Σ be a weak fan which is strongly compatible with Γ ([12] 2.2.6).

Let $D'_{\Sigma'}$ and D_Σ be the sets of nilpotent orbits. The quotients $\Gamma' \setminus D'_{\Sigma'}$ and $\Gamma \setminus D_\Sigma$ are endowed with the structures of the objects in the category $\mathcal{B}(\log)$ ([13] 3.2.4).

Assume the following condition on Σ:

$(*)$ The image in $\mathfrak{g}'_{\mathbf{Q}}$ of any cone in Σ is contained in σ'.

Then, we have the natural map

$$\mathrm{gr}^W_{-1} \colon \Gamma \setminus D_\Sigma \to \Gamma' \setminus D'_{\Sigma'}$$

induced by the natural map $\check{D} \to \check{D}'; F \mapsto \mathrm{gr}^W_{-1}(F)$.

Let

$$\varphi \colon S \to \Gamma' \setminus D'_{\Sigma'}$$

be a strict morphism in the category $\mathcal{B}(\log)$, where a morphism is said to be *strict* if the pullback of the log structure on the target is naturally isomorphic to that on the source.

Let J_Σ be the fiber product of

$$S \to \Gamma' \setminus D'_{\Sigma'} \leftarrow \Gamma \setminus D_\Sigma.$$

Then, a series of main results in [12] say that J_Σ is a nice space in various senses; for instance, by 2.5.5 of [12], J_Σ is Hausdorff if S is Hausdorff.

1.7. By another main theorem 2.6.6 of [12], J_Σ represents the following functor.

Let H' be a polarized log Hodge structure on S endowed with a Γ'-level structure μ' of type $(-1,(h^{p,q})_{p+q=-1},H'_{\mathbf{Z}},\langle\ ,\ \rangle_{-1},\Gamma',\Sigma')$ corresponding to $\varphi\colon S\to\Gamma'\setminus D'_{\Sigma'}$ by the main theorem B in [13] 4.2.1.

Then, the functor represented by J_Σ associates to $T\in\mathcal{B}(\log)/S$ the set of isomorphism classes of a log mixed Hodge structure (LMH, for short) H on T with polarized graded quotients endowed with a Γ-level structure μ ([12] 2.6.2) satisfying the following conditions (1) and (2) (see [13] 2.6, [7] 2.3, 2.5, [12] 1.3 for the definition of LMH; recall that, roughly speaking, an LMH is a pre-LMH satisfying the following three conditions pointwise, i.e., the admissibility, the Griffiths transversality, and that it yields a mixed Hodge structure in the usual sense after a sufficiently twisted specialization).

(1) $\mathrm{gr}^W_w(H)$ is isomorphic to the pullback of H', $\mathbf{Z}$ (unit Hodge structure), and 0 if $w=-1$, $w=0$, and $w\neq 0,-1$, respectively.

(2) For any $t\in T^{\log}$, if $\tilde{\mu}_t\colon H_t\xrightarrow{\sim}H_{\mathbf{Z}}$ (H here denotes the lattice of H by abuse of notation) is a representative of the germ of μ at t, then there exists a $\sigma\in\Sigma$ such that $\exp(\sigma)$ contains the image of the induced map $\pi_1^+(\tau^{-1}\tau(t))\to\mathrm{Aut}(H_t)\overset{\text{by }\tilde{\mu}_t}{\to}\mathrm{Aut}(H_{\mathbf{Z}})$, and such that $(\sigma,\tilde{\mu}_t(\mathbf{C}\otimes F_t))$ generates a nilpotent orbit.

Here $\pi_1^+(\tau^{-1}\tau(t)):=\mathrm{Hom}((M_T/\mathcal{O}_T)^\times_{\tau(t)},\mathbf{N})\subset\mathrm{Hom}((M_T/\mathcal{O}_T)^\times_{\tau(t)},\mathbf{Z})=\pi_1(\tau^{-1}\tau(t))$ and F is the Hodge filtration of H.

We have an embedding

$$\mathrm{Mor}(-,J_\Sigma)\subset\mathcal{E}xt^1(\mathbf{Z},H')$$

of functors from the category $\mathcal{B}(\log)/\mathcal{S}$ to the category of sets. Here $\mathcal{E}xt^1$ is the sheaf $T\mapsto\mathrm{Ext}^1_T(\mathbf{Z},H'|_T)$ in the category of log mixed Hodge structures with polarized graded quotients (cf. [7] 3.1.4). The image of this embedding consists of H satisfying the following (3).

(3) For any $t\in T^{\log}$, if $\widetilde{\mu'_t}\colon H'_t\xrightarrow{\sim}H'_{\mathbf{Z}}$ (H' here denotes the lattice of H' by abuse of notation) is a representative of the germ of μ' at t, then there exists a $\sigma\in\Sigma$ such that $\exp(\sigma)$ contains the image of the induced map $\pi_1^+(\tau^{-1}\tau(t))\to\mathrm{Aut}(H_t)\overset{\text{by }\widetilde{\mu'_t}\oplus\mathrm{id}}{\to}\mathrm{Aut}(H_{\mathbf{Z}})$, and such that $(\sigma,(\widetilde{\mu'_t}\oplus\mathrm{id})(\mathbf{C}\otimes F_t))$ generates a nilpotent orbit.

Now we proceed to state the results. Fix a σ' and let Γ', Γ, and Σ' be as in 1.6. Let φ be as in 1.6 and H' as in 1.7.

We first state results in 1-dimensional base case for the reader's convenience, though they are essentially included in those in 2-dimensional base case (cf. [10] §7, [12] 7.5.3 (3)).

We say that, for any object $T \in \mathcal{B}(\log)$, J_Σ *graphs* an extension $a \in \mathrm{Ext}^1_T(\mathbf{Z}, H')$ if a belongs to $\mathrm{Mor}(T, J_\Sigma) \subset \mathrm{Ext}^1_T(\mathbf{Z}, H')$.

PROPOSITION 1.8. *Assume that* $\dim \sigma' \leqq 1$. *Let* σ *be a nilpotent cone in* $\mathfrak{g}_\mathbf{Q}$ *whose image in* $\mathfrak{g}'_\mathbf{Q}$ *is* σ'. *Assume that* σ *is admissible, i.e., its action on* $H_\mathbf{Q}$ *is admissible. Then, there is a finite subdivision of* σ *such that, for each member* τ *of this subdivision, the translations* $\mathrm{Ad}(\gamma)(\upsilon)$ *of all the faces* υ *of* τ *by all the elements* γ *of* Γ *(the translations by* Γ*, for short) form a fan.*

We prove this proposition in the last section after preparations.

REMARK 1.9. The main theorem in [10] essentially claims more strongly as follows. (For its proof, see also [12] 7.4.) Let σ' be as in 1.8. Then, there is a fan Σ in $\mathfrak{g}_\mathbf{Q}$ which is strongly compatible with Γ, and all of whose cones are admissible and have the images σ' or $\{0\}$ in $\mathfrak{g}'_\mathbf{Q}$, satisfying the condition that, for any σ as in 1.8, there is a finite subdivision of σ such that each member of this subdivision is contained in some cone in Σ.

COROLLARY 1.10. *Assume that* $\dim \sigma' \leqq 1$. *For any object* T *of* $\mathcal{B}(\log)$ *over* S *and any* $a \in \mathrm{Ext}^1_T(\mathbf{Z}, H')$, *locally on* T, *there is a log modification* $T' \to T$ ([13] 3.6) *and, locally on* T', *there is a fan* Σ *(being strongly compatible with* Γ *and satisfying* $(*)$ *in* 1.6*) such that* J_Σ *graphs* a, *which means that the restriction of* a *in* $\mathrm{Ext}^1_{T'}(\mathbf{Z}, H')$ *belongs to* $\mathrm{Mor}(T', J_\Sigma) \subset \mathrm{Ext}^1_{T'}(\mathbf{Z}, H')$.

We call such J_Σ *a log Néron model for* a.

PROOF. Let H be the LMH corresponding to a. Let $t \in T^{\log}$, and we work around $\tau(t)$. Let $\widetilde{\mu'_t}$ be as in 1.7 (3). Then, via $\widetilde{\mu'_t} \oplus \mathrm{id}$, the monoid $\pi_1^+(\tau^{-1}\tau(t))$ acts on $H_\mathbf{Z}$.

Let σ be the local monodromy cone of H at $\tau(t)$, that is, the cone in $\mathfrak{g}_\mathbf{Q}$ generated by the logarithms of the actions of the elements of $\pi_1^+(\tau^{-1}\tau(t))$

(cf. [13] 2.5.11). This is determined up to the translation by an element of Γ. Clearly, the image of σ in $\mathfrak{g}'_{\mathbf{Q}}$ is contained in σ'.

By localizing T and taking a chart around $\tau(t)$, we may assume that there is a chart $P \to \Gamma(T, M_T)$ with a sharp fs monoid P such that $P \to (M_T/\mathcal{O}_T^{\times})_t$ is bijective. Then, for any $t' \in T^{\log}$, the monoid $\pi_1^+(\tau^{-1}\tau(t'))$ is regarded as a face of $\pi_1^+(\tau^{-1}\tau(t))$, and the action of $\pi_1^+(\tau^{-1}\tau(t'))$ on $H_{\mathbf{Z}}$ factors through the action of $\pi_1^+(\tau^{-1}\tau(t))$ on $H_{\mathbf{Z}}$ modulo the translation by an element of Γ.

Now we apply 1.8 and make a log modification T' of T according to the finite subdivision of $\mathrm{Hom}(P, \mathbf{Q}_{\geq 0})$ induced by that of σ in 1.8. Let $\tau \subset \sigma$ be a member of this subdivision. Then, all the translations of τ with their faces form a fan Σ by 1.8, which is easily seen to be strongly compatible with Γ. By localizing T', we may assume that Σ contains the set of the local monodromy cones of $H|_{T'}$.

We will show that J_Σ graphs a. For this, it suffices to verify the condition (3) in 1.7. But, since the set of the local monodromy cones of H is now contained in Σ, we can take the local monodromy cone itself as the desired cone σ in the condition (3) in 1.7. □

COROLLARY 1.11. *Assume that* $\dim \sigma' \leqq 1$. *For an fs log analytic space T over S which is log smooth over* $\mathbf{C}$ ([13] 2.1.11), *let U be the open subspace of T where the log structure is trivial. Let $a \in \mathrm{Ext}^1_U(\mathbf{Z}, H')$ be an extension of gradedly polarized variations of MHS, which is admissible with respect to T. Then, locally on T, there is a log modification $T' \to T$ and, locally on T', there is a fan Σ (being strongly compatible with Γ and satisfying* $(*)$ *in* 1.6*) such that the morphism $U \xrightarrow{a} J_\Sigma$ extends to a morphism $T' \to J_\Sigma$. (Note that, by the definition of a log modification, the open subspace U of T can be regarded also as an open subspace of T'.)*

PROOF. By the assumption of the admissibility, a extends to an element of $\mathrm{Ext}^1_T(\mathbf{Z}, H')$. Hence this corollary is reduced to the previous one. □

COROLLARY 1.12. *Assume that* $\dim \sigma' \leqq 1$ *and that S is a complex analytic manifold endowed with the log structure defined by a smooth divisor Z. Then any normal function on $S - Z$ which is admissible with respect to S, locally on S, extends to a section of J_Σ for a fan Σ which is strongly compatible with Γ and satisfies* $(*)$ *in* 1.6.

PROOF. Let $U = S - Z$. A normal function on U is nothing but an element of $\mathrm{Ext}^1_U(\mathbf{Z}, H')$. Since, in this case, a log modification over S have to be trivial, this corollary is reduced to the previous one. □

We remark that in the last case 1.12, the log Néron model in the sense of [12] exists as a best model and satisfies the same conclusion. See [12] 6.1.8.

1.13. We proceed to the surface base case.

We introduce some terms to state the result. Let Σ be a finite set of nilpotent cones in $\mathfrak{g}_{\mathbf{Q}}$ (not necessarily a fan). Let $\{\text{fan}\}$ be the set of the fans in $\mathfrak{g}_{\mathbf{Q}}$. A map $s\colon \Sigma \to \{\text{fan}\}$ is a *finite multi-subdivision of* Σ *of length one* if for any $\sigma \in \Sigma$, the image of σ by s is a finite subdivision of σ. Let Σ_s be the union of $s(\sigma)$ for all $\sigma \in \Sigma$. A *finite multi-subdivision* of Σ is a sequence $s_1, \dots, s_n$ of finite multi-subdivisions of length one such that s_1 is a finite multi-subdivision of Σ of length one, and for any j with $2 \leqq j \leqq n$, s_j is a finite multi-subdivision of $\Sigma_{s_{j-1}}$ of length one.

We call an element of Σ_{s_n} a *member* of this finite multi-subdivision.

A *finite multi-subdivision* of a nilpotent cone σ is a finite multi-subdivision of the set of all faces of σ.

The next is the main theorem in this paper, which is proved in Section 6 after the necessary preparations.

THEOREM 1.14. *Assume that* $\dim \sigma' = 2$. *Let* σ *be a nilpotent cone in* $\mathfrak{g}_{\mathbf{Q}}$ *whose image in* $\mathfrak{g}'_{\mathbf{Q}}$ *is* σ'. *Assume that* σ *is admissible. Then, there is a finite multi-subdivision of* σ *such that, for each member* τ *of this multi-subdivision, the translations* $\mathrm{Ad}(\gamma)(\upsilon)$ *of all the faces* υ *of* τ *by all the elements* γ *of* Γ *form a weak fan.*

We expect that the conclusion in 1.14 would hold without the assumption $\dim \sigma' = 2$. Another problem is whether we can find a finite subdivision instead of a finite multi-subdivision.

COROLLARY 1.15. *Assume that* $\dim \sigma' = 2$. *For any object* T *of* $\mathcal{B}(\log)$ *over* S *and any* $a \in \mathrm{Ext}^1_T(\mathbf{Z}, H')$, *there is a finite set of surjective, strict (1.6), local isomorphisms* $T'_j \to T_j$ $(0 \leqq j \leqq n-1)$ *with* $T_0 = T$ *and a set of log modifications* $T_j \to T'_{j-1}$ $(1 \leqq j \leqq n)$ *such that, locally on* T_n, *there* *is a weak fan* Σ *(being strongly compatible with* Γ *and satisfying* $(*)$ *in* 1.6*) such that* J_Σ *graphs the restriction of* a.

We call such J_Σ *a log Néron model for* a.

PROOF. The proof is parallel to that of 1.10.

Let H be the LMH corresponding to a. Let $t \in T^{\log}$. The monoid $\pi_1^+(\tau^{-1}\tau(t))$ acts on $H_{\mathbf{Z}}$ via $\widetilde{\mu'_t}\oplus\mathrm{id}$, where $\widetilde{\mu'_t}$ is as in 1.7 (3).

Let σ be the local monodromy cone of H at $\tau(t)$.

In the same way as in 1.10, we may assume that there is a chart $P \to \Gamma(T, M_T)$ such that $P \to (M_T/\mathcal{O}_T^\times)_t$ is bijective. Then, for any $t' \in T^{\log}$, the action of $\pi_1^+(\tau^{-1}\tau(t'))$ on $H_{\mathbf{Z}}$ factors through the action of $\pi_1^+(\tau^{-1}\tau(t))$ on $H_{\mathbf{Z}}$ modulo the translation by Γ.

