Clifford Modules, Finite-Dimensional Approximation and Twisted K-Theory

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Abstract. A twisted version of Furuta’s generalized vector bundle provides a finite-dimensional model of twisted K-theory. We generalize this fact involving actions of Clifford algebras. As an application, we show that an analogy of the Atiyah-Singer map for the generalized vector bundles is bijective. Furthermore, a finite-dimensional model of twisted K-theory with coefficients \( \mathbb{Z}/p \) is given.

1. Introduction

Furuta’s generalized vector bundle [9], which we call a vectorial bundle in this paper, arises naturally as a geometric object approximating a family of Fredholm operators. This means that there is a natural homomorphism of groups

\[ \alpha : [X, \mathcal{F}(\mathcal{H})] \longrightarrow KF(X), \]

where \([X, \mathcal{F}(\mathcal{H})]\) is the group of homotopy classes of continuous maps from a topological space \( X \) to the space \( \mathcal{F}(\mathcal{H}) \) of Fredholm operators on a separable Hilbert space \( \mathcal{H} \), and \( KF(X) \) is the group of homotopy classes of \((\mathbb{Z}/2\text{-graded})\) vectorial bundles on \( X \). Usual vector bundles are examples of vectorial bundles, so that there exists a natural homomorphism from the \( K \)-group \( K(X) \) to \( KF(X) \). It is shown [9] that this homomorphism \( K(X) \rightarrow KF(X) \) is an isomorphism on a compact Hausdorff space \( X \). In this case, the \( K \)-group of \( X \) is also realized as \([X, \mathcal{F}(\mathcal{H})]\), as is well-known [1]. Hence the homomorphism \( \alpha \), coming from a “finite-dimensional approximation”, turns out to be bijective.

In [10], the construction above is generalized to

\[ \alpha : K^\tau(X) \longrightarrow KF^\tau(X), \]

where \( K^\tau(X) \) stands for the twisted \( K \)-group [5, 7] twisted by a principal bundle \( \tau \) over \( X \) whose structure group is the projective unitary group of \( \mathcal{H} \),

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and $KF^\tau(X)$ consists of homotopy classes of $\tau$-twisted vectorial bundles on $X$. The homomorphism $\alpha$ again comes from an idea of finite-dimensional approximation of a family of Fredholm operators, and turns out to be bijective for any CW complex $X$. It should be noticed that a general description of a class in $K^\tau(X)$ usually involves some infinite-dimensional objects. The isomorphism above provides a way to describe $K^\tau(X)$ in terms of finite-dimensional objects.

The aim of this paper is to generalize the isomorphisms $\alpha$ involving actions of Clifford algebras: let $Cl(n) = Cl(\mathbb{R}^n)$ be the Clifford algebra associated to $\mathbb{R}^n$ equipped with the standard metric, $\mathcal{H}_n$ a separable infinite-dimensional $\mathbb{Z}/2$-graded Hilbert space which contains each irreducible $\mathbb{Z}/2$-graded module of $Cl(n)$ infinitely many, and $\mathcal{F}_n$ the non-contractible connected component of the space of self-adjoint Fredholm operators on $\mathcal{H}_n$ which are degree 1 (i.e. switching the gradings) and anti-commute with the actions of generators of $Cl(n)$. As is known [6], $\mathcal{F}_n$ classifies the $K$-cohomology $K^{-n}$, so that $[X, \mathcal{F}_n] \cong K^{-n}(X)$. On the other hand, vectorial bundles with $Cl(n)$-actions are also introduced in [9]. Their homotopy classes constitute a group $KF_{Cl(n)}(X)$, providing a model of the $K$-cohomology $K^{-n}(X)$. As before, we can construct a natural homomorphism

$$\alpha : [X, \mathcal{F}_n] \longrightarrow KF_{Cl(n)}(X).$$

Taking a “twist” into account, we also have a natural homomorphism

$$K^{-n}(X) \longrightarrow KF^\tau_{Cl(n)}(X).$$

Then we will prove:

**Theorem 1.** For any twist $\tau$ on a CW complex $X$, the homomorphism $K^{-n}(X) \rightarrow KF^\tau_{Cl(n)}(X)$ is bijective.

The idea of the proof of Theorem 1 is parallel to that in [10]: we lift $K^{-n}(X)$ and $KF^\tau_{Cl(n)}(X)$ to certain generalized cohomology theories, and compare these theories by using a natural transformation induced from $\alpha$. Then the problems reduce to the case of a single point: The key fact that the natural transformation is bijective in this case again relies on a result of Furuta [9].

The main result in [10] allows us to describe classes in $K^{-n}(X)$ by using ordinary twisted vectorial bundles on $X \times [0, 1]^n$, whereas Theorem
1 provides a different way to describe classes in $K^{\tau-n}(X)$. The equivalence of these two options is useful in studying $K^{\tau-n}(X)$, and will be applied to a construction of twisted differential $K$-cohomology in a forthcoming paper.

A more simple application of Theorem 1 is the bijectivity of a homomorphism

$$\text{AS} : KF_{Cl(n)}^{\tau}(X) \rightarrow KF_{Cl(n-1)}^{\tau-1}(X),$$

whose construction is similar to that of the homotopy equivalence $\mathcal{F}_n \rightarrow \Omega F_{n-1}$ of Atiyah-Singer [6]. Another application of Theorem 1 is an introduction of a finite-dimensional model of twisted mod $p$ $K$-theory, or twisted $K$-theory with coefficients in $\mathbb{Z}/p$, based on twisted vectorial bundles with Clifford action.

The organization of this paper is as follows: In Section 2, we recall Clifford modules [2, 8], and the classifying space $\mathcal{F}_n$ of the $K$-cohomology constructed out of the space of Fredholm operators [6]. In Section 3, we briefly review a definition of twisted $K$-theory, and summarize axioms of the induced cohomology theory. In Section 4, we introduce twisted vectorial bundles with Clifford action, generalizing an idea in [9]. The definition is quite parallel to that of twisted vectorial bundles without Clifford action [10]. In this section, we also summarize axioms of certain cohomology theory induced from $KF_{Cl(n)}^{\tau}(X)$: its proof is skipped, because the argument in [10] is straightly generalized to the present case. Then, in Section 5, we introduce the homomorphisms $\alpha$ and prove our main theorem (Theorem 5.2), from which Theorem 1 is derived as a corollary. In the proof of the main theorem, we refrain from reproducing the same argument as that in [10], and only details a proof of a key proposition. Finally, in Section 6, we introduce the counterpart of the Atiyah-Singer map to twisted vectorial bundles with Clifford action, and prove its bijectivity. Our finite-dimensional model of twisted mod $p$ $K$-theory is also provided in this section.

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2. Review of Clifford Modules and Fredholm Operators

2.1. Clifford modules

For $n > 0$, we let $Cl(n) = Cl(\mathbb{R}^n)$ be the Clifford algebra associated to the standard $\mathbb{R}^n$, that is, the algebra over $\mathbb{R}$ generated by the generators $e_i$,
$(i = 1, \ldots, n)$ subject to the relation $e_i e_j + e_j e_i = -2\delta_{i,j}$.

By a (unitary) module of $Cl(n)$, we mean a $\mathbb{Z}/2$-graded Hermitian vector space $V = V^0 \oplus V^1$ over $\mathbb{C}$ equipped with an algebra homomorphism $\rho : Cl(n) \to \text{End}_\mathbb{C}(V)$ such that $\rho(e_i) : V \to V$, $(i = 1, \ldots, n)$ are skew-Hermitian maps of degree 1. (As a convention of this paper, we put a hat on the symbol of the direct sum to distinguish the grading of a $\mathbb{Z}/2$-graded vector space $V$: the even part appears on the left of $\oplus$ and the odd part on the right.)

Finite-rank irreducible modules of $Cl(n)$ are classified as follows: if $n$ is odd, then $Cl(n)$ has essentially a unique irreducible module $\Delta_n$; if $n$ is even, then $Cl(n)$ has essentially two distinct irreducible modules $\Delta_+^n$, $\Delta_-^n$. One irreducible module is obtained by switching the grading of the other. These irreducible modules are distinguished by the action of the volume element, that is,

$$\rho_{\Delta_+^n}(e_1 \cdots e_n) = \pm (\sqrt{-1})^{n/2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with respect to the decomposition $\Delta_\pm^n = (\Delta_\pm^n)^0 \oplus (\Delta_\pm^n)^1$. For convenience, we put $\Delta_n = \Delta_+^n \oplus \Delta_-^n$.

Under the natural isomorphism $Cl(n) \otimes Cl(n') \cong Cl(n+n')$, a $Cl(n)$-module $V$ and a $Cl(n')$-module $V'$ give a $Cl(n+n')$-module $V \otimes V'$, where the tensor product is taken in the $\mathbb{Z}/2$-graded sense. If $n$ or $n'$ is even, and both $V$ and $V'$ are irreducible, then $V \otimes V'$ is also irreducible. In particular, $\Delta_+^{2m} \otimes \Delta_+^{2m'} \cong \Delta_+^{2(m+m')}$.

The above behaviour of irreducible modules under tensor products implies:

**Lemma 2.1 ([8, 9]).** Let $n$ and $m$ be positive integers.

1. The category of $Cl(n)$-modules and that of $Cl(n+2m)$-modules are equivalent under the functor assigning $V \otimes \Delta_+^{2m}$ to a $Cl(n)$-module $V$ and $f \otimes \text{id}$ to a homomorphism $f$ of $Cl(n)$-modules.

