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A Remark on Malliavin Calculus : Uniform Estimates and Localization

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Abstract. The author [1] showed precise estimates for the regularity on heat operators associated with degenerate elliptic operators. In the present paper, he shows that these estimates can be uniformized and localized similarly to the heat operators associated with Hörmander type degenerate elliptic operators.

1. Introduction

Let $W_0 = \{w \in C([0,\infty); \mathbf{R}^d);\ w(0) = 0\}$, $\mathcal{F}$ be the Borel algebra over W_0 and μ be the standard Wiener measure on $(W_0, \mathcal{F})$. Let $B^i : [0,\infty) \times W_0 \to \mathbf{R}$, $i = 1, \dots, d$, be given by $B^i(t,w) = w^i(t)$, $(t,w) \in [0,\infty) \times W_0$. Then $\{(B^1(t), \dots, B^d(t)); t \in [0,\infty)\}$ is a d-dimensional Brownian motion under μ. Let $B^0(t) = t$, $t \in [0,\infty)$. Let $\mathcal{F}_s^t$, $t \geqq s \geqq 0$, be the sub-σ-algebra generated by $\{B^i(r) - B^i(s); r \in [s,t],\ i = 1, \dots, d\}$. Then $\{\mathcal{F}_0^t\}_{t \geqq 0}$ is the Brownian filtration. Also, let H be the Cameron-Martin subspace relative to the Wiener space (W_0, μ), i.e.

$$H = \{w \in W_0; w(t) \text{ is absolutely continuous in } t,\ \sum_{k=1}^d \int_0^\infty |\frac{dw^k}{dt}(t)|^2 dt < \infty\}$$

and its inner product $(\cdot, *)_H$ is given by

$$(w, \tilde{w})_H = \sum_{k=1}^d \int_0^\infty \frac{dw^k}{dt}(t) \frac{d\tilde{w}^k}{dt}(t) dt, \qquad w, \tilde{w} \in H.$$

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Let Λ be a set. We denote by $U_\Lambda C_b^\infty(\mathbf{R}^N;\mathbf{R}^M)$, $N, M \geqq 1$, the set of families of smooth functions $\{f_\lambda\}_{\lambda\in\Lambda}$ from $\mathbf{R}^N$ to $\mathbf{R}^M$ such that

$$\sup_{\lambda\in\Lambda, x\in\mathbf{R}^N} |\frac{\partial^\alpha}{\partial x^\alpha} f_\lambda(x)| < \infty$$

for any multi-index $\alpha \in \mathbf{Z}^N_{\geqq 0}$.

Let $\{V_i^\lambda\}_{\lambda\in\Lambda} \in U_\Lambda C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$, $i = 0, 1, \dots, d$. We regard V_i^λ's as vector fields on $\mathbf{R}^N$. Let $X^\lambda(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, $\lambda \in \Lambda$, be the solution to the Stratonovich stochastic integral equation

$$X^\lambda(t,x) = x + \sum_{i=0}^d \int_0^t V_i^\lambda(X^\lambda(s,x)) \circ dB^i(s). \tag{1}$$

Then there is a unique strong solution to this equation. Moreover we may assume that $X^\lambda(t,x)$ is continuous in t and smooth in x, and that $X^\lambda(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$, $t \in [0,\infty)$, is a diffeomorphism with probability one.

Let $A = A_d = \{v_0, v_1, \dots, v_d\}$, be an alphabet, a set of letters, and A^* be the set of words consisting of A including the empty word which is denoted by 1. For $u = u^1 \cdots u^k \in A^*$, $u^j \in A$, $j = 1, \dots, k$, $k \geqq 0$, we denote by $n_i(u)$, $i = 0, \dots, d$, the cardinal of $\{j \in \{1, \dots, k\}; u^j = v_i\}$. Let $|u| = n_0(u) + \ldots + n_d(u)$, a length of u, and $\| u \| = |u| + n_0(u)$ for $u \in A^*$. Let $\mathbf{R}\langle A\rangle$ be the $\mathbf{R}$-algebra of noncommutative polynomials on A, $\mathbf{R}\langle\langle A\rangle\rangle$ be the $\mathbf{R}$-algebra of noncommutative formal series on A, $\mathcal{L}(A)$ be the free Lie algebra over $\mathbf{R}$ on the set A, and $\mathcal{L}((A))$ be the $\mathbf{R}$-Lie algebra of free Lie series on the set A.

Let $r : A^* \setminus \{1\} \to \mathcal{L}(A)$ denote the right normed bracketing operator inductively given by

$$r(v_i) = v_i, \qquad i = 0, 1, \dots, d,$$

and

$$r(v_i u) = [v_i, r(u)], \qquad i = 0, 1, \dots, d,\ u \in A^* \setminus \{1\}.$$

For any $w_1 = \sum_{u\in A^*} a_{1u} u, \in \mathbf{R}\langle\langle A\rangle\rangle$ and $w_2 = \sum_{u\in A^*} a_{2u} u, \in \mathbf{R}\langle A\rangle$, we define a kind of an inner product $\langle w_1, w_2\rangle$ by

$$\langle w_1, w_2\rangle = \sum_{u\in A^*} a_{1u} a_{2u} \in \mathbf{R}.$$

We can regard vector fields V_i^λ, $i = 0, 1, \dots, d$, $\lambda \in \Lambda$, as first differential operators over $\mathbf{R}^N$. Let $\mathcal{DO}(\mathbf{R}^N)$ denote the set of smooth differential operators over $\mathbf{R}^N$. Then $\mathcal{DO}(\mathbf{R}^N)$ is a noncommutative algebra over $\mathbf{R}$. Let $\Phi^\lambda : \mathbf{R}\langle A\rangle \to \mathcal{DO}(\mathbf{R}^N)$, $\lambda \in \Lambda$, be a homomorphism given by

$$\Phi^\lambda(1) = Identity, \qquad \Phi^\lambda(v_{i_1}\cdots v_{i_n}) = V_{i_1}^\lambda \cdots V_{i_n}^\lambda,$$

for any $n \geqq 1$, $i_1, \dots, i_n = 0, 1, \dots, d$, $\lambda \in \Lambda$. Then we see that

$$\Phi^\lambda(r(v_i u)) = [V_i^\lambda, \Phi^\lambda(r(u))], \qquad i = 0, 1, \dots, d, \ u \in A^* \setminus \{1\}.$$

Let $A_m^* = \{u \in A^*;\ \| u \| = m\}$, $m \geqq 0$, and let $\mathbf{R}\langle A\rangle_m = \sum_{u\in A_m^*} \mathbf{R}u$, and $\mathbf{R}\langle A\rangle_{\leqq m} = \sum_{k=0}^m \mathbf{R}\langle A\rangle_k$, $m \geqq 0$. Let $\mathcal{L}(A)_m = \mathcal{L}(A) \cap \mathbf{R}\langle A\rangle_m$, and $\mathcal{L}(A)_{\leqq m} = \mathcal{L}(A) \cap \mathbf{R}\langle A\rangle_{\leqq m}$, $m \geqq 1$. Let $A^{**} = \{u \in A^*;\ u \neq 1, v_0\}$, and $A_{\leqq m}^{**} = \{u \in A^{**};\ \| u \| \leqq m\}$, $m \geqq 1$.

Now we introduce a condition (U_ΛFG) on the family of vector field $\{V_i^\lambda$, $i = 0, 1, \dots, d$, $\lambda \in \Lambda\}$, as follows.

(U_ΛFG) *There are an integer* ℓ_0 *and* $\{\varphi_{u,u'}^\lambda\}_{\lambda\in\Lambda} \in U_\Lambda C_b^\infty(\mathbf{R}^N; \mathbf{R})$, $u \in A_{\leqq \ell_0+2}^{**}$, $u' \in A_{\leqq \ell_0}^{**}$, *satisfying the following equation.*

$$\Phi^\lambda(r(u)) = \sum_{u'\in A_{\leqq \ell_0}^{**}} \varphi_{u,u'}^\lambda \Phi^\lambda(r(u')), \qquad u \in A_{\leqq \ell_0+2}^{**}.$$

Now let us define a semigroup of linear operators $\{P_t^\lambda\}_{t\geqq 0}$ on $C_b^\infty(\mathbf{R}^N)$ by

$$(P_t^\lambda f)(x) = E^\mu[f(X^\lambda(t, x))], \qquad f \in C_b^\infty(\mathbf{R}^N).$$

We prove the following in this paper.

Theorem 1. *Assume* (U_ΛFG) *holds. Then for any* $n, m \geqq 0$ *with* $n + m \geqq 1$ *and* $u_1, \dots, u_{n+m} \in A^{**}$, *there exists a* $C > 0$ *such that*

$$\sup_{\lambda\in\Lambda} ||\Phi^\lambda(r(u_1)\cdots r(u_n))P_t^\lambda \Phi^\lambda(r(u_{n+1})\cdots r(u_{n+m}))f||_{L^p(\mathbf{R}^N;dx)} \leqq Ct^{-(||u_1||+\cdots+||u_{n+m}||)/2}||f||_{L^p(\mathbf{R}^N;dx)}$$

for any $p \in [1, \infty]$ *and* $f \in C_0^\infty(\mathbf{R}^N)$.

