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Quasilinear Parabolic Equation and Its Applications to Fourth Order Equations with Rough Initial Data

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Abstract. The main part of this paper is devoted to establishing existence and uniqueness results for a class of abstract quasilinear parabolic equations by using the theory of continuous maximal regularity. The abstract results are then applied to some fourth order quasilinear parabolic equations (such as the surface diffusion flow and the Willmore flow) with rough initial data.

1. Introduction

In this paper we study the abstract quasilinear parabolic equation of the form

$$\begin{cases} \dot{u} + A(u)u = f(u), \\ u(0) = x \end{cases} \tag{1.1}$$

on a Banach space E_0, where $-A(\cdot)$ is the infinitesimal generator of the strongly continuous analytic semigroup on E_0 with dense domain E_1. The main purpose of this paper is to establish the existence and uniqueness of a local solution of the equation (1.1) and to apply this result to solve fourth order equations with rough initial data. Typical examples of fourth order equations include evolution of hypersurfaces $\{\Gamma_t\}$ in $\mathbf{R}^n$, such as the surface diffusion flow,

$$V = -\Delta_\Gamma H, \tag{1.2}$$

and, more generally, the anisotropic surface diffusion flow

$$V = -\Delta_\Gamma H_\mu. \tag{1.3}$$

Here V denotes the normal velocity of the evolving hypersurface Γ_t and Δ_Γ denotes the Laplace-Beltrami operator on $\Gamma = \Gamma_t$. The mean curvature is denoted by H while H_μ is an anisotropic mean curvature of Γ.

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We also consider the Willmore flow (cf. [6])

$$V = -\Delta_\Gamma H - \frac{1}{2}H^3 + HR. \tag{1.4}$$

Here R denotes the scalar curvature. If $n = 3$, $R = 2K$ where K is the Gaussian curvature. Another example is a parabolic approximation of the evolution equation for the height of a crystal growth

$$\frac{\partial u}{\partial t} = -B\nabla \cdot \left\{ \mathbf{\Lambda} \cdot \nabla \left[\nabla \cdot \left(\frac{\nabla u}{|\nabla u|} \right) + \frac{g_3}{g_1} \nabla \cdot (|\nabla u| \nabla u) \right] \right\}, \tag{1.5}$$

for nonnegative $B > 0$. Here ∇ denotes the gradient, $\nabla = (\partial/\partial x, \partial/\partial y)$, and $\mathbf{\Lambda}$ is a 2×2 matrix.

Let h^s denote the little Hölder spaces of order $s > 0$, that is, the closure of BUC^∞ in BUC^s, the latter space being the Banach space of all bounded and uniformly Hölder continuous functions of order s. The aim of the second half of this paper is to apply our abstract result to solve the above fourth order equations with $h^{1+\beta}$-initial data ($0 < \beta < 1$), which is less regular than those treated in the literature.

Now we state the precise settings of the problem (1.1). We say that a mapping $f : X \to Y$ between two metric spaces X, Y is locally Lipschitz continuous, and we use the notation $f \in C^{1-}(X, Y)$, if every point $x \in X$ has a neighborhood W such that $f|_W$ is Lipschitz continuous. Let E_α be the continuous interpolation space (cf. [14, Chapter 1]) between E_0 and E_1 with parameter α ($0 < \alpha < 1$) and $\mathbb{B}_{E_\alpha}(y, r)$ an open ball centered at y with radius $r > 0$ on E_α. A pair (A, f) denotes a mapping from U into $\mathcal{H}(E_1, E_0) \times E_0$. Here U is a nonempty subset of E_0 and $\mathcal{H}(E_1, E_0)$ denotes the set of bounded linear operators B such that $-B$ is the infinitesimal generator of a strongly continuous analytic semigroup on E_0. The definitions of $BUC_{1-\alpha}, BUC^1_{1-\alpha}$ and $\mathcal{M}_\alpha$ shall be given in Section 2. We state the main result of this paper.

Theorem 1.1. *Let $\alpha \in (0, 1)$ be fixed and let $E_\alpha = (E_0, E_1)^0_{\alpha,\infty}$ be a continuous interpolation space. Assume that $U_\alpha \subset E_\alpha$ is open.*

For the operator A assume that

$$A \in C^{1-}(U_\alpha, \mathcal{M}_\alpha(E_1, E_0)). \tag{1.6}$$

For the nonlinear term f assume that for $M > 0$ there exist $C_M > 0, \theta \in [\alpha, 1)$ and $p \in (0,1)$ which satisfy $q := p + (\theta - \alpha)/(1-\alpha) < 1$ such that

$$\|f(z_1) - f(z_2)\|_{E_0} \leq C_M \left(\|z_1\|_{E_1}^p + \|z_2\|_{E_1}^p + 1 \right) \|z_1 - z_2\|_{E_\theta}, \tag{1.7}$$

for $z_1, z_2 \in E_1 \cap \overline{\mathbb{B}}_{E_\alpha}(x_0, M)$.

Then for every $x_0 \in V_\alpha$, there exist positive constants $\tau = \tau(x_0), \varepsilon = \varepsilon(x_0)$ and $c = c(x_0)$ such that (1.1) *has a unique solution*

$$u(\cdot, x) \in BUC^1_{1-\alpha}([0,\tau], E_0) \cap BUC_{1-\alpha}([0,\tau], E_1)$$

for any initial value $x \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon)$. Moreover,

$$\|u(\cdot, x) - u(\cdot, y)\|_{C([0,\tau], E_\alpha)} \leq c\|x - y\|_{E_\alpha}^{(\theta-\alpha)/(1-\alpha)}, \quad x, y \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon).$$

Here $\overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon)$ is the closed ball in E_α centered at x_0 with radius $\varepsilon > 0$.