We apply 1.14 and have a sequence

$$T_n \to T'_{n-1} \to T_{n-1} \to T'_{n-2} \to \cdots \to T_1 \to T'_0 \to T_0 = T$$

as in the statement according to the finite multi-subdivision in 1.14. By further localization of T_n, we may assume that $n = 0$ and that all the translations of σ with their faces form a weak fan Σ, which is strongly compatible with Γ.

Then, J_Σ graphs a, as is seen in the same way as in 1.10. □

COROLLARY 1.16. *Assume that* $\dim\sigma' = 2$. *For an fs log analytic space* T *over* S *which is log smooth over* $\mathbf{C}$, *let* U *be the open subspace of* T *where the log structure is trivial. Let* $a \in \mathrm{Ext}^1_U(\mathbf{Z}, H')$ *be an extension of gradedly polarized variations of MHS, which is admissible with respect to* T. *Then, there is a finite set of surjective, strict, local isomorphisms* $T'_j \to T_j$ $(0 \leqq j \leqq n-1)$ *with* $T_0 = T$ *and a set of log modifications* $T_j \to T'_{j-1}$ $(1 \leqq j \leqq n)$ *such that, locally on* T_n, *a weak fan* Σ *(being strongly compatible with* Γ *and satisfying* $(*)$ *in* 1.6*) exists and the morphism* $U_n \to U \xrightarrow{a} J_\Sigma$ *extends to a morphism* $T_n \to J_\Sigma$. *Here* U_n *is the inverse image of* U *in* T_n.

PROOF. Similarly to 1.11, this is reduced to the previous corollary. □

1.17. Here we explain the idea of the proof of the main theorem 1.14. The full proof will be given in Section 6.

Consider the following three toy models (A1), (A2), and (B). Notation here is temporary.

Let $H' = \mathbf{Q}$. Let $\Gamma = \mathbf{Z}$. Let $N'_1, N'_2 \in \mathrm{Hom}(\Gamma, H')$. Let $\sigma' = \mathbf{Q}^2_{\geq 0}$. Let $\mathfrak{g} = \sigma' \times H'$. Let Γ act on $\mathfrak{g}$ by $\gamma\colon (a_1, a_2, h') \mapsto (a_1, a_2, h' + (a_1N'_1 + a_2N'_2)(\gamma))$ ($\gamma \in \Gamma$). Let $\sigma \subset \mathfrak{g}$ be a finitely generated sharp cone. Assume that the projection $\sigma \hookrightarrow \mathfrak{g} \to \sigma'$ is surjective and that $\sigma \cap (\{(0,0)\} \times H') = \{0\}$.

We consider the following three conditions.

(A1) $N'_1 = N'_2 = 1$. (Here we naturally identify $\mathrm{Hom}(\Gamma, H')$ with $\mathbf{Q}$.)

(A2) $N'_1 = 0, N'_2 = 1$ and $\sigma \cap (\{(1,0)\} \times H')$ is a singleton, say, $\{(1, 0, h'_1)\}$.

(B) $N'_1 = 0, N'_2 = 1$ and $\sigma \cap (\{(1,0)\} \times H')$ is not a singleton but is 1-dimensional.

In each case, we ask if there exists a finite subdivision of σ such that, for each member τ of this subdivision generates a fan, i.e., all the translations of τ by Γ with their faces form a fan. We observe that it is affirmative only in the first two cases:

In (A1), we subdivide σ into the set Σ of all faces of σ_j ($j \in \frac{1}{2}\mathbf{Z}$), where

$$\sigma_j = \{(a_1, a_2, h') \in \sigma \mid j(a_1 + a_2) \leqq h' \leqq (j + \tfrac{1}{2})(a_1 + a_2)\}.$$

Then, for any $\gamma \neq 0$ and any j, we have $\gamma(\sigma_j) \cap \sigma_j = \{0\}$. Hence any $\tau \in \Sigma$ generates a fan.

In (A2), we subdivide σ into the set Σ of all faces of σ_j ($j \in \frac{1}{2}\mathbf{Z}$), where

$$\sigma_j = \{(a_1, a_2, h') \in \sigma \mid a_1h'_1 + a_2j \leqq h' \leqq a_1h'_1 + a_2(j + \tfrac{1}{2})\}.$$

Then, again, for any $\gamma \neq 0$ and any j, we have $\gamma(\sigma_j) \cap \sigma_j = \{0\}$. Hence any $\tau \in \Sigma$ generates a fan.

In (B), we cannot resolve the overlapping, i.e., for any finite subdivision Σ of σ, there exists a $\tau \in \Sigma$ which does not generate a fan. In fact, there exists a 3-dimensional τ whose intersection with $\{(1,0)\} \times H'$ is 1-dimensional. Then, for any $\gamma \in \Gamma - \{0\}$, the intersection $\gamma(\tau) \cap \tau$ is not a face of τ.

Now we return from toy models to the explanation of the idea of the proof of 1.14. Let H' be the polarized nilpotent orbit of weight -1 as in 1.7.

To prove 1.14, roughly, we carefully choose a decreasing filtration $(J^j)_j$ of the unipotent part (cf. 6.1) $\Gamma_u = H'_{\mathbf{Z}}$ of Γ such that for any j and for any $\gamma \in J^j \smallsetminus J^{j+1}$, the action of γ modulo J^{j+1} looks like either that of γ in

(A1), (A2), or (B). The existence of such a nice filtration is proved in 3.21 below, based on a property 3.15 of the polarized nilpotent orbit H'.

We subdivide σ according to the above nice filtration.

In the case (A1) or (A2), we subdivide the given cone to resolve the overlapping by generalizing the procedures in the above toy models. The precise procedures for (A1) and (A2) are provided in 2.3 and 2.8 below, respectively.

In the case (B), we cannot resolve the overlapping. Instead, we prove that the given cone generates a weak fan after a finite subdivision. The key observation in this step is 4.3 below; the other lemmas in Sections 4–5 are rather standard.

2. Subdivision of Cones

In this section, we prepare the lemmas of type (A1) and of type (A2) (see 1.17), which show how to subdivide cones according to the nice filtration explained in 1.17.

2.1. Let H be a finite dimensional $\mathbf{Q}$-vector space. Let

$$X = \mathbf{Q}^2_{\geq 0} \times H.$$

Let $\sigma \subset X$ be a finitely generated sharp cone. Assume the following condition:

$$\sigma \cap (\{(0,0)\} \times H) = \{0\}. \tag{1}$$

Let L be a finitely generated free $\mathbf{Z}$-module. Let $N_1, N_2 \in \mathrm{Hom}(L, H)$. Let L act on X by

$$l\colon (a_1, a_2, h) \mapsto (a_1, a_2, h + (a_1N_1 + a_2N_2)(l)) \quad (l \in L).$$

Note that, in applying the results in this section to the proof of the main theorem, we take $H'_{\mathbf{Q}}$ in the main theorem as the H here.

We introduce the following notation. For a rational number ε with $0 \leqq \varepsilon \leqq 1$, let

$$H_{1+\varepsilon} = \mathbf{Q}_{\geq 0}(1-\varepsilon, \varepsilon) \times H.$$

In particular,

$$H_1 = \mathbf{Q}_{\geq 0} \times \{0\} \times H, \text{ and}$$
$$H_2 = \{0\} \times \mathbf{Q}_{\geq 0} \times H.$$

LEMMA 2.2. *Let the notation and the assumption be as in* 2.1. *Let* $b_1, b_2 \in \mathbf{Q}_{>0}$. *Then, the subset*

$$S_{b_1,b_2} := \sigma \cap (\{(a_1, a_2) \mid a_1b_1 + a_2b_2 = 1\} \times H)$$

of X *is bounded. In particular, each fiber of the projection* $\sigma \hookrightarrow X \to \mathbf{Q}^2_{\geq 0}$ *is bounded.*

PROOF. First we show that

$$M := \{(a_1, a_2) \in \mathbf{Q}^2 \mid a_1b_1 + a_2b_2 = 0\} \times H \subset \mathbf{Q}^2 \times H$$

is a supporting hyperplane of $\{0\}$ of the cone $\sigma \subset X \subset \mathbf{Q}^2 \times H$. Let $(a_1, a_2, h) \in M \cap \sigma$. Then, (a_1, a_2) is in the image of $\sigma \subset X$, so $a_1, a_2 \geqq 0$. Since $b_1, b_2 > 0$, we have $a_1 = a_2 = 0$. Hence, $(0, 0, h) = (a_1, a_2, h)$ belongs to σ. By the condition 2.1 (1), we have $h = 0$. Thus, $M \cap \sigma = \{0\}$, i.e., M is a supporting hyperplane of $\{0\}$ of σ.

Since our set S_{b_1,b_2} is the intersection of σ and the translation of M by a vector in $\mathbf{Q}^2 \times H$, it is bounded. □

The next is the lemma for the situation of type (A1).

LEMMA 2.3. *Let the notation and the assumption be as in* 2.1. *Then we have the following.*

(1) *Assume that* $a_1N_1 + a_2N_2 \colon L \to H$ *is injective for any* $(a_1, a_2) \in \mathbf{Q}^2_{\geq 0} - \{(0, 0)\}$. *Then, there is a finite subdivision of* σ *such that, for each member* τ *of this subdivision and for any* $l \in L - \{0\}$, *we have* $l(\tau) \cap \tau = \{0\}$.

(2) *Assume that* N_1 *is injective. Then there is a positive* $\varepsilon_0 \leqq 1$ *such that for any positive rational number* $\varepsilon \leqq \varepsilon_0$, *there is a finite subdivision of*

$$\sigma \cap (H_1 + H_{1+\varepsilon}) = \sigma \cap \Big(\big(\mathbf{Q}_{\geq 0}(1, 0) + \mathbf{Q}_{\geq 0}(1 - \varepsilon, \varepsilon)\big) \times H\Big)$$

such that, for each member τ of this subdivision and for any $l \in L - \{0\}$, we have $l(\tau) \cap \tau = \{0\}$.

(3) *Assume that we are given an identification $L \otimes \mathbf{Q} = H$. We regard N_1 and N_2 as elements of* $\mathrm{End}(H)$ *via this identification. Let $J \subset H$ be a $\mathbf{Q}$-subspace satisfying $N_1^{-1}N_2J \subset J$ (cf. Notation and Terminology). Then there is a positive $\varepsilon_0 \leqq 1$ such that for any positive rational number $\varepsilon \leqq \varepsilon_0$, there is a finite subdivision of*

$$\sigma \cap (H_1 + H_{1+\varepsilon}) = \sigma \cap \Big(\big(\mathbf{Q}_{\geq 0}(1,0) + \mathbf{Q}_{\geq 0}(1-\varepsilon,\varepsilon)\big) \times H\Big)$$

such that, for each member τ of this subdivision and for any $l \in L \smallsetminus J$, we have $l(\tau) \cap \tau = \{0\}$.

Before the proof, we briefly review the pullback of the subdivision. The next fact is well-known and is seen by considering the dual statement.

Lemma 2.4. *Let σ, τ be two polyhedral cones in a vector space. Let υ be a face of the cone $\sigma \cap \tau$. Let σ_1 and τ_1 be the faces of σ and τ spanned by υ, respectively. Then, we have $\upsilon = \sigma_1 \cap \tau_1$.*

2.5. Let Σ be a fan in a vector space V. Let σ be a polyhedral cone. Let $p\colon \sigma \to V$ be a map induced by a linear map. Assume that for each $\tau \in \Sigma$, the pullback $p^{-1}(\tau)$ is sharp. (This holds, for example, when σ is sharp or when p is injective.)

Then, it is easily seen from 2.4 that the set Σ' of the cones of the form $\sigma_1 \cap p^{-1}(\tau_1)$, where σ_1 is a face of σ and τ_1 is an element of Σ, makes a fan. We call Σ' the *pullback* of Σ by p.

Proof of 2.3. To prove (1), we may replace σ with a larger finitely generated sharp cone containing σ and satisfying 2.1 (1). Hence, by 2.2, we may assume that there is a convex polytope C in H such that σ is generated (as a cone) by the set $\{(1,0,h) \,|\, h \in C\} \cup \{(0,1,h) \,|\, h \in C\}$. (Concretely, we can take as C the image in H of the subset $S_{1,1}$ in 2.2 for the original σ.) Then, a subdivision Σ of C naturally induces a subdivision of σ. That is, for each $C' \in \Sigma$, the set $\{(1,0,h) \,|\, h \in C'\} \cup \{(0,1,h) \,|\, h \in C'\}$ generates a subcone of σ and these cones together with their faces form a subdivision of σ.

On the other hand, by the assumption of the injectivity, we have

$$\inf_{a_1+a_2=1} \inf_{l\in L-\{0\}} |(a_1N_1 + a_2N_2)l| > 0,$$

where we fix a metric $|-|$ on H. Hence, by subdividing the polytope C sufficiently finely, we may assume that $(C + (a_1N_1 + a_2N_2)l) \cap C$ is empty for any $a_1, a_2 > 0$ with $a_1 + a_2 = 1$ and any $l \in L - \{0\}$. The last condition implies the desired property $l(\sigma) \cap \sigma = \{0\}$, which completes the proof of (1). See the remark 2.6 below for an alternative proof of (1).

(2) Since N_1 is injective, there is a positive $\varepsilon_0 \leqq 1$ such that for any positive rational $\varepsilon \leqq \varepsilon_0$, the operator $(1-\varepsilon)N_1 + \varepsilon N_2$ is injective. Hence, by replacing N_2 by $(1-\varepsilon)N_1 + \varepsilon N_2$, and X by $H_1 + H_{1+\varepsilon}$, (2) is reduced to (1).

(3) Let $A := N_1J + N_2J$. Then, the action of L on X induces the action of $L/(J \cap L)$ on

$$\overline{X} := \mathbf{Q}^2_{\geq 0} \times (H/A)$$

because we have $a_1N_1l + a_2N_2l \in A$ for $a_1, a_2 \in \mathbf{Q}_{\geq 0}$ if $l \in J$. Further, we have the operators

$$\overline{N_j} \colon L/(J \cap L) \to H/A$$

induced by N_j for $j = 1, 2$.

Let $\overline{\sigma}$ be the image of σ in $\overline{X}$. Then, $\overline{\sigma} \cap (\{(0,0)\} \times (H/A))$ is trivial and we are in the situation in 2.1 with H/A, $L/(J \cap L), \overline{N}_1, \overline{N}_2, \overline{X}$, and $\overline{\sigma}$ for H, L, N_1, N_2, X, and σ there.

We prove that $\overline{N_1}$ is injective. Let $l \in L$ and assume $N_1(l) \in A$. Since $A = N_1J + N_2J$, there are $j_1, j_2 \in J$ such that $N_1(l) = N_1(j_1) + N_2(j_2)$. From this, $N_1(l - j_1) \in N_2J$, so $l - j_1 \in N_1^{-1}N_2J \subset J$ by the assumption. Hence $l \in J$, and $\overline{N_1}$ is injective.