Now let $\tilde{V}_i^\lambda \in C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$, $\lambda \in \Lambda$, $i = 0,\dots,d$, and let $\tilde{X}^\lambda(t,x)$, $t \in [0,\infty)$, $x \in \mathbf{R}^N$, be a solution to the following SDE

$$\tilde{X}^\lambda(t,x) = x + \sum_{i=0}^{d} \int_0^t \tilde{V}_i^\lambda(\tilde{X}^\lambda(s,x)) \circ dB^i(s). \tag{2}$$

Let us define a semigroup of linear operators $\{\tilde{P}_t^\lambda\}_{t \geqq 0}$ on $C_b^\infty(\mathbf{R}^N)$ by

$$(\tilde{P}_t^\lambda f)(x) = E^\mu[f(\tilde{X}^\lambda(t,x))], \qquad f \in C_b^\infty(\mathbf{R}^N).$$

Then we have the following localization result.

Theorem 2. *Let $x_0 \in \mathbf{R}^N$ and $\varepsilon_0 > 0$. Assume that $\{V_i^\lambda\}_{\lambda\in\Lambda}$, $i = 0,1,\dots,d$, belongs to $U_\Lambda C_b^\infty(\mathbf{R}^N;\mathbf{R}^N)$ and satisfies* (U_ΛFG). *Assume moreover that*

$$\tilde{V}_i^\lambda(x) = V_i^\lambda(x), \qquad x \in B(x_0;2\varepsilon_0),\ \lambda \in \Lambda,\ i = 0,1,\dots,d.$$

Then for any $\varphi \in C_0^\infty(B(x_0;\varepsilon_0))$ and $u_1,\dots,u_n \in A^{**}$, $n \geqq 1$, there exists a $C > 0$ such that

$$\sup_{\lambda\in\Lambda, x\in\mathbf{R}^N} |(\Phi^\lambda(r(u_1)\cdots r(u_n))(\varphi\tilde{P}_t^\lambda f))(x)| \leqq Ct^{-(\|u_1\|+\cdots+\|u_n\|)/2} \sup_{x\in\mathbf{R}^N} |f(x)|$$

and

$$\sup_{\lambda\in\Lambda, x\in\mathbf{R}^N} |(\tilde{P}_t^\lambda\Phi^\lambda(r(u_1)\cdots r(u_n))(\varphi f))(x)| \leqq Ct^{-(\|u_1\|+\cdots+\|u_n\|)/2} \sup_{x\in\mathbf{R}^N} |f(x)|$$

for any $f \in C_b^\infty(\mathbf{R}^N;\mathbf{R})$. Here $B(x_0,\varepsilon_0)$ denotes the open ε_0-neighborhood of x_0.

We use Malliavin calculus to prove above theorems, and use the notation in Shigekawa [5] for Malliavin calculus. We regard $(W_0,\mathcal{F},\mu,\{\mathcal{F}_0^t\}_{t\geqq 0})$ as a filtered probability space, and use the following notation. $\mathcal{S}$ denotes the set of continuous $\{\mathcal{F}_0^t\}_{t \geqq 0}$-semimartingales. $S : \mathcal{S} \times A^* \to \mathcal{S}$ and $\hat{S} : \mathcal{S} \times A^* \to \mathcal{S}$ are defined inductively by

$$S(Z;1)(t) = Z(t), \quad t \geqq 0, \quad \hat{S}(Z;1)(t) = Z(t), \quad t \geqq 0, \qquad Z \in \mathcal{S},$$

and

$$S(Z;uv_i)(t) = \int_0^t S(Z,u)(s) \circ dB^i(s),$$
$$\hat{S}(Z;v_iu)(t) = -\int_0^t \tilde{S}(Z,u)(s) \circ dB^i(s), \quad t \geqq 0,$$

for any $Z \in \mathcal{S},\ i = 0, 1, \dots, d,\ u \in A^*$.

Also, we denote $S(1,u)(t)$ and $\hat{S}(1,u)$, $u \in A^*$, by $B(t;u)$ and $\hat{B}(t;u)$ respectvely.

2. Semimartingale on $\mathbf{R}\langle\langle A \rangle\rangle$

We say that $X : [0,\infty) \times W_0 \to \mathbf{R}\langle\langle A \rangle\rangle$ is an $\mathbf{R}\langle\langle A \rangle\rangle$-valued continuous semimartingale, if there are continuous semimartingales X_u, $u \in A^*$, such that $X(t) = \sum_{u \in A^*} X_u(t)u$. For $\mathbf{R}\langle\langle A \rangle\rangle$-valued continuous semimartingale $X(t), Y(t)$, we can define $\mathbf{R}\langle\langle A \rangle\rangle$-valued continuous semimartingales $\int_0^t X(s) \circ dY(s)$ and $\int_0^t \circ dX(s)Y(s)$ by

$$\int_0^t X(s) \circ dY(s) = \sum_{u,w \in A^*} (\int_0^t X_u(s) \circ dY_w(s))uw,$$
$$\int_0^t \circ dX(s)Y(s) = \sum_{u,w \in A^*} (\int_0^t Y_w(s) \circ dX_u(s))uw,$$

where

$$X(t) = \sum_{u \in A^*} X_u(t)u, \qquad Y(t) = \sum_{w \in A^*} Y_w(t)w.$$

Then we have

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s) \circ dY(s) + \int_0^t \circ dX(s)Y(s).$$

Since $\mathbf{R}$ is regarded a vector subspace in $\mathbf{R}\langle\langle A \rangle\rangle$, we can define $\int_0^t X(s) \circ dB^i(s)$, $i = 0, 1, \dots, d$, naturally.

Now let us consider the following SDE on $\mathbf{R}\langle\langle A\rangle\rangle$

$$\hat{X}(t) = 1 + \sum_{i=0}^{d} \int_0^t \hat{X}(s)v_i \circ dB^i(s), \qquad t \geqq 0. \tag{3}$$

One can easily solve this SDE and obtains

$$\hat{X}(t) = \sum_{u \in A^*} B(t;u)u. \tag{4}$$

We also have the following (c.f. [2]).

Proposition 3. $\log \hat{X}(t) \in \mathcal{L}((A))$, $t \geqq 0$, *with probability one.*

Note that

$$d(\hat{X}(t)^{-1}) = -\hat{X}(t)^{-1} d\hat{X}(t)\hat{X}(t)^{-1} = -\sum_{i=0}^{d} v_i \hat{X}(t)^{-1} \circ dB^i(t)$$

and so

$$\hat{X}(t)^{-1} = 1 - \sum_{i=0}^{d} \int_0^t v_i \hat{X}(s)^{-1} \circ dB^i(s).$$

3. Uniform Estimates

We assume the condition (U_ΛFG) for $\{V_i^\lambda;\ i = 0, 1, \dots, d,\ \lambda \in \Lambda\}$ throughout this paper. The argument in this section is essentially the same as in Sections 2 and 3 in [1], or [2], and so we state results sometimes without proofs.

Proposition 4. *There are* $\{\varphi_{u,u'}^\lambda\}_{\lambda \in \Lambda} \in U_\Lambda C_b^\infty(\mathbf{R}^N)$, $u \in A^{**}$, $u' \in A_{\leqq \ell_0}^{**}$ *such that*

$$\Phi^\lambda(r(u)) = \sum_{u' \in A_{\leqq \ell_0}^{**}} \varphi_{u,u'}^\lambda \Phi^\lambda(r(u')), \qquad u \in A^{**}.$$

PROOF. It is obivious that our assetion is valid for $u \in A^{**}_{\leqq \ell_0+2}$. Suppose that our assertion is valid for any $u \in A^{**}_{\leqq m}$, $m \geqq \ell_0 + 2$. Then we have for any $i = 0, 1, \ldots, d$ and $u \in A^{**}_{\leqq m}$,

$$\Phi^\lambda(r(v_i u)) = [V_i^\lambda, \Phi^\lambda(r(u))] = \sum_{u' \in A^{**}_{\leqq \ell_0}} [V_i^\lambda, \varphi^\lambda_{u,u'} \Phi^\lambda(r(u'))]$$

$$= \sum_{u' \in A^{**}_{\leqq \ell_0}} (V_i^\lambda \varphi^\lambda_{u,u'}) \Phi^\lambda(r(u')) + \sum_{u', u'' \in A^{**}_{\leqq \ell_0}} \varphi^\lambda_{u,u'} \varphi^\lambda_{v_i u', u''} \Phi^\lambda(r(u'')).$$

So we see that our assertion is valid for any $u \in A^{**}_{\leqq m+1}$. Thus by induction we have our assertion. □

For any C^∞ vector field W on $\mathbf{R}^N$, we see that

$$d(X^\lambda(t)_*^{-1} W)(x) = \sum_{i=0}^{d} (X^\lambda(t)_*^{-1} [V_i^\lambda, W])(x) \circ dB^i(t),$$

where $X^\lambda(t)_*$ is a push-forward operator with respect to the diffeomorphism $X^\lambda(t, \cdot) : \mathbf{R}^N \to \mathbf{R}^N$. So we have

$$d(X^\lambda(t)_*^{-1} \Phi^\lambda(r(u)))(x)$$

$$= \sum_{i=0}^{d} ((X(t)_*^\lambda)^{-1} \Phi^\lambda(r(v_i u)))(x) \circ dB^i(t)$$

for any $u \in A^* \setminus \{1\}$.