Theorem 1.1 is motivated by Clément and Simonett [7]. The assumption on the operator $A(\cdot)$ in (1.6) is the same as that of [7, Theorem 3.1]. The new ingredient of Theorem 1.1, however, is the assumption on the nonlinear term $f(\cdot)$, which reflects the structure of the lower order terms of the fourth order equations considered in this paper. Theorem 1.1 is based on maximal regularity results of Da Prato-Grisvard [8] and Angenent [3]. Maximal regularity results of [8] and [3] are useful for showing the smoothing property of the equation. In [7] the assumption on the nonlinear term $f(\cdot)$ is so restrictive that [7, Theorem 3.1] cannot be applied to prove the unique existence of a local solution of a fourth order equation with less regular initial data. In this paper, however, by imposing a new estimate (1.7) for the nonlinear term $f(\cdot)$, we can deal with fourth order equations with $h^{1+\beta}$-initial data.

In our previous paper [4], we applied the analytic semigroup theory of Buttu [5] to solve the fourth order equations with $h^{1+\beta}$-initial data. However, the assumption on the nonlinear term was more complicated than our new assumption (1.7). Thanks to maximal regularity results of [8] and [3], the assumption on the nonlinear term are more general and simpler. There is also a significant difference between the assumptions on the nonlinear term $f(\cdot)$ of this paper and that of [4]. The parameter q, introduced in Theorem 1.1, is less than one. This q denotes the sum of necessary powers of E_1-norm of the right hand side of (1.7) if E_θ-norm is estimated by the interpolation inequality with E_α and E_1. On the other hand, in our previous paper [4] the sum of necessary powers of $E_1(= D(A))$-norm for the nonlinear term is one when interpolating $E_0(= X)$ and $E_1 = D(A)$ in [4]. Hence we are forced to take E_1 to be a little Hölder space larger than $h^{4+\beta}$ when applying the result to the fourth order quasilinear parabolic equations.

We now state a local existence result for the fourth order quasilinear parabolic equations with rough initial data, which is the goal of the second half of this paper.

Theorem 1.2. *Let the initial hypersurface Γ_0 be given as the graph of a function $u_0 \in h^{1+\beta}(\mathbf{R}^{n-1})$ with $\beta \in (0,1)$, i.e., $\Gamma_0 = \{x_n = u_0(x'); x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}\}$. Then for $\gamma \in (0,\beta)$ there exist positive constants $T = T(u_0) > 0$ and $\varepsilon = \varepsilon(u_0) > 0$ such that the problem (1.k) with $k = 2,4$ has a unique classical solution*

$$u(\cdot, w) \in BUC_{1-\alpha}([0,T], h^{4+\gamma}) \cap BUC^1_{1-\alpha}([0,T], h^\gamma),$$

for any initial value $w \in \overline{\mathbb{B}}_{h^{1+\beta}(\mathbf{R}^{n-1})}(u_0, \varepsilon)$. Here $\alpha = (1 + \beta - \gamma)/4$.

Remark 1.3. The structure of problems (1.3) and (1.5) is the same as the structure of problems (1.2) and (1.4), and therefore we can state similar theorems for problems (1.3) and (1.5).

Several local and global existence results for the surface diffusion flow and the Willmore flow were obtained by Escher and Simonett [9], [16]. They dealt with the surface diffusion flow and the Willmore flow of multi-dimensional closed hypersurfaces. Their regularity assumption on the initial hypersurface was $h^{2+\beta}$. In this paper, we deal with the surface diffusion flow and the Willmore flow of multi-dimensional hypersurfaces represented by a graph. The assumption on the initial data in Theorem 1.2 is $h^{1+\beta}$, that is, the curvature of the initial hypersurface is not necessarily continuous.

Recently there have been several papers which deal with fourth order equations with rough initial data. Escher and Mucha [10] proved the unique solvability for the the surface diffusion flow with rough initial data by using the theory of harmonic analysis. The class of the initial data considered there is a Besov space $B^{5/2-4/p}_{p,2}$ and they impose the condition $p > (2n+8)/3$ on the parameter p, where n is the space dimension. In other words, the field of normal vectors to the initial hypersurface is Hölder continuous in space and time, hence the space $B^{5/2-4/p}_{p,2}$ is a subset of $C^{1+\alpha}$. However, the Besov space $B^{5/2-4/p}_{p,2}$ of [10] is not comparable with the class $h^{1+\beta}$ considered in this paper. In [12], Koch and Lamm prove the existence of solutions of the graphical Willmore and mean curvature flows with Lipschitz initial data. Their method relies on a technique introduced by Koch and Tataru [13]. The class of the initial data in [12] is larger than the class of $h^{1+\beta}$ in this paper. However they impose the smallness assumption on the Lipschitz norm of the initial data. Since they show the existence and uniqueness of a solution by a fixed-point argument on the function space X^{δ}_{T} which is introduced in [12] they need to choose the Lipschitz norm of the initial data to be sufficently small. So our result is not included in their result. The purpose of this paper is to establish an abstract theorem—Theorem 1.1. This abstract theorem is flexible enough to be applied to various fourth order problems. However, the space where the time continuity on the solution $u(\cdot, w)$ holds is slightly smaller than the space of the initial data u_0 since we arrange parameters β and γ as $0 < \gamma < \beta < 1$. Our abstract theorem is new because of the new assumption (1.7) on the lower order term. Moreover, to the extend of our knowledge, the application to fourth order parabolic problems with less regular initial data has not appeared in the literature.

Let us give an outline of the main proofs. For the proof of Theorem 1.1, we follow the idea of the proof of [7, Theorem 3.1] by Clément and Simonett. The assumption (1.7) is different from the assumption of [7, Theorem 3.1], and therefore, we more carefully estimate the nonlinear term $f(\cdot)$. We prove Theorem 1.2 by applying the abstract result of Theorem 1.1.

The content of this paper is as follows. In Section 2, we introduce some notation and state maximal regularity results of the linear theory, which shall be used in Section 3 to prove Theorem 1.1. In the final section, Section 4, we finally prove the local existence and uniqueness of fourth order equations with $h^{1+\beta}$-initial data via the unique local existence result of Theorem 1.1.