Therefore, by (2), there is a positive $\varepsilon_0 \leqq 1$ such that for any positive rational $\varepsilon \leqq \varepsilon_0$, there is a finite subdivision of $\overline{\sigma} \cap \Big(\big(\mathbf{Q}_{\geq 0}(1,0) + \mathbf{Q}_{\geq 0}(1-\varepsilon, \varepsilon)\big) \times (H/A)\Big)$ such that, for each member $\overline{\tau}$ of this subdivision and for any $l \in (L/(J \cap L)) - \{0\}$, we have $l(\overline{\tau}) \cap \overline{\tau} = \{0\}$. To pull back this subdivision (2.5) gives a subdivision of $\sigma \cap (H_1 + H_{1+\varepsilon})$. Let τ be a member of it and $l \in L \smallsetminus J$. By construction, $l(\tau) \cap \tau \subset \{(0,0)\} \times A$. Together with the condition 2.1 (1), we have $l(\tau) \cap \tau = \{0\}$. Hence, this is a desired subdivision. □

Remark 2.6. We sketch another proof for 2.3 (1). In general, the following holds. Let an abstract group G act linearly on a $\mathbf{Q}$-vector space V of finite dimension. Let V' be a G-stable $\mathbf{Q}_{\geq 0}$-submonoid of V. Assume that the action of G on $V' - \{0\}$ is proper and free. Let σ be a finitely generated sharp cone contained in V'. Then, there is a finite subdivision of σ such that, for each member τ of this subdivision and for any $g \in G - \{1\}$, we have $g(\tau) \cap \tau = \{0\}$.

This is seen by considering the projection $p\colon V' - \{0\} \to G \backslash (V' - \{0\})$ and observing that any $x \in \sigma - \{0\}$ admits a neighborhood S satisfying that $S \to p(S)$ is a homeomorphism and that $p^{-1}p(S)$ is isomorphic to $G \times p(S)$ as G-torsors over $p(S)$.

We apply this with $G = L$, $V = \mathbf{Q}^2 \times H$, and $V' = X - (\{(0,0)\} \times H) \cup \{0\}$. The freeness is direct by the injectivity assumption. The properness is deduced from the fact that if N_λ (λ runs over a directed set) converges in the space of the injective homomorphisms from L to V, and if $N_\lambda l_\lambda$ ($l_\lambda \in L$) converges, then l_λ converges, that is, eventually is constant.

The next is a 1-dimensional variant of 2.3.

Lemma 2.7. *Let the notation and the assumption be as in* 2.1. *Let* $Y = \mathbf{Q}_{\geq 0} \times H$. *Let* L *act on* Y *by* $l\colon (a,h) \mapsto (a, h + aN_1(l))$ $(l \in L)$. *Let* $\tau \subset Y$ *be a finitely generated sharp cone. Assume that* $\tau \cap (\{0\} \times H) = \{0\}$. *Then, there is a finite subdivision of* τ *such that, for each member* υ *of this subdivision, we have that* $l(\upsilon) \cap \upsilon = \{0\}$ *in the case* $l \in L \smallsetminus N_1^{-1}(0)$, *and that* l *acts trivially on* υ *in the case* $l \in L \cap N_1^{-1}(0)$.

Proof. Let $C := \tau \cap (\{1\} \times H)$. Then, similarly to 2.2, the assumption $\tau \cap (\{0\} \times H) = \{0\}$ implies that C is bounded. Further, τ is spanned by C, and a subdivision of C naturally induces a subdivision of τ.

We regard C as a subset of H. Fix a metric on H. Since the image $N_1 L$ is discrete in H, we have $\inf_{x \in N_1 L - \{0\}} |x| > 0$. Hence, we can take a finite subdivision of C such that each member C' of this subdivision satisfies the condition that $(C' + N_1 l) \cap C'$ is empty for any $l \in L \smallsetminus N_1^{-1}(0)$.

It is clear that the subdivision of τ induced by this subdivision of C satisfies the desired condition. □

The next is the lemma of type (A2).

LEMMA 2.8. *Let the notation and the assumption be as in* 2.1. *Then we have the following.*

(1) *Assume that the dimension of* $\sigma \cap H_1$ *is* 0 *or* 1. *Assume also* $N_1 L = 0$. *Then, there is a finite subdivision of* σ *such that, for each member* τ *of this subdivision and for any* $l \in L \smallsetminus N_2^{-1}(0)$, *we have* $l(\tau) \cap \tau \subset H_1$.

(2) *Assume that we are given an identification* $L \otimes \mathbf{Q} = H$. *We regard* N_1 *and* N_2 *as elements of* $\mathrm{End}(H)$ *via this identification. Let* J, V *be two* $\mathbf{Q}$*-subspaces of* H. *Assume that there is an* $h \in H$ *such that* $\sigma \cap H_1$ *is contained in the cone generated by* $(1, 0, h + V)$. *Then there is a finite subdivision of* σ *such that, for each member* τ *of this subdivision and for any* $l \in (J \cap L) \smallsetminus N_2^{-1}(N_1 J + V)$, *we have either* $l(\tau) \cap \tau = \{0\}$ *or* $l(\tau) \cap \tau = \tau \cap H_1$.

PROOF. To prove (1), we may replace σ with a larger finitely generated sharp cone containing σ satisfying the same condition in (1) and the condition 2.1 (1). Hence, we may assume that there are a vector $v \in H$ and a convex polytope C in H such that σ is generated by the set $\{(1, 0, v)\} \cup \{(0, 1, h) \mid h \in C\}$.

Since the image $N_2 L$ is discrete in H, there is a finite subdivision $\{C_j\}_j$ of C such that for any j and any $l \in N_2 L - \{0\}$, the intersection $C_j \cap (C_j + l)$ is empty. This subdivision of C naturally induces a subdivision of σ, that is, the cones σ_j generated by the set $\{(1, 0, v)\} \cup \{(0, 1, h) \mid h \in C_j\}$ together with their faces form a subdivision of σ.

We prove that this is the desired one. Take an element x of σ_j, which we can write $x = (a, b, av + bh)$ with $a, b \in \mathbf{Q}_{\geq 0}, h \in C_j$. Let $l \in L$ act on x and we have $l(x) = (a, b, av + bh + aN_1(l) + bN_2(l)) = (a, b, av + bh + bN_2(l))$ by the assumption $N_1 L = 0$. Hence, if an element of $l(\sigma_j) \cap \sigma_j$ does not belong to H_1, there are $b > 0$ and $h, h' \in C_j$ such that $bh = bh' + bN_2(l)$ so $h = h' + N_2(l)$. By the choice of the subdivision $\{C_j\}_j$, we have $N_2(l) = 0$. Hence, $l(\sigma_j) \cap \sigma_j \subset H_1$ for any $l \in L \smallsetminus N_2^{-1}(0)$, which means that our subdivision satisfies the desired condition.

(2) Let $\tau := \sigma \cap H_1$. First we prove this (2) under the additional assumption that for any $l \in (J \cap L) \smallsetminus N_1^{-1}(0)$, we have $l(\tau) \cap \tau = \{0\}$. In this case, let $A := N_1 J + V$. Apply (1) with H/A for H, with L for $J \cap L$, with the induced operators $J \cap L \to H/A$ by N_1, N_2 for N_1, N_2, and with the image $\overline{\sigma}$ of σ in $\mathbf{Q}_{\geq 0}^2 \times (H/A)$ for σ. Note that N_1 sends $J \cap L$ to A so that the operator induced by N_1 is zero. Then, (1) gives a finite subdivision Σ'

of $\overline{\sigma}$ such that for any $v' \in \Sigma'$ and $l \in (J \cap L) \smallsetminus N_2^{-1}(A)$, the intersection $l(v') \cap v'$ is contained in $\mathbf{Q}_{\geq 0} \times \{0\} \times (H/A)$.

To pull back this subdivision gives a subdivision Σ of σ. For any $v \in \Sigma$, we have $l(v) \cap v \subset H_1$ for any $l \in (J \cap L) \smallsetminus N_2^{-1}(A)$. We prove that the last inclusion implies that $l(v) \cap v$ coincides with either $\{0\}$ or $v \cap H_1$. In fact, by this inclusion, we have $l(v) \cap v = l(v \cap H_1) \cap (v \cap H_1) \subset l(\tau) \cap \tau$. Hence, if $N_1(l) \neq 0$, the assumption $l(\tau) \cap \tau = \{0\}$ gives $l(v) \cap v = \{0\}$. Otherwise, l acts trivially on H_1 and $l(v \cap H_1) = v \cap H_1$, which coincides with $l(v) \cap v$.

In the general case, we first apply 2.7 to the cone τ with H_1 as Y there and subdivide τ. Take any finite subdivision $\Sigma = \{\sigma_j\}$ of σ which induces this subdivision of τ. Then, we can apply the above proof for the special case to each σ_j because for any j and any $l \in L \smallsetminus N_1^{-1}(0)$, we have $l(\sigma_j \cap H_1) \cap (\sigma_j \cap H_1) = \{0\}$. We denote by Σ_j the resulting subdivision of σ_j for each j. (We remark that these Σ_j already give a multi-subdivision of σ satisfying the desired condition. Actually, it will suffice for the main theorem in this paper.) Take a finite subdivision Σ' of the fan Σ which induces a finite subdivision of Σ_j on each σ_j. Then, it is clear that this Σ' satisfies the desired condition. □

3. Polarized Nilpotent Orbits

One of the key facts which we will use later in the proof of the main theorem is the following proposition 3.2 on a pure nilpotent orbit. This might be known, but we include a proof for completeness.

3.1. Let $H_{\mathbf{Z}}$ be a free $\mathbf{Z}$-module of finite rank, let w be an integer, and let $\langle\ ,\ \rangle$ be a non-degenerate $(-1)^w$-symmetric pairing on $H_{\mathbf{Z}}$. Let $(h^{p,q})_{p+q=w}$ be non-negative integers satisfying $h^{p,q} = h^{q,p}$ and such that almost all of them are zero. Let $D = D(H_{\mathbf{Z}}, (h^{p,q}), \langle\ ,\ \rangle)$ be the classifying space of polarized Hodge structures, and $\check{D}$ its compact dual.

Let $G_{\mathbf{Q}}$ be the group of the $\mathbf{Q}$-automorphisms of $(H_{\mathbf{Q}}, \langle\ ,\ \rangle)$, and $\mathfrak{g}_{\mathbf{Q}}$ the associated Lie algebra.

Let $N_1, N_2 \in \mathfrak{g}_{\mathbf{Q}} \subset \mathrm{End}(H_{\mathbf{Q}})$ be mutually commutative nilpotent elements. Let $F \in \check{D}$. Assume that (N_1, N_2, F) generates a nilpotent orbit ([13] 5.4.1).

Note that, in applying the results in this section to the main theorem in Section 6, $H_{\mathbf{Q}}$ here is $H'_{\mathbf{Q}}$ in the main theorem. Note also that we will use only the case $w = -1$ in Section 6.

We assume that the associated weight filtrations $W(N_1 + N_2)[-w]$ and $W(N_2)[-w]$ coincide. We denote by M this filtration.

The main result in this section is the following, which will be proved in 3.5–3.14 below after preparations.

Proposition 3.2. *Let the notation and the assumption be as in* 3.1. *For any* $n \geqq 0$, *we have*

$$M_{w-1} \cap \bigcap_{j=0}^{\infty} \big(M_{w-2} + (N_2^j)^{-1}(\operatorname{Im} N_1^{j+1})\big) \cap (N_2 N_1^{-1})^n (M_{w-2}) \subset M_{w-2}.$$

3.3. For the proof, first we review the direct sum decomposition of $H_{\mathbf{R}} = H_{\mathbf{Q}} \otimes_{\mathbf{Q}} \mathbf{R}$ associated to (N_1, N_2, F).

Though this direct sum decomposition can be described in terms of the associated SL(2)-orbit (see 3.4 below), we define it here without that theory as follows (cf. [9] 2.5).

Let $(M, \hat{F}_{(2)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure (M, F). Let $s^{(2)} \colon \mathrm{gr}^M \to H_{\mathbf{R}}$ be the splitting of M by $(M, \hat{F}_{(2)})$. Next, let $(M(N_1), \hat{F}_{(1)})$ be the $\mathbf{R}$-split mixed Hodge structure associated to the mixed Hodge structure $(M, \exp(iN_2)\hat{F}_{(2)})$. Then $M(N_1)$ coincides with $W(N_1)[-w]$. Let $s^{(1)} \colon \mathrm{gr}^{M(N_1)} \to H_{\mathbf{R}}$ be the splitting of $M(N_1)$ by $(M(N_1), \hat{F}_{(1)})$.

For any $j, k \in \mathbf{Z}$, let $H_{\mathbf{R}}^{[j,k]} = s^{(1)}(\mathrm{gr}_j^{M(N_1)}) \cap s^{(2)}(\mathrm{gr}_k^M)$. Then, we have a direct sum decomposition

$$H_{\mathbf{R}} = \bigoplus_{j,k \in \mathbf{Z}} H_{\mathbf{R}}^{[j,k]}.$$

In particular, for any $j, k \in \mathbf{Z}$,

$$M(N_1)_j = \bigoplus_{k \in \mathbf{Z}, j' \leqq j} H_{\mathbf{R}}^{[j',k]} \text{ and } M_k = \bigoplus_{j \in \mathbf{Z}, k' \leqq k} H_{\mathbf{R}}^{[j,k']}.$$

In the following, we denote by $h^{[j,k]}$ the $[j,k]$-component of an element h of $H_{\mathbf{R}}$.

Further, this direct sum decomposition naturally induces that of $\mathfrak{g}_{\mathbf{R}}$ as follows. Let $\mathfrak{g}_{\mathbf{R}}^{[j,k]}$ be the subspace of $\mathfrak{g}_{\mathbf{R}}$ consisting of the operators N satisfying $N(H_{\mathbf{R}}^{[j',k']}) \subset H_{\mathbf{R}}^{[j+j',k+k']}$ for any $j', k' \in \mathbf{Z}$. Then, we have a direct sum decomposition

$$\mathfrak{g}_{\mathbf{R}} = \bigoplus_{j,k \in \mathbf{Z}} \mathfrak{g}_{\mathbf{R}}^{[j,k]}.$$

We denote by $N^{[j,k]}$ the $[j,k]$-component of an element N of $\mathfrak{g}_{\mathbf{R}}$.

By [3] 4.20 (cf. [9] 2.7), we know that

$$N_1 \in \mathfrak{g}_{\mathbf{R}}^{[-2,-2]}, \text{and } N_2 \in \bigoplus_{j \leqq 0} \mathfrak{g}_{\mathbf{R}}^{[j,-2]}.$$

We define

$$\hat{N}_2 := N_2^{[0,-2]}.$$

3.4. As mentioned in the above, these all are incorporated into the theory of the associated SL(2)-orbit ([3] 4.20). Though it is not necessary in the sequel, we explain it briefly, basing on the formulation in [9] §2.

Let (ρ, φ) be the SL(2)-orbit in two variables associated to (N_1, N_2, F), where ρ is a homomorphism of algebraic groups $\mathrm{SL}(2, \mathbf{C})^2 \to G_{\mathbf{C}}$ and φ is a holomorphic map $\mathbf{P}^1(\mathbf{C})^2 \to \check{D}$.

Then, $H_{\mathbf{R}}^{[j,k]}$ is the part of $H_{\mathbf{R}}$ on which $\rho\left(\begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$ $(\lambda \in \mathbf{R}^\times)$ acts via the multiplication by λ^{j-w} and $\rho\left(\begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$ $(\lambda \in \mathbf{R}^\times)$ acts via the multiplication by λ^{k-w}.

Similarly, $\mathfrak{g}_{\mathbf{R}}^{[j,k]}$ is the part of $\mathfrak{g}_{\mathbf{R}}$ on which $\mathrm{Ad}\left(\rho\left(\begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\right)$ $(\lambda \in \mathbf{R}^\times)$ acts via the multiplication by λ^j and $\mathrm{Ad}\left(\rho\left(\begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1/\lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\right)$ $(\lambda \in \mathbf{R}^\times)$ acts via the multiplication by λ^k.