Let $m \geqq 3\ell_0$. Let $\{c_i^{\lambda,m}(\cdot, u, u')\}_{\lambda \in \Lambda} \in U_\Lambda C_b^\infty(\mathbf{R}^N, \mathbf{R})$, $i = 0, 1, \ldots, d$, $u, u' \in A^{**}_{\leqq m}$, be given by

$$c_i^{\lambda,m}(x; u, u') = \begin{cases} 1, & \text{if } ||v_i u|| \leqq m \text{ and } u' = v_i u, \\ \varphi^\lambda_{v_i u, u'}(x), & \text{if } ||v_i u|| > m \text{ and } ||u'|| \leqq \ell_0, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\varphi^\lambda_{u,u'}$'s are as in Proposition 4. Then we have

$$d(X^\lambda(t)_*^{-1} \Phi^\lambda(r(u)))(x)$$

$$= \sum_{i=0}^{d} \sum_{u' \in A^{**}_{\leqq m}} (c_i^{\lambda,m}(X^\lambda(t, x); u, u')(X^\lambda(t)_*^{-1} \Phi^\lambda(r(u')))(x) \circ dB^i(t)$$

for any $u \in A^{**}_{\leqq m}$.

Let $a^{\lambda,m}(t,x;u,u')$, $u,u' \in A^{**}_{\leqq m}$, be the solution to the following SDE

$$\begin{aligned}
&da^{\lambda,m}(t,x;u,u') \\
&= \sum_{i=0}^{d} \sum_{u'' \in A^{**}_{\leqq m}} c_i^{\lambda,m}(X^\lambda(t,x);u,u'')a^{\lambda,m}(t,x;u'',u')dB^i(t) \\
&\quad + \frac{1}{2}\sum_{i=1}^{d} \sum_{u'' \in A^{**}} (V_i^\lambda c_i^{\lambda,m})(X^\lambda(t,x);u'',u')a^{\lambda,m}(t,x;u'',u')dt \\
&\quad + \frac{1}{2}\sum_{i=1}^{d} \sum_{u_1,u_2' \in A^{**}_{\leqq m}} (c_i^{\lambda,m}(X^\lambda(t,x);u,u_1) \\
&\quad \times c_i^{\lambda,m}(X^\lambda(t,x);u_1,u_2)a^{\lambda,m}(t,x;u_2,u')dt, \\
&a^{\lambda,m}(0,x;u,u') = \langle u,u' \rangle.
\end{aligned}$$

Such a solution exists uniquely, and moreover, we may assume that $a^{\lambda,m}(t,x;u,u')$ is smooth in x with probability one. Then we have

$$\sup_{\lambda\in\Lambda, x\in\mathbf{R}^N} E^\mu[\sup_{t\in[0,T]} |\frac{\partial^{|\alpha|}}{\partial x^\alpha} a^{\lambda,m}(t,x;u,u')|^p] < \infty, \qquad p\in[1,\infty),\ T>0$$

for any multi-index α. One can easily see that

$$\begin{aligned}
(5) \qquad &da^{\lambda,m}(t,x;u,u') \\
&= \sum_{i=0}^{d} \sum_{u'' \in A^{**}_{\leqq m}} (c_i^{\lambda,m}(X^\lambda(t,x);u,u'')a^{\lambda,m}(t,x;u'',u')) \circ dB^i(t).
\end{aligned}$$

Then the uniqueness of SDE implies

$$(X^\lambda(t)_*^{-1}\Phi^\lambda(r(u)))(x) = \sum_{u' \in A^{**}_{\leqq m}} a^{\lambda,m}(t,x;u,u')\Phi^\lambda(r(u'))(x),\ u \in A^{**}_{\leqq m}.$$

Similarly we see that there exists a unique solution $b^{\lambda,m}(t,x;u,u')$,

$u, u' \in A^{**}_{\leqq m}$, to the SDE

$$b^{\lambda,m}(t, x; u, u') \tag{6}$$
$$= \langle u, u' \rangle - \sum_{i=0}^{d} \sum_{u'' \in A^{**}_{\leqq m}} \int_0^t (b^{\lambda,m}(s, x; u, u''))(c_i^{\lambda,m}(X^\lambda(s, x); u'', u')) \circ dB^i(t).$$

Then we see that

$$\sum_{u'' \in A^{**}_{\leqq m}} a^{\lambda,m}(t, x, u, u'') b^{\lambda,m}(t, x, u'', u') = \langle u, u' \rangle, \qquad u, u' \in A^{**}_{\leqq m},$$
$$\Phi^\lambda(r(u))(x) = \sum_{u' \in A^{**}_{\leqq m}} b^{\lambda,m}(t, x; u, u')(X^\lambda(t)_*^{-1} \Phi^\lambda(r(u')))(x), \ u \in A^{**}_{\leqq m},$$

and

$$\sup_{\lambda \in \Lambda, x \in \mathbf{R}^N} E^\mu[\sup_{t \in [0,T]} |\frac{\partial^{|\alpha|}}{\partial x^\alpha} b^{\lambda,m}(t, x; u, u')|^p] < \infty, \qquad p \in [1, \infty), \ T > 0$$

for any multi-index α. Let

$$R^*_m = \{v_0 u; u \in A^*, ||u|| = m - 1\} \cup \bigcup_{i=0}^{d} \{v_i u; u \in A^*, ||u|| = m\}.$$

Then we have the following.

Proposition 5. *For any $m \geqq 3\ell_0$,*

$$a^{\lambda,m}(t, x, u, u')$$
$$= \sum_{u_1 \in A^*_{\leqq m}} \langle u_1 u, u' \rangle B(t, u_1)$$
$$+ \sum_{u_1 \in A^* : u_1 u \in R^*_m} \sum_{u_2 \in A^*_{\leqq \ell_0}} S(\varphi_{u_1 u, u_2}(X^\lambda(\cdot, x)) a^{\lambda,m}(\cdot, x, u_2, u'), u_1)(t)$$

for any $t \in [0, \infty)$, $x \in \mathbf{R}^N$, and $u, u' \in A^{**}_{\leqq m}$.

Proof. The assertion is obvious from the definition, if $||u|| = m$. Note that

$$a^{\lambda,m}(t,x;u,u') = \langle u,u'\rangle + \sum_{i=0}^{d} \sum_{u_1 \in A^{**}_{\leqq m}} S(c_i^{\lambda,m}(X^{\lambda}(\cdot,x);u,u_1)a^{\lambda,m}(\cdot,x;u_1,u'),v_i)(t).$$

Therefore, if $||u|| = m-1$, we have

$$\begin{aligned} a^{\lambda,m}(t,x;u,u') &= \langle u,u'\rangle + \sum_{i=1}^{d} S(\langle v_i u,u'\rangle a^{\lambda,m}(\cdot,x;v_i u,u'),v_i)(t) \\ &\quad + \sum_{u_1 \in A^{**}_{\leqq \ell_0}} S(\varphi_{v_0 u,u_1}(X^{\lambda}(\cdot,x))a^{\lambda,m}(\cdot,x,u_1,u'),v_0)(t) \\ &= \langle u,u'\rangle + \sum_{i=1}^{d} \langle v_i u,u'\rangle B(t,v_i) \\ &\quad + \sum_{i=1}^{d}\sum_{j=0}^{d} \sum_{u_1 \in A^{**}_{\leqq \ell_0}} S(S(\varphi_{v_j v_i u,u_1}(X^{\lambda}(\cdot,x))a^{\lambda,m}(\cdot,x,u_1,u'),v_j),v_i)(t) \\ &\quad + \sum_{u_1 \in A^{**}_{\leqq \ell_0}} S(\varphi_{v_0 u,u_1}(X^{\lambda}(\cdot,x))a^{\lambda,m}(\cdot,x,u_1,u'),v_0)(t). \end{aligned}$$

So we have our assertion. Similarly by induction in $m - ||u||$ we have our assertion. □

Then by Equation (4), we have the following.

Corollary 6. *For any $m \geqq 3\ell_0$,*

$$\begin{aligned} a^{\lambda,m}(t,x;u,u') &= \langle \hat{X}(t)u,u'\rangle \\ &\quad + \sum_{u_1 \in A^*: u_1 u \in R^*_m} \sum_{u_2 \in A^*_{\leqq \ell_0}} S(\varphi_{u_1 u,u_2}(X^{\lambda}(\cdot,x))a^{\lambda,m}(\cdot,x;u_2,u'),u_1)(t) \end{aligned}$$

for any $t \in [0,\infty)$, $x \in \mathbf{R}^N$, *and* $u, u' \in A^{**}_{\leqq m}$. *In particular,*

$$a^{\lambda,m}(t,x;v_i,u) = \langle \hat{X}(t)v_i, u\rangle$$
$$+ \sum_{u_1\in A^*:u_1v_i\in R^*_m} \sum_{u_2\in A^*_{\leqq \ell_0}} S(\varphi^\lambda_{u_1v_i,u_2}(X^\lambda(\cdot,x))\langle \hat{X}(\cdot)u_2, v_i\rangle, u_1)(t)$$
$$+ \sum_{u_1\in A^*:u_1u\in R^*_m} \sum_{u_2\in A^*_{\leqq \ell_0}} \sum_{u_3\in A^*:u_3u_2\in R^*_m} \sum_{u_4\in A^*_{\leqq \ell_0}}$$
$$S(\varphi^\lambda_{u_1v_i,u_2}(X^\lambda(\cdot,x))S(\varphi^\lambda_{u_3u_2,u_4}(X^\lambda(\cdot,x))a^{\lambda,m}(\cdot,x,u_4,u),u_3),u_1)(t).$$

Here $\hat{X}(t)$ *is a solution to SDE* (3).