2. Maximal Regularity

First, we recall interpolation spaces and the notion of maximal regularity. Throughout this paper, we use the notation of [7]. Let E_0, E_1 be two Banach spaces such that E_1 is continuously embedded in E_0. Let $\mathcal{H}(E_1, E_0)$

be the space of all bounded linear operators $B \in \mathcal{L}(E_1, E_0)$ which have the additional property that $-B$, considered as an unbounded operator in E_0, generates a strongly continuous analytic semigroup on E_0. For $T > 0$ set $J = [0, T]$, $\dot{J} := J \setminus \{0\}$. Let $0 < \alpha < 1$ be fixed. Let

$$\begin{aligned} BUC_{1-\alpha}(J, E) &:= \Big\{ u \in C(\dot{J}, E); [t \mapsto t^{1-\alpha} u] \in BUC(\dot{J}, E), \\ &\qquad\qquad \lim_{t \to 0+} t^{1-\alpha} \|u(t)\|_E = 0 \Big\}, \\ BUC^1_{1-\alpha}(J, E) &:= \{ u \in C^1(\dot{J}, E); u, \dot{u} \in BUC_{1-\alpha}(J, E) \}, \end{aligned}$$

where E is a (real or complex) Banach space. Then we set

$$\begin{aligned} \mathbb{E}_1(J) &= BUC^1_{1-\alpha}(J, E_0) \cap BUC_{1-\alpha}(J, E_1), \\ \mathbb{E}_0(J) &= BUC_{1-\alpha}(J, E_0), \end{aligned}$$

where the norms of $\mathbb{E}_1(J)$ and $\mathbb{E}_0(J)$ are defined by

$$\begin{aligned} \|u\|_{\mathbb{E}_1(J)} &= \sup_{t \in \dot{J}} t^{1-\alpha} (\|u'(t)\|_{E_0} + \|u(t)\|_{E_1}), \\ \|u\|_{\mathbb{E}_0(J)} &= \sup_{t \in \dot{J}} t^{1-\alpha} \|u(t)\|_{E_0}. \end{aligned}$$

The vector spaces $\mathbb{E}_1(J)$ and $\mathbb{E}_0(J)$ are Banach spaces with respect to the above norms. In the following we will use the notation

$$E_\alpha = (E_0, E_1)^0_{\alpha, \infty}$$

for the continuous interpolation spaces. See also [7, Section 2], [14, Chapter 1].

Let us recall the notion of maximal regularity. The class $\mathcal{M}_\alpha(E_1, E_0)$ is defined for $B \in \mathcal{H}(E_1, E_0)$ as

$$B \in \mathcal{M}_\alpha(E_1, E_0) \Longleftrightarrow \left(\frac{d}{dt} + B, \gamma \right) \in \mathrm{Isom}(\mathbb{E}_1(J), \mathbb{E}_0(J) \times E_\alpha),$$

where $\gamma : \mathbb{E}_0(J) \to E_\alpha; v \mapsto \gamma v = v(0)$. The norm of $\mathcal{M}_\alpha(E_1, E_0)$ is equivalent to the norm of $\mathcal{L}(E_1, E_0)$.

If $B \in \mathcal{M}_\alpha(E_1, E_0)$, then $(\mathbb{E}_0(J), \mathbb{E}_1(J))$ is called *a pair of maximal regularity* for B.

3. Abstract Quasilinear Parabolic Equations

In this section we study the existence of solutions to the quasilinear parabolic equation of the form

$$\begin{cases} \dot{u} + A(u)u = f(u), \\ u(0) = x. \end{cases} \tag{3.1}$$

For this purpose we shall first recall the notion of a solution. We assume that

$$(A, f) : U \to \mathcal{H}(E_1, E_0) \times E_0,$$

where U is a nonempty subset of E_0. Let $x \in U$ be given and let $J \subset \mathbf{R}^+ := [0, \infty)$ be an interval which contains 0. By a *solution* of (3.1) on J we mean a function

$$u \in C^1(\dot{J}, E_0) \cap C(\dot{J}, E_1) \cap C(J, U)$$

which satisfies

$$\begin{cases} \dot{u}(t) + A(u(t))u(t) = f(u(t)), \quad t \in \dot{J}, \\ u(0) = x, \end{cases} \tag{3.2}$$

where $\dot{J} = J \setminus \{0\}$. We shall prove our main unique local existence result of the problem (3.1), that is, Theorem 1.1 in Section 1.

Proof of Theorem 1.1. The proof is parallel to the proof of [7, Theorem 3.1] except in how it handles the nonlinear term $f(\cdot)$. We set

$$\mathbb{E}_0(J) := BUC_{1-\alpha}(J, E_0),$$
$$\mathbb{E}_1(J) := BUC^1_{1-\alpha}(J, E_0) \cap BUC_{1-\alpha}(J, E_1).$$

We rewrite the problem (3.1) into

$$\begin{cases} \dot{u} + Au = B(u) + f(u), \\ u(0) = x, \end{cases} \tag{3.3}$$

where $A := A(x_0)$ and $B(z) := A(x_0) - A(z)$ for $z \in U_\alpha$. We conclude that $B \in C^{1-}(U_\alpha, \mathcal{L}(E_1, E_0))$ and that $B(x_0) = 0$. We may assume that E_α is equipped with the (equivalent) norm

$$\| \cdot \|_{E_\alpha} := \sup_{s>0} s^{1-\alpha} \|(\omega + A)e^{-s(\omega+A)} \cdot \|_{E_0}$$

where ω is a fixed number such that $\operatorname{type}(-(\omega + A)) < 0$. Let $T > 0$ be fixed and let $J := [0, T]$. It follows from [7, Lemma 2.2] that there exists a constant $M_1 \geq 1$ such that

$$\begin{aligned} &\|u\|_{C(J_\tau, E_\alpha)} \leq M_1 \|u\|_{\mathbb{E}_1(J_\tau)}, \\ &u \in \mathbb{E}_1(J_\tau), \quad u(0) = 0, \quad J_\tau = [0, \tau] \subset J. \end{aligned} \tag{3.4}$$