Finally, $\hat{N}_2$ is the image of $\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$ by the homomorphism of Lie algebras $\mathfrak{sl}(2, \mathbf{R})^2 \to \mathfrak{g}_{\mathbf{R}}$ induced by ρ.

3.5. We begin the proof of 3.2. The proof goes through several reduction steps, which completes in 3.14. We work over $\mathbf{R}$ and prove the inclusion in 3.2 after tensoring $\mathbf{R}$. We regard N_1 and N_2 as elements of

$\mathrm{End}(H_{\mathbf{R}})$ and, for a subspace V of $H_{\mathbf{Q}}$, we use the same symbol to denote the subspace $V \otimes_{\mathbf{Q}} \mathbf{R}$ of $H_{\mathbf{R}}$.

First we have

$$N_1^{-1} M_{w-2} \subset M_w + \mathrm{Ker}\, N_1. \tag{1}$$

In fact, let h be any element of $H_{\mathbf{R}}$. Then, we can write it as $h = h_1 + h_2 \in H_{\mathbf{R}}$, where $h_1 \in M_w$ and $h_2 \in \bigoplus_{j \in \mathbf{Z}, w+1 \leqq k} H_{\mathbf{R}}^{[j,k]}$. Assume $N_1(h) \in M_{w-2}$. Since $N_1 \in \mathfrak{g}_{\mathbf{R}}^{[-2,-2]}$, we have $N_1(h_1) \in M_{w-2}$ and $N_1(h_2) \in \bigoplus_{j \in \mathbf{Z}, w-1 \leqq k} H_{\mathbf{R}}^{[j,k]}$. Hence $N_1(h_2) = N_1(h) - N_1(h_1) \in M_{w-2}$ and $N_1(h_2)$ must be 0, which implies $h \in M_w + \mathrm{Ker}\, N_1$, and (1) is proved.

Hence, for 3.2, it is enough to prove a slightly stronger statement as follows.

$$\begin{aligned} M_{w-1} \cap \bigcap_{j=0}^{\infty} \bigl(M_{w-2} + (N_2^j)^{-1}(\mathrm{Im}\, N_1^{j+1})\bigr) \\ \cap N_2(N_1^{-1}N_2)^n(M_w + \mathrm{Ker}\, N_1) \subset M_{w-2}. \end{aligned} \tag{2}$$

LEMMA 3.6. *Let x_j, y_j be elements of $H_{\mathbf{R}}$ $(0 \leqq j \leqq k)$. Assume $N_1(y_j) = x_{j-1}$ $(j > 0)$, $N_2(y_j) = x_j$ $(j \geqq 0)$, and $x_k \in M_{w-1}$, that is,* $y_0 \overset{N_2}{\mapsto} x_0 \overset{N_1}{\mapsfrom} y_1 \overset{N_2}{\mapsto} \cdots \overset{N_2}{\mapsto} x_{k-1} \overset{N_1}{\mapsfrom} y_k \overset{N_2}{\mapsto} x_k \in M_{w-1}$.

Then, for any j, we have $x_j \in M_{w-1}$ and $y_j \in M_{w+1}$.

PROOF. We prove this by downward induction on j. First, $x_k \in M_{w-1}$ by assumption. Hence, it is enough to prove the following two implications (i) for any j and (ii) for $j > 0$.

(i) $x_j \in M_{w-1} \Rightarrow y_j \in M_{w+1}$.

(ii) $y_j \in M_{w+1} \Rightarrow x_{j-1} \in M_{w-1}$.

We prove (i). Suppose $x_j \in M_{w-1}$. Since $N_2(y_j) = x_j$, we see that y_j belongs to $N_2^{-1} M_{w-1}$, which coincides with M_{w+1} because $M = W(N_2)[-w]$.

We prove (ii). Let $j > 0$. Suppose $y_j \in M_{w+1}$. Since $x_{j-1} = N_1(y_j)$, we see that x_{j-1} belongs to $N_1(M_{w+1})$, which is included in M_{w-1} because $N_1 \in \mathfrak{g}_{\mathbf{R}}^{[-2,-2]}$. □

3.7. We reduce (2) in 3.5 to the following statement for all k with $0 \leqq k \leqq n$:

$(3)_k$ Let x_j, y_j be elements of $H_{\mathbf{R}}$ $(0 \leqq j \leqq k)$. Assume the following $(*)$.
$(*)$ $N_1(y_j) = x_{j-1}$ $(j > 0)$, $N_2(y_j) = x_j$ $(j \geqq 0)$, and $x_k \in M_{w-1} \cap \bigcap_{j=0}^{\infty} \big(M_{w-2} + (N_2^j)^{-1}(\operatorname{Im} N_1^{j+1})\big)$, that is,
$y_0 \overset{N_2}{\mapsto} x_0 \overset{N_1}{\hookleftarrow} y_1 \overset{N_2}{\mapsto} \cdots \overset{N_2}{\mapsto} x_{k-1} \overset{N_1}{\hookleftarrow} y_k \overset{N_2}{\mapsto} x_k \in M_{w-1} \cap \bigcap_{j=0}^{\infty} \big(M_{w-2} + (N_2^j)^{-1}(\operatorname{Im} N_1^{j+1})\big)$.
The claim is this: if $y_0 \in M_w + \operatorname{Ker} N_1$, then, $y_0 \in M_w$.

We prove this reduction of (2) in 3.5 to $(3)_k$. Let x_n be an element of the left hand side of (2) in 3.5. Then, there are x_j $(0 \leqq j < n)$ and y_j $(0 \leqq j \leqq n)$ together with x_n satisfying all the assumptions in $(3)_n$. (In particular, $y_0 \in M_w + \operatorname{Ker} N_1$.) Hence $y_0 \in M_w$ by $(3)_n$. Then, $N_2(y_0) = x_0$ belongs to M_{w-2} because $N_2(M_w) \subset M_{w-2}$.

We have to show $x_n \in M_{w-2}$, which is the right hand side of (2) in 3.5. If $n = 0$, we have already proved it. If $n > 0$, we note $N_1(y_1) = x_0 \in M_{w-2}$. By this and by (1) in 3.5, we see $y_1 \in M_w + \operatorname{Ker} N_1$. Hence, by $(3)_{n-1}$ in this time, we have $y_1 \in M_w$, which implies in the same way that $N_2(y_1) = x_1$ belongs to M_{w-2}. If $n > 1$, again by (1) in 3.5 and by $N_1(y_2) = x_1 \in M_{w-2}$, we have $y_2 \in M_w + \operatorname{Ker} N_1$. By $(3)_{n-2}$, we have $y_2 \in M_w$. Thus, inductively, we show that all y_j's belong to M_w and all x_j's belong to M_{w-2}. In particular, $x_n \in M_{w-2}$.

Therefore, it is enough to show $(3)_k$.

3.8. Further, we reduce $(3)_k$ in 3.7 to the following statement $(4)_k$. In the following, we denote $y^{[j]}$ for $y^{[j,w+1]}$ for simplicity.

$(4)_k$ Let $l \in \mathbf{Z}$. Let x_j, y_j be elements of $H_{\mathbf{R}}$ $(0 \leqq j \leqq k)$. Assume $(*)$ in 3.7 $(3)_k$. Assume that for each j with $0 \leqq j \leqq k$, the component $y_j^{[a]}$ is zero whenever $a > l + 2j$. The claim is this: if $y_0^{[l]} \in \operatorname{Ker} N_1$, then $y_0^{[l]} = 0$.

We prove the reduction of $(3)_k$ in 3.7 to $(4)_k$. Assume $(*)$ in 3.7 $(3)_k$. Then, by 3.6, all y_j's belong to M_{w+1} and all x_j's belong to M_{w-1}. In particular, $y_0 \in M_{w+1}$.

Therefore, under the assumption $(*)$ in 3.7 $(3)_k$, $y_0 \in M_w + \operatorname{Ker} N_1$ if and only if $y_0^{[a]} \in \operatorname{Ker} N_1$ for all $a \in \mathbf{Z}$, and $y_0 \in M_w$ if and only if $y_0^{[a]} = 0$ for all $a \in \mathbf{Z}$. Hence the claim in 3.7 $(3)_k$ is equivalent to the following one: if $y_0^{[a]} \in \operatorname{Ker} N_1$ for all $a \in \mathbf{Z}$, then, $y_0^{[a]} = 0$ for all $a \in \mathbf{Z}$.

We prove the following slightly stronger claim than this one by using $(4)_{k'}$ for $k' \leqq k$. Let l be an integer.

Claim $(3)_{k,l}$. Let x_j, y_j be elements of $H_{\mathbf{R}}$ $(0 \leqq j \leqq k)$. Assume $(*)$ in 3.7 $(3)_k$. If $y_0^{[a]} \in \operatorname{Ker} N_1$ for $a \geqq l$, then, $y_0^{[a]} = 0$ for $a \geqq l$.

To prove $(3)_k$ in 3.7, it is enough to prove this for all l (under the assumption $(*)$ in 3.7 $(3)_k$).

Proof of Claim $(3)_{k,l}$. We prove this by induction on k. First note that the claim trivially holds if l is sufficiently large so that $H_{\mathbf{R}}^{[a,w+1]} = 0$ for $a \geqq l$.

Let $k = 0$. Then, $(4)_0$ says that, for any $l \in \mathbf{Z}$, if $y_0^{[l+a]} = 0$ $(a > 0)$ and $y_0^{[l]} \in \operatorname{Ker} N_1$, then $y_0^{[l]} = 0$. Hence, by downward induction on a, the claim $(3)_{0,l}$ holds for any l.

Next, let $k > 0$. Assume that the claim holds for any (k', l) with $k' < k$ and for $(k, l+1)$. Then, we can show $(3)_{k,l}$ by using $(4)_k$ as follows.

By assumption, $y_0^{[a]} \in \operatorname{Ker} N_1$ $(a > l)$. Then, by the induction hypothesis $(3)_{k,l+1}$, we have $y_0^{[a]} = 0$ $(a > l)$. So $x_0^{[a,w-1]} = 0$ $(a > l)$ and $y_1^{[a]} \in \operatorname{Ker} N_1$ $(a > l+2)$. By the induction hypothesis $(3)_{k-1,l+3}$, the last fact implies $y_1^{[a]} = 0$ $(a > l+2)$. So $x_1^{[a,w-1]} = 0$ $(a > l+2)$ and $y_2^{[a]} \in \operatorname{Ker} N_1$ $(a > l+4)$ if $k > 1$. By $(3)_{k-2,l+5}$, we can deduce $y_2^{[a]} = 0$ $(a > l+4)$, and, inductively, $y_j^{[a]} = 0$ $(a > l+2j)$ for all $0 \leqq j \leqq k$. Further, $y_0^{[l]} \in \operatorname{Ker} N_1$ by assumption. Therefore, $(4)_k$ shows $y_0^{[l]} = 0$. The claim follows. □

It remains to show $(4)_k$.

3.9. First, the case $l > w$ of 3.8 $(4)_k$ is easy. In fact, the homomorphism $\operatorname{gr}_l^{M(N_1)} \to \operatorname{gr}_{l-2}^{M(N_1)}$ induced by N_1 is injective if $l > w$. Hence the kernel of N_1 on $H_{\mathbf{R}}^{[l,w+1]}$ is zero for any $l > w$. This proves the case $l > w$.

3.10. We reduce $(4)_k$ with $l \leqq w$ to the following $(5)_{k,l}$. Let $k \geqq 0$ and $l \leqq w$.

$(5)_{k,l}$ Let x_j, y_j be elements of $H_{\mathbf{R}}$ $(0 \leqq j \leqq k)$. Assume $y_j \in H_{\mathbf{R}}^{[l+2j,w+1]}$ $(0 \leqq j \leqq k)$. Assume the following condition $(*)'$, which is similar to $(*)$ in 3.7 $(3)_k$.

$(*)'$ $N_1(y_j) = x_{j-1}$ $(j > 0)$, $\hat{N}_2(y_j) = x_j$ $(j \geqq 0)$, and $x_k \in \bigcap_{j=0}^{\infty} (\hat{N}_2^j)^{-1}(\operatorname{Im} N_1^{j+1})$, that is,

$y_0 \overset{\hat{N}_2}{\mapsto} x_0 \overset{N_1}{\hookleftarrow} y_1 \overset{\hat{N}_2}{\mapsto} \cdots \overset{\hat{N}_2}{\mapsto} x_{k-1} \overset{N_1}{\hookleftarrow} y_k \overset{\hat{N}_2}{\mapsto} x_k \in \bigcap_{j=0}^{\infty}(\hat{N}_2^j)^{-1}(\operatorname{Im} N_1^{j+1})$.

The claim is this: if $y_0 \in \operatorname{Ker} N_1$, then $y_0 = 0$.

The proof of the reduction is as follows. In the following, we denote $x_j^{[a]}$ for $x_j^{[a,w-1]}$ $(0 \leqq j \leqq k, a \in \mathbf{Z})$ for simplicity. Let x_j, y_j be as in 3.8 $(4)_k$. Take the components $x_j^{[l+2j]}$ and $y_j^{[l+2j]}$ as new x_j, y_j $(0 \leqq j \leqq k)$. Then, the assumptions in $(5)_{k,l}$ are satisfied, that is,

$$N_1(y_j^{[l+2j]}) = x_{j-1}^{[l+2j-2]} \text{ for } j > 0. \tag{6}$$

$$\hat{N}_2(y_j^{[l+2j]}) = x_j^{[l+2j]} \text{ for } j \geqq 0. \tag{7}$$

$$x_k^{[l+2k]} \in \bigcap_{j=0}^{\infty}(\hat{N}_2^j)^{-1}(\operatorname{Im} N_1^{j+1}). \tag{8}$$

$$y_0^{[l]} \in \operatorname{Ker} N_1. \tag{9}$$

We check them. First, (9) is a part of the assumption in $(4)_k$. The remaining ones are deduced from the corresponding assumption in $(*)$ in 3.7 $(3)_k$, respectively. In fact, (6) is by $N_1(y_j) = x_{j-1}$ and by the fact $N_1 \in \mathfrak{g}_{\mathbf{R}}^{[-2,-2]}$. Next, (7) is by $N_2(y_j) = x_j$ and by the assumption $y_j^{[a]} = 0$ for $a > l + 2j$ in 3.8 $(4)_k$. Finally, we verify (8). Let $j \geqq 0$. Since $x_k \in M_{w-2} + (N_2^j)^{-1}(\operatorname{Im} N_1^{j+1})$, the element $N_2^j(x_k)$ belongs to $M_{w-2-2j} + \operatorname{Im} N_1^{j+1}$. We consider the $[l + 2k, w - 1 - 2j]$-component of this element. Since $N_1^{j+1} \in \mathfrak{g}_{\mathbf{R}}^{[-2j-2,-2j-2]}$, this component is in the image of N_1^{j+1}. On the other hand, since $y_k^{[a]} = 0$ for $a > l+2k$, we have $x_k^{[a]} = 0$ for $a > l+2k$. Hence the concerned component is $\hat{N}_2^j(x_k^{[l+2k]})$. Thus, $\hat{N}_2^j(x_k^{[l+2k]})$ belongs to $\operatorname{Im}(N_1^{j+1})$, which completes the verification of (8).

Now, $(5)_{k,l}$ implies $y_0^{[l]} = 0$, and 3.8 $(4)_k$ follows.

3.11. We prove 3.10 $(5)_{k,l}$ $(k \geqq 0, l \leqq w)$.

First, by the assumption in 3.10 $(*)'$, for any $a \geqq 0$, we have

$$\hat{N}_2^a(x_k) \in \operatorname{Im} N_1^{a+1}. \tag{10}$$

Take a sufficiently large m such that $\hat{N}_2^m(x_k) = 0$.