Proposition 7. *Let* $m \geqq 3\ell_0$.
(1) *For any* $u \in A^{**}_{\leqq m}$, $u' \in A^*$, $i = 0,1,\ldots,d$ *with* $v_iu' \in A^{**}_{\leqq m}$, *if* $||v_iu'|| > \ell_0$, *then*

$$b^{\lambda,m}(t,x,u,v_iu') = \tilde{S}(b^{\lambda,m}(\cdot,x,u,u');v_i) + \langle u, v_iu'\rangle,$$

and if $||v_iu'|| \leqq \ell_0$, *then*

$$b^{\lambda,m}(t,x,u,v_iu') = \tilde{S}(b^{\lambda,m}(\cdot,x,u,u');v_i)(t) + \langle u, v_iu'\rangle$$
$$+\sum_{j=0}^{d} \sum_{u_1\in A^{**}_{\leqq m}, v_ju_1\in R^*_m} \tilde{S}(b^{\lambda,m}(\cdot,x,u,v_ju_1)\varphi^\lambda_{v_ju_1,v_iu'}(X^\lambda(\cdot,x));v_j)(t)$$

for any $t \in [0,\infty)$, $x \in \mathbf{R}^N$, *and* $\lambda \in \Lambda$.
(2) *For any* $u, u_2 \in A^{**}_{\leqq m}$, $u_1 \in A^*$ *with* $||u_2|| \geqq \ell_0$, $||u|| \leqq ||u_2||$ *and* $||u_1u_2|| \leqq m$,

$$b^{\lambda,m}(t,x,u,u_1u_2) = \tilde{S}(b^{\lambda,m}(\cdot,x,u,u_2);u_1).$$

Proof. Since we have

$$b^{\lambda,m}(t,x,u,v_iu')$$
$$= \langle u, v_iu'\rangle + \sum_{j=0}^{d} \sum_{u_1\in A^{**}_{\leqq m}} \tilde{S}(b^{\lambda,m}(\cdot,x,u,u_1)c^{\lambda,m}_j(X^\lambda(\cdot,x));u_1,v_iu');v_j)(t),$$

we have the assertion (1) from the definition of $c_j^{\lambda,m}$.

The assertion (2) is an easy consequence of the first part of the assertion (1). □

Let E be a separable real Hilbert space and $r \in \mathbf{R}$. Let us denote by $W^{\infty,\infty-}(E) \bigcap_{s \geqq 0, p\in(1,\infty)} W^{s,p}(E)$. Let $\mathcal{K}_\Lambda(E)$ denote the set of families $\{f_\lambda\}_{\lambda\in\Lambda}$ of functionals $f_\lambda : (0,1] \times \mathbf{R}^N \to W^{\infty,\infty-}(E)$ satisfying the following two conditions.
(1) $f_\lambda(t,x)$ is smooth in x and $\frac{\partial^\alpha}{\partial x^\alpha} f_\lambda(t,x)$ is continuous in $(t,x) \in (0,1]\times\mathbf{R}^N$ for any multi-index α.
(2) $\sup_{\lambda\in\Lambda, t\in(0,1], x\in\mathbf{R}^N} \| \frac{\partial^\alpha}{\partial^\alpha x} f_\lambda(t,x) \|_{W^{s,p}(E)} < \infty$, for any multi-index α,$s \in \mathbf{R}$ and $p \in (1,\infty)$.
We denote $\mathcal{K}_\Lambda(\mathbf{R})$ by $\mathcal{K}_\Lambda$.

By checking carefully the estimates discussed in Chapter 6 in Shigekawa [5], we see that $\{a^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ and $\{b^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ belong to $\mathcal{K}_\Lambda$ for any $u,u' \in A^*_{\leqq m}$.

Then by Corollary 6, we have the following.

PROPOSITION 8. *For any* $u,u' \in A^*_{\leqq m}$, $\{t^{-m/2}(a^{\lambda,m}(t,x;u,u') - \langle \hat{X}(t)u,u'\rangle)\}_{\lambda\in\Lambda}$ *belong to* $\mathcal{K}_\Lambda$. *In particular,* $\{t^{-((\|u'\|-\|u\|)\vee 0)/2} a^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ *belong to* $\mathcal{K}_\Lambda$.

Similarly by Proposition 7 we have the following.

PROPOSITION 9. *For any* $u,u' \in A^*_{\leqq m}$, $\{t^{-((\|u'\|-\|u\|)\vee 0)/2} b^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ *belong to* $\mathcal{K}_\Lambda$.

Now let $k^{\lambda,m}(t,x;u) \in H$, $\lambda \in \Lambda$, $(t,x) \in [0,\infty) \times \mathbf{R}^N$, $u \in A^{**}_{\leqq m}$, be given by

$$k^{\lambda,m}(t,x;u) = (\int_0^{t\wedge\cdot} a^{\lambda,m}(s,x;v_i,u)ds)_{i=1,\dots,d}.$$

Then we have the following.

PROPOSITION 10. *For any* $u \in A^*_{\leqq m}$, $\{t^{-\|u\|/2} k^{\lambda,m}(t,x;u)\}_{\lambda\in\Lambda}$ *belong to* $\mathcal{K}_\Lambda(H)$.

Let $M^{\lambda,m}(t,x;u,u')$, $(t,x) \in [0,\infty) \times \mathbf{R}^N$, $u,u' \in A^{**}_{\leqq m}$, be given by

$$M^{\lambda,m}(t,x;u,u') = t^{-(||u||+||u'||)/2}(k^{\lambda,m}(t,x;u), k^{\lambda,m}(t,x;u'))_H \qquad (7)$$
$$= t^{-(||u||+||u'||)/2} \sum_{i=1}^{d} \int_0^t a^{\lambda,m}(s,x;v_i,u) a^{\lambda,m}(s,x;v_i,u') ds.$$

Also, let $\hat{M}^{(m)}(t;u,u')$, $(t,x) \in [0,\infty) \times \mathbf{R}^N$, $u,u' \in A^{**}_{\leqq m}$, be given by

$$\hat{M}^{(m)}(t;u,u') = t^{-(||u||+||u'||)/2} \sum_{i=1}^{d} \int_0^t \langle \hat{X}(t)v_i, u\rangle \langle \hat{X}(t)v_i, u'\rangle. \qquad (8)$$

We can prove the following from Propositions 8 and 9 by the exactly same method as in [1] Section 4 .

Proposition 11. (1) *For any* $p \in (1,\infty)$,

$$\sup_{\lambda\in\Lambda, t\in(0,1], x\in\mathbf{R}^N} E^{\mu}[det(M^{\lambda,m}(t,x;u,u'))^{-p}_{u,u'\in A^{**}_{\leqq m}}] < \infty.$$

(2) *For any* $p \in (1,\infty)$,

$$\sup_{t\in(0,1]} E^{\mu}[det(\hat{M}^{(m)}(t;u,u'))^{-p}_{u,u'\in A^{**}_{\leqq m}}] < \infty.$$

(3) $\{t^{-1/2}(M^{\lambda,m}(t,x;u,u') - \hat{M}^{(m)}(t;u,u'))\}_{\lambda\in\Lambda}$ *belong to* $\mathcal{K}_\Lambda$ *for any* $u,u' \in A^{**}_{\leqq m}$.

Let $(\check{M}^{\lambda,m}(t,x;u,u'))_{u,u'\in A^{**}_{\leqq m}}$ be the inverse matrix of $(M^{\lambda,m}(t,x;u,u'))_{u,u'\in A^{**}_{\leqq m}}$ and $(\tilde{M}^{(m)}(t;u,u'))_{u,u'\in A^{**}_{\leqq m}}$ be the inverse matrix of $(\hat{M}^{(m)}(t,x;u,u'))_{u,u'\in A^{**}_{\leqq m}}$.

Then we have the following.

Corollary 12. $\{\check{M}^{\lambda,m}(t,x;u,u')\}_{\lambda\in\Lambda}$ *and* $\{\tilde{M}^{(m)}(t;u,u')\}_{\lambda\in\Lambda}$ *belong to* $\mathcal{K}_\Lambda$ *for any* $u,u' \in A^{**}_{\leqq m}$. *Moreover,* $\{t^{-1/2}(\check{M}^{\lambda,m}(t,x;u,u') - \tilde{M}^{(m)}(t;u,u'))\}_{\lambda\in\Lambda}$ *belong to* $\mathcal{K}_\Lambda$ *for any* $u,u' \in A^{**}_{\leqq m}$.