Moreover, we obtain

$$\|e^{-tA} z\|_{\mathbb{E}_1(J_\tau)} \leq c(\omega) e^{\omega T} \sup_{s>0} s^{1-\alpha} \|(\omega + A) e^{-s(\omega + A)} z\|_{E_0} \leq M_2 \|z\|_{E_\alpha} \tag{3.5}$$

for $z \in E_\alpha$ and $J_\tau \subset J$. Let $\|K_A\| := \|K_A\|_{\mathcal{L}(\mathbb{E}_0(J), \mathbb{E}_1(J))}$, where the operator K_A is defined by

$$(K_A f)(t) := \int_0^t e^{-(t-\tau)A} f(\tau)\, d\tau,$$

for $f \in \mathbb{E}_1(J)$. See also [7, Section 2]. Then from our assumptions (1.6) and (1.7) there exist positive constants ρ_0, b, C_{ρ_0} and $L \geq 1$ such that $\mathbb{B}_{E_\alpha}(x_0, 2\rho_0) \subset U_\alpha$ and such that

$$\|B(z)\|_{\mathcal{L}(E_1, E_0)} \leq \frac{1}{4\|K_A\| M_1}, \quad z \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \rho_0), \tag{3.6}$$

$$\|B(z_1) - B(z_2)\|_{\mathcal{L}(E_1, E_0)} \leq L \|z_1 - z_2\|_{E_\alpha}, \tag{3.7}$$

$$\|f(z_1) - f(z_2)\|_{E_0} \leq C_{\rho_0} (\|z_1\|_{E_1}^p + \|z_2\|_{E_1}^p + 1) \|z_1 - z_2\|_{E_\theta}, \tag{3.8}$$

where $z_1, z_2 \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \rho_0)$. Let $\varepsilon_0 := \min(\rho_0, (4\|K_A\| M_1 L)^{-1})$. Then we find a number $T_1 \in J$ such that

$$\begin{aligned} &\|e^{-tA} x_0 - x_0\|_{E_\alpha} \leq \varepsilon_0/2, \quad t \in J_1 := [0, T_1], \\ &\|e^{-tA} x_0\|_{\mathbb{E}_1(J_1)} \leq \varepsilon_0/2. \end{aligned} \tag{3.9}$$

The first inequality in (3.9) follows from the strong continuity of the semigroup e^{-tA} on E_α, whereas the second one is a consequence of [7, Remark 2.1]. Let $\tau \leq T_1$ be given and set $J_\tau = [0, \tau]$. For $x \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon)$ with $2M_2 \varepsilon \leq \varepsilon_0$ we set

$$W_x(J_\tau) := \{v \in \mathbb{E}_1(J_\tau); v(0) = x, \|v - x_0\|_{C(J_\tau, E_\alpha)} \leq \varepsilon_0\} \cap \overline{\mathbb{B}}_{\mathbb{E}_1(J_\tau)}(0, \varepsilon_0)$$

and we equip this set with the topology of $\mathbb{E}_1(J_\tau)$. It follows from [7, Lemma 2.2] that $W_x(J_\tau)$ is a closed subset of $\mathbb{E}_1(J_\tau)$ and thus is a complete metric space. The estimates (3.5) and (3.9) yield $[t \mapsto e^{-tA}x] \in W_x(J_\tau)$. This shows that $W_x(J_\tau)$ is nonempty. Let $v \in W_x(J_\tau)$ be given. We invoke the interpolation inequality,

$$\|v\|_{E_\theta} \le c\|v\|_{E_\alpha}^{(1-\theta)/(1-\alpha)}\|v\|_{E_1}^{(\theta-\alpha)/(1-\alpha)} \qquad \text{for } v \in E_1,$$

and we estimate the constant $C_{\rho_0}\|v\|_{E_\alpha}^{(1-\theta)/(1-\alpha)}$ by a constant (from above) depending only on ρ_0 (which is still denoted by C_{ρ_0}) since the norm $\|v\|_{E_\alpha}$ can be estimated from above by $\rho_0 > 0$. Then we obtain from (3.6)–(3.8)

$$\begin{aligned}
(3.10)\quad & t^{1-\alpha}\|B(v(t))v(t) + f(v(t))\|_{E_0} \\
&\le \|B(v(t))\|_{\mathcal{L}(E_1,E_0)}t^{1-\alpha}\|v(t)\|_{E_1} + t^{1-\alpha}\|f(v(t))\|_{E_0} \\
&\le \frac{1}{4\|K_A\|M_1}\|v\|_{\mathbb{E}_1(J_\tau)} + C_{\rho_0}t^{1-\alpha}(\|v(t)\|_{E_1}^p + 1)\|v(t)\|_{E_\theta} \\
&\le \frac{\varepsilon_0}{4\|K_A\|M_1} \\
&\quad + C_{\rho_0}t^{1-\alpha}(\|v(t)\|_{E_1}^p + 1)\|v(t)\|_{E_\alpha}^{(1-\theta)/(1-\alpha)}\|v(t)\|_{E_1}^{(\theta-\alpha)/(1-\alpha)} \\
&\le \frac{\varepsilon_0}{4\|K_A\|M_1} + C_{\rho_0}\Big[(t^{1-\alpha}\|v(t)\|_{E_1})^{p+(\theta-\alpha)/(1-\alpha)} \\
&\quad \times t^{(1-\alpha)\left(1-p-\frac{\theta-\alpha}{1-\alpha}\right)} + (t^{1-\alpha}\|v(t)\|_{E_1})^{(\theta-\alpha)/(1-\alpha)} \times t^{(1-\alpha)(\frac{1-\theta}{1-\alpha})}\Big] \\
&\le \frac{\varepsilon_0}{4\|K_A\|M_1} + C_{\rho_0}\Big[\|v\|_{\mathbb{E}_1(J_\tau)}^q\tau^{(1-\alpha)(1-q)} \\
&\quad + \|v\|_{\mathbb{E}_1(J_\tau)}^{(\theta-\alpha)/(1-\alpha)}\tau^{(1-\alpha)(\frac{1-\theta}{1-\alpha})}\Big] \\
&\le \frac{\varepsilon_0}{4\|K_A\|M_1} + C_{\rho_0}\left[\varepsilon_0^q\tau^{(1-\alpha)(1-q)} + \varepsilon_0^{(\theta-\alpha)/(1-\alpha)}\tau^{1-\theta}\right].
\end{aligned}$$