2. The functor induces an isomorphism $H_{\mathbb{Z}/2}(V) \cong H_{\mathbb{Z}/2}(V \otimes \Delta_+^{2m})$, where $H_{\mathbb{Z}/2}(V)$ is the following vector space introduced to any $Cl(n)$-module $V$:

$$H_{\mathbb{Z}/2}(V) = \left\{ \gamma : V \to V \middle| \begin{array}{l} \text{degree 1, Hermitian,} \\
\rho_V(e_i)\gamma + \gamma\rho_V(e_i) = 0 \text{ for } i = 1, \ldots, n \end{array} \right\}. \]
and $H_{\mathbb{Z}/2}(V \otimes \Delta^+_m)$ is defined similarly.

Notice that this lemma also makes sense in the case of $n = 0$. (In this case, we forget Clifford actions, and regard a $Cl(0)$-module $V$ as just a $\mathbb{Z}/2$-graded Hermitian vector space, and $H_{\mathbb{Z}/2}(V)$ as the space of degree 1 Hermitian maps on $V$.)

For $n = 1, 2$, we describe the irreducible $Cl(n)$-modules explicitly. In the case of $n = 1$, the irreducible module is $\Delta_1 = \mathbb{C} \oplus \mathbb{C}$ and $\rho_{\Delta_1}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In the case of $n = 2$, the irreducible $Cl(2)$-module $\Delta_2^+$ is $\Delta_2^+ = \mathbb{C} \oplus \mathbb{C}$ and

$$
\rho_{\Delta_2^+}(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \rho_{\Delta_2^+}(e_2) = \begin{pmatrix} 0 & -\sqrt{-1} \\ -\sqrt{-1} & 0 \end{pmatrix}.
$$

We easily see $H_{\mathbb{Z}/2}(\Delta_1) = \mathbb{C}$, with its basis $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $H_{\mathbb{Z}/2}(\Delta_2^+) = 0$.

### 2.2. Fredholm operators

For $n > 0$, let $H_n$ be a separable infinite-dimensional $\mathbb{Z}/2$-graded Hilbert space which contains each irreducible $Cl(n)$-modules infinitely many. A particular construction of $H_n$ is $H_n = H \otimes \Delta_n$, where $H$ is an ungraded separable Hilbert space of infinite-dimension. We also let $\tilde{F}_n$ be the space of degree 1 self-adjoint Fredholm operators on $H_n$ anti-commuting with the actions of $e_i \in Cl(n)$, $(i = 1, \ldots, n)$:

$$
\tilde{F}_n = \left\{ A : H_n \rightarrow H_n \middle| \begin{array}{l}
\text{degree 1, Fredholm, } A^* = A \\
Ae_i + e_i A = 0 \text{ for } i = 1, \ldots, n
\end{array} \right\}.
$$

We topologize this space by the operator norm. In the case that $n$ is odd, $\tilde{F}_n$ has three connected components [6]. Two of them are contractible, and we will denote the remaining non-trivial component by $F_n(H_n) = F_n$. In the case that $n$ is even, we put $F_n(H_n) = F_n = \tilde{F}_n$. In the case of $n = 0$, we also define $F_0 = \tilde{F}_0$ to be the space of degree 1 self-adjoint Fredholm operators on a separable infinite-dimensional $\mathbb{Z}/2$-graded Hilbert space.

Notice that there exists a homotopy equivalence [6]:

$$
\text{AS} : F_n(H_n) \longrightarrow \Omega F_{n-1}(H_n),
$$
where $\Omega \mathcal{F}_{n-1}(\mathcal{H}_n)$ stands for the space of maps $\tilde{A} : [-1, 1] \to \mathcal{F}_{n-1}(\mathcal{H}_n)$ such that $\tilde{A}(\pm 1)$ are invertible. For $A \in \mathcal{F}_n(\mathcal{H}_n)$, an explicit description of the map $AS(A) : [-1, 1] \to \mathcal{F}_{n-1}(\mathcal{H}_n)$ is

$$AS(A)(t) = A + \sqrt{-1}te_n.$$  

Notice also that there is a homeomorphism $\mathcal{F}_n \cong \mathcal{F}_{n+2m}$, ([6]). This homeomorphism $\mathcal{F}_n(\mathcal{H}_n) \to \mathcal{F}_{n+2m}(\mathcal{H}_{n+2m})$ is given by $A \mapsto A \otimes \text{id}$ under the identification $\mathcal{H}_{n+2m} \cong \mathcal{H}_n \otimes \Delta^+_{2m}$.

Because of the homotopy equivalence $\mathcal{F}_n \to \Omega \mathcal{F}_{n-1}$, the space $\mathcal{F}_n$ provides a model of the classifying space of the $K$-theory of degree $-n$. Put differently, we may define the $K$-group $K^{-n}(X)$ of a CW complex $X$ to be the homotopy classes of continuous maps from $X$ to $\mathcal{F}_n$. Under this realization of $K^{-n}$, the homeomorphism $\mathcal{F}_n \cong \mathcal{F}_{n+2n}$ induces the Bott periodicity.

\textbf{Remark 1.} As a model of the classifying space of $K^{-n}$, the space of Fredholm operators $\mathcal{F}_n$ is chosen in this paper. We can also choose the model provided in [5]. With this choice, the subsequent argument is still valid.

3. Twisted $K$-Theory

3.1. Twisted $K$-Theory

To twist usual topological $K$-theory, we will use a principal bundle whose structure group is a projective unitary group: For a separable infinite-dimensional Hilbert space $H$, the projective unitary group $PU(H)$ is defined by the quotient $PU(H) = U(H)/U(1)$. We topologize $PU(H)$ by using the operator norm topology on $U(H)$. Then, for $n \geq 0$, the group $PU(H)$ acts on $\mathcal{F}_n(\mathcal{H}_n) = \mathcal{F}_n(H \otimes \Delta_n)$ by conjugation, and we can associate a fiber bundle $\mathcal{F}_n(\tau) = \tau \times_{PU(H)} \mathcal{F}_n$ to a given principal $PU(H)$-bundle $\tau$ over a space $X$. (In the case that we employ the model of the classifying space of $K$-theory in [5], we give $PU(H)$ a compact open topology.)

Let $\Gamma(X, \mathcal{F}_n(\tau))$ be the space of sections of this fiber bundle $\mathcal{F}_n(\tau) \to X$. For a section $A \in \Gamma(X, \mathcal{F}_n(\tau))$, we define the support of $A$ to be the closure of the set of points $x \in X$ at which $A_x$ is not invertible:

$$\text{Supp}(A) = \{x \in X | A_x \text{ is not invertible}\}.$$
For a closed subspace $Y \subset X$, we denote by $\Gamma(X, Y, \mathcal{F}_n(\tau))$ the set of sections $A \in \Gamma(X, \mathcal{F}_n(\tau))$ such that $\text{Supp}(A) \cap Y = \emptyset$.

Now, we define $K^\tau_{Cl(n)}(X, Y)$ to be the homotopy classes of $A \in \Gamma(X, Y, \mathcal{F}_n(\tau))$. Two sections $A_0, A_1 \in \Gamma(X, Y, \mathcal{F}_n(\tau))$ are said to be homotopic if there exists a section $\hat{A} \in \Gamma(X \times I, Y \times I, \mathcal{F}_n(\tau) \times I)$ such that $\hat{A}|_{X \times \{i\}} = A_i$, $(i = 0, 1)$. (We denote by $I = [0, 1]$ the unit interval.) A choice of an identification $\mathcal{H}_n \oplus \mathcal{H}_n \cong \mathcal{H}_n$ makes $K^\tau_{Cl(n)}(X, Y)$ into an abelian group. In view of the homotopy equivalence $\mathcal{F}_n \to \Omega \mathcal{F}_{n-1}$, the group $K^\tau_{Cl(n)}(X, Y)$ is isomorphic to $K^{\tau-n}(X, Y) = K^\tau(X \times I^n, Y \times I^n \cup X \times \partial I^n)$, the $\tau$-twisted $K$-group [5, 7] of degree $-n$.

### 3.2. Axioms of twisted $K$-theory

To lift the group $K^\tau_{Cl(n)}(X, Y)$ into a generalized cohomology theory, we introduce a category $\hat{\mathcal{C}}$ as follows: an object in $\hat{\mathcal{C}}$ is a triple $(X, Y, \tau)$ consisting of a CW pair $(X, Y)$ and a principal $PU(H)$-bundle $\tau \to X$. A morphism $(f, F) : (X', Y', \tau') \to (X, Y, \tau)$ in $\hat{\mathcal{C}}$ consists of a continuous map $f : X' \to X$ such that $f(Y') \subset Y$ and a bundle isomorphism $F : \tau' \to f^*\tau$ covering the identity of $X'$.