Note that

$$X^{\lambda}(t)_*^{-1}DX^{\lambda}(t,x) = (\int_0^{t\wedge\cdot} (X^{\lambda}(s)_*^{-1}V_i^{\lambda})(x)ds)_{i=1,\dots,d}$$
$$= \sum_{u\in A^{**}_{\leqq m}} k^{\lambda,m}(t,x;u)\Phi^{\lambda}(r(u))(x)$$

for $(t,x)\in[0,\infty)\times\mathbf{R}^N$ (c.f.[3]). Let $f\in C_b^{\infty}(\mathbf{R}^N)$. Since we have

$$D(f(X^{\lambda}(t,x))) = {}_{T_x^*}\langle (X^{\lambda}(t)^* df)(x), X^{\lambda}(t)_*^{-1}DX^{\lambda}(t,x)\rangle_{T_x},$$

we see that

$$(D(f(X^{\lambda}(t,x))), k^{\lambda,m}(t,x;u))_H$$
$$= \sum_{u'\in A^{**}_{\leqq m}} \langle (X^{\lambda}(t)^* df)(x), \Phi^{\lambda}(r(u'))\rangle_x t^{(||u||+||u'||)/2} M^{\lambda,m}(t,x;u,u').$$

So we have

(9) $$t^{||u||/2}\Phi^{\lambda}(r(u))(f(X^{\lambda}(t,\cdot)))(x) = {}_{T_x^*}\langle (X^{\lambda}(t)^* df)(x), \Phi^{\lambda}(r(u))\rangle_{T_x}$$
$$= \sum_{u'\in A^{**}_{\leqq m}} \check{M}^{\lambda,m}(t,x;u,u')t^{-||u'||/2}(D(f(X^{\lambda}(t,x)), k^{\lambda,m}(t,x;u'))_H$$

and

(10) $$t^{||u||/2}(\Phi^{\lambda}(r(u))f)(X^{\lambda}(t,x))$$
$$= \sum_{u_1,u_2\in A^{**}_{\leqq m}} \check{M}^{\lambda,m}(t,x;u_1,u_2)t^{-(||u_1||-||u||)/2} b^{\lambda,m}(t,x;u,u_1)$$
$$\times t^{-||u_2||/2}(D(f(X^{\lambda}(t,x)), k^{\lambda,m}(t,x;u_2))_H.$$

Therefore we have the following.

Theorem 13. *Let $f\in C_b^{\infty}(\mathbf{R}^N)$. Then we have the following.*
(1) *For any $u\in A^{**}_{\leqq m}$, $p\in(1,\infty)$ and $r>0$,*

$$\sup_{t\in(0,1],\lambda\in\Lambda,x\in\mathbf{R}^N} ||t^{||u||/2}(\Phi^{\lambda}(r(u))f)(X^{\lambda}(t,\cdot))(x)||_{W^{r,p}} < \infty.$$

(2) *For any* $F \in W^{\infty,\infty-}$ *and* $u \in A^{**}_{\leqq m}$, *we have*

$$t^{||u||/2}\Phi^{\lambda}(r(u))(E^{\mu}[Ff(X^{\lambda}(t,\cdot))](x) = E^{\mu}[(\mathcal{R}^{\lambda}_{0}(t,x;u)F)f(X^{\lambda}(t,x))]$$

and

$$E^{\mu}[Ft^{||u||/2}(\Phi^{\lambda}(r(u))f)(X^{\lambda}(t,x))] = E^{\mu}[(\mathcal{R}^{\lambda}_{1}(t,x;u)F)f(X^{\lambda}(t,x))].$$

Here

$$\mathcal{R}^{\lambda}_{0}(t,x;u)F = \sum_{u' \in A^{**}_{\leqq m}} D^{*}(\check{M}^{\lambda,m}(t,x;u,u')t^{-||u'||/2}k^{\lambda,m}(t,x;u')F)$$

and

$$\mathcal{R}^{\lambda}_{1}(t,x;u)F = \sum_{u_1,u_2 \in A^{**}_{\leqq m}} D^{*}(\check{M}^{\lambda,m}(t,x;u_1,u_2)t^{-(||u_1||-||u||)/2}b^{\lambda,m}(t,x;u,u_1) \times t^{-||u_2||/2}k^{\lambda,m}(t,x;u_2)F).$$

One can easily prove the following.

Proposition 14. *If* $\{F_{\lambda}(t,x)\}_{\lambda\in\Lambda}$ *belongs to* $\mathcal{K}_{\Lambda}$, *then* $\{\mathcal{R}^{\lambda}_{0}(t,x;u)(F_{\lambda}(t,x))\}_{\lambda\in\Lambda}$ *and* $\{\mathcal{R}^{\lambda}_{1}(t,x;u)(F_{\lambda}(t,x))\}_{\lambda\in\Lambda}$ *belong to* $\mathcal{K}_{\Lambda}$.

Then by Theorem 13 and Proposition 14, we have the following.

Proposition 15. *For any* $n, m \geqq 0$ *with* $n+m \geqq 1$ *and* $u_1, \dots, u_{n+m} \in A^{**}$, *there exists a* $C_0 > 0$ *such that*

$$\sup_{\lambda\in\Lambda} ||\Phi^{\lambda}(r(u_1)\cdots r(u_n))P^{\lambda}_{t}\Phi^{\lambda}(r(u_{n+1})\cdots r(u_{n+m}))f||_{L^{\infty}(\mathbf{R}^N;dx)} \leqq C_0 t^{-(||u_1||+\cdots+||u_{n+m}||)/2}||f||_{L^{\infty}(\mathbf{R}^N;dx)}$$

for any $f \in C^{\infty}_{b}(\mathbf{R}^N)$.

Now let us define $\{g^{\lambda}_{k}\}_{\lambda\in\Lambda}, \in U_{\Lambda}C^{\infty}_{b}(\mathbf{R}^N;\mathbf{R})$, $k = 0, 1 \dots, d$, $\{\hat{V}^{\lambda}_{k}\}_{\lambda\in\Lambda} \in U_{\Lambda}C^{\infty}_{b}(\mathbf{R}^N;\mathbf{R}^N)$, $k = 0, 1, \dots, d$, and $\{c^{\lambda}\}_{\lambda\in\Lambda} \in U_{\Lambda}C^{\infty}_{b}(\mathbf{R}^N;\mathbf{R})$ by

$$g^{\lambda}_{k}(x) = \sum_{j=1}^{N} \frac{\partial}{\partial x^j} V^{\lambda j}_{i}(x) \qquad k = 0, 1, \dots, d,$$

$$\hat{V}_0^\lambda(x) = -V_0^\lambda(x) + \sum_{i=1}^d g_i^\lambda(x) V_i^\lambda(x),$$
$$\hat{V}_k^\lambda(x) = V_i^\lambda(x), \ k = 1, \dots, d,$$
$$c^\lambda(x) = -g_0^\lambda(x) + \frac{1}{2}\sum_{j=1}^d (g_j^\lambda(x)^2 + (V_j^\lambda(x) g_j^\lambda)(x)).$$

Then we see that $\{\hat{V}_i^\lambda; i = 0, 1, \dots, d, \lambda \in \Lambda\}$ satisfies the condition (U_ΛFG). Let $\hat{X}^\lambda(t, x)$, $t \geqq 0$, $x \in \mathbf{R}^N$, $\lambda \in \Lambda$, be the solution to the following stochastic differenntial equation

$$\hat{X}^\lambda(t, x) = x + \sum_{i=0}^d \int_0^t \hat{V}_i^\lambda(\hat{X}^\lambda(s, x)) \circ dB^i(s).$$

Let $\hat{P}_t^\lambda$, $t \geqq 0$, be a linear operator in $C_b^\infty(\mathbf{R}^N)$ given by

$$(\hat{P}_t^\lambda f)(x) = E[K_\lambda(t, x) f(\hat{X}^\lambda(t, x))], \qquad f \in C_b^\infty(\mathbf{R}^N),$$

where

$$K_\lambda(t, x) = \exp(\int_0^t c(\hat{X}^\lambda(s, x)) ds), \qquad x \in \mathbf{R}^N, \ t \geqq 0.$$

Note that $\{K_\lambda(t, x)\}_{\lambda \in \Lambda}$ belongs to $\mathcal{K}_\Lambda$. So again by Theorem 13, Proposition 14 and definition for $\hat{V}_k^\lambda$, we have the following.

Proposition 16. *For any $n, m \geqq 0$ with $n + m \geqq 1$ and $u_1, \dots, u_{n+m} \in A^{**}$, there exists a $C_1 > 0$ such that*

$$\sup_{\lambda \in \Lambda} ||\Phi^\lambda(r(u_1) \cdots r(u_n)) \hat{P}_t^\lambda \Phi^\lambda(r(u_{n+1}) \cdots r(u_{n+m})) f||_{L^\infty(\mathbf{R}^N; dx)}$$
$$\leqq C_1 t^{-(||u_1|| + \cdots + ||u_{n+m}||)/2} ||f||_{L^\infty(\mathbf{R}^N; dx)}$$

for any $f \in C_b^\infty(\mathbf{R}^N)$.