The estimate (3.10) shows that $B(v)v + f(v) \in \mathbb{E}_0(J_\tau)$ for any $v \in W_x(J_\tau)$. Thus, the mapping

$$G_x : W_x(J_\tau) \to \mathbb{E}_1(J_\tau), \quad G_x(v) := e^{-tA}x + K_A(B(v)v + f(v))$$

is well-defined for any $x \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon)$.

(i) It follows from (3.4), (3.9), (3.10) and from the strong continuity of the semigroup e^{-tA} on E_α that

$$\begin{aligned} \|G_x(v) - x_0\|_{C(J_\tau,E_\alpha)} & \\ \leq \|e^{-tA}(x - x_0)\|_{C(J_\tau,E_\alpha)} & \\ \quad + \|e^{-tA}x_0 - x_0\|_{C(J_\tau,E_\alpha)} + \|K_A(B(v)v + f(v))\|_{C(J_\tau,E_\alpha)} & \\ \leq c\|x - x_0\|_{E_\alpha} + \frac{\varepsilon_0}{2} & \\ \quad + M_1\|K_A\|\Bigg[\frac{\varepsilon_0}{4\|K_A\|M_1} & \\ \quad + C_{\rho_0}\left(\varepsilon_0^q\tau^{(1-\alpha)(1-q)} + \varepsilon_0^{(\theta-\alpha)/(1-\alpha)}\tau^{1-\theta}\right)\Bigg] & \\ \leq \varepsilon_0, & \end{aligned} \tag{3.11}$$

provided that $\|x-x_0\|_{E_\alpha} \leq \varepsilon$ for a sufficiently small number ε and provided that τ is small enough. We can always arrange τ smaller since the relevant constants and $\|K_A\|_{\mathcal{L}(\mathbb{E}_0(J_\tau),\mathbb{E}_1(J_\tau))}$ are independent of $J_\tau \subset J$. Additionally, we also obtain

$$\begin{aligned} \|G_x(v)\|_{\mathbb{E}_1(J_\tau)} & \\ \leq \|e^{-tA}(x - x_0)\|_{\mathbb{E}_1(J_\tau)} + \|e^{-tA}x_0\|_{\mathbb{E}_1(J_\tau)} & \\ \quad + \|K_A(B(v)v + f(v))\|_{\mathbb{E}_1(J_\tau)} & \\ \leq M_2(x - x_0)\|_{E_\alpha} + \frac{\varepsilon_0}{2} & \\ \quad + \|K_A\|\left[\frac{\varepsilon_0}{4\|K_A\|M_1} + C_{\rho_0}\left(\varepsilon_0^q\tau^{(1-\alpha)(1-q)} + \varepsilon_0^{(\theta-\alpha)/(1-\alpha)}\tau^{1-\theta}\right)\right] & \\ \leq \varepsilon_0 & \end{aligned} \tag{3.12}$$

if ε and τ are small enough. Lastly, observe that $G_x(v)(0) = x$. We have shown that $G_x(W_x(J_\tau)) \subset W_x(J_\tau)$ for all $x \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon)$, provided that ε and τ are sufficiently small.

(ii) Let $x_1, x_2 \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon)$ be given and pick $v_1 \in W_{x_1}(J_\tau)$ and $v_2 \in W_{x_2}(J_\tau)$. It follows from (3.5) that

$$\|e^{-tA}(x_1 - x_2)\|_{\mathbb{E}_1(J_\tau)} \leq M_2\|x_1 - x_2\|_{E_\alpha}. \tag{3.13}$$

Moreover, we obtain from (3.4) that

$$\begin{aligned} \|(v_1 - v_2) - e^{-tA}(x_1 - x_2)\|_{C(J_\tau,E_\alpha)} & \\ \leq M_1\|(v_1 - v_2) - e^{-tA}(x_1 - x_2)\|_{\mathbb{E}_1(J_\tau)}. & \end{aligned} \tag{3.14}$$

This estimate together with (3.13) immediately yields

$$\|v_1 - v_2\|_{C(J_\tau, E_\alpha)} \leq M_1 \|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)} + M_2(1 + M_1)\|x_1 - x_2\|_{E_\alpha}, \tag{3.15}$$

Next, observe that

$$\|B(v_1)(v_1 - v_2)\|_{\mathbb{E}_0(J_\tau)} \leq \frac{1}{4\|K_A\| M_1} \|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)}, \tag{3.16}$$

$$\begin{aligned} \|(B(v_1) - B(v_2))v_2\|_{\mathbb{E}_0(J_\tau)} &\leq L\|v_1 - v_2\|_{C(J_\tau, E_\alpha)} \|v_2\|_{\mathbb{E}_1(J_\tau)} \\ &\leq \varepsilon_0 L \|v_1 - v_2\|_{C(J_\tau, E_\alpha)}. \end{aligned} \tag{3.17}$$