We divide into three cases: (i) $l \leqq w-m-k$; (ii) $w-m-k < l \leqq w-k$; (iii) $w-k < l \leqq w$, and prove them one by one.

3.12. We prove the case 3.11 (i) $l \leqq w-m-k$.

This case is easy. First we prove by downward induction on j that for any j with $0 \leqq j \leqq k$,

$$\hat{N}_2^{m+k-j}(x_j) = 0, \text{ and} \tag{11}$$

$$\hat{N}_2^{m+k-j+1}(y_j) = 0. \tag{12}$$

In fact, (11) for $j = k$ holds by the choice of m. Next, (11) and (12) are equivalent because $\hat{N}_2(y_j) = x_j$. Finally, assume (12) for $j > 0$. Then,

$$\hat{N}_2^{m+k-j+1}(x_{j-1}) = \hat{N}_2^{m+k-j+1}N_1(y_j) = N_1\hat{N}_2^{m+k-j+1}(y_j) = N_1(0) = 0.$$

Thus we proved (11) and (12). In particular, $\hat{N}_2^{m+k+1}y_0 = 0$. Recall that y_0 belongs to $H_{\mathbf{R}}^{[l,w+1]}$. Since

$$\hat{N}_2^{m+k+1}\colon H_{\mathbf{R}}^{[l,w+1]} \to H_{\mathbf{R}}^{[l,w-2m-2k-1]}$$

is injective because $(w+1)+(w-2m-2k-1) = 2(w-m-k) \geqq 2l$, we have $y_0 = 0$.

3.13. We proceed to the case 3.11 (ii) $w-m-k < l \leqq w-k$.

In this case, we use the full assumptions. First we prove

CLAIM. $\hat{N}_2^{w-l}(x_0) \in \operatorname{Im} N_1^{w-l+1}$.

PROOF OF CLAIM. We prove $\hat{N}_2^{w-l-j}(x_j) \in \operatorname{Im} N_1^{w-l-j+1}$ $(0 \leqq j \leqq k)$ by the downward induction on j. The case $j = 0$ is our claim. First, by 3.11 (10) with $a = w-l-k \geqq 0$, the case for $j = k$ follows. Let $j > 0$. Assume that the case for j holds. Then, $N_1(\hat{N}_2^{w-l-j}(x_j))$ belongs to $\operatorname{Im} N_1^{w-l-j+2}$, and this element coincides with

$$\hat{N}_2^{w-l-j}N_1(x_j) = \hat{N}_2^{w-l-j}N_1\hat{N}_2(y_j) = \hat{N}_2^{w-l-j+1}N_1(y_j) = \hat{N}_2^{w-l-j+1}(x_{j-1}),$$

which completes the proof of Claim. □

In the following, we prove $y_0 = 0$ by using Claim. Note that N_1-weight of $\hat{N}_2^{w-l}(x_0)$ is the same as that of y_0, which is l: $\hat{N}_2^{w-l}(x_0) = (\hat{N}_2^{w-l}(x_0))^{[l,2l-w-1]}$.

On the other hand, since $y_0 \in \operatorname{Ker} N_1$, we see that x_0 and hence $\hat{N}_2^{w-l}(x_0)$ also belong to $\operatorname{Ker} N_1$. Since $\operatorname{Ker} N_1 \cap \operatorname{Im} N_1^{w-l+1} \subset M(N_1)_{l-1}$, the element $\hat{N}_2^{w-l}(x_0) = (\hat{N}_2^{w-l}(x_0))^{[l,2l-w-1]}$ belongs to $M(N_1)_{l-1}$ by Claim. Hence, this is zero. Then, $\hat{N}_2^{w-l+1}(y_0) = \hat{N}_2^{w-l}(\hat{N}_2(y_0)) = \hat{N}_2^{w-l}(x_0) = 0$.

Recall that y_0 belongs to $H_{\mathbf{R}}^{[l,w+1]}$. Since $\hat{N}_2^{w-l+1} \colon H_{\mathbf{R}}^{[l,w+1]} \to H_{\mathbf{R}}^{[l,2l-w-1]}$ is injective because $(w+1)+(2l-w-1) = 2l$, we have $y_0 = 0$.

3.14. Finally, we prove the case 3.11 (iii) $w-k < l \leqq w$.

This case is similarly treated in the previous case. In fact, since the argument of the reduction to Claim in 3.13 does not use the assumption $w-m-k < l \leqq w-k$, it is enough to show the statement of Claim in 3.13, that is, $\hat{N}_2^{w-l}(x_0) \in \operatorname{Im} N_1^{w-l+1}$.

To see it, we prove the equality

$$\hat{N}_2^{j}(x_0) = N_1^{j+1}(y_{j+1}) \quad (0 \leqq j \leqq w-l). \tag{13}$$

(Note that $w-l+1 \leqq k$.) The case for $j = w-l$ is what we need. The case for $j = 0$ is by assumption. Assume (13) for some $j < w-l$. Then, sending (13) by $\hat{N}_2$, we see

$$\hat{N}_2^{j+1}(x_0) = \hat{N}_2 N_1^{j+1}(y_{j+1}) = N_1^{j+1}\hat{N}_2(y_{j+1}) = N_1^{j+1}(x_{j+1}) = N_1^{j+2}(y_{j+2}),$$

which is (13) for $j+1$. This completes the proof of 3.2. □

In the rest of this section, we discuss several consequences of 3.2. For the main theorem, actually we use only the following corollary, which is proved in 3.16–3.20. We return to the convention that we work over $\mathbf{Q}$ unless stated otherwise (cf. Notation and Terminology).

Corollary 3.15. *Let the notation and the assumption be as in* 3.1. *Let* $V := \operatorname{Ker} N_1 \cap \operatorname{Im} N_1 \cap \operatorname{Ker} N_2$. *Then, we have the following.*

(1) *Let* Y_j *be the increasing filtration defined by* $Y_0 = 0$ *and* $Y_{j+1} = N_1 N_2^{-1}(Y_j + V)$. *Then, for any* $n \geqq 0$, *we have*

$$\Big(\bigcup_{j=0}^{\infty} Y_j\Big) \cap (N_2 N_1^{-1})^n(0) \subset M_{w-2}.$$

(2) *Let* X_j *be the increasing filtration defined by* $X_0 = 0$ *and* $X_{j+1} = N_2^{-1}(N_1 X_j + V)$. *Then, for any* $n \geqq 0$, *we have*

$$(\bigcup_{j=0}^{\infty} X_j) \cap (N_1^{-1} N_2)^n(0) \subset N_1^{-1} M_{w-2}.$$

The Y_j and the X_j are actually increasing. This is shown by the induction on j. In fact, $Y_0 \subset Y_1$ is trivial, and the correspondence $Y \mapsto N_1 N_2^{-1}(Y + V)$ from the set of the subspaces of $H_{\mathbf{Q}}$ to itself is order-preserving. Hence $Y_j \subset Y_{j+1}$ implies $Y_{j+1} \subset Y_{j+2}$. Similarly, $X \mapsto N_2^{-1}(N_1 X + V)$ is order-preserving so that the X_j is increasing.

Before starting the proof of the corollary, we prove some lemmas.

LEMMA 3.16. *Let* V *be a* $\mathbf{Q}$*-vector space of finite dimension. Let* W *be an increasing filtration of subspaces. Let* N *be a nilpotent endomorphism of* V *preserving* W. *Assume that the relative monodromy filtration* $M(N, W)$ *exists. Then we have* $\operatorname{Ker}(N) \cap W_w \subset M(N, W)_w$ *for any* $w \in \mathbf{Z}$.

This is well-known. For the proof, see, for instance, [12] 1.2.1.3.

LEMMA 3.17. $N_2^k Y_j \subset N_1^k Y_{j-k}$ *for any* $j \geqq k \geqq 0$.

PROOF. If $k = 0$, then this is trivial. Assume that the case for k holds. Let $j \geqq k + 1$. Then,

$$\begin{aligned} N_2^{k+1} Y_j &\subset N_2 N_1^k Y_{j-k} = N_1^k N_2 Y_{j-k} \\ &= N_1^k N_2 N_1 N_2^{-1}(Y_{j-k-1} + V) \\ &= N_1^{k+1} N_2 N_2^{-1}(Y_{j-k-1} + V) \\ &\subset N_1^{k+1}(Y_{j-k-1} + V) = N_1^{k+1}(Y_{j-k-1}), \end{aligned}$$

where the last equality is by $V \subset \operatorname{Ker} N_1$. Hence, the case for $k+1$ holds. □

LEMMA 3.18. $N_2^j Y_j = 0$ *for any* $j \geqq 0$.

Proof. The case $j = 0$ is trivial. Assume that the case for j holds. Then,

$$\begin{aligned} N_2^{j+1} Y_{j+1} &= N_2^{j+1}(N_1 N_2^{-1}(Y_j + V)) \\ &= N_2^j(N_1 N_2 N_2^{-1}(Y_j + V)) \\ &\subset N_2^j N_1(Y_j + V) = N_2^j N_1 Y_j \quad (\text{by } V \subset \operatorname{Ker} N_1) \\ &= N_1 N_2^j Y_j = N_1(0) = 0, \end{aligned}$$

which is the case for $j + 1$. □

3.19. We prove 3.15 (1).

By 3.2, it is enough to show the following two inclusions for any j.

$(*)$ $Y_j \subset M_{w-1}$.

$(**)$ $Y_j \subset (N_2^k)^{-1}(\operatorname{Im} N_1^{k+1})$ for any $k \geqq 0$.

We prove $(*)$ by induction on j. The case $j = 0$ is trivial. Assume that $Y_j \subset M_{w-1}$. By the inclusion $\operatorname{Ker} N_1 \cap \operatorname{Im} N_1 \subset M(N_1)_{w-1}$ and by the fact that $M = M(N_1 + N_2)$ is the relative monodromy filtration $M(N_2, M(N_1))$, the lemma 3.16 shows that

$$V = (\operatorname{Ker} N_1 \cap \operatorname{Im} N_1) \cap \operatorname{Ker} N_2 \subset M(N_1)_{w-1} \cap \operatorname{Ker} N_2 \subset M_{w-1}.$$

Together with the induction hypothesis, we have $Y_j + V \subset M_{w-1}$. Hence $N_2^{-1}(Y_j + V) \subset M_{w+1}$, and $N_1 N_2^{-1}(Y_j + V) \subset N_1 M_{w+1} \subset M_{w-1}$, which is $(*)$ for $j + 1$.

We prove $(**)$. First assume $j > k$. Then, by the lemma 3.17, $N_2^k Y_j \subset N_1^k Y_{j-k} \subset N_1^k(\operatorname{Im} N_1) = \operatorname{Im} N_1^{k+1}$.

Next assume $j \leqq k$. Then, by the lemma 3.18, $N_2^k Y_j \subset N_2^k Y_k = 0$.

Hence, in any case, $(**)$ holds.

3.20. We prove 3.15 (2). The case $n = 0$ is trivial. Assume $n > 0$.

First we show $Y_j = N_1 X_j$ by induction on j. The case $j = 0$ is trivial. Assume $Y_j = N_1 X_j$. Then, $N_1 X_{j+1} = N_1 N_2^{-1}(N_1 X_j + V) = N_1 N_2^{-1}(Y_j + V) = Y_{j+1}$.

Using this, we see

$$\begin{aligned} N_1(X_j \cap (N_1^{-1} N_2)^n(0)) &= N_1(X_j \cap N_1^{-1}(N_2 N_1^{-1})^{n-1}(0)) \\ &\subset N_1 X_j \cap (N_2 N_1^{-1})^{n-1}(0) \\ &= Y_j \cap (N_2 N_1^{-1})^{n-1}(0) \subset M_{w-2}. \end{aligned}$$

Here the last inclusion is by 3.15 (1) already proved. Hence, $X_j \cap (N_1^{-1}N_2)^n(0) \subset N_1^{-1}M_{w-2}$. This completes the proof of 3.15. □

Now, using 3.15 just proved, we get a nice filtration on $H_{\mathbf{Q}}$ as follows.

PROPOSITION 3.21. *Let the notation and the assumption be as in* 3.1. *Let* $V := \operatorname{Ker} N_1 \cap \operatorname{Im} N_1 \cap \operatorname{Ker} N_2$ *as in* 3.15. *Then there are subspaces* $J^0, J^1, \ldots, J^n$ *of* $H_{\mathbf{Q}}$ *satisfying the following two conditions*:
(i) $N_1^{-1}N_2J^0 \subset J^0$.
(ii) $H_{\mathbf{Q}} = (H_{\mathbf{Q}} \setminus J^0) \cup \bigcup_{j=0}^{n} \big(J^j \setminus N_2^{-1}(N_1J^j + V)\big) \cup (N_1^{-1}M_{w-2} \cap M_w)$.

PROOF. Consider the sequence of subspaces $\{0\}, (N_1^{-1}N_2)(0), (N_1^{-1}N_2)^2(0), (N_1^{-1}N_2)^3(0), \ldots$. This sequence is increasing, which is seen as in the same way as we saw right after 3.15 that the Y_j is increasing. Since the whole space $H_{\mathbf{Q}}$ is finite dimensional, the sequence is eventually stable. We take as J^0 the stable subspace:

$$J^0 = (N_1^{-1}N_2)^k(0)$$

with a sufficiently large $k > 0$. Then, $N_1^{-1}N_2J^0 = (N_1^{-1}N_2)^{k+1}(0) = J^0$, and (i) is satisfied.

Next we define a decreasing sequence $J^1, J^2, J^3, \ldots$ of subspaces inductively by the formula

$$J^{j+1} = J^j \cap N_2^{-1}(N_1J^j + V).$$

This sequence is eventually stable again, and we take an $n \geqq k$ such that $J^n = J^{n+1}$.

To prove (ii), which is equivalent to $H_{\mathbf{Q}} = (H_{\mathbf{Q}} \setminus J^0) \cup \bigcup_{i=0}^{n}(J^j \setminus J^{j+1}) \cup (N_1^{-1}M_{w-2} \cap M_w)$, it is enough to show that J^n is contained in $N_1^{-1}M_{w-2} \cap M_w$, that is,

$$J^n \subset N_1^{-1}M_{w-2}, \text{ and} \tag{1}$$

$$J^n \subset M_w. \tag{2}$$

First we prove

$$J^j \subset (N_1^{-1}N_2)^{k-j}(0) + X_j \tag{3}$$

for any $j \leqq k$.

The case $j = 0$ of (3) holds by the definition of J^0. We assume (3) for j. Then, for $k \geqq j + 1$,

$$\begin{aligned} N_1 J^j + V &\subset N_1((N_1^{-1}N_2)^{k-j}(0) + X_j) + V \\ &\subset N_2(N_1^{-1}N_2)^{k-j-1}(0) + N_1 X_j + V. \end{aligned}$$

By pulling it back by N_2, we see

$$\begin{aligned} N_2^{-1}(N_1 J^j + V) &\subset (N_1^{-1}N_2)^{k-j-1}(0) + N_2^{-1}(N_1 X_j + V) \\ &= (N_1^{-1}N_2)^{k-j-1}(0) + X_{j+1}. \end{aligned}$$

Since J^{j+1} is contained in $N_2^{-1}(N_1 J^j + V)$, the inclusion (3) for $j + 1$ is proved. Hence, (3) for any $j \leqq k$ is proved.