Observe that

$$\int_{\mathbf{R}^N} (P_t^\lambda f_1)(x) f_2(x) dx = \int_{\mathbf{R}^N} f_1(x) (\hat{P}_t^\lambda f_2)(x) dx, \qquad t \geqq 0,$$

for any $f_1, f_2 \in C_0^\infty(\mathbf{R}^N)$.

Now Theorem 1 is an easy consequence of this fact and Propositions 15 and 16.

4. Localization

First, we remind the following result (c.f. Stroock-Varadhan [6] Theorem 2.1.3).

Proposition 17. *Let E be a mormed space. Let $T, B > 0$ $\beta \in (0,1)$, and $p \in (2/\beta, \infty)$. Suppose that a continuous function $\phi : [0,T] \to E$ satisfies*

$$\int_0^T \int_0^T (\frac{||\phi(t) - \phi(s)||_E}{|t-s|^\beta})^p ds dt \leqq B.$$

Then we have

$$||\phi(t) - \phi(s)||_E \leqq \frac{8\beta(4B)^{1/p}}{\beta - 2/p}|t-s|^{\beta - 2/p}, \qquad t, s \in [0,T].$$

Now let $x_0 \in \mathbf{R}^N$, $\varepsilon_0 > 0$. $\tilde{V}_i^\lambda : \mathbf{R}^N \to \mathbf{R}^N$, and $V_i^\lambda : \mathbf{R}^N \to \mathbf{R}^N$, $\lambda \in \Lambda$, $i = 0, \ldots, d$, be as in Theorem 2. Also, let $X^\lambda(t,x)$ and $\tilde{X}^\lambda(t,x)$ be solutions to Equation (1) and (2) respectively. We may assume that $x_0 = 0$, and $\varepsilon_0 < 1/2$.

By checking the computation in Shigekawa [5] Section 6, we see that for any $n \geqq 1$, $k \geqq 0$ and multi-index $\alpha \in \mathbf{Z}^N_{\geqq 0}$, there is a $C > 0$ such that

$$\sup_{\lambda \in \Lambda, x \in \mathbf{R}^N} E^\mu[||D^k \frac{\partial^\alpha}{\partial x^\alpha} X^\lambda(t,x) - D^k \frac{\partial^\alpha}{\partial x^\alpha} X^\lambda(s,x)||^{2n}_{H^{\otimes k} \otimes (\mathbf{R}^N)^{\otimes k+1}}] \leqq C|t-s|^n$$

for all $t, s \in [0,1]$.

Let $\tilde{Y}^\lambda(T) : W_0 \to [0,\infty)$, $T \in (0,1]$ given by

$$\tilde{Y}^\lambda(T) = \int_0^T \int_0^T dt\, ds$$
$$\times \int_{|x|<2} dx \frac{|X^\lambda(t,x) - X^\lambda(s,x)|^{2(N+2)} + |\nabla_x X^\lambda(t,x) - \nabla_x X^\lambda(s,x)|^{2(N+2)}}{|t-s|^{N+2}}$$

$\tilde{Y}^\lambda(T)$ is $\mathcal{F}_0^T$ measurable. Also, we see that for any $k \geqq 0$ and $p \in (1,\infty)$ there is a $C > 0$ such that

$$\sup_{\lambda \in \Lambda} ||\tilde{Y}^\lambda(T)||_{W^{k,p}} \leqq CT^2, \qquad T \in (0,1].$$

Thus we see that

$$\sup_{\lambda\in\Lambda, T\in(0,1]} T^{-2}||\tilde{Y}^{\lambda}(T)||_{W^{r,p}} < \infty \tag{11}$$

for any $r > 0$ and $p \in (1, \infty)$.

Let us take a $\rho \in C_0^{\infty}(\mathbf{R}; \mathbf{R})$ such that $0 \leqq \rho \leqq 1$, $\rho(z) = 1$, $|z| \leqq 1$, and $\rho(z) = 0$, $|z| > 2$.

Then we have the following.

PROPOSITION 18. (1) *There is a* $C_0 > 0$ *such that*

$$E^{\mu}[\rho(T^{-1}\tilde{Y}^{\lambda}(T)), \sup_{x\in B(0,2), t\in[0,T]} |X^{\lambda}(t,x) - x| \geqq C_0 T^{1/3}] = 0$$

for any $\lambda \in \Lambda$, $T \in (0, 1]$.
(2) *For any* $r > 1$

$$\sup_{\lambda\in\Lambda, T\in(0,1]} T^{-r} E^{\mu}[1 - \rho(T^{-1}\tilde{Y}^{\lambda}(T))] < \infty.$$

(3) *For any* $n \geqq 1$ $p \in (1, \infty)$ *and* $r > 1$,

$$\sup_{\lambda\in\Lambda, T\in(0,1]} T^{-r} (\sum_{k=1}^{n} E^{\mu}[||D^k(\rho(T^{-1}\tilde{Y}^{\lambda}(T)))||_{H^{\otimes k}}^p]^{1/p}) < \infty.$$

PROOF. Let E_N be a normed space such that $E_N = C^{\infty}(B(0,2); \mathbf{R}^N)$ as a set and the norm $|| \ ||_{E_N}$ of E_N is given by

$$||f||_{E_N} = (\int_{B(0,2)} (|f(x)|^{2(N+2)} + |\nabla f(x)|^{2(N+2)}) dx)^{1/(2(N+2))}, \qquad f \in E_N.$$

Then by Sobolev's iequality, there is a constant $C_N > 0$ such that

$$\sup_{x\in B(0,2)} |f(x)| \leqq C_N ||f||_{E_N}, \qquad f \in E_N.$$

Note that

$$\tilde{Y}^{\lambda}(T) = \int_0^T \int_0^T dt\, ds (\frac{||X^{\lambda}(t,\cdot) - X^{\lambda}(s,\cdot)||_{E_N}}{|t-s|^{1/2}})^{2(N+2)}.$$

So, applying Proposition 17 for $p = 2(N+2)$, $B = T$, and $\beta = 1/2$, we see that if $\tilde{Y}^\lambda(T) \leqq 2T$, then

$$\sup_{x\in B(0,2)} |X^\lambda(t,x) - X^\lambda(s,x)| \leqq C_N ||X^\lambda(t,\cdot) - X^\lambda(s,\cdot)||_{E_N}$$
$$\leqq \frac{4C_N(8T)^{1/p}}{\beta - 2/p}|t-s|^{\beta - 2/p}, \qquad t,s, \in [0,T],$$

which implies

$$\sup_{x\in B(0,2), t\in[0,T]} |X^\lambda(t,x) - x| \leqq \frac{4C_N 8(2N+4)}{N} T^{(N+1)/(2N+4)}.$$

Since $(N+1)/(2N+4) \geqq 1/3$, we have the assetion (1).

Note that

$$E^\mu[1 - \rho(T^{-1}\tilde{Y}^\lambda(T))] \leqq \mu(T^{-1}\tilde{Y}^\lambda(T) \geqq 1) \leqq T^r E^\mu[(T^{-2}\tilde{Y}^\lambda(T))^r].$$

This and Equation (11) imply the assertion (2).

Since we have

$$D(\rho(T^{-1}\tilde{Y}^\lambda(T))) = T^{-1}\rho'(T^{-1}\tilde{Y}^\lambda(T))D\tilde{Y}^\lambda(T),$$

we see that

$$E^\mu[||D(\rho(T^{-1}\tilde{Y}^\lambda(T)))||_H^p]^{1/p}$$
$$\leqq T(\sup_{z\in\mathbf{R}} |\rho'(z)|)\mu(T^{-1}\tilde{Y}^\lambda(T) > 1)^{1/p} T^{-2} ||\tilde{Y}^\lambda(T)||_{W^{1,p}}.$$

So we have the assetion (3) for $n = 1$.