For $(X, Y, \tau) \in \hat{\mathcal{C}}$, we define the group $K^{\tau-j}_{Cl(n)}(X, Y)$ by

$$K^{\tau-j}_{Cl(n)}(X, Y) = \begin{cases} K^{\tau \times I^j}_{Cl(n)}(X \times I^j, Y \times I^j \cup X \times \partial I^j), & (j \geq 0) \\ K^{\tau+j}_{Cl(n)}(X, Y), & (j < 0) \end{cases}$$

A morphism $(f, F) : (X', Y', \tau') \to (X, Y, \tau)$ clearly induces a homomorphism $(f, F)^* : K^{\tau-j}_{Cl(n)}(X, Y) \to K^{\tau-j}_{Cl(n)}(X', Y')$. Thus, the assignment $(X, Y, \tau) \to K^{\tau-j-n}(X, Y)$ gives rise to a functor from $\hat{\mathcal{C}}$ to the category of abelian groups. Since $K^{\tau-j}_{Cl(n)}(X, Y) \cong K^{\tau-j-n}(X, Y)$, we see the following properties from [7]:

**Proposition 3.1.** The functors assigning $K^{\tau+j}_{Cl(n)}(X, Y)$ to $(X, Y, \tau) \in \hat{\mathcal{C}}$, $(j \in \mathbb{Z})$ have the following properties:

1. *(Homotopy axiom)* If $(f_i, F_i) : (X', Y', \tau') \to (X, Y, \tau)$, $(i = 0, 1)$ are homotopic, then the induced homomorphisms coincide: $(f_0, F_0)^* = (f_1, F_1)^*$. 
(2) (Excision axiom) For subcomplexes $A, B \subset X$, the inclusion map induces the isomorphism:

$$K^\tau_{Cl(n)} (A \cup B, B) \cong K^\tau_{Cl(n)} (A, A \cap B).$$

(3) (Exactness axiom) There is the natural long exact sequence:

$$\cdots \to K^{\tau+j-1}_{Cl(n)} (Y) \xrightarrow{\delta_{j-1}} K^{\tau+j}_{Cl(n)} (X, Y) \to K^{\tau+j}_{Cl(n)} (X) \to K^{\tau+j}_{Cl(n)} (Y) \xrightarrow{\delta_j} \cdots.$$  

(4) (Additivity axiom) For any family $\{(X_\lambda, Y_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda}$ in $\hat{\mathcal{C}}$, the inclusion maps $X_\lambda \to \bigsqcup_{\lambda} X_\lambda$ induce the natural isomorphism:

$$K^{\tau_j}_{Cl(n)} (\bigsqcup_{\lambda} X_\lambda, \bigsqcup_{\lambda} Y_\lambda) \cong \prod_{\lambda} K^{\tau_{\lambda,j}}_{Cl(n)} (X_\lambda, Y_\lambda).$$

We notice that the proof of the exactness axiom uses the Bott periodicity

$$K^{\tau-j}_{Cl(n)} (X, Y) \cong K^{\tau-j-2}_{Cl(n)} (X, Y).$$

This isomorphism is given by multiplying a generator of $K^{-2}(pt) = K^0 (D^2, S^1) \cong \mathbb{Z}$. (For $k > 0$, we denote by $D^k$ the unit disk in $\mathbb{R}^k$, and by $S^{k-1} = \partial D^k$ the unit sphere.) In general, there exists a multiplication

$$K^{\tau-j}_{Cl(n)} (X, Y) \times K^{\tau-k}_{Cl(m)} (X, Y') \longrightarrow K^{\tau-j-k}_{Cl(n+m)} (X, Y \cup Y').$$

This is induced from the map $\mathcal{F}_n (\mathcal{H}_n) \times \mathcal{F}_m (\mathcal{H}_m) \to \mathcal{F}_{n+m} (\mathcal{H}_n \otimes \mathcal{H}_m)$ given by $(A, A') \mapsto A \otimes 1 + 1 \otimes A'$, where the tensor products are taken in the graded sense.

4. Vectorial Bundles with Clifford Actions

4.1. Definitions

Definition 4.1. Let $n$ be a positive integer and $X$ a topological space. For a subset $U \subset X$, we define the category $\mathcal{H}F_{Cl(n)} (U)$ as follows. An object in $\mathcal{H}F_{Cl(n)} (U)$ is a pair $(E, h)$ consisting of a finite-rank $\mathbb{Z}/2$-graded Hermitian vector bundle $E \to U$ equipped with bundle maps $e_i : E \to E$, $(i = 1, \ldots, n)$ of degree 1 satisfying $e_i e_j + e_j e_i = -2 \delta_{i,j}$ and of a Hermitian
map $h : E \to E$ of degree 1 satisfying $hc_i + e_i h = 0$, ($i = 1, \ldots, n$). The homomorphisms in $\mathcal{H}_C_{Cl(n)}(U)$ are

$$\text{Hom}_{\mathcal{H}_C_{Cl(n)}(U)}((E, h), (E', h')) = \left\{ \phi : E \to E' \middle| \text{degree 0, } \phi h = h' \phi, \quad e_i \phi = \phi e_i \text{ for } i = 1, \ldots, n \right\} / \equiv;$$

where the meaning of the equivalence relation $\phi \equiv \phi'$ is as follows:

For each point $x \in U$, there are a positive number $\mu > 0$ and an open subset $V \subset U$ containing $x$ such that: for all $y \in V$ and $\xi \in (E, h)_y, \mu$, we have $\phi(\xi) = \phi'(\xi)$.

In the above, we put

$$(E, h)_y, \mu = \bigoplus_{\lambda < \mu} \text{Ker}(h^2_y - \lambda) = \bigoplus_{\lambda < \mu} \{ \xi \in E_y | h^2_y \xi = \lambda \xi \}.$$ 

We will just write $\phi$ to mean the homomorphism in the category $\mathcal{H}_C_{Cl(n)}(U)$ represented by $\phi : (E, h) \to (E', h')$.

**Definition 4.2.** Let $X$ be a topological space, $\tau \to X$ a principal $PU(H)$-bundle, and $U \subset X$ a subset.

(a) We define the category $P^\tau(U)$ as follows. The objects in $P^\tau(U)$ consist of sections $s : U \to \tau|_U$. The morphisms in $P^\tau(U)$ are defined by

$$\text{Hom}_{P^\tau(U)}(s, s') = \{ g : U \to U(\mathcal{H}) | s' \pi(g) = s \},$$

where $\pi : PU(H) \to U(H)$ is the projection. The composition of morphisms is defined by the pointwise multiplication.

(b) We define the category $\mathcal{H}F^\tau_{Cl(n)}(U)$ as follows. The objects in $\mathcal{H}F^\tau_{Cl(n)}(U)$ are the same as those in $P^\tau(U) \times \mathcal{H}F_{Cl(n)}(U)$:

$$\text{Obj}(\mathcal{H}F^\tau_{Cl(n)}(U)) = \text{Obj}(P^\tau(U)) \times \text{Obj}(\mathcal{H}F_{Cl(n)}(U)).$$

The homomorphisms in $\mathcal{H}F^\tau_{Cl(n)}(U)$ are defined by:

$$\text{Hom}_{\mathcal{H}F^\tau_{Cl(n)}(U)}((s, (E, h)), (s', (E', h')))$$

$$= \text{Hom}_{P^\tau(U)}(s, s') \times \text{Hom}_{\mathcal{H}F_{Cl(n)}(U)}((E, h), (E', h')) / \sim,$$
where the equivalence relation \(\sim\) identifies \((g, \phi)\) with \((g\zeta, \phi\zeta)\) for any \(U(1)\)-valued map \(\zeta : U \to U(1)\).

**Definition 4.3.** For a positive integer \(n\) and a principal \(PU(H)\)-bundle \(\tau\) over a topological space \(X\), we define the category \(\mathcal{K} \mathcal{F}_{\text{Cl}(n)}^\tau(X)\) as follows.

1. An object \((\mathcal{U}, \mathcal{E}_\alpha, \Phi_{\alpha\beta})\) in \(\mathcal{K} \mathcal{F}_{\text{Cl}(n)}^\tau(X)\) consists of an open cover \(\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathfrak{A}}\) of \(X\), objects \(\mathcal{E}_\alpha\) in \(\mathcal{H} \mathcal{F}_{\text{Cl}(n)}^\tau(U_\alpha)\), and homomorphisms \(\Phi_{\alpha\beta} : \mathcal{E}_\beta \to \mathcal{E}_\alpha\) in \(\mathcal{H} \mathcal{F}_{\text{Cl}(n)}^\tau(U_{\alpha\beta})\) such that:

\[
\Phi_{\alpha\beta} \Phi_{\beta\alpha} = 1 \quad \text{in} \quad \mathcal{H} \mathcal{F}_{\text{Cl}(n)}^\tau(U_{\alpha\beta});
\]

\[
\Phi_{\alpha\beta} \Phi_{\beta\gamma} = \Phi_{\alpha\gamma} \quad \text{in} \quad \mathcal{H} \mathcal{F}_{\text{Cl}(n)}^\tau(U_{\alpha\beta\gamma}),
\]

where \(U_{\alpha\beta} = U_\alpha \cap U_\beta\) and \(U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma\) as usual. We call an object in the category \(\mathcal{K} \mathcal{F}_{\text{Cl}(n)}^\tau(X)\) a \(\tau\)-twisted \(\text{Cl}(n)\)-vectorial bundle over \(X\).

2. A homomorphism \((\{U'_\alpha\}, \mathcal{E}'_\alpha, \Phi'_{\alpha'\beta'})\) \(\to\) \((\{U_\alpha\}, \mathcal{E}_\alpha, \Phi_{\alpha\beta})\) consists of homomorphisms \(\Psi_{\alpha\alpha'} : \mathcal{E}'_{\alpha'} \to \mathcal{E}_\alpha\) in \(\mathcal{H} \mathcal{F}_{\text{Cl}(n)}^\tau(U_{\alpha\alpha'}\cap U'_{\alpha'})\) such that the following diagrams commute in \(\mathcal{H} \mathcal{F}_{\text{Cl}(n)}^\tau(U_{\alpha\beta} \cap U'_{\alpha'} \cap U'_{\beta'})\) and \(\mathcal{H} \mathcal{F}_{\text{Cl}(n)}^\tau(U_{\alpha\beta} \cap U'_{\alpha'} \cap U'_{\beta'})\), respectively.

In the case of \(n = 0\), we can identify \(\mathcal{K} \mathcal{F}_{\text{Cl}(0)}^\tau(X) = \mathcal{K} \mathcal{F}^\tau(X)\) with the category of \(\tau\)-twisted vectorial bundles ([10]) on \(X\). Also, in the case that \(\tau\) is the trivial \(PU(H)\)-bundle \(\tau = X \times PU(H)\), we can identify \(\mathcal{K} \mathcal{F}_{\text{Cl}(n)}^\tau(X) = \mathcal{K} \mathcal{F}_{\text{Cl}(0)}(X)\) with the category of \((\mathbb{Z}/2\text{-graded})\) \(\text{Cl}(n)\)-vectorial bundles ([9]) on \(X\).