We prove (1). We have $J^k \subset X_k$ by (3) for $j = k$. Hence,

$$\begin{aligned} J^n \subset J^k &= J^0 \cap J^k \\ &\subset (N_1^{-1}N_2)^k(0) \cap X_k \\ &\subset N_1^{-1}M_{w-2} \end{aligned}$$

by 3.15 (2). The inclusion (1) is proved.

To see (2), first, by the choice of n and by (1) just proved, we have

$$\begin{aligned} J^n &= J^n \cap N_2^{-1}(N_1 J^n + V) \\ &\subset N_1^{-1}M_{w-2} \cap N_2^{-1}(M_{w-2} + V). \end{aligned}$$

Hence, (2) is reduced to the inclusion

$$N_1^{-1}M_{w-2} \cap N_2^{-1}(M_{w-2} + V) \subset N_2^{-1}M_{w-2} = M_w,$$

which is still reduced to

(4) $$N_2(N_1^{-1}M_{w-2}) \cap (M_{w-2} + V) \subset M_{w-2}.$$

We prove (4). By 3.2, it suffices to show that

(5) $M_{w-2} + V$ is contained in M_{w-1}, and

(6) V is contained in $(N_2^j)^{-1}(\operatorname{Im} {N_1}^{j+1})$ for any $j \geqq 0$.

First, we already saw $V \subset M_{w-1}$ in 3.19. Hence (5) follows.

Next, $V \subset \operatorname{Im} N_1$ by definition. This is the case $j = 0$ of (6). Further, if $j \geqq 1$, $(N_2^j)^{-1}(\operatorname{Im} N_1^{j+1})$ contains $(N_2^j)^{-1}(0)$, which contains $\operatorname{Ker} N_2$. Hence it also contains V. Thus (6) follows, and (4) is proved. □

The next is not indispensable to prove the main theorem, but enable us to simplify the construction; see 3.24 below.

Proposition 3.22. *Under the same assumption as in* 3.15*, let* $J = (N_1^{-1}N_2)^k(0)$ $(k \gg 0)$*. Then, we have the following.*

(1) $\operatorname{Ker} N_2 \subset J$.

(2) *Let* $A = N_1 J + N_2 J$*. Then,* $A = N_2 J$*. Further, for any* $a_1, a_2 \in \mathbf{Q}^2_{\geq 0} - \{(0,0)\}$*, the homomorphism* $H_{\mathbf{Q}}/J \to H_{\mathbf{Q}}/A$ *induced by* $a_1N_1 + a_2N_2$ *is injective.*

Proof. We continue to work over $\mathbf{Q}$.

(1) Take an element $x_1 \in \operatorname{Ker} N_2$. Since $\operatorname{Ker} N_2 \subset M_w$, we have $N_1(x_1) \in N_1(M_w) \subset M_{w-2} \subset N_2(M_w)$. Hence, there is an element $x_2 \in M_w$ such that $N_1(x_1) = N_2(x_2)$. Similarly, $N_1(x_2) \in N_1(M_w) \subset N_2(M_w)$. Hence, there is an element $x_3 \in M_w$ such that $N_1(x_2) = N_2(x_3)$. Inductively, we can take a sequence $x_2, x_3, \ldots$ of elements of $H_{\mathbf{Q}}$ such that $N_1(x_j) = N_2(x_{j+1})$ for any $j \geqq 1$.

Since $H_{\mathbf{Q}}$ is finite dimensional, there is an $n \geqq 1$ and $c_1, \ldots, c_{n-1} \in \mathbf{Q}$ such that

$$x_n = c_1x_1 + \cdots + c_{n-1}x_{n-1}.$$

We may assume $x_n = 0$. In fact, for $j = 1, 2, \ldots, n$, let

$$x'_j = x_j - c_{n-1}x_{j-1} - \cdots - c_{n-j+1}x_1.$$

In particular, $x'_1 = x_1$ and $x'_n = 0$.

Then, since $N_2(x_1) = 0$, we have

$$\begin{aligned} N_2(x'_j) &= N_2(x_j) - c_{n-1}N_2(x_{j-1}) - \cdots - c_{n-j+1}N_2(x_1) \\ &= N_1(x_{j-1}) - c_{n-1}N_1(x_{j-2}) - \cdots - c_{n-j+2}N_1(x_1) - c_{n-j+1}\cdot 0 \\ &= N_1(x_{j-1} - c_{n-1}x_{j-2} - \cdots - c_{n-j+2}x_1) \\ &= N_1(x'_{j-1}) \end{aligned}$$

for any $j = 2, \ldots, n$. Hence, we can replace x_j by x'_j, and we may assume $x_n = 0$.

Then, $N_1(x_{n-1}) = N_2(x_n) = 0$, and $x_{n-1} \in N_1^{-1}(0)$. Similarly, $N_1(x_{n-2}) = N_2(x_{n-1}) \in N_2N_1^{-1}(0)$, and $x_{n-2} \in N_1^{-1}N_2N_1^{-1}(0) = (N_1^{-1}N_2)^2(0)$. Inductively, we see $x_1 \in (N_1^{-1}N_2)^{n-1}(0)$, which is contained in J.

(2) First, we prove the equality $N_1J + N_2J = N_2J$. Since $J = N_1^{-1}N_2J$, we have $N_1J = N_1(N_1^{-1}N_2J) \subset N_2J$. From this, the equality follows.

We prove that the homomorphism $H_{\mathbf{Q}}/J \to H_{\mathbf{Q}}/A$ induced by $a_1N_1 + a_2N_2$ is injective. First we assume $a_2 = 0$. Then, $(a_1N_1)^{-1}(A) = N_1^{-1}N_2J = J$, and the concerned map is injective. Next, the case where $(a_1, a_2) = (0, 1)$ is reduced to (1) because $N_2^{-1}A = N_2^{-1}N_2J = J + \operatorname{Ker} N_2 \subset J$ by (1). To reduce the other case to the case where $(a_1, a_2) = (0, 1)$, let $N_2' := a_1N_1 + a_2N_2$. It is enough to show

$$J = J' := (N_1^{-1}N_2')^k(0) \quad (k \gg 0)\text{ , and} \tag{3}$$

$$A = A' := N_2'J'. \tag{4}$$

To see (3), we prove

$$J_k := (N_1^{-1}N_2)^k(0) = J_k' := (N_1^{-1}N_2')^k(0) \tag*{$(5)_k$}$$

by induction on k. The case $k = 0$ is trivial. We assume $(5)_k$. Then, $J_{k+1}' = N_1^{-1}N_2'J_k' = N_1^{-1}(a_1N_1 + a_2N_2)J_k$ by the definition of N_2' and $(5)_k$, and it is contained in $N_1^{-1}(N_1J_k + N_2J_k) = J_k + N_1^{-1}N_2J_k = J_k + J_{k+1} \subset J_{k+1}$ because $(J_k)_k$ is increasing. By symmetry, $J_{k+1} \subset J_{k+1}'$. Hence $(5)_{k+1}$ follows.

Finally, we prove (4). By (3), $A' = N_2'J' = (a_1N_1 + a_2N_2)J$, which is contained in $N_1J + N_2J = A$. Thus $A' \subset A$. By symmetry, $A \subset A'(= N_1J + N_2'J)$. Hence (4) follows, and (2) is proved. □

Together with the results in the previous section, we obtain

PROPOSITION 3.23. *Let the notation and the assumption be as in* 3.1. *Let* $L = H_{\mathbf{Z}}$, $H = H_{\mathbf{Q}}$, *and regard* N_1, N_2 *as elements of* $\operatorname{End}(H) = \operatorname{Hom}(L, H)$. *Let* X *be as in* 2.1 *on which* L *acts. Let* σ, H_1, *and* $H_{1+\varepsilon}$ *be also as in* 2.1. *Let* $V := \operatorname{Ker} N_1 \cap \operatorname{Im} N_1 \cap \operatorname{Ker} N_2$. *Assume that there is an* $h \in H$ *such that* $\sigma \cap H_1$ *is contained in the cone generated by* $(1, 0, h+V)$.

Then there is a positive $\varepsilon_0 \leqq 1$ *such that for any positive rational number* $\varepsilon \leqq \varepsilon_0$*, there is a finite subdivision of*

$$\sigma \cap (H_1 + H_{1+\varepsilon}) = \sigma \cap \Big(\big(\mathbf{Q}_{\geq 0}(1,0) + \mathbf{Q}_{\geq 0}(1-\varepsilon, \varepsilon) \big) \times H \Big)$$

such that, for each member τ *of this subdivision and for any* $l \in L \setminus (N_1^{-1}M_{w-2} \cap M_w)$*, we have either* $l(\tau) \cap \tau = \{0\}$ *or* $l(\tau) \cap \tau = \tau \cap H_1$*.*

PROOF. Take a sequence of subspaces $J^0, J^1, \dots, J^n$ of H as in 3.21. Apply 2.3 (3) with $J = J^0$. Then we see that there is a positive $\varepsilon_0 \leqq 1$ such that for any positive rational number $\varepsilon \leqq \varepsilon_0$, there is a finite subdivision Σ of $\sigma \cap (H_1 + H_{1+\varepsilon})$ such that, for any member τ of Σ and for any $l \in L \setminus J^0$, we have $l(\tau) \cap \tau = \{0\}$. Fix such an ε.

Next, take an index j with $0 \leqq j \leqq n$. Apply 2.8 (2) with $J = J^j$. Then we see that there is a finite subdivision Σ_j of σ such that, for any member τ of Σ_j and for any $l \in (J^j \cap L) \setminus N_2^{-1}(N_1 J^j + V)$, we have either $l(\tau) \cap \tau = \{0\}$ or $l(\tau) \cap \tau = \tau \cap H_1$.

Considering a common subdivision of Σ and the pullbacks of Σ_j's ($0 \leqq j \leqq n$) to $H_1 + H_{1+\varepsilon}$, we see that there is a finite subdivision Υ of $\sigma \cap (H_1 + H_{1+\varepsilon})$ such that for any member τ of Υ and for any $l \in (L \setminus J^0) \cup \bigcup_{j=0}^{n} \big((J^j \cap L) \setminus N_2^{-1}(N_1 J^j + V) \big)$, we have either $l(\tau) \cap \tau = \{0\}$ or $l(\tau) \cap \tau = \tau \cap H_1$.

By the condition 3.21 (ii), $(L \setminus J^0) \cup \bigcup_{j=0}^{n} \big((J^j \cap L) \setminus N_2^{-1}(N_1 J^j + V) \big)$ contains $L \setminus (N_1^{-1}M_{w-2} \cap M_w)$.

Hence, for any $l \in L \setminus (N_1^{-1}M_{w-2} \cap M_w)$, we have either $l(\tau) \cap \tau = \{0\}$ or $l(\tau) \cap \tau = \tau \cap H_1$. $\square$

REMARK 3.24. In this proposition, actually, we can take $\varepsilon_0 = 1$ by 3.22.

4. Combinatorial Lemmas

In this section, we add more lemmas to be used to care for the situation of type (B) (cf. 1.17).

The first is well-known.

Recall that a cone σ is said to be *simplicial* if it is spanned by $\dim \sigma$ vectors. A subdivision of a cone or a fan is said to be *simplicial* if it consists of simplicial cones.

Lemma 4.1. *Let Σ be a finite fan in a vector space. Then there is a finite subdivision Σ' of Σ consisting of simplicial cones such that the set of 1-faces of Σ' coincides with that of Σ.*

For the proof, see, for example, [1].

Lemma 4.2. *Let the notation and the assumption be as in* 2.5. *Assume the following two conditions.*

(1) *The image of $p\colon \sigma \to V$ coincides with the support of Σ.*

(2) *For any face σ_1 of σ and for any element τ of Σ, the intersection $p(\sigma_1) \cap \tau$ is a face of τ.*

Then, Σ coincides with the set of the cones of the form $p(\sigma')$, where σ' is an element of Σ'.

Proof. Let $\tau \in \Sigma$. By the assumption (1), $\tau = p(p^{-1}(\tau))$. Since $p^{-1}(\tau)$ is an element of Σ', we have one inclusion. To see the other inclusion, let $\sigma' \in \Sigma'$, and it is enough to show that $p(\sigma') \in \Sigma$. By definition of Σ', there are a face σ_1 of σ and an element τ of Σ such that $\sigma' = \sigma_1 \cap p^{-1}(\tau)$. Then, $p(\sigma') = p(\sigma_1) \cap \tau$. By the assumption (2), the right hand side of the last equality is a face of τ so that it belongs to Σ. □

The next is a key observation. The proof is also not hard.

Lemma 4.3. *Let the situation be as in* 2.1. *Let $M \subset H$ be a subspace of H.*

Then there is a positive $\varepsilon_0 \leqq 1$ such that for any positive rational number $\varepsilon \leqq \varepsilon_0$, there is a finite subdivision Σ of the cone

$$\sigma \cap (H_1 + H_{1+\varepsilon}) = \sigma \cap \Big(\big(\mathbf{Q}_{\geq 0}(1,0) + \mathbf{Q}_{\geq 0}(1-\varepsilon, \varepsilon) \big) \times H \Big)$$

satisfying the following two conditions.

(a) *Any 1-cone in Σ is contained either in H_1 or in $H_{1+\varepsilon}$.*

(b) *For any element of Σ, its image in $\mathbf{Q}^2_{\geq 0} \times (H/M)$ is simplicial.*

Proof. Consider the projection

$$p\colon \mathbf{Q}^2_{\geq 0} \times H \to \mathbf{Q}^2_{\geq 0} \times (H/M) =: \overline{X}.$$

The first step of the proof is similar to the proof of 4.3.8 of [13]: For each face σ_1 of σ, subdivide its image $p(\sigma_1)$ in $\overline{X}$ into finitely many sharp cones. Let B be the set of all these sharp cones. For each $\tau \in B$, take a finite fan Σ_τ in $\overline{X}$ such that $\bigcup_{\tau' \in \Sigma_\tau} \tau' = \overline{X}$ and $\tau \in \Sigma_\tau$. Consider the set Σ_0 of all cones of the form $\bigcap_{\tau \in B} \sigma(\tau)$, where $\sigma(\tau)$ is an element of Σ_τ for each $\tau \in B$. Let Σ' be the set of the elements of Σ_0 which are contained in some element of B. Then Σ' is a fan whose support is $p(\sigma)$. For each face σ_1 of σ, there is a subfan of Σ' whose support coincides with $p(\sigma_1)$.

This fan Σ' satisfies the following property.

(1) For any element $\sigma' \in \Sigma'$ and any face σ_1 of σ, the intersection $\sigma' \cap p(\sigma_1)$ is a face of σ'.

In fact, since $p(\sigma_1)$ is the union of some elements σ'_j of Σ', the cone $\sigma' \cap p(\sigma_1)$ is the union of faces $\sigma' \cap \sigma'_j$ of σ', so $\sigma' \cap p(\sigma_1)$ itself is a face of σ'.

Note that any subdivision of Σ' still satisfies (1). This is easily seen from the fact that for any subcone σ'' of σ', the subset $\sigma'' \cap p(\sigma_1)$ of σ'' is the intersection of the face $\sigma' \cap p(\sigma_1)$ of σ' and σ''.

Note also that for any positive rational number $\varepsilon \leqq 1$, the pullback Σ'_ε of Σ' to $\overline{C} := \big(\mathbf{Q}_{\geq 0}(1,0) + \mathbf{Q}_{\geq 0}(1-\varepsilon, \varepsilon)\big) \times (H/M)$ satisfies a similar condition:

$(1)_\varepsilon$ For any element $\sigma' \in \Sigma'_\varepsilon$ and any face σ_1 of $\sigma \cap (H_1 + H_{1+\varepsilon})$, the intersection $\sigma' \cap p(\sigma_1)$ is a face of σ'.