Similarly, we have the assertion (3) for $n \geqq 2$ also. □

Proposition 19. *Suppose that $U_j \in W^{\infty,\infty-}$, $j = 1,\dots,m$, and assume that $|U_j| \leqq 1$ $\mu - a.s.$ $j = 1,\dots,m$. Then for any $n \geqq 1$*

$$||D^n(\prod_{j=1}^m U_j)||_{H^{\otimes n}} \leqq n^{n+1} \sum_{k=1}^n (\sum_{j=1}^m ||D^k U_j||_{H^{\otimes k}})^{n/k}.$$

PROOF. Note that

$$\begin{aligned}
&||D^n(\prod_{j=1}^m U_j)||_{H^{\otimes n}} \\
&\leqq \sum_{k=1}^n \sum_{1 \leqq i_1 < \cdots < i_k \leqq m} \sum_{\ell_1, \dots, \ell_k \geqq 1, \ell_1 + \cdots \ell_k = n} \frac{n!}{\ell_1! \dots \ell_k!} \\
&\quad \times (\prod_{j \neq i_1, \dots i_k} |U_j|) ||D^{\ell_1} U_{i_1}||_{H^{\otimes \ell_1}} \cdots ||D^{\ell_k} U_{i_k}||_{H^{\otimes \ell_k}} \\
&\leqq \sum_{k=1}^n \sum_{\ell_1, \dots, \ell_k \geqq 1, \ell_1 + \cdots \ell_k = n} \frac{n!}{\ell_1! \dots \ell_k!} (\sum_{i=1}^m ||D^{\ell_1} U_i||_{H^{\otimes \ell_1}}) \cdots (\sum_{i=1}^m ||D^{\ell_k} U_i||_{H^{\otimes \ell_k}}) \\
&\leqq \sum_{k=1}^n \sum_{\ell_1, \dots, \ell_k \geqq 1, \ell_1 + \cdots \ell_k = n} \frac{n!}{\ell_1! \dots \ell_k!} ((\sum_{i=1}^m ||D^{\ell_1} U_i||_{H^{\otimes \ell_1}})^{n/\ell_1} \\
&\qquad + \cdots + (\sum_{i=1}^m ||D^{\ell_k} U_i||_{H^{\otimes \ell_k}})^{n/\ell_k}).
\end{aligned}$$

Here we use the fact

$$x_1^{\ell_1} \cdots x_k^{\ell_k} \leqq \max\{x_1, \dots, x_k\}^{\ell_1 + \cdots + \ell_k}, \qquad x_1, \dots, x_k \geqq 0.$$

This implies our assertion. □

Let $\theta_T : W_0 \to W_0$, $T \geqq 0$, be given by

$$\theta_T(w)(t) = w(T + t) - w(T), \qquad w \in W_0.$$

Then $\mu \circ \theta_T^{-1} = \mu$.

Let $T_n = \sum_{k=n}^\infty 8^{-k} = 8^{-n+1}/7$, $n \geqq 0$, and let $Z^\lambda_{n,m} \in W^{\infty,\infty-}$, $n > m \geqq 1$, by

$$Z^\lambda_{n,m} = \prod_{k=m}^n \rho(8^k \tilde{Y}(8^{-k}; \theta_{T_{k+1}} w)).$$

Note that $\rho(8^k \tilde{Y}(8^{-k}; \theta_{T_{k+1}} w))$ is $\mathcal{F}^{T_k}_{T_{k+1}}$-measurable and so $Z^\lambda_{n,m}$ is $\mathcal{F}^{T_m}_{T_{n+1}}$-measurable.

PROPOSITION 20. (1) *Let $C_0 > 0$ be as in Proposition 18. and m_0 be n integer such that $C_0 2^{-m_0+1} < 1$. Then for any $n > m \geqq m_0$,*

$$E^\mu[Z^\lambda_{n,m}, \sup_{x \in B(0,1), t \in [0, T_m - T_n]} |X^\lambda(t, x; \theta_{T_n} w) - x| > C_0 2^{-m+1}] = 0.$$

(2) *For any* $r > 0$ *and* $p \in (1, \infty)$ *we see that*

$$\sup_{\lambda \in \Lambda, n > m \geqq 1} ||Z^{\lambda}_{n,m}||_{W^{r,p}} < \infty.$$

Proof. Note that

$$X^{\lambda}(t + s, x; \theta_{T_{n+1}} w) = X^{\lambda}(t, X^{\lambda}(s, x; \theta_{T_{n+1}} w); \theta_{T_n + s} w).$$

Thereofore we have

$$\begin{aligned} &\sup_{x \in B(0,1), t \in [0, T_m - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| \\ &\leqq \sup_{x \in B(0,1), t \in [0, T_{m+1} - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| \\ &+ \sup_{x \in B(0,1), t \in [0, 8^{-m}]} |X^{\lambda}(t, X^{\lambda}(T_{m+1} - T_n, x; \theta_{T_n} w); \theta_{T_{m+1}} w) \\ &\qquad - X^{\lambda}(T_{m+1} - T_n, x; \theta_{T_n} w)|. \end{aligned}$$

and so if $n > m \geqq m_0$

$$\begin{aligned} &\{\sup_{x \in B(0,1), t \in [0, T_m - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| > C_0 2^{-m+1}\} \\ \subset &\{\sup_{x \in B(0,1), t \in [0, T_{m+1} - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| > C_0 2^{-(m+1)+1}\} \\ &\cup \{\sup_{x \in B(0,2), t \in [0, 8^{-m}]} |X^{\lambda}(t, x; \theta_{T_{m+1}} w) - x| > C_0 2^{-m}\}. \end{aligned}$$

Therefore we see that

$$\begin{aligned} &E^{\mu}[Z^{\lambda}_{n,m}, \sup_{x \in B(0,1), t \in [0, T_m - T_n]} |X^{\lambda}(t, x; \theta_{T_n} w) - x| > C_0 2^{-m+1}] \\ &\leqq \sum_{k=m}^{n} E^{\mu}[Z^{\lambda}_{n,m}, \sup_{x \in B(0,2), t \in [0, 8^{-k}]} |X^{\lambda}(t, x; \theta_{T_{k+1}} w) - x| > C_0 2^{-k}] = 0. \end{aligned}$$

This and Propositon 18 (1) imply the assertion (1).

By Proposition 18 (3) we see that

$$\sum_{k=1}^{\infty} \sup_{\lambda \in \Lambda} E^{\mu}[||D^{\ell} \rho(8^k \tilde{Y}^{\lambda}(8^{-k}; \theta_{T_{k+1}} w))||^p_{H^{\otimes k}}] < \infty$$

for any $\ell \geqq 1$ and $p \in (1,\infty)$. Since $0 \leqq \rho \leqq 1$, we see by Propositions 19 that

$$\sum_{k=1}^{\ell} \sup_{\lambda\in\Lambda, n>m\geqq 1} E^{\mu}[||D^k Z_{n,m}^{\lambda}||_{H^{\otimes k}}^p] < \infty$$

for any $\ell \geq 1$ and $p \in (1,\infty)$. Since $|Z_{n,m}^{\lambda}| \leqq 1$, we have the assertion (2). □

Let $Z_m^{\lambda} = \lim_{n\to\infty} Z_{n,m}^{\lambda}$ for $\lambda \in \Lambda$ and $m \geqq 1$.
Then we have the following.

Proposition 21. (1) *Let $C_0 > 0$ be as in Proposition 18 and m_0 be an integer such that $C_0 2^{-m_0+1} < 1/2$. Then for any $m \geqq m_0$,*

$$E^{\mu}[Z_m^{\lambda}, \sup_{x\in B(0,1), t\in[0,T_m]} |X^{\lambda}(t,x) - x| > C_0 2^{-m+1}] = 0.$$

(2) *$Z_m^{\lambda} \in W^{\infty,\infty-}$ for any $\lambda \in \Lambda$ and $m \geqq 1$, and moreover we see that for any $r > 0$ and $p \in (1,\infty)$*

$$\sup_{\lambda\in\Lambda, m\geqq 1} ||Z_m^{\lambda}||_{W^{r,p}} < \infty.$$

Let $\hat{Z}_{n,m}^{\lambda} \in W^{\infty,\infty-}$, $n > m \geqq 1$, be given by

$$\hat{Z}_{n,m}^{\lambda}(w) = \prod_{k=m}^{n} \rho(8^k \tilde{Y}(8^{-k}; \theta_{T_m - T_k} w)),$$

and let $\hat{Z}_m^{\lambda} = \lim_{n\to\infty} \hat{Z}_{n,m}^{\lambda}$, $m \geqq 1$.
Then we have the following similarly to Proposition 21.

Proposition 22. (1) *Let $C_0 > 0$ be as in Proposition 18 and m_0 be an integer such that $C_0 2^{-m_0+1} < 1/2$. Then for any $m \geqq m_0$,*

$$E^{\mu}[\hat{Z}_m^{\lambda}, \sup_{x\in B(0,1), t\in[0,T_m]} |X^{\lambda}(t,x) - x| > C_0 2^{-m+1}] = 0.$$

(2) *$\hat{Z}_m^{\lambda} \in W^{\infty,\infty-}$ for any $\lambda \in \Lambda$ and $m \geqq 1$, and moreover we see that for any $r > 0$ and $p \in (1,\infty)$*

$$\sup_{\lambda\in\Lambda, m\geqq 1} ||\hat{Z}_m^{\lambda}||_{W^{r,p}} < \infty.$$

Let $\tilde{X}^\lambda(t,x)$ bethe solution to Equation (2). Then we can take good version such that $\tilde{X}^\lambda(\cdot,*) : [0,\infty)\times\mathbf{R}^N \to \mathbf{R}^N$ is continuous and $\tilde{X}^\lambda(t,\cdot) : \mathbf{R}^N \to \mathbf{R}^N$ is a homeomorphism for any $t > 0$ with probability one. We take such a version. Also, we remind that

$$\tilde{V}_i^\lambda(x) = V_i^\lambda(x), \qquad x \in B(0,2\varepsilon_0),\ i = 0,1,\dots,d.$$

Then we have the following.