By definition, we can specify an object \(\mathcal{E} \in \mathcal{K} \mathcal{F}_{\text{Cl}(n)}^\tau(X)\) by the data

\[
(\mathcal{U}, s_\alpha, g_{\alpha\beta}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})
\]

consisting of:
• an open cover $\mathcal{U} = \{ U_\alpha \}$ of $X$;

• local sections $s_\alpha : U_\alpha \to \tau|_{U_\alpha}$, which define the transition functions $\tilde{g}_{\alpha \beta} : U_{\alpha \beta} \to PU(H)$ by $s_\alpha \tilde{g}_{\alpha \beta} = s_\beta$;

• functions $g_{\alpha \beta} : U_{\alpha \beta} \to U(H)$ such that $\pi \circ g_{\alpha \beta} = \tilde{g}_{\alpha \beta}$, which define $z_{\alpha \beta \gamma} : U_{\alpha \beta \gamma} \to U(1)$ by $g_{\alpha \beta} g_{\beta \gamma} = z_{\alpha \beta \gamma} g_{\alpha \gamma}$;

• $\mathbb{Z}/2$-graded Hermitian vector bundles $E_\alpha \to U_\alpha$ of finite rank whose fibers are $Cl(n)$-modules by means of bundle maps $e_i : E_\alpha \to E_\alpha$, $(i = 1, \ldots, n)$ of degree 1 satisfying $e_i e_j + e_j e_i = -2\delta_{i,j}$.

• Hermitian maps $h_\alpha : E_\alpha \to E_\alpha$ of degree 1 such that $h_\alpha e_i + e_i h_\alpha = 0$ for all $i = 1, \ldots, n$;

• maps $\phi_{\alpha \beta} : E_\beta|_{U_{\alpha \beta}} \to E_\alpha|_{U_{\alpha \beta}}$ of degree 0 such that $h_\alpha \phi_{\alpha \beta} = \phi_{\alpha \beta} h_\beta$, $e_i \phi_{\alpha \beta} = \phi_{\alpha \beta} e_i$ for $i = 1, \ldots, n$ and:

$$
\phi_{\alpha \beta} \phi_{\beta \gamma} \equiv 1 \quad \text{on } U_{\alpha \beta};
$$

$$
\phi_{\alpha \beta} \phi_{\beta \gamma} \equiv z_{\alpha \beta \gamma} \phi_{\alpha \gamma} \quad \text{on } U_{\alpha \beta \gamma}.
$$

The support of $E$ is defined by

$$
\text{Supp}(E) = \{ x \in X | (h_\alpha)_x \text{ is not invertible for some } \alpha \}.
$$

For a subspace $Y \subset X$, we define $K\mathcal{F}^\tau_{Cl(n)}(X, Y)$ to be the full subcategory consisting of the objects $E \in K\mathcal{F}^\tau_{Cl(n)}(X)$ such that $\text{Supp}(E) \cap Y = \emptyset$.

Now, for $(X, Y, \tau) \in \hat{\mathcal{C}}$, we define $KF^\tau_{Cl(n)}(X, Y)$ to be the homotopy classes of $\tau$-twisted $Cl(n)$-vectorial bundles $E \in K\mathcal{F}^\tau_{Cl(n)}(X, Y)$: we say $E_0$ and $E_1$ are homotopic if there exists $E \in K\mathcal{F}^\tau_{Cl(n)}(X \times I, Y \times I)$ such that $E|_{X \times \{ i \}}$ is isomorphic to $E_i$ in $K\mathcal{F}^\tau_{Cl(n)}(X, Y)$ for each $i = 0, 1$. In the same way as in the case without $Cl(n)$-actions [9, 10], $KF^\tau_{Cl(n)}(X, Y)$ gives rise to an abelian group.

4.2. Axioms

For $(X, Y, \tau) \in \hat{\mathcal{C}}$ and $j \geq 0$, we put:

$$
KF^\tau_{Cl(n)}(X, Y) = KF^\tau_{Cl(n)}(X \times I^j, Y \times I^j \cup X \times \partial I^j).
$$
We also put $KF_{\text{Cl}(n+1)}(X,Y) = KF_{\text{Cl}(n)}(X,Y)$. Then, the argument in [10] applies to Cl$(n)$-vectorial bundles, and we have a “cohomology theory”:

**Proposition 4.4.** The functors assigning $KF_{\text{Cl}(n)}(X,Y)$ to $(X,Y,\tau) \in \hat{C}$, $(j \leq 1)$ have the following properties:

1. (Homotopy axiom) If $(f_i, F_i) : (X', Y', \tau') \to (X, Y, \tau)$, $(i = 0, 1)$ are homotopic, then the induced homomorphisms coincide: $(f_0, F_0)^* = (f_1, F_1)^*$.

2. (Excision axiom) For subcomplexes $A, B \subset X$, the inclusion map induces the isomorphism:

$$KF_{\text{Cl}(n+1)}(A \cup B) \cong KF_{\text{Cl}(n)}(A, A \cap B).$$

3. (“Exactness” axiom) There is the natural complex of groups:

$$\cdots \delta_{-1} : KF_{\text{Cl}(n+0)}(X,Y) \to KF_{\text{Cl}(n)}(X) \to KF_{\text{Cl}(n)}(Y) \delta_0 : KF_{\text{Cl}(n)}(X,Y).$$

This complex is exact except at the term $KF_{\text{Cl}(n)}(Y)$.

4. (Additivity axiom) For a family $\{(X_\lambda, Y_\lambda, \tau_\lambda)\}_{\lambda \in \Lambda}$ in $\hat{C}$, the inclusion maps $X_\lambda \to \bigsqcup \lambda X_\lambda$ induce the natural isomorphism:

$$KF_{\text{Cl}(n)}(\bigsqcup \lambda (X_\lambda, Y_\lambda), (\bigsqcup \lambda X_\lambda, \bigsqcup \lambda Y_\lambda)) \cong \prod \lambda KF_{\text{Cl}(n)}(X_\lambda, Y_\lambda).$$

Notice that, in constructing $\delta_0$ above, we use the multiplication

$$KF_{\text{Cl}(n)}(X,Y) \times KF(D^2, S^1) \to KF_{\text{Cl}(n)}(X \times D^2, Y \times D^2 \cup X \times S^1).$$

In general, we can define a multiplication

$$\otimes : KF_{\text{Cl}(n)}(X,Y) \times KF_{\text{Cl}(m)}(X,Y') \to KF_{\text{Cl}(n+m)}(X, Y \cup Y').$$

This is induced from the functor $\otimes : \mathcal{H} \mathcal{F}_{\text{Cl}(n)}(U) \times \mathcal{H} \mathcal{F}_{\text{Cl}(m)}(U) \to \mathcal{H} \mathcal{F}_{\text{Cl}(n+m)}(U)$ given by $(E, h) \otimes (E', h') = (E \otimes E', h \otimes 1 + 1 \otimes h')$, where the tensor products are taken in the $\mathbb{Z}/2$-graded sense.
5. Main Theorem

5.1. Finite-dimensional approximation

To begin with, we construct the following homomorphism via a “finite-dimensional approximation”:

\[ \alpha : K_{\text{Cl}(n)}^\tau(X) \rightarrow KF_{\text{Cl}(n)}^\tau(X). \]

The construction is exactly the same as that performed in [10]: let \( A \in \Gamma(X, F_n(\tau)) \) be a section given. We then make the following choice:

- an open cover \( \{ U_\alpha \} \) of \( X \);
- local sections \( s_\alpha : U_\alpha \rightarrow \tau|_{U_\alpha} \) of \( \tau \), which define the transition functions \( g_{\alpha\beta} : U_{\alpha\beta} \rightarrow PU(H) \) by \( s_\alpha g_{\alpha\beta} = s_\beta \);
- lifts \( g_{\alpha\beta} : U_{\alpha\beta} \rightarrow U(H) \) of the transition functions \( \bar{g}_{\alpha\beta} : U_{\alpha\beta} \rightarrow PU(H) \);
- positive numbers \( \mu_\alpha \) such that the family of vector spaces

\[ E_\alpha = \bigcup_{x \in U_\alpha} (H_n, (A_\alpha)_x)^{<\mu_\alpha} = \bigcup_{x \in U_\alpha} \bigoplus_{\lambda < \mu_\alpha} \{ \xi \in H_n | (A_\alpha)_x^2 \xi = \lambda \xi \} \]

becomes a vector bundle of finite rank.

By means of the trivializations \( s_\alpha \), the section \( A \) induces maps \( A_\alpha : U_\alpha \rightarrow F_n(H_n) \) such that \( g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} = A_\alpha \). Now, for \( i = 1, \ldots, n \), the action of \( e_i \in Cl(n) \) on \( H_n \) induces a vector bundle map \( e_i : E_\alpha \rightarrow E_\alpha \) of degree 1 satisfying \( e_i e_j + e_j e_i = -2\delta_{i,j} \). The restriction of \( A_\alpha \) defines a Hermitian map \( h_\alpha : E_\alpha \rightarrow E_\alpha \) of degree 1 anti-commuting with \( e_i \). Finally, we define a map \( \phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \rightarrow E_\alpha|_{U_{\alpha\beta}} \) by the composition of the following maps:

\[ E_\beta|_{U_{\alpha\beta}} \xrightarrow{\text{inclusion}} U_{\alpha\beta} \times H_n \xrightarrow{\text{id} \times g_{\alpha\beta}} U_{\alpha\beta} \times H_n \xrightarrow{\text{projection}} E_\alpha|_{U_{\alpha\beta}}. \]

Then \( E = (\{ U_\alpha \}, s_\alpha, g_{\alpha\beta}, (E_\alpha, h_\alpha), \phi_{\alpha\beta}) \) is a \( \tau \)-twisted \( Cl(n) \)-vectorial bundle on \( X \), and a well-defined homomorphism \( \alpha : K_{\text{Cl}(n)}^\tau(X) \rightarrow KF_{\text{Cl}(n)}^\tau(X) \) is induced from the assignment \( A \mapsto E \).