To obtain the estimate for $\|f(v_1) - f(v_2)\|_{\mathbb{E}_0(J_\tau)}$, we observe that

$$\begin{aligned} & t^{1-\alpha}\|f(v_1(t)) - f(v_2(t))\|_{E_0} \\ &\leq C_{\rho_0} t^{1-\alpha} \left(\|v_1(t)\|_{E_1}^p + \|v_2(t)\|_{E_1}^p + 1 \right) \\ &\quad \times \|v_1(t) - v_2(t)\|_{E_\alpha}^{(1-\theta)/(1-\alpha)} \|v_1(t) - v_2(t)\|_{E_1}^{(\theta-\alpha)/(1-\alpha)} \\ &\leq C_{\rho_0} \Big[t^{(1-\alpha)(1-q)} \left(t^{1-\alpha}\|v_1(t)\|_{E_1} \right)^p \\ &\qquad + t^{(1-\alpha)(1-q)} \left(t^{1-\alpha}\|v_2(t)\|_{E_1} \right)^p + t^{(1-\alpha)(\frac{1-\theta}{1-\alpha})} \Big] \\ &\quad \times \left(M_1\|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)} + (M_1 M_2 + M_2)\|x_1 - x_2\|_{E_\alpha} \right)^{(1-\theta)/(1-\alpha)} \\ &\qquad \times \left(t^{1-\alpha}\|v_1(t) - v_2(t)\|_{E_1} \right)^{(\theta-\alpha)/(1-\alpha)} \\ &\leq C_{\rho_0} \left(\|v_1\|_{\mathbb{E}_1(J_\tau)}^p \tau^{(1-\alpha)(1-q)} + \|v_2\|_{\mathbb{E}_1(J_\tau)}^p \tau^{(1-\alpha)(1-q)} + \tau^{1-\theta} \right) \\ &\quad \times \left(M_1\|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)} + (M_1 M_2 + M_2)\|x_1 - x_2\|_{E_\alpha} \right)^{(1-\theta)/(1-\alpha)} \\ &\qquad \times \|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)}^{(\theta-\alpha)/(1-\alpha)} \\ &\leq C_{\rho_0} \left(2\varepsilon_0^p \tau^{(1-\alpha)(1-q)} + \tau^{1-\theta} \right) \left(M_1^{(1-\theta)/(1-\alpha)} \|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)}^{(1-\theta)/(1-\alpha)} \right. \\ &\quad \left. + (M_1 M_2 + M_2)^{(1-\theta)/(1-\alpha)} \|x_1 - x_2\|_{E_\alpha}^{(1-\theta)/(1-\alpha)} \right) \\ &\qquad \times \|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)}^{(\theta-\alpha)/(1-\alpha)}. \end{aligned} \tag{3.18}$$

In order to derive the last inequality in (3.18), we use

$$(a+b)^r \leq a^r + b^r,$$

for a fixed $r \in (0,1]$, and any positive numbers $a > 0, b > 0$.

It follows from the definition of ε_0 and from (3.13)–(3.18) that there exists a constant $c_2 > 0$ such that

$$
\begin{aligned}
(3.19)\qquad & \|G_{x_1}(v_1) - G_{x_2}(v_2)\|_{\mathbb{E}_1(J_\tau)} \\
&\le c_2\|x_1 - x_2\|_{E_\alpha}^{(\theta-\alpha)/(1-\alpha)} + \bigg[\frac{1}{2}\|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)} \\
&\quad + \|K_A\| M_1 C_{\rho_0} M_1^{(1-\theta)/(1-\alpha)} \\
&\quad \times \left(2\varepsilon_0^p \tau^{(1-\alpha)(1-q)} + \tau^{1-\theta}\right) \|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)}\bigg] \\
&\le c_2\|x_1 - x_2\|_{E_\alpha}^{(\theta-\alpha)/(1-\alpha)} + \frac{3}{4}\|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)},
\end{aligned}
$$

provided that τ is chosen small enough.

(iii) As a particular case we obtain from (3.19) that

$$
\begin{aligned}
&\|G_x(v_1) - G_x(v_2)\|_{\mathbb{E}_1(J_\tau)} \le \frac{3}{4}\|v_1 - v_2\|_{\mathbb{E}_1(J_\tau)}, \\
&x \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon), \quad v_1, v_2 \in W_x(J_\tau).
\end{aligned}
$$

(iv) It follows from (i)–(iii) and Banach's fixed point theorem that the mapping G_x has a unique fixed point

$$u(\cdot, x) \in W_x(J_\tau) \subset BUC^1_{1-\alpha}(J_\tau, E_0) \cap BUC_{1-\alpha}(J_\tau, E_1)$$

for each $x \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon)$. This u is the unique mild solution. By a standard argument we observe that $u \in BUC^1_{1-\alpha}(J_\tau, E_0)$, which is a solution.

(v) We infer from (3.19) that

$$\|u(\cdot, x) - u(\cdot, y)\|_{\mathbb{E}_1(J_\tau)} \le 4c_2\|x - y\|_{E_\alpha}^{(\theta-\alpha)/(1-\alpha)}, \quad x, y \in \overline{\mathbb{B}}_{E_\alpha}(x_0, \varepsilon).$$

Thus the estimate

$$\|u(\cdot, x) - u(\cdot, y)\|_{C([0,\tau], E_\alpha)} \le c\|x - y\|_{E_\alpha}^{(\theta-\alpha)/(1-\alpha)}$$