PROPOSITION 23. *Let $C_0 > 0$ be as in Proposition 18 and m_0 be an integer such that $C_0 2^{-m_0+1} < \varepsilon_0/2$. Then we have the following.*
(1) *For any $m \geqq m_0$,*

$$E^\mu[Z_m^\lambda, \sup_{x\in B(0,3\varepsilon_0/2), t\in[0,T_m]} |\tilde{X}^\lambda(t,x) - X^\lambda(t,x)| > 0] = 0,$$

and

$$E^\mu[\hat{Z}_m^\lambda, \sup_{x\in B(0,3\varepsilon_0/2), t\in[0,T_m]} |\tilde{X}^\lambda(t,x) - X^\lambda(t,x)| > 0] = 0.$$

(2) *For any $m \geqq m_0$,*

$$E^\mu[\hat{Z}_m^\lambda, \inf_{y\in\mathbf{R}^N\setminus B(0,3\varepsilon_0/2), t\in[0,T_m]} |\tilde{X}^\lambda(t,y)| < \varepsilon_0] = 0.$$

In particular, for any $f \in C_0(B(0,\varepsilon_0))$,

$$E^\mu[\hat{Z}_m^\lambda f(\tilde{X}^\lambda(t,y))] = E^\mu[\hat{Z}_m^\lambda f(X^\lambda(t,y))], \qquad t\in[0,T_m],\ y\in\mathbf{R}^N.$$

PROOF. The assertion (1) is an immediate consequence of the uniqueness of the solution to Equation (2) and Propositions 21 and 22. Then we see that

$$E^\mu[\hat{Z}_m^\lambda, \inf_{x\in\partial B(0,3\varepsilon_0/2), t\in[0,T_m]} |\tilde{X}^\lambda(t,x)| < \varepsilon_0] = 0.$$

and

$$E^\mu[\hat{Z}_m^\lambda, \sup_{t\in[0,T_m]} |\tilde{X}^\lambda(t,0)| > \varepsilon_0/2] = 0.$$

Here $\partial B(0,3\varepsilon_0/2)$ is the bounadary of $B(0,3\varepsilon_0/2)$.

Since $\tilde{X}^\lambda(t,\cdot):\mathbf{R}^N\to\mathbf{R}^N$, $t\in[0,T_m]$, is homeomorphism with probability one, we see that for $\mu-a.s.w\in\{\hat{Z}^\lambda_m>0\}$, $B(0,\varepsilon_0)\subset\tilde{X}^\lambda(t,B(0,3\varepsilon_0/2),w)$, and so

$$\tilde{X}^\lambda(t,\mathbf{R}^N\setminus B(0,3\varepsilon_0/2),w)\cap B(0,\varepsilon_0)=\emptyset,\qquad t\in[0,T_m].$$

So we see that

$$E^\mu[\hat{Z}^\lambda_m,\tilde{X}^\lambda(t,y)\in B(0,\varepsilon_0)]=E^\mu[\hat{Z}^\lambda_m,X^\lambda(t,y)\in B(0,\varepsilon_0)]=0$$

for all $t\in[0,T_m]$ and $y\in\mathbf{R}^N\setminus B(0,3\varepsilon_0/2)$. Therefore we have our assetion. □

Now let us prove Theorem 2.

Let $C_0>0$ be as in Proposition 18 and m_0 be an integer such that $C_0 2^{-m_0+1}<\varepsilon_0/2<1/2$. Let

$$g_k(x;f,\lambda)=E^\mu[(1-\rho(8^{k-1}\tilde{Y}^\lambda(8^{-(k-1)};w))f(\tilde{X}^\lambda(t-T_k,x))],\qquad x\in\mathbf{R}^N,\ k\geqq m_0$$

for any $f\in C^\infty_b(\mathbf{R}^N)$.

Then we see that

$$|g_k(x;f,\lambda)|\leqq E^\mu[(1-\rho(8^{k-1}\tilde{Y}^\lambda(8^{-(k-1)};w))^2]^{1/2}\sup_{x\in\mathbf{R}^N}|f(x)|. \tag{12}$$

By Proposition 18 (2) we see that that

$$\sup_{k\geqq 0,\ \lambda\in\Lambda}8^{k\gamma}E^\mu[(1-\rho(8^{k-1}\tilde{Y}^\lambda(8^{-(k-1)};w))^2]^{1/2}<\infty \tag{13}$$

for any $\gamma>0$. This implies $Z^\lambda_m\to 1$ as $m\to\infty$ $\mu-a.s.$ and so

$$1=Z^\lambda_m+\sum_{k=m+1}^\infty Z^\lambda_k(1-\rho(8^{k-1}\tilde{Y}^\lambda(8^{-(k-1)};\theta_{T_k}w)))\qquad\mu-a.s.\qquad m\geqq 1.$$

For each $t\in(0,1]$, let $m=m(t)$ be a minimum integer m such that $m\geqq m_0$ and $T_m<t$. Then we see that $T_m\geqq T_{m_0}\wedge(t/8)$. Note that for any

$\varphi \in C_0^\infty(B(0, \varepsilon_0))$

$$
\begin{aligned}
(\varphi \tilde{P}_t^\lambda f)(x) &= \varphi(x) E^\mu[f(\tilde{X}^\lambda(t,x))] \\
&= \varphi(x) E^\mu[Z_m^\lambda f(\tilde{X}^\lambda(t,x))] \\
+ \sum_{k=m+1}^{\infty} \varphi(x) E^\mu[Z_k^\lambda &(1 - \rho(8^{k-1}\tilde{Y}^\lambda(8^{-(k-1)}; \theta_{T_k} w))) f(\tilde{X}^\lambda(t,x))]. \\
&= \varphi(x) E^\mu[Z_m^\lambda (\tilde{P}_{t-T_m} f)(X^\lambda(T_m, x))] \\
&+ \sum_{k=m+1}^{\infty} \varphi(x) E^\mu[Z_k^\lambda g_k(X^\lambda(T_k, x); f, \lambda)].
\end{aligned}
$$

Then by Theorem 13 and Proposition 21 we see that for any $u_1, u_2, \ldots, u_n \in A^{**}$ there is a constant $C > 0$ independent of $\lambda \in \Lambda$ or $t \in (0,1]$ such that

$$
\begin{aligned}
&\sup_{x \in \mathbf{R}^N} |(\Phi^\lambda(r(u_1) \ldots r(u_n)) \varphi \tilde{P}_t^\lambda f)(x)| \\
&\leqq C t^{-||u_1 u_2 \ldots u_n||/2} \sup_{x \in \mathbf{R}^N} |f(x)| + \sum_{k=m+1}^{\infty} C T_k^{-||u_1 u_2 \ldots u_n||/2} \sup_{x \in \mathbf{R}^N} |g_k(x; f, \lambda)|.
\end{aligned}
$$

Then Equations (12) and (13) imply the first part of Theorem 2.

Since

$$
\hat{Z}_m^\lambda(\theta_{t-T_m} w) = \prod_{k=m}^{\infty} \rho(8^k \tilde{Y}^\lambda(8^{-k}; \theta_{t-T_k} w)),
$$

we have

$$
1 = \hat{Z}_m^\lambda(\theta_{t-T_m} w) + \sum_{k=m}^{\infty} \hat{Z}_{k+1}^\lambda(\theta_{t-T_{k+1}} w)(1 - \rho(8^k \tilde{Y}^\lambda(8^{-k}; \theta_{t-T_k} w)).
$$

So we see from Propositions 22 and 23 that for $f \in C_0^\infty(B(0, \varepsilon_0))$

$$
\begin{aligned}
(\tilde{P}_t^\lambda f)(x) &= E^\mu[f(\tilde{X}^\lambda(t,x))] \\
&= E^\mu[\hat{Z}_m^\lambda(\theta_{t-T_m} w) f(\tilde{X}^\lambda(t,x))] \\
+ \sum_{k=m}^{\infty} E^\mu[\hat{Z}_{k+1}^\lambda &(\theta_{t-T_{k+1}} w)(1 - \rho(8^k \tilde{Y}(8^{-k}; \theta_{t-T_k} w)) f(\tilde{X}^\lambda(t,x))] \\
&= E^\mu[E^\mu[\hat{Z}_m^\lambda f(X^\lambda(T_m, y))]|_{y = \tilde{X}^\lambda(t-T_m, x))}]
\end{aligned}
$$

$$+\sum_{k=m}^{\infty} E^{\mu}[(1-\rho(8^k\tilde{Y}(8^{-k};\theta_{t-T_k}w))$$
$$\times E^{\mu}[\hat{Z}^{\lambda}_{k+1} f(X^{\lambda}(T_{k+1},y))]|_{y=\tilde{X}^{\lambda}(t-T_{k+1},x))}].$$

So by Proposition 23 we have for any $f \in C_0^{\infty}(B(0,\varepsilon_0))$

$$\sup_{x\in\mathbf{R}^N} |(\tilde{P}^{\lambda}_t f)(x)|$$
$$\leqq \sup_{x\in\mathbf{R}^N} |E^{\mu}[\hat{Z}^{\lambda}_m f(X^{\lambda}(T_m,x))]|$$
$$+\sum_{k=m}^{\infty} E^{\mu}[1-\rho(8^k\tilde{Y}(8^{-k}))] \sup_{x\in\mathbf{R}^N} |E^{\mu}[\hat{Z}^{\lambda}_{k+1} f(X^{\lambda}(T_{k+1},x))]|.$$

So we have the last part of Theorem 2 by Theorem 13 and a similar argument.

This completes the proof of Theorem 2.

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