The construction above also induces \( \alpha : K_{\text{Cl}(n)}^\tau(X,Y) \rightarrow KF_{\text{Cl}(n)}^\tau(X,Y) \) as well as \( \alpha_j : K_{\text{Cl}(n)}^{\tau+j}(X,Y) \rightarrow KF_{\text{Cl}(n)}^{\tau+j}(X,Y) \) for any \( (X,Y,\tau) \in \hat{C} \) and \( j \leq 1 \). Then, in the same way as in [10], we get:
Proposition 5.1. The homomorphisms \( \alpha_j : K_{Cl(n)}^{\tau+j}(X,Y) \to KF_{Cl(n)}^{\tau+j}(X,Y) \), \( (j \leq 1) \) give rise to natural transformations from the functors in Proposition 3.1 to those in Proposition 4.4.

5.2. Main theorem and its corollary

Theorem 1 in Section 1 is a corollary (Corollary 5.4) to:

Theorem 5.2. Let \( \tau \) be any principal \( PU(H) \)-bundle over a CW complex \( X \). For any \( n,j \geq 0 \), the homomorphism \( \alpha_{-j} : K_{Cl(n)}^{\tau-j}(X) \to KF_{Cl(n)}^{\tau-j}(X) \) is bijective.

The key to this theorem is the following proposition, which we will prove in the next subsection:

Proposition 5.3. For any \( k,j \geq 0 \), the following homomorphism is bijective:
\[
\alpha_{-j} : K_{Cl(n)}^{\tau-j}(D^k,S^{k-1}) \to KF_{Cl(n)}^{\tau-j}(D^k,S^{k-1}),
\]
where \( (D^k,S^{k-1}) \) means \( (pt,\emptyset) \) when \( k = 0 \).

Proof of Theorem 5.2. In view of Proposition 3.1, 4.4, 5.1 and 5.3, the proof is exactly the same as that of the main result of [10]: First, in the case that \( X \) is a finite CW complex, we prove the bijectivity of \( \alpha_{-j} \) by an induction on the number of cells in \( X \). Then, the bijectivity of \( \alpha_{-j} \) in the general case follows from that in the finite case through an argument by using the telescope of \( X \). \( \square \)

Corollary 5.4. Suppose \( (X,Y,\tau) \in \hat{\mathcal{C}} \) and \( j \geq 0 \) are given.

(a) The finite-dimensional approximation induces the bijection:
\[
\alpha_{-j} : K_{Cl(n)}^{\tau-j}(X,Y) \to KF_{Cl(n)}^{\tau-j}(X,Y).
\]

(b) The multiplication of a generator of \( K(D^2,S^1) \cong KF(D^2,S^1) \cong \mathbb{Z} \) induces the bijection:
\[
KF_{Cl(n)}^{\tau-j}(X,Y) \to KF_{Cl(n)}^{\tau-j-2}(X,Y).
\]

(c) There exists a natural isomorphism
\[
K^{\tau-j-n}(X,Y) \cong KF_{Cl(n)}^{\tau-j}(X,Y).
\]
5.3. Key proposition

This subsection is devoted to the proof of Proposition 5.3, which is clearly equivalent to:

**Proposition 5.5.** For any $n, k \geq 0$, the following homomorphism is bijective:

$$\alpha : K_{Cl(n)}(D^k, S^{k-1}) \longrightarrow KF_{Cl(n)}(D^k, S^{k-1}).$$

Notice that the principal $PU(H)$-bundle $\tau$ is absent (or trivial) in the present case. Therefore $K_{Cl(n)}(D^k, S^{k-1})$ is identified with the homotopy classes of maps from the $k$-dimensional disk $D^k$ to $F_n$ which carry all points in the sphere $S^{k-1} = \partial D^k$ into the subspace $F^*_n \subset F_n$ consisting of invertible operators:

$$K_{Cl(n)}(D^k, S^{k-1}) = [(D^k, S^{k-1}), (F_n, F^*_n)].$$

To prove Proposition 5.5, recall the homeomorphism $F_n(\mathcal{H}_n) \rightarrow F_{n+2m}(\mathcal{H}_{n+2m})$ given by $A \mapsto A \otimes \text{id}$ under the identification $\mathcal{H}_n \otimes \Delta^+_{2m} \cong \mathcal{H}_{n+2m}$. Consequently, for any CW pair $(X, Y)$, we have a natural isomorphism

$$K_{Cl(n)}(X, Y) \longrightarrow K_{Cl(n+2m)}(X, Y).$$

There is a similar “periodicity” for vectorial bundles:

**Lemma 5.6.** Let $n$ be a non-negative integer. For any CW pair $(X, Y)$ and $m > 0$, the tensor product of the irreducible $Cl(2m)$-module $\Delta^+_{2m}$ induces a natural isomorphism $KF_{Cl(n)}(X, Y) \rightarrow KF_{Cl(n+2m)}(X, Y)$ fitting in the commutative diagram:

$$\begin{array}{ccc}
K_{Cl(n)}(X, Y) & \longrightarrow & KF_{Cl(n)}(X, Y) \\
\alpha \downarrow & & \alpha \\
K_{Cl(n+2m)}(X, Y) & \longrightarrow & KF_{Cl(n+2m)}(X, Y).
\end{array}$$

**Proof.** The first part of this lemma, which is shown in [9], follows from Lemma 2.1. The second part is clear by construction. □

As a consequence of this lemma, it suffices to consider the case of $n = 0$ and $n = 1$ only in Proposition 5.5. In the case of $n = 0$, the proposition is
established in [10]. Hence we are left with the case of \( n = 1 \). To deal with this case, we use the following fact (Remark 10.29 (2), [9]):

**Proposition 5.7 ([9]).** For \( n, k > 0 \), there is a natural isomorphism

\[
K_{F_{Cl(n)}}(pt) \rightarrow K_{F_{Cl(k+n)}}(D^k, S^{k-1})
\]

given by the multiplication of the “symbol of the \( k \)-dimensional supersymmetric harmonic oscillator”.

The symbol of the 1-dimensional supersymmetric harmonic oscillator ([8]) is the \( Cl(1) \)-vectorial bundle \((F, h) \in K_{F_{Cl(1)}}(I, \partial I)\) defined by:

\[
F = I \times \Delta_1 = I \times (\mathbb{C} \oplus \mathbb{C}), \quad h = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \quad (t \in I = [-1, 1]).
\]

The symbol of the \( k \)-dimensional supersymmetric harmonic oscillator is the \( Cl(k) \)-vectorial bundle \( \otimes_{i=1}^{k} \pi_i^*(F, h) \in K_{F_{Cl(k)}}(I^k, \partial I^k) \), where \( \pi_i : I^k \rightarrow I \) is the projection onto the \( i \)th factor.

Proposition 5.7 leads to the following computational result:

**Corollary 5.8.** For \( k \geq 0 \), we have:

\[
K_{F_{Cl(1)}}(D^k, S^{k-1}) \cong \begin{cases} \mathbb{Z}, & (k : \text{odd}) \\ 0, & (k : \text{even}) \end{cases}
\]

**Proof.** First, we consider the case that \( k \) is an odd integer \( k = 2m + 1 \). By means of Lemma 5.6 and Proposition 5.7, we have

\[
K_{F_{Cl(1)}}(D^{2m+1}, S^{2m}) \cong K_{F_{Cl(2m+1)}}(D^{2m+1}, S^{2m}) \cong K_F(pt) \cong \mathbb{Z}.
\]

In the even case \( k = 2m \), we use Lemma 5.6 and Proposition 5.7 again to have

\[
K_{F_{Cl(1)}}(D^{2m}, S^{2m-1}) \cong K_{F_{Cl(2m+1)}}(D^{2m}, S^{2m-1}) \cong K_{F_{Cl(1)}}(pt).
\]

That \( K_{F_{Cl(1)}}(pt) = 0 \) is shown as follows: any element in \( K_{F_{Cl(1)}}(pt) \) can be represented by a pair \((E, h)\) of a \( Cl(1) \)-module \( E \) and a Hermitian map \( h : E \rightarrow E \) of degree 1 anti-commuting with the action of \( e_1 \in Cl(1) \).
Since the irreducible $C\ell(1)$-module is unique up to an equivalence, we can express $E$ as $E = V \otimes \Delta_1$, where $V$ is a vector space of finite rank. Now we define $(\tilde{E}, \tilde{h}) \in K_{C\ell(1)}([0, 1])$ by setting $\tilde{E} = I \times E$ and $\tilde{h}_t = (\cos \frac{\pi t}{2}) h + \sqrt{-1}(\sin \frac{\pi t}{2}) \gamma$, where $\gamma$ is a basis of $H_{\mathbb{Z}/2}(\Delta_1) = \mathbb{C}$ such that $\gamma^2 = 1$. Then $(\tilde{E}, \tilde{h})$ is a homotopy between $(E, h)$ and a $C\ell(1)$-vectorial bundle representing $0 \in K_{C\ell(1)}(pt)$. □

As is well-known, we have

$$K_{C\ell(1)}(D^k, S^{k-1}) = [(D^k, S^{k-1}), (F_1, F_1^s)] = \pi_k(F_1) = \begin{cases} \mathbb{Z}, & (k : \text{odd}) \\ 0, & (k : \text{even}) \end{cases}$$

Therefore $\alpha : K_{C\ell(1)}(D^k, S^{k-1}) \to K_{C\ell(1)}(D^k, S^{k-1})$ is apparently bijective in the case of $k$ even.