This is seen as follows. By 2.4, σ_1 is the intersection of a face σ_2 of σ and a face c of $H_1 + H_{1+\varepsilon}$. We have $p^{-1}p(c) = c$. On the other hand, by the definition of Σ'_ε, the cone σ' is the intersection of an element τ of Σ' and a face c' of $\overline{C}$. Then, the concerned set $\sigma' \cap p(\sigma_1)$ coincides with

$$\begin{aligned}\tau \cap c' \cap p(\sigma_2 \cap c) &= \tau \cap c' \cap p(\sigma_2 \cap p^{-1}(p(c))) \\ &= \tau \cap c' \cap p(\sigma_2) \cap p(c) \\ &= (\tau \cap p(\sigma_2)) \cap (c' \cap p(c)).\end{aligned}$$

Since $\tau \cap p(\sigma_2)$ is a face of τ by (1), and since $c' \cap p(c)$ is a face of c', its intersection is a face of $\tau \cap c' = \sigma'$.

Since there are only finitely many 1-cones in Σ', we can take a positive $\varepsilon_0 \leqq 1$ such that for any positive rational number $\varepsilon \leqq \varepsilon_0$, the pullback Σ'_ε additionally satisfies the following condition:

(a)$'$ Any 1-cone in Σ'_ε is contained either in $\mathbf{Q}_{\geq 0}(1,0) \times (H/M)$ or in $\mathbf{Q}_{\geq 0}(1-\varepsilon, \varepsilon) \times (H/M)$.

Further, for each ε, by 4.1, there is a finite simplicial subdivision Σ''_ε of Σ'_ε which still satisfies $(1)_\varepsilon$ and (a)$'$ (with Σ'_ε replaced by Σ''_ε).

We prove that the pullback Σ of Σ''_ε to $\sigma \cap (H_1 + H_{1+\varepsilon})$ satisfies (a) and (b).

First, by 4.2, $(1)_\varepsilon$ implies the following (2).

(2) The set of the images in $\overline{C}$ of all cones of Σ coincides with Σ''_ε.

Together with (a)$'$, we get (a).

Second, again by (2), the image in $\overline{C}$ of each cone in Σ belongs to Σ''_ε, and simplicial. Thus (b) is satisfied, which proves the lemma. □

The next is also for the situation of type (B).

Lemma 4.4. *Let H be a finite dimensional vector space. Let V_1, V_2 be two subspaces. Let C_j be a polytope in V_j $(j = 1, 2)$. Let σ_j be the cone in $\mathbf{Q}^2 \times H$ generated by (e_j, C_j), where e_j is the j-th unit vector $(j = 1, 2)$. Let $\sigma = \sigma_1 + \sigma_2$.*

(1) *Assume that V_j is generated by the set $\{c - d \mid c, d \in C_j\}$ as a vector space $(j = 1, 2)$ and that σ is simplicial. Then, we have $V_1 \cap V_2 = \{0\}$.*

(2) *Assume that $V_1 \cap V_2 = \{0\}$. Let Σ_j be a finite subdivision of σ_j for $j = 1, 2$. Then,*

$$\Sigma := \{\tau_1 + \tau_2 \mid \tau_1 \in \Sigma_1, \tau_2 \in \Sigma_2\}$$

is a subdivision of $\sigma_1 + \sigma_2$. All the 1-faces of Σ are contained either in σ_1 or in σ_2.

Proof. (1) Let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be the vertices of C_1 and of C_2 respectively. Then, $(1,0,v_1), \ldots, (1,0,v_m), (0,1,w_1), \ldots, (0,1,w_n)$ are vertices of the simplicial cone $\sigma_1 + \sigma_2$, and hence, are linearly independent.

Let $v \in V_1 \cap V_2$. Then, by assumption, v is written as $v = \sum_{j=1}^m c_j v_j$ with $c_j \in \mathbf{Q}$ and $\sum_{j=1}^m c_j = 0$, and also written as $v = \sum_{j=1}^n d_j w_j$ with $d_j \in \mathbf{Q}$ and $\sum_{j=1}^n d_j = 0$. Hence,

$$\sum_{j=1}^m c_j v_j = \sum_{j=1}^n d_j w_j.$$

From this, we have

$$\sum_{j=1}^{m} c_j(1, 0, v_j) = \sum_{j=1}^{n} d_j(0, 1, w_j).$$

By the linear independency, all c_j and d_j are zero so that $v = 0$. Therefore, $V_1 \cap V_2 = \{0\}$.

(2) We may assume that $V_1 + V_2 = H$. Then, the natural isomorphism

$$(\mathbf{Q} \times \{0\} \times V_1) \times (\{0\} \times \mathbf{Q} \times V_2) \xrightarrow{\simeq} \mathbf{Q}^2 \times H; (x_1, x_2) \mapsto x_1 + x_2$$

induces an isomorphism from the product cone $\sigma_1 \times \sigma_2$ to $\sigma_1 + \sigma_2$, and Σ is nothing but the image by this map of the product fan $\Sigma_1 \times \Sigma_2$. This proves the first assertion.

For the second assertion, note that any 1-face of Σ is either a 1-face of Σ_1 or that of Σ_2. Hence it is contained in $\sigma_1 \cup \sigma_2$, which completes the proof. □

5. Admissibility

In this section, we return to the mixed situation in Section 1. We gather in this section a few consequences of admissibility to be used in the proof of the main results.

Let the situation be as in 1.6. Let σ be an admissible nilpotent cone in $\mathfrak{g}_{\mathbf{Q}}$.

Convention. Below, we adopt the following general convention: For any element $N \in \mathfrak{g}_{\mathbf{Q}}$, we denote its image in $\mathfrak{g}'_{\mathbf{Q}}$ with the prime: N'.

We identify $\mathfrak{g}_{\mathbf{Q}}$ with $\mathfrak{g}'_{\mathbf{Q}} \times H'_{\mathbf{Q}}$ as a $\mathbf{Q}$-vector space.

Proposition 5.1. $\sigma \cap (\{0\} \times H'_{\mathbf{Q}}) = \{0\}$.

Proof. Let e be the standard generator 1 of $\mathbf{Q} \subset H'_{\mathbf{Q}} \oplus \mathbf{Q} = H_{\mathbf{Q}}$ of weight 0. In general, $N \in \mathfrak{g}_{\mathbf{Q}}$ is zero if and only if $N' = 0$ and $N(e) = 0$.

Let $N \in \sigma \cap (\{0\} \times H'_{\mathbf{Q}})$. Since N' is already zero, it is enough to show that $N(e)$ is zero. By the admissibility (1.4), $N(e) \in M(0)_{-2} = W_{-2} = \{0\}$. (See 3.3 for $M(-)$.) Hence $N(e) = 0$ and $N = 0$. □

Proposition 5.2. *Let N' be an element of σ'. Let H'_1 be the fiber of the projection $\mathfrak{g}_{\mathbf{Q}} \to \mathfrak{g}'_{\mathbf{Q}}$ over N', which we identify with $H'_{\mathbf{Q}}$ via $N \leftrightarrow N(e)$ (e as in the proof of* 5.1*). Let τ be the polytope $\sigma \cap H'_1$. Then, we have the following.*

(1) *For $N_1, N_2 \in \tau$, the difference of N_1 and N_2 regarded as an element of $H'_{\mathbf{Q}}$ belongs to* $\operatorname{Ker} N'$.

(2) *τ is contained in* $\operatorname{Im} N'$.

Proof. (1) Let $N_1, N_2 \in \tau$. Since $N_1N_2 = N_2N_1$, we have

$$N'(N_2(e)) = N'(N_1(e)).$$

Hence $N_2(e) - N_1(e) \in \operatorname{Ker} N'$.

(2) Let $N \in \tau$. Take an element $e + h$ ($h \in H'_{\mathbf{Q}}$) of $M(N, W)_0$. By the admissibility (1.4), $N(e) + N(h) \in M(N, W)_{-2} \cap W_{-1} = (W(N')[1])_{-1} \subset \operatorname{Im} N'$. Hence, $N(e) \in \operatorname{Im} N'$ and $\tau \subset \operatorname{Im} N'$. □

In the rest of this section, we assume that $\dim \sigma' = 2$. Fix a set of generators N'_1, N'_2 of σ'. Let $H'_{(1,0)}$ be the fiber of the projection $\mathfrak{g}_{\mathbf{Q}} \to \mathfrak{g}'_{\mathbf{Q}}$ over N'_1, which we identify with $H'_{\mathbf{Q}}$ via $N \leftrightarrow N(e)$. Let τ be the polytope $\sigma \cap H'_{(1,0)}$.

Proposition 5.3. *Assume that σ is not contained in τ. Then, the subset $S = \{t_1 - t_2 \mid t_1, t_2 \in \tau\}$ of $H'_{\mathbf{Q}}$ is contained in $V := \operatorname{Ker} N'_1 \cap \operatorname{Im} N'_1 \cap \operatorname{Ker} N'_2$.*

Proof. First, by 5.2 (1), we have $S \subset \operatorname{Ker} N'_1$.

Similarly, by 5.2 (2), we have $\tau \subset \operatorname{Im} N'_1$. Hence, $S \subset \operatorname{Im} N'_1$.

Let $N_1, L_1 \in \tau$. Recall that e is the standard generator 1 of $\mathbf{Q}$ of weight 0. We show $L_1(e) - N_1(e) \in \operatorname{Ker} N'_2$, which completes the proof. Since σ is not contained in τ, there is an N whose image in σ' is of the form $aN'_1 + bN'_2$ with $b \neq 0$. Since $NL_1 = L_1N$, we have $N(L_1(e)) = L_1(N(e))$, and

$$(aN'_1 + bN'_2)(L_1(e)) = N'_1(N(e)).$$

Similarly, since $NN_1 = N_1N$, we have

$$(aN'_1 + bN'_2)(N_1(e)) = N'_1(N(e)).$$

Hence, $(aN_1' + bN_2')(L_1(e)) = (aN_1' + bN_2')(N_1(e))$, which implies

$$L_1(e) - N_1(e) \in \operatorname{Ker}(aN_1' + bN_2').$$

Since we already know that $L_1(e) - N_1(e) \in \operatorname{Ker} N_1'$, we deduce $L_1(e) - N_1(e) \in \operatorname{Ker} N_2'$. Thus, we proved $S \subset \operatorname{Ker} N_2'$. □

6. Proofs of Main Results

Here we prove 1.8 and 1.14.

6.1. Let Γ_u be the kernel of the natural projection $\Gamma \to \Gamma'$. Then Γ_u is naturally isomorphic to the additive group $H'_{\mathbf{Z}}$, via the correspondence $\gamma \leftrightarrow \gamma(e)$. Here e is the standard generator $1 \in \mathbf{Z} \subset H'_{\mathbf{Z}} \oplus \mathbf{Z} = H_{\mathbf{Z}}$. We identify Γ_u and $H'_{\mathbf{Z}}$ via this isomorphism. The group Γ is isomorphic to a semi-direct product of Γ_u and Γ'.

We begin the proof of 1.8.

6.2. First we claim that, in the statement of 1.8, we can replace "Γ" with "Γ_u".

We prove this claim till the end of this paragraph 6.2. We may assume $\dim \sigma' = 1$. Let N' be the generator of the monoid $\{N'' \in \sigma' \mid \exp(N'') \in \Gamma'\}$, which is isomorphic to $\mathbf{N}$. Fix a point N of σ whose image in $\mathfrak{g}'_{\mathbf{Q}}$ is N' and let $h := N(e)$.

Since N' is nilpotent, there is an integer $M > 0$ such that all the elements $h, \frac{N'h}{2}, \frac{N'^2h}{6}, \ldots, \frac{N'^{k-1}h}{k!}, \ldots$ belong to the lattice $\frac{1}{M}H'_{\mathbf{Z}}$.

Thus, we have

(∗) $\frac{N'^{k-1}h}{k!} \in \frac{1}{M}H'_{\mathbf{Z}}$ for any $k \geqq 1$.

If we replace the lattice $H'_{\mathbf{Z}}$ with $\frac{1}{M}H'_{\mathbf{Z}}$, the groups Γ and Γ_u become larger. We may assume that the (larger) Γ_u-version of 1.8 for $\frac{1}{M}H'_{\mathbf{Z}}$ holds. Hence, it is enough to show that the action on σ of any element of the original Γ coincides with that of some element of the larger Γ_u because a subset of a fan is a fan if it is closed under the operation of taking a face.

Because the original Γ is a semi-direct product of Γ_u and Γ', it is enough to prove that the action on σ of any element of the original Γ' coincides with that of some element of the larger Γ_u.

We identify the fiber of $\sigma \hookrightarrow \mathfrak{g}_{\mathbf{Q}} \to \mathfrak{g}'_{\mathbf{Q}}$ over N' with a subset of $H'_{\mathbf{Q}}$ via $N \leftrightarrow N(e)$. Then the action of any element $\exp(nN')$ ($n \in \mathbf{Z}$) of Γ' is

$$h+x \mapsto e^{nN'}(h+x) = h+x+(e^{nN'}-1)(h) = h+x+N'\Big(\sum\nolimits_{k \geqq 1} n^k \tfrac{N'^{k-1}h}{k!}\Big),$$

where $h + x$ ($x \in H'_{\mathbf{Q}}$) is any element of the concerned fiber. Here we use the fact that $N'(x) = 0$, which is by 5.2 (1).

Since $\sum_{k \geqq 1} n^k \frac{N'^{k-1}h}{k!}$ is in $\frac{1}{M}H'_{\mathbf{Z}}$ by $(*)$, this action is certainly realized by that of the corresponding element of the larger Γ_u, which completes the proof of our claim.

Remark 6.3. Here we explain another proof of the claim in 6.2. (This may be simpler, but we prefer the above because it is easier to be generalized.) We use the above notation. Instead of 5.2 (1), we use 5.2 (2). By 5.2 (2), there is an $h' \in H'_{\mathbf{Q}}$ such that $h = N'h'$. Take an integer $M > 0$ such that $h' \in \frac{1}{M}H'_{\mathbf{Z}}$. Let γ be the element of the larger Γ_u corresponding to h'. Then, replacing σ by $\gamma^{-1}\sigma$, we may assume that $h' = h = 0$. In this case, the action of Γ' is trivial, and our claim follows.

6.4. Now 1.8 is direct from 2.7. In fact, we take $H'_{\mathbf{Q}}$ as H there, $\Gamma_u \cong H'_{\mathbf{Z}}$ as L, and σ as τ. By 5.1, we have $\sigma \cap (\{0\} \times H'_{\mathbf{Q}}) = \{0\}$. Hence we can apply 2.7, and we may assume that for any element $\gamma \in \Gamma_u$, either $\gamma(\sigma) \cap \sigma = \{0\}$ or the action of γ on σ is trivial. Then, all the translations by Γ_u of all the faces of σ form a fan.

6.5. We prove the main theorem 1.14 in some steps till the end of this section.

Similarly to the case of 1-dimension in 6.2, we first claim that, in the statement of 1.14, we can replace "Γ" with "Γ_u".

We prove this claim till the end of this paragraph 6.5. Let $N'_1, \dots, N'_m$ be a set of generators of the fs monoid $\{N'' \in \sigma' \mid \exp(N'') \in \Gamma'\}$. For each j with $1 \leqq j \leqq m$, take a point N_j of σ whose image in $\mathfrak{g}'_{\mathbf{Q}}$ is N'_j. Let $h_j := N_j(e)$.