follows. □

4. Applications to Fourth Order Equations with Rough Initial Data

4.1. New criterion for local existence and uniqueness for fourth order quasilinear parabolic PDEs

In this section, we solve various kinds of fourth order parabolic equations with rough initial data. To be more precise, we shall derive the existence and uniqueness for $h^{1+\beta}$-initial data ($0 < \beta < 1$). With this assumption, the curvature or the second derivative of initial data of the graph may not be continuous. For this purpose we apply the abstract theorem which was established in Section 3. Thus, we have to verify that the structures of the principal parts and the lower order terms of several fourth order equations indeed satisfy the assumptions of the abstract theorem. A direct application of Theorem 1.1 for each equation is rather complicated. Thus we establish a statement which bridges the gap between the abstract theory and its application to PDE problems. The fourth order quasilinear equations which we treat in this paper have a common structure of the principal parts and lower order terms. We extract the essence of the structure and establish a new criterion for local existence with rough initial data which is easy to check. We now consider an equation of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} + \mathscr{A}(\nabla u)u = \mathscr{F}(\nabla u, \nabla^2 u, \nabla^3 u), \\ \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where $\nabla^k u$ denote the k-th order derivative for $k = 1, 2, \dots$ in $\mathbf{R}^n$. This is a special quasilinear equation for $u = u(x,t), x \in \mathbf{R}^n$ but it includes interesting problems. We impose several continuity conditions for the operator $\mathscr{A}(\cdot)$ and the lower order term $\mathscr{F}(\cdot,\cdot,\cdot)$. Let $\mathscr{A}$ be a fourth order differential operator whose coefficients depends on $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \dots, \partial u/\partial x_l)$ for $u \in h^{1+\beta}$ on h^γ ($0 < \gamma < \beta < 1$). We denote the coefficients of $\mathscr{A}$ by a_α, i.e.,

$$\mathscr{A}(\nabla u) = \sum_{|\alpha|=4} a_\alpha(\nabla u) D^\alpha.$$

Here for $k = 1, 2, \dots, 0 < \nu < 1$, $h^{k+\nu}(\mathbf{R}^l, \mathbf{R})$ denotes the little Hölder space and it is identified with

$$h^{k+\nu}(\mathbf{R}^l, \mathbf{R}) := \Big\{ \varphi \in BUC^{k+\nu}(\mathbf{R}^l, \mathbf{R}); \lim_{\delta \to 0} \max_{|\alpha|=k} \sup_{\substack{x \neq y \\ |x-y|<\delta}} \frac{|D^\alpha \varphi(x) - D^\alpha \varphi(y)|}{|x-y|^\nu} = 0 \Big\}, \tag{4.2}$$

where α denotes the multi index.

(A1) (Uniform ellipticity) For any $Z > 0$ there is a positive constant $\mathcal{V} = \mathcal{V}_Z$ such that

$$\sum_{|\alpha|=4} a_\alpha(\boldsymbol{\zeta}) \xi^\alpha \geq \mathcal{V} |\boldsymbol{\xi}|^4,$$

for all $\boldsymbol{\zeta}$ satisfying $|\boldsymbol{\zeta}| \leq Z$.

(A2) (Regularity) All coefficients a_α are of C^1-class.

(F) $\mathscr{F}(\cdot,\cdot,\cdot)$ is of C^1-class. For $\widetilde{M} > 0$ there exists $C_{\widetilde{M}} > 0$ such that for $\boldsymbol{\zeta}_i \in \mathbf{R}^l, \boldsymbol{\eta}_i \in \mathbf{R}^m, \boldsymbol{\xi}_i \in \mathbf{R}^n$ $(i = 1, 2)$

$$\begin{aligned} &|\mathscr{F}(\boldsymbol{\zeta}_1, \boldsymbol{\eta}_1, \boldsymbol{\xi}_1) - \mathscr{F}(\boldsymbol{\zeta}_2, \boldsymbol{\eta}_2, \boldsymbol{\xi}_2)| \\ &\leq C_{\widetilde{M}} \{ 1 + (|\boldsymbol{\eta}_1|^2 + |\boldsymbol{\eta}_2|^2)|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| \\ &\qquad + (|\boldsymbol{\xi}_1| + |\boldsymbol{\xi}_2|)|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2| + (|\boldsymbol{\eta}_1| + |\boldsymbol{\eta}_2|)|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2| \}, \end{aligned} \tag{4.3}$$

for $|\boldsymbol{\zeta}_1|, |\boldsymbol{\zeta}_2| \leq \widetilde{M}$. Here $|\cdot|$ denotes the usual Euclidean norm, i.e., for $\boldsymbol{x} = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N$, $|\boldsymbol{x}|$ is defined by $|\boldsymbol{x}| := (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$.

Theorem 4.1. *Assume* (A1), (A2) *and* (F). *Let* $0 < \gamma < \beta < 1$. *Then for* $u_0 \in h^{1+\beta}$ *there exist* $\tau = \tau(u_0) > 0$ *and* $\varepsilon = \varepsilon(u_0) > 0$ *such that the problem* (4.1) *has a unique solution*

$$u(\cdot, w) \in BUC_{1-\alpha}([0,\tau], h^{4+\gamma}) \cap BUC^1_{1-\alpha}([0,\tau], h^\gamma)$$

for any initial value $w \in \overline{\mathbb{B}}_{h^{1+\beta}}(u_0, \varepsilon)$. *Here* $\alpha = (1 + \beta - \gamma)/4$.

Proof. Take $\nu \in (0, \gamma)$ and set two Banach spaces (F_0, F_1) as

$$F_0 := h^\nu, \quad F_1 := h^{4+\nu}.$$

Then, take the parameter $\theta = (\gamma - \nu)/4$ and set two Banach spaces (E_0, E_1) as

$$E_0 := (F_0, F_1)^0_{\theta,\infty} = h^\gamma, \quad E_1 := h^{4+\gamma}.$$

We take $\alpha := (1+\beta-\gamma)/4$ and observe by the reiteration Theorem (cf. [14]) that

$$E_\alpha = (E_0, E_1)^0_{\alpha,\infty} = h^{1+\beta}.$$

The reiteration Theorem also asserts that

$$h^{2+\gamma} = (E_\alpha, E_1)^0_{\omega_1,\infty}, \quad h^{3+\gamma} = (E_\alpha, E_1)^0_{\omega_2,\infty},$$

where $\omega_1 = (1+\gamma-\beta)/(3+\gamma-\beta), \quad \omega_2 = (2+\gamma-\beta)/(3+\gamma-\beta)$.