Now, it remains the case of $k$ odd. Since we have $K_{C\ell(1)}(D^k, S^{k-1}) \cong \mathbb{Z}$ and $K_{C\ell(1)}(D^k, S^{k-1}) \cong \mathbb{Z}$ in this case, it suffices to see the correspondence of generators through $\alpha$. As is well-known [4], a self-adjoint Fredholm operator whose spectral flow is 1 generates $[(I, \partial I), (F_1, F_1^s)] \cong \pi_1(F_1) \cong \mathbb{Z}$. Hence the bijectivity of $\alpha$ in the case of $k = 1$ (and $n = 1$) follows from:

**Lemma 5.9.** There is a continuous map $A : (I, \partial I) \to (F_1, F_1^s)$ such that:

1. its spectral flow is 1;
2. $\alpha([A]) = [(F, h)]$ in $K_{C\ell(1)}(I, \partial I)$.

**Proof.** Let $H$ be the Hilbert space with its complete orthonormal basis $\{e_\ell\}_{\ell \in \mathbb{Z}}$. For $t \in \mathbb{R}$, we define a bounded self-adjoint operator $a_t : H \to H$ by $a_te_\ell = (t + \ell)/\sqrt{(t + \ell)^2 + 1}$. A computation shows

$$\| (a_t - a_{t'}) e_\ell \| \leq \left| \frac{t + \ell}{\sqrt{(t + \ell)^2 + 1}} - \frac{t + \ell}{\sqrt{(t' + \ell)^2 + 1}} \right| + \left| \frac{t + \ell}{\sqrt{(t' + \ell)^2 + 1}} - \frac{t' + \ell}{\sqrt{(t' + \ell)^2 + 1}} \right| \leq \frac{|t - t'|}{\sqrt{(t' + \ell)^2 + 1}} \left( \frac{|t + \ell|}{\sqrt{(t + \ell)^2 + 1}} + \frac{|t - t'|}{\sqrt{(t' + \ell)^2 + 1}} \right) \leq 2|t - t'|.$$
Thus, we get \( \| (a_t-a_{t'})u \| \leq 2|t-t'|\|u\| \) for \( u \in H \), so that \( \|a_t-a_{t'}\| \leq 2|t-t'| \). This means the map \( \alpha : \mathbb{R} \to B(H) \) is continuous. (Here \( B(H) \) is topologized by the operator norm. In the case where the topology of \( B(H) \) is the compact-open topology in the sense of [5], the map \( \alpha : \mathbb{R} \to B(H) \) is still continuous, since \( \mathbb{R} \times H \to H, ((t,u) \mapsto a_tu) \) is.) Now, we choose \( \epsilon > 0 \) sufficiently small. Then, setting \( \mathcal{H}_1 = H \otimes \Delta_1 = H \oplus H \), \( A_t = \begin{pmatrix} 0 & a_t \\ a_t & 0 \end{pmatrix} \) and \( I = [-\epsilon, \epsilon] \), we get \( A : (I, \partial I) \to (\mathcal{F}_1, \mathcal{F}_1^*) \) such that its spectral flow is 1 and its finite-dimensional approximation is a \( Cl(1) \)-vectorial bundle on \( (I, \partial I) \) homotopic to \( (F, h) \). \( \Box \)

To establish the bijectivity of \( \alpha \) in the case of general odd number \( k = 2m+1 \), we recall the map \( \mathcal{F}_p(\mathcal{H}_p) \times \mathcal{F}_q(\mathcal{H}_q) \to \mathcal{F}_{p+q}(\mathcal{H}_p \otimes \mathcal{H}_q) \) inducing the ring structure on the \( K \)-cohomology theory: the explicit description of the map is \( (A, B) \mapsto A \otimes 1 + 1 \otimes B \). From this description and that of vectorial bundles, we see the commutative diagram

\[
\begin{array}{ccc}
K_{Cl(p)}(X, Y) \times K_{Cl(q)}(X, Y') & \longrightarrow & K_{Cl(p+q)}(X, Y \cup Y') \\
\alpha \times \alpha & & \alpha \\
K_{FCl(p)}(X, Y) \times K_{FCl(q)}(X, Y') & \otimes & K_{FCl(p+q)}(X, Y \cup Y').
\end{array}
\]

This induces the following commutative diagram:

\[
\begin{array}{ccc}
\prod_{i=1}^{2m+1} K_{Cl(1)}(I, \partial I) & \longrightarrow & K_{Cl(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \\
\Pi \alpha & & \alpha \\
\prod_{i=1}^{2m+1} K_{FCl(1)}(I, \partial I) & \longrightarrow & K_{FCl(2m+1)}(I^{2m+1}, \partial I^{2m+1}).
\end{array}
\]

By Proposition 5.7, \( K_{FCl(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \cong \mathbb{Z} \) is generated by the symbol of the \((2m+1)\)-dimensional supersymmetric harmonic oscillator, which is the product of \( 2m+1 \) copies of \((F, h) \in K_{FCl(1)}(I, \partial I) \). Thus, by Lemma 5.9 and the commutative diagram above, the homomorphism

\[
\alpha : K_{Cl(2m+1)}(I^{2m+1}, \partial I^{2m+1}) \longrightarrow K_{FCl(2m+1)}(I^{2m+1}, \partial I^{2m+1})
\]

is surjective. Since any surjective homomorphism \( \mathbb{Z} \to \mathbb{Z} \) is bijective, we conclude that \( \alpha \) above is bijective. Therefore the following homomorphism is also bijective by Lemma 5.6:

\[
\alpha : K_{Cl(1)}(I^{2m+1}, \partial I^{2m+1}) \longrightarrow K_{FCl(1)}(I^{2m+1}, \partial I^{2m+1}),
\]
which completes the proof of Proposition 5.5.

6. Applications

6.1. The Atiyah-Singer map

As is mentioned, the map of Atiyah-Singer [6]

\[ \text{AS} : \mathcal{F}_n(\mathcal{H}_n) \longrightarrow \Omega \mathcal{F}_{n-1}(\mathcal{H}_n) \]

is a homotopy equivalence for \( n > 0 \), and induces the natural isomorphism

\[ \text{AS} : K_{\tau}^{n-j}(X, Y) \longrightarrow K_{\tau}^{n-j-1}(X, Y). \]

The aim of this subsection is to introduce a counterpart of this construction to twisted \( Cl(n) \)-vectorial bundles: For any space \( U \) and \((E, h) \in \mathcal{H}F_{Cl(n)}(U)\), we can define an object \((\tilde{E}, \tilde{h}) \in \mathcal{H}F_{Cl(n)}(U \times I)\) by setting

\[ \tilde{E} = E \times I, \quad \tilde{h}(x, t) = h(x) + \sqrt{-1}te_n, \]

where \( I = [-1, 1] \). The assignment \((E, h) \mapsto (\tilde{E}, \tilde{h})\) gives rise to a functor

\[ \text{AS} : \mathcal{H}F_{Cl(n)}(U) \longrightarrow \mathcal{H}F_{Cl(n-1)}(U \times I). \]

It is easy to globalize this construction to get the following functor for any principal \( PU(H) \)-bundle \( \tau \) over a space \( X \) and its subspace \( Y \subset X \):

\[ \text{AS} : \mathcal{K}F_{Cl(n)}^\tau(X, Y) \longrightarrow \mathcal{K}F_{Cl(n-1)}^\tau(X \times I, Y \times I \cup X \times \partial I). \]

This then induces a natural homomorphism for any \( j \geq 0 \).

\[ \text{AS} : KF_{Cl(n)}^{\tau-j}(X, Y) \longrightarrow KF_{Cl(n-1)}^{\tau-j-1}(X, Y). \]

**Lemma 6.1.** For any positive integer \( n > 0 \) and any principal \( PU(H) \)-bundle \( \tau \) over a space \( X \) and its subspace \( Y \subset X \), the following diagram is commutative:

\[
\begin{array}{ccc}
K_{Cl(n)}^\tau(X, Y) & \xrightarrow{\text{AS}} & K_{Cl(n-1)}^\tau(X, Y) \\
\downarrow{\alpha} && \downarrow{\alpha} \\
KF_{Cl(n)}^\tau(X, Y) & \xrightarrow{\text{AS}} & KF_{Cl(n-1)}^\tau(X, Y).
\end{array}
\]
Suppose that we apply the construction in Subsection 5.1 to \( A \) to have a vectorial bundle
\[
E = \{ (U_\alpha)_{\alpha \in \mathfrak{A}}, s_\alpha, g_{\alpha\beta}, (E_\alpha, h_\alpha), \phi_{\alpha\beta} \} \in \mathcal{K}\mathcal{F}_{CL(n)}^T(X, Y).
\]
Hence we have \( E_\alpha = \bigcup_{x \in U_\alpha} (H_n, (A_\alpha)_x) \leq \mu_\alpha \) under a choice of a positive number \( \mu_\alpha \). Without loss of generality, we can assume that there is \( \varepsilon_\alpha > 0 \) satisfying
\[
\lambda_1(x) \leq \lambda_2(x) \leq \cdots \leq \lambda_r(x) < \mu_\alpha - \varepsilon_\alpha < \mu_\alpha + \varepsilon_\alpha < \lambda_{r+1}(x)
\]
for all \( x \in U_\alpha \), where \( r \) is the rank of the vector bundle \( E_\alpha \), and \( \lambda_i(x) \) is the \( i \)th eigenvalue of \( (A_\alpha)_x^2 \). Then the twisted \( Cl(n-1) \)-vectorial bundle
\[
\text{AS}(E) = \{ (\tilde{U}_\alpha)_{\alpha \in \mathfrak{A}}, \tilde{s}_\alpha, \tilde{g}_{\alpha\beta}, (\tilde{E}_\alpha, \tilde{h}_\alpha), \tilde{\phi}_{\alpha\beta} \}
\]
\[
\in \mathcal{K}\mathcal{F}_{Cl(n-1)}^T(X \times I, Y \times I \cup X \times \partial I)
\]
is given by setting \( \tilde{U}_\alpha = \pi^{-1}(U_\alpha) = U_\alpha \times I \), \( \tilde{s}_\alpha = \pi^* s_\alpha \), \( \tilde{g}_{\alpha\beta} = \pi^* g_{\alpha\beta} \), \( \tilde{E} = \pi^* E_\alpha \), \( \tilde{h}_\alpha(x, t) = h_\alpha(x) + \sqrt{-1}te_n \) and \( \tilde{\phi}_{\alpha\beta} = \pi^* \phi_{\alpha\beta} \), where \( \pi : X \times I \to X \) is the projection. Then \( \text{AS}(E) \) represents the image \( \text{AS}(\alpha([A])) \).