Since N'_j is nilpotent, the set

$$S := \left\{ \frac{N'_{j_1} \cdots N'_{j_{k-1}}(h_{j_k})}{k!} \;\middle|\; k \geqq 1, 1 \leqq j_1, \dots, j_k \leqq m \right\}$$

is finite, and there is an integer $M > 0$ such that the lattice $\frac{1}{M} H'_{\mathbf{Z}}$ contains this finite set.

Thus, we have

(∗) $S \subset \frac{1}{M} H'_{\mathbf{Z}}$.

Similarly to 6.2, it is enough to show that the action on σ of any element of Γ' coincides with that of some element of the larger Γ_u because a subset of a weak fan is a weak fan if it is closed under the operation of taking a face.

We see that, for any element N of σ, there are non-negative rational numbers a_j ($1 \leqq j \leqq m$) and $x \in \bigcap \mathrm{Ker}\,(N'_j) \subset H'_{\mathbf{Q}}$ such that $N' = \sum a_j N'_j$ (cf. the Convention in §5) and that $N(e) = \sum a_j h_j + x$. In fact, N' is written as $\sum a_j N'_j$. Consider the element $\sum a_j N_j$. This is in σ. Hence, by 5.3, $x := N(e) - (\sum a_j N_j)(e) = N(e) - \sum a_j h_j$ is annihilated by N'_k for any k.

Then the action of any element $\exp(L')$ ($L' = \sum m_l N'_l, m_l \in \mathbf{Z}$) of Γ' on the $H'_{\mathbf{Q}}$-component of N is described as

$$\begin{aligned}\textstyle\sum a_j h_j + x \mapsto \sum a_j h_j + x + (e^{L'} - 1)(\sum a_j h_j) \\ = \textstyle\sum a_j h_j + x + \sum_{k \geqq 1} \frac{L'^{k-1}}{k!} L'(\sum a_j h_j).\end{aligned}$$

But, we have

$$\begin{aligned}L'(\textstyle\sum_j a_j h_j) &= (\textstyle\sum_l m_l N'_l)(\sum_j a_j h_j) \\ &= \textstyle\sum_{j,l} a_j m_l (N'_l(h_j)) \\ &= \textstyle\sum_{j,l} a_j m_l (N'_j(h_l)) \quad (\text{by } N_l N_j = N_j N_l) \\ &= (\textstyle\sum_j a_j N'_j)(\sum_l m_l h_l).\end{aligned}$$

Hence the action is

$$\textstyle\sum a_j h_j + x \mapsto \sum a_j h_j + x + (\sum_j a_j N'_j)\big(\sum_{k \geqq 1} \frac{L'^{k-1}}{k!}(\sum_l m_l h_l)\big).$$

Since $\sum_{k \geqq 1} \frac{L'^{k-1}}{k!}(\sum_l m_l h_l)$ is in $\frac{1}{M} H'_{\mathbf{Z}}$ by (∗), this action is certainly realized by that of the corresponding element of the larger Γ_u, which completes the proof of our claim.

6.6. In the rest, we prove the Γ_u-version of 1.14. To prove it, we can replace σ by each member of a finite subdivision of σ and replace σ' by the image of the member. Further, if the image of the member in $\mathfrak{g}'_{\mathbf{Q}}$ is of one dimension, such a member can be treated by 1.8. Hence, in the replacement, it is enough to consider only the member whose image in $\mathfrak{g}'_{\mathbf{Q}}$ is 2-dimensional.

Take a set of generators N'_1, N'_2 of σ'. In the following, let $H'_{\mathbf{Z}}$ act on $\mathfrak{g}_{\mathbf{Q}}$ via the isomorphism $H'_{\mathbf{Z}} \cong \Gamma_u$ in 6.1. Let H_j $(j = 1, 2)$ be the pullback of the cone generated by N'_j in $\mathfrak{g}'_{\mathbf{Q}}$ by the projection $\mathfrak{g}_{\mathbf{Q}} \to \mathfrak{g}'_{\mathbf{Q}}$.

First we use 3.23, and prove that we may assume

(1) For any $l \in H'_{\mathbf{Z}} \smallsetminus (N_1'^{-1}M_{-3} \cap M_{-1})$, we have $l(\sigma) \cap \sigma \subset H_1$.

We let $w = -1$, and take H' as H in 3.23. Let $V = \operatorname{Ker} N'_1 \cap \operatorname{Im} N'_1 \cap \operatorname{Ker} N'_2$. By 5.3, there is an $h \in H'_{\mathbf{Q}}$ such that $\sigma \cap H_1$ is contained in the cone generated by $(1, 0, h + V)$. Hence, we can apply 3.23 and, if we replace N'_2 by $(1-\varepsilon)N'_1 + \varepsilon N'_2$ for a sufficiently small $\varepsilon > 0$, we may assume (1).

In general, we can work by a compactness argument as follows. For each rational number a with $0 \leqq a < 1$, take $(1-a)N'_1 + aN'_2$ as N_1 in 3.23 and $\frac{1-a}{2}N'_1 + \frac{1+a}{2}N'_2$ as N_2 in 3.23. (Note that then the condition $W(N_1 + N_2) = W(N_2)$ in 3.1 is satisfied.) Then we can apply 3.23 in virtue of 5.3. Apply 3.23, and let $b_a = (1-\varepsilon_0)a + \varepsilon_0\frac{1+a}{2}$, where ε_0 is what the proposition gives. Let $b_1 = 1$.

Similarly, for each rational number a with $0 < a \leqq 1$, take $(1-a)N'_1 + aN'_2$ as N_1 there and $(1-\frac{a}{2})N'_1 + \frac{a}{2}N'_2$ as N_2 there. Apply 5.3 and 3.23, and let $c_a = (1-\varepsilon_0)a + \varepsilon_0\frac{a}{2}$. Let $c_0 = 0$.

Consider the set of the intervals $I_a := [c_a, b_a]$ $(a \in [0, 1])$.

We prove that there is a sequence

$$e_0 = d_0 = 0 < e_1 < d_1 < e_2 < \cdots < e_k < d_k = e_{k+1} = 1$$

such that for each $j = 0, 1, \ldots, k$, the interval $[e_j, e_{j+1}]$ is contained in I_{d_j}.

Consider the set S of all sequences $e_0 = d_0 = 0 < e_1 < d_1 < e_2 < \cdots < e_{k-1} < d_{k-1} < e_k$ with various k such that for each $j = 0, 1, \ldots, k-1$, the interval $[e_j, e_{j+1}]$ is contained in I_{d_j}. Let e be the supremum of such e_k. Then, since $c_e < e$, there is a sequence $e_0 = d_0 = 0 < e_1 < d_1 < e_2 < \cdots < e_{k-1} < d_{k-1} < e_k$ belonging to S with $c_e < e_k$. By replacing e_k by $\max\{c_e, (d_{k-1} + e_k)/2\}$ and by defining $d_k = e$ and $e_{k+1} = b_e$, we obtain another sequence in S whose largest member e_{k+1} is strictly larger than e (a contradiction) unless $e = 1$. Thus we see $e = 1$ and $d_k = e_{k+1} = 1$.

Subdivide σ' into the $2k$ cones with their faces spanned by two elements $(1-d_j)N'_1+d_jN'_2$ and $(1-e_l)N'_1+e_lN'_2$ with $l=j$ or $l=j+1$. Subdivide σ by their pullbacks, and replace σ with each pullback and further replace it with each member of the subdivision which 3.23 gives. Then, by construction, we see that the condition (1) is satisfied. (We take $(1-d_j)N'_1+d_jN'_2$ as new N'_1 and $(1-e_l)N'_1+e_lN'_2$ as new N'_2.)

Note that this property (1) is preserved by further subdivision and by the replacement of N'_2.

6.7. Let C_j $(j=1,2)$ be the inverse image of N'_j by $\sigma\hookrightarrow\mathfrak{g}_{\mathbf{Q}}\to\mathfrak{g}'_{\mathbf{Q}}$, which we regard as a subset in $H'_{\mathbf{Q}}$. Let V_j be the subspace generated by the set $\{c-d\,|\,c,d\in C_j\}$.

Next we enhance the argument in 6.6 and show that we may assume further that the following two conditions:

(2) σ is generated by $(\sigma\cap H_1)\cup(\sigma\cap H_2)$, and

(3) $(V_1+M_{-3})\cap(V_2+M_{-3})=M_{-3}$.

To see it, by the compactness argument in 6.6, we can work around N'_1, that is, it suffices to show that after replacing N'_2 by $(1-\varepsilon)N'_1+\varepsilon N'_2$ for any sufficiently small $\varepsilon\in\mathbf{Q}_{>0}$, and after subdividing σ, (2) and (3) are satisfied.

We use 4.3. By this lemma with $M=M_{-3}$, we may assume the above (2) and the following $(*)$.

$(*)$ The image $\overline{\sigma}$ of σ in $\sigma'\times(H'_{\mathbf{Q}}/M_{-3})$ is simplicial.

By 4.4 (1), $(*)$ implies that the intersection of the images of V_1 and V_2 in $H'_{\mathbf{Q}}/M_{-3}$ is $\{0\}$, and hence, we get the above (3).

Note that the property (3) is preserved by further subdivision. Hereafter we always assume (3). (We will not replace N'_2 any more.)

6.8. Hereafter we always assume 6.6 (1) and 6.7 (3).

Next, by 4.1, without loss of (2) in 6.7, we may assume that σ is simplicial.

Then, by 4.4 (1), the intersection of V_1 and V_2 is $\{0\}$. Hence, by 4.4 (2), a pair of finite subdivisions of $\sigma\cap H_1$ and of $\sigma\cap H_2$ induce a subdivision of σ.

Apply 2.7 by taking $H=H'_{\mathbf{Q}}$, $L=H'_{\mathbf{Z}}$, $N_1=N'_1$, and $\tau=\sigma\cap H_1$. Then, it gives a subdivision of $\sigma\cap H_1$.

Apply 2.7 by taking $H = H'_{\mathbf{Q}}$, $L = H'_{\mathbf{Z}}$, $N_1 = N'_2$, and $\tau = \sigma \cap H_2$. Then, it gives a subdivision of $\sigma \cap H_2$.

By 4.4 (2), these two subdivisions induce a subdivision of σ, and, after replacing σ with each member of this subdivision, we may assume further the following two conditions.

(4) In the case $l \in H'_{\mathbf{Z}} \smallsetminus N_1'^{-1}(0)$ we have $l(\sigma \cap H_1) \cap (\sigma \cap H_1) = \{0\}$, and in the case $l \in H'_{\mathbf{Z}} \cap N_1'^{-1}(0)$ the action of l is trivial on $\sigma \cap H_1$, and

(5) In the case $l \in H'_{\mathbf{Z}} \smallsetminus N_2'^{-1}(0)$ we have $l(\sigma \cap H_2) \cap (\sigma \cap H_2) = \{0\}$, and in the case $l \in H'_{\mathbf{Z}} \cap N_2'^{-1}(0)$ the action of l is trivial on $\sigma \cap H_2$.

In this process, (2) in 6.7 is still preserved by the last statement of 4.4 (2).

6.9. Thus, we may assume (1)–(5) in 6.6–6.8. Under these assumptions, we prove the following (6), which completes the proof of the main theorem.

(6) Let $l \in H'_{\mathbf{Z}}$. Let σ_1 and σ_2 be faces of σ. Assume that $l(\sigma_1)$ and σ_2 have a common interior point x and that there is an $F \in \check{D}$ such that both $(l(\sigma_1), F)$ and (σ_2, F) generate nilpotent orbits. Then, $l(\sigma_1) = \sigma_2$.

First assume that $l \notin N_1'^{-1}(M_{-3}) \cap M_{-1}$. Then, by (1) in 6.6, $l(\sigma) \cap \sigma = l(\sigma \cap H_1) \cap (\sigma \cap H_1)$. By (4) in 6.8, this cone coincides with $\{0\}$ if $N'_1(l) \neq 0$, and coincides with $\sigma \cap H_1$ if $N'_1(l) = 0$. In both cases, this is a common face of $l(\sigma)$ and of σ. Hence, $l(\sigma_1) \cap \sigma_2$ is a common face of $l(\sigma_1)$ and of σ_2. Since $l(\sigma_1)$ and σ_2 have a common interior point, $l(\sigma_1) = l(\sigma_1) \cap \sigma_2 = \sigma_2$.

Thus we may and will assume $l \in N_1'^{-1}(M_{-3}) \cap M_{-1}$ in the following. For $j = 1, 2$, let

$$\tau_j := \sigma_j \cap H_1 \quad \text{and} \quad \upsilon_j := \sigma_j \cap H_2.$$

Then, by (2) in 6.7,

$$\sigma_j = \tau_j + \upsilon_j.$$

Further, there are interior points t_j of τ_j and u_j of υ_j such that

$$x = l(t_1 + u_1) = t_2 + u_2.$$

Since $l(t_1 + u_1) = t_1 + u_1 + aN'_1(l) + bN'_2(l)$, where aN'_1 and bN'_2 are the images of t_1 and u_1 in $\mathfrak{g}'_{\mathbf{Q}}$ respectively, we have

$$t_1 - t_2 + aN'_1(l) = u_2 - u_1 - bN'_2(l).$$

Since $N'_1(l) \in M_{-3}$, the left hand side belongs to $V_1 + M_{-3}$. Similarly, since $N'_2(l) \in N'_2M_{-1} = M_{-3}$, the right hand side belongs to V_2+M_{-3}. Hence, (3) in 6.7 implies that both sides are in M_{-3}. On the other hand, by Griffiths transversality, both sides are in $F^{-1} \cap \overline{F}^{-1}$. In fact, take $e+h \in F^0$ ($h \in H'_{\mathbf{C}}$). Since $(l(\sigma_1), F)$ generates a nilpotent orbit, we have $l(t_1) + N'_1(h) \in F^{-1}$. Similarly, since (σ_2, F) generates a nilpotent orbit, we have $t_2 + N'_1(h) \in F^{-1}$. Hence, $l(t_1) - t_2 \in F^{-1}$. Since this element is real, it is also in $\overline{F}^{-1}$. This element is the left hand side of the above equality.

Since $F^{-1} \cap \overline{F}^{-1} \cap M_{-3} = \{0\}$, both sides are zero. Here we use the fact that (M, F) is a mixed Hodge structure.

Now we have the equality

$$t_1 + aN'_1(l) = t_2.$$

The left hand side of this belongs to $l(\sigma \cap H_1)$ and the right hand side belongs to $\sigma \cap H_1$. Hence, the condition (4) in 6.8 implies that, if $N'_1(l)$ is not zero, then both sides of this equality are zero. Then, t_1 is also zero by 5.1. Since t_1 is an interior point of τ_1, the cone τ_1 is $\{0\}$ on which l acts trivially. On the other hand, if $N'_1(l)$ is zero, then l trivially acts on τ_1 again. Thus, in any case, l acts on τ_1 trivially. Similarly, the equality

$$u_1 + bN'_2(l) = u_2$$

and the condition (5) in 6.8 imply that l acts on υ_1 trivially. Hence, l acts on $\sigma_1 = \tau_1 + \upsilon_1$ trivially, and $l(\sigma_1) = \sigma_1$. Since $l(\sigma_1) = \sigma_1$ and σ_2 are faces of σ, and since they have a common interior point, they coincide: $l(\sigma_1) = \sigma_1 = \sigma_2$, which completes the proof of (6) and hence the proof of 1.14, that is, that all the translations of σ with their faces form a weak fan. □

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(Received March 29, 2012)
(Revised October 4, 2012)

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