Next, we describe the image \( \alpha(\text{AS}([A])) \) applying the construction in Subsection 5.1 to \( A \in \Gamma(X, Y, \mathcal{F}_n(\tau)) \). By means of the local trivialization \( \pi^* s_\alpha \) of \( \pi^* \tau = \tau \times I \), the section \( \text{AS}(A) \) defines a map \( \bar{A}_\alpha : \tilde{U}_\alpha \to \mathcal{F}_{n-1}(H_n) \). By our definition of the Atiyah-Singer map, we have \( (\bar{A}_\alpha)(x, t) = (A_\alpha)_x + \sqrt{-1}te_n \). We here define an open cover \( \{ V(s; \varepsilon_\alpha) \}_{s \in I} \) of \( I = [-1, 1] \) by
\[
V(s; \varepsilon_\alpha) = \{ t \in I \mid s - \varepsilon_\alpha < t^2 < s + \varepsilon_\alpha \}.
\]
Then, for any \( (x, t) \in U_\alpha \times V(s; \varepsilon_\alpha) \), the eigenvalues \( \tilde{\lambda}_i(x, t) \) of \( (\bar{A}_\alpha)_x^2 \) satisfy
\[
\tilde{\lambda}_1(x, t) \leq \tilde{\lambda}_2(x, t) \leq \cdots \leq \tilde{\lambda}_r(x, t) < \mu_\alpha + s < \tilde{\lambda}_{r+1}(x, t),
\]
since \( \tilde{\lambda}_i(x, t) = \lambda_i(x) + t^2 \). This implies
\[
\bigcup_{(x, t) \in U_\alpha \times V(s; \varepsilon_\alpha)} (H_n, \bar{A}_\alpha(x, t)) \leq \mu_\alpha + s = \tilde{E}_\alpha |_{U_\alpha \times V(s; \varepsilon_\alpha)}.
\]
Thus, $\alpha(\text{AS}([A]))$ is represented by the twisted $\text{Cl}(n-1)$-vectorial bundle obtained from $\text{AS}(\mathbb{E})$ through the refinement $\{U_\alpha \times V(s; \varepsilon_\alpha)\}$ of the open cover $\{\tilde{U}_\alpha\}$, which is isomorphic to $\text{AS}(\mathbb{E})$ itself. Hence $\text{AS}(\alpha([A])) = \alpha(\text{AS}([A]))$. □

**Theorem 6.2.** For any $(X, Y, \tau) \in \hat{C}$, $j \in \mathbb{Z}$ and $n > 0$, the homomorphism

$$\text{AS} : KF_{\text{Cl}(n)}^{\tau-j}(X, Y) \longrightarrow KF_{\text{Cl}(n-1)}^{\tau-j-1}(X, Y)$$

is bijective.

**Proof.** Lemma 6.1 provides us the commutative diagram

$$
\begin{array}{ccc}
K_{\text{Cl}(n)}^{\tau-j}(X, Y) & \xrightarrow{\text{AS}} & K_{\text{Cl}(n-1)}^{\tau-j-1}(X, Y) \\
\quad \quad \alpha \downarrow & & \quad \quad \alpha \downarrow \\
KF_{\text{Cl}(n)}^{\tau-j}(X, Y) & \xrightarrow{\text{AS}} & KF_{\text{Cl}(n-1)}^{\tau-j-1}(X, Y).
\end{array}
$$

Since AS in the upper row is bijective by [6], Theorem 5.2 implies the conclusion. □

Lemma 5.6 is generalized to the twisted case, so that we have a natural isomorphism $KF_{\text{Cl}(n)}^{\tau-j}(X, Y) \to KF_{\text{Cl}(n+2m)}^{\tau-j}(X, Y)$. The composition of maps

$$KF_{\text{Cl}(n)}^{\tau-j}(X, Y) \longrightarrow KF_{\text{Cl}(n+2)}^{\tau-j}(X, Y) \xrightarrow{\text{AS}^2} KF_{\text{Cl}(n)}^{\tau-j-2}(X, Y)$$

is readily identified with the multiplication of a generator of $K(D^2, S^1)$. Thus, Theorem 6.2 reproduces Corollary 5.4 (b).

**6.2. Twisted K-theory with coefficients $\mathbb{Z}/p$**

Let $p$ be a positive integer. The aim of this subsection is to provide a model of twisted $K$-theory with its coefficients $\mathbb{Z}/p$, or twisted mod $p K$-theory by using twisted vectorial bundles. For this aim, we begin with a formulation of twisted mod $p K$-theory based on an idea in [3].

**Definition 6.3.** Let $\tau$ be a principal $PU(H)$-bundle over a space $X$.  

(a) For a non-negative integer \( n \), we define a \( \tau \)-twisted mod \( p \) \( K \)-cocycle of degree \( -n - 1 \) on \( X \) to be a pair \((A, T)\) consisting of \( A \in \Gamma(X, \tau \times \rho \mathbb{P}U(H)) \mathcal{F}_n(\mathcal{H}_n)\) and \( T \in \Gamma(X \times [0, 1], (\tau \times [0, 1]) \times \rho \mathbb{P}U(H) \mathcal{F}_n(\mathcal{H}_n^{\otimes p})) \) such that \( T|_{t=0} = A^{\otimes p} \) and \( \text{Supp}(T|_{t=1}) = \emptyset \).

(b) We define a homotopy between \( \tau \)-twisted mod \( p \) \( K \)-cocycles \((A_0, T_0)\) and \((A_1, T_1)\) of degree \( -n - 1 \) on \( X \) to be a \( \tau \)-twisted mod \( p \) \( K \)-cocycle \((\tilde{A}, \tilde{T})\) of degree \( -n - 1 \) on \( X \times [0, 1] \) such that \((\tilde{A}, \tilde{T})|_{t=0} = (A_0, T_0)\) and \((\tilde{A}, \tilde{T})|_{t=1} = (A_1, T_1)\) for \( i = 0, 1 \).

(c) We define \( K_{\text{Cl}(n)}^{\tau -1}(X; \mathbb{Z}/p) \) to be the group of homotopy classes of mod \( p \) \( K \)-cocycles of degree \( -n - 1 \) on \( X \). (The group structure is defined in the same way as \( K_{\text{Cl}(n)}^{\tau}(X) \).)

**Lemma 6.4.** There exists a natural exact sequence:

\[
K_{\text{Cl}(n)}^{\tau -1}(X) \xrightarrow{m_p} K_{\text{Cl}(n)}^{\tau -1}(X) \xrightarrow{\rho_p} K_{\text{Cl}(n)}^{\tau -1}(X; \mathbb{Z}/p) \xrightarrow{\delta_p} K_{\text{Cl}(n)}^{\tau}(X) \xrightarrow{m_p} K_{\text{Cl}(n)}^{\tau}(X).
\]

**Proof.** We define \( \delta_p \) by \( \delta_p([(A, T)]) = [A] \) and \( m_p \) by \( m_p([A]) = [A^{\otimes p}] = p[A] \). To define \( \rho_p \), we represent an element in \( K_{\text{Cl}(n)}^{\tau -1}(X) \) by a section \( B \in \Gamma(X \times I, X \times \partial I, (\tau \times I) \times \rho \mathbb{P}U(H) \mathcal{F}_n(\mathcal{H}_n^{\otimes p})) \), where \( I = [0, 1] \). The section \( B|_{t=0} \) takes values in the space of invertible operators in \( \mathcal{F}_n(\mathcal{H}_n^{\otimes p}) \). Hence we can assume \( B|_{t=0} = J^{\otimes p} \) for some invertible operator \( J \in \mathcal{F}_n^{\star}(\mathcal{H}_n) \).

If we put \( \rho_p([B]) = [(J, B)] \), then \( \rho_p \) gives rise to a well-defined a homomorphism.

Now, if \([B] \in K_{\text{Cl}(n)}^{\tau -1}(X)\) is such that \( \rho_p([B]) = 0 \), then there exists a homotopy \((\tilde{A}, \tilde{T})\) between \((J, B)\) and \((J, J^{\otimes p})\). By a reparametrization of \( \tilde{T} \), we can construct a homotopy connecting \( B \) and \( \tilde{A}^{\otimes p} \), so that the exactness at the second term \( K_{\text{Cl}(n)}^{\tau -1}(X) \) holds. To see the exactness at the third term \( K_{\text{Cl}(n)}^{\tau -1}(X; \mathbb{Z}/p) \), let \((A, T)\) be such that \([A] = 0\) in \( K_{\tau}(X) \). Then there is a homotopy \( H \) between \( A \in \mathcal{F}_n(\mathcal{H}_n) \) and an invertible operator \( J \in \mathcal{F}_n^{\star}(\mathcal{H}_n) \). Concatenating \( H^{\otimes p} \) and \( T \), we have \( B \) such that \( \rho_p([B]) = [(A, T)] \). The exactness at the forth term \( K_{\tau}(X) \) directly follows from the definitions of \( \delta_p \) and \( m_p \). □
Since $K^\tau_{Cl(n)}(X) \cong K^{\tau-n-1}(X)$, the group $K^\tau_{Cl(n)}(X; \mathbb{Z}/p)$ fits into

$$K^{\tau-n-1}(X) \xrightarrow{m_p} K^{\tau-n-1}(X) \xrightarrow{\rho_p} K^\tau_{Cl(n)}(X; \mathbb{Z}/p) \xrightarrow{\delta_p} K^{\tau-n}(X) \xrightarrow{m_p} K^{\tau-n}(X).$$

Thus, the $\tau$-twisted mod $p$ $K$-theory $K^{\tau-n-1}(X; \mathbb{Z}/p)$ of $X$ of degree $-n-1$ can be defined as $K^{\tau-n-1}(X; \mathbb{Z}/p) = K^\tau_{Cl(n)}(X; \mathbb{Z}/p)$. (By the help of the Bott periodicity, we can actually give an isomorphism between $K^\tau_{Cl(n)}(X; \mathbb{Z}/p)$ and the group $K^{\tau-n-1}(X; \mathbb{Z}/p)$ constructed out of the so-called Moore space.)

Now, we introduce our finite-dimensional model of $K^\tau_{Cl(n)}(X; \mathbb{Z}/p)$.

**Definition 6.5.** Let $\tau$ be a principal $PU(H)$-bundle over a space $X$.

(a) For a non-negative integer $n$, we define a $\tau$-twisted mod $p$ $Cl(n)$-vectorial bundle on $X$ to be a pair $(\mathbb{E}, \mathbb{H})$ consisting of $\mathbb{E} \in K^\tau_{Cl(n)}(X)$ and $\mathbb{H} \in K^\tau_{Cl(n)}(X \times I)$ such that $\mathbb{H}|_{t=0}$ is isomorphic to $\mathbb{E}$ and $\text{Supp}(\mathbb{H}|_{t=1}) = \emptyset$.

(b) We define a homotopy between $\tau$-twisted mod $p$ $Cl(n)$-vectorial bundles $(\mathbb{E}_0, \mathbb{H}_0)$ and $(\mathbb{E}_1, \mathbb{H}_1)$ on $X$ to be a $(\tau \times I)$-twisted mod $p$ $Cl(n)$-vectorial bundle $(\mathbb{E}, \mathbb{H})$ on $X \times I$ such that $\mathbb{E}|_{t=i}$ and $\mathbb{H}|_{t=i}$ are isomorphic to $\mathbb{E}_i$ and $\mathbb{H}_i$, respectively, for $i = 0, 1$.

(c) We define $KF^\tau_{Cl(n)}(X)$ to be the group of homotopy classes of $\tau$-twisted mod $p$ $Cl(n)$-vectorial bundles on $X$.

**Lemma 6.6.** There exists a natural exact sequence:

$$KF^\tau_{Cl(n)}(X) \xrightarrow{m_p} K^\tau_{Cl(n)}(X) \xrightarrow{\rho_p} K^\tau_{Cl(n)}(X; \mathbb{Z}/p) \xrightarrow{\delta_p} K^\tau_{Cl(n)}(X) \xrightarrow{m_p} K^\tau_{Cl(n)}(X).$$

**Proof.** We define $\delta_p$ by $\delta_p([\mathbb{E}, \mathbb{H}]) = [\mathbb{E}]$ and $m_p([\mathbb{F}]) = [\mathbb{F} \oplus \mathbb{F}] = p[\mathbb{F}]$. To define $\rho_p$, let $\mathbb{F} \in K^\tau_{Cl(n)}(X \times I, X \times \partial I)$ represent an element in $KF^\tau_{Cl(n)}(X)$. Then $\text{Supp}(\mathbb{F}|_{t=0}) = \emptyset$, so that $\mathbb{F}|_{t=0}$ is isomorphic to $\mathbb{O} \oplus \mathbb{O}$, where $\mathbb{O} \in K^\tau_{Cl(n)}(X)$ is such that $\text{Supp}([\mathbb{O}]) = \emptyset$, or equivalently $[\mathbb{O}] = 0$ in $KF^\tau_{Cl(n)}(X)$. If we put $\rho_p([\mathbb{F}]) = [(\mathbb{O}, \mathbb{F})]$, then $\rho_p$ is a well-defined homomorphism. Now, the exactness of the sequence can be shown by using the argument in the proof of Lemma 6.4: The only thing to notice is that we
apply a Mayer-Vietoris construction (Lemma 4.2, [10]) to a “concatenation” of twisted $Cl(n)$-vectorial bundles. □

**Lemma 6.7.** There exists a natural homomorphism

$$\alpha : K^\tau_{Cl(n)}(X; \mathbb{Z}/p) \longrightarrow K^\tau_{Cl(n)}(X; \mathbb{Z}/p)$$

making the following diagram commutative:

$$
\begin{array}{ccc}
K^\tau_{Cl(n)}(X) & \xrightarrow{\rho_p} & K^\tau_{Cl(n)}(X; \mathbb{Z}/p) \\
\downarrow & & \downarrow \alpha \\
K^\tau_{Cl(n)}(X) & \xrightarrow{\delta_p} & K^\tau_{Cl(n)}(X)
\end{array}
$$

$$
\begin{array}{ccc}
K^\tau_{Cl(n)}(X) & \xrightarrow{\rho_p} & K^\tau_{Cl(n)}(X; \mathbb{Z}/p) \\
\downarrow & & \downarrow \alpha \\
K^\tau_{Cl(n)}(X) & \xrightarrow{\delta_p} & K^\tau_{Cl(n)}(X)
\end{array}
$$

where the vertical maps other than $\alpha$ are those constructed in Subsection 5.1.

**Proof.** We define $\alpha$ in question based on the construction in Subsection 5.1: Suppose that a $\tau$-twisted mod $p$ $K$-cocycle $(A, T)$ of degree $-n-1$ on $X$ is given. By definition, $A_x = T_{(x,0)}$ holds for all $x \in X$. To have a finite-dimensional approximation of $A$, we choose an open cover $\{U_\alpha\}$ of $X$, local trivializations $s_\alpha$ of $\tau$, lifts of transition functions $g_{\alpha\beta}$ and positive numbers $\mu_\alpha$ so that $\bigcup_{x \in U_\alpha} (\mathcal{H}_n, (A_\alpha)_x)^{<\mu_\alpha}$ gives rise to a vector bundle. Also, to have a finite-dimensional approximation of $T$, we choose an open cover $\{\tilde{U}_\tilde{\alpha}\}$ of $X \times I$, local trivializations $\tilde{s}_\tilde{\alpha}$ of $\tau \times I$, lifts of transition functions $\tilde{g}_{\tilde{\alpha}\tilde{\beta}}$, and positive numbers $\tilde{\mu}_\tilde{\alpha}$ so that $\bigcup_{(x,t) \in \tilde{U}_\tilde{\alpha}} (\mathcal{H}_{\mathbb{Z}/p} \otimes (T_{\tilde{\alpha}})_{(x,t)})^{<\tilde{\mu}_\tilde{\alpha}}$ gives rise to a vector bundle. We can choose these data for $T$ in a way compatible with the data for $A$, that is,

- the open cover $\{U_\alpha\}$ agrees with the open cover $\{\tilde{U}_{\tilde{\alpha}}|_{t=0}\}$ of $X \times \{0\}$;
- If $U_\alpha = \tilde{U}_{\tilde{\alpha}}|_{t=0}$, then $s_\alpha = \tilde{s}_{\tilde{\alpha}}|_{t=0}$, $g_{\alpha\beta} = \tilde{g}_{\tilde{\alpha}\tilde{\beta}}|_{t=0}$ and $\mu_\alpha = \tilde{\mu}_{\tilde{\alpha}}$.

Such a choice is possible because the eigenvalues of $(T_{\tilde{\alpha}})^2_{(x,t)}$ are continuous in $(x,t)$. Under the choice above, we get a $\tau$-twisted mod $p$ $Cl(n)$-vectorial bundle $(\mathbb{E}, \mathbb{H})$ as a finite-dimensional approximation of $(A, T)$. We put $\alpha([(A, T)]) = [(\mathbb{E}, \mathbb{H})]$ and define the homomorphism $\alpha$. Once $\alpha$ is defined, the commutativity of the diagram is obvious from the construction. □
THEOREM 6.8. For any \((X, \emptyset, \tau) \in \hat{C}\), the homomorphism in Lemma 6.6
\[
\alpha : K_{Cl(n)}^\tau(X; \mathbb{Z}/p) \to KF_{Cl(n)}^{\tau-1}(X; \mathbb{Z}/p)
\]
is bijective, so that there is an isomorphism \(K^{\tau-n-1}(X; \mathbb{Z}/p) \cong KF_{Cl(n)}^{\tau-1}(X; \mathbb{Z}/p)\).

PROOF. The theorem follows from Lemma 6.4, 6.6, 6.7 and Theorem 5.2. □

Though will not be detailed anymore, we can take into account additional support conditions to define the relative versions \(K_{Cl(n)}^{\tau-1}(X,Y; \mathbb{Z}/p)\) as well as \(KF_{Cl(n)}^{\tau-1}(X,Y; \mathbb{Z}/p)\) for any \((X,Y,\tau) \in \hat{C}\). Then, in the same way as above, we get isomorphisms \(K_{Cl(n)}^{\tau-1}(X,Y; \mathbb{Z}/p) \cong KF_{Cl(n)}^{\tau-1}(X,Y; \mathbb{Z}/p)\) and
\[
K^{\tau-j-n-1}(X,Y; \mathbb{Z}/p) \cong KF_{Cl(n)}^{\tau-1}(X \times I^j, Y \times I^j \cup X \times \partial I^j; \mathbb{Z}/p).
\]

References


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