By our assumption (A1), the generation theorem of analytic semigroup on little Hölder space (cf. [14, Chapter 3]) asserts that for $u \in h^{1+\beta}$, $-\mathscr{A}(\nabla u)$ generates a strongly continuous analytic semigroup on F_0. Then, by the maximal regularity results of Da Prato-Grisvard [8] and Angenent [3],

$$\mathscr{A}(\nabla u) \in \mathcal{M}_\alpha(E_1, E_0).$$

Now we study the mapping properties of $[u \mapsto a_\alpha(\nabla u)]$. From the assumption (A2) and the fundamental theorem of calculus,

$$a_\alpha(\nabla u_1) - a_\alpha(\nabla u_2) = \sum_{j=1}^{l} \left(\int_0^1 \partial_j a_\alpha(s\nabla u_1 + (1-s)\nabla u_2)\, ds \right) \cdot (\partial_j u_1 - \partial_j u_2),$$

where $\partial_j = \partial/\partial x_j$. Thus we have

$$\begin{aligned} (4.4)\quad \|a_\alpha(\nabla u_1) - a_\alpha(\nabla u_2)\|_{h^\gamma} &\le \sum_{j=1}^{l} \left(\int_0^1 \|\partial_j a_\alpha(s\nabla u_1 + (1-s)\nabla u_2)\|\, ds \right) \\ &\quad \times \|\partial_j u_1 - \partial_j u_2\|_{h^\gamma} \\ &\le c\|u_1 - u_2\|_{h^{1+\gamma}} \le c\|u_1 - u_2\|_{h^{1+\beta}}. \end{aligned}$$

From (4.4) and the Banach algebra property of h^γ, the Lipschitz continuity of $[u \mapsto \mathscr{A}(\nabla u)]$ from $h^{1+\beta}$ to $\mathcal{L}(h^{4+\gamma}, h^\gamma)$ follows.

We conclude that

$$\mathscr{A} \in C^{1-}(U_\alpha, \mathcal{M}_\alpha(E_1, E_0)).$$

For $u_1, u_2 \in \mathbb{B}_{E_\alpha}(u_0, M)$, define $\widetilde{M} > 0$ as

$$\|u_i\|_{E_\alpha} \le \|u_i - u_0\|_{E_\alpha} + \|u_0\|_{E_\alpha} \le M + \|u_0\|_{E_\alpha} := \widetilde{M}.$$

The assumption (F) implies that $\mathscr{F}(\nabla u, \nabla^2 u, \nabla^3 u)$ for $u \in h^{4+\gamma}$ is in E_0 since $\mathscr{F}(\cdot,\cdot,\cdot)$ is a composite of a smooth function and a little Hölder function (see the characterization of $h^{k+\nu}$ in (4.2) just before (A1)). Thus we observe that

$$\begin{aligned}(4.5)\quad &\|\mathscr{F}(\nabla u_1, \nabla^2 u_1, \nabla^3 u_1) - \mathscr{F}(\nabla u_2, \nabla^2 u_2, \nabla^3 u_2)\|_{E_0} \\ &\le C_{\widetilde{M}}\big\{1 + \big(\|\nabla^2 u_1\|_{h^\gamma}^2 + \|\nabla^2 u_2\|_{h^\gamma}^2\big)\,\|\nabla^2 u_1 - \nabla^2 u_2\|_{h^\gamma} \\ &\quad + \big(\|\nabla^3 u_1\|_{h^\gamma} + \|\nabla^3 u_2\|_{h^\gamma}\big)\,\|\nabla^2 u_1 - \nabla^2 u_2\|_{h^\gamma} \\ &\quad + \big(\|\nabla^2 u_1\|_{h^\gamma} + \|\nabla^2 u_2\|_{h^\gamma}\big)\,\|\nabla^3 u_1 - \nabla^3 u_2\|_{h^\gamma}\big\}.\end{aligned}$$

We shall estimate each term of the right hand side of (4.5). Observe that

$$\begin{aligned}(4.6)\qquad &\big(\|\nabla^2 u_1\|_{h^\gamma}^2 + \|\nabla^2 u_2\|_{h^\gamma}^2\big)\,\|\nabla^2 u_1 - \nabla^2 u_2\|_{h^\gamma} \\ &\le (\|u_1\|_{h^{2+\gamma}}^2 + \|u_2\|_{h^{2+\gamma}}^2)\|u_1 - u_2\|_{h^{2+\gamma}},\end{aligned}$$

and apply the interpolation inequality in order to estimate $\|u_1\|_{h^{2+\gamma}}$ and $\|u_2\|_{h^{2+\gamma}}$. That is,

$$(4.7)\qquad \|u_i\|_{h^{2+\gamma}} \le c\|u_i\|_{E_\alpha}^{1-\omega_1}\|u_i\|_{E_1}^{\omega_1} \le c\widetilde{M}^{1-\omega_1}\|u_i\|_{E_1}^{\omega_1}, \quad (i = 1, 2).$$

Applying (4.7) to (4.6) we have

$$\begin{aligned}(4.8)\qquad &(\|\nabla^2 u_1\|_{h^\gamma}^2 + \|\nabla^2 u_2\|_{h^\gamma}^2)\|\nabla^2 u_1 - \nabla^2 u_2\|_{h^\gamma} \\ &\le C_{\widetilde{M}}(\|u_1\|_{E_1}^{\omega_1} + \|u_2\|_{E_1}^{\omega_1})\|u_1 - u_2\|_{h^{2+\gamma}}.\end{aligned}$$

A similar argument enables us to conclude that

$$\begin{aligned}(4.9)\qquad &(\|\nabla^3 u_1\|_{h^\gamma} + \|\nabla^3 u_2\|_{h^\gamma})\|\nabla^2 u_1 - \nabla^2 u_2\|_{h^\gamma} \\ &\le C_{\widetilde{M}}(\|u_1\|_{E_1}^{\omega_2} + \|u_2\|_{E_1}^{\omega_2})\|u_1 - u_2\|_{h^{2+\gamma}}.\end{aligned}$$

$$\begin{aligned}(4.10)\qquad &(\|\nabla^2 u_1\|_{h^\gamma} + \|\nabla^2 u_2\|_{h^\gamma})\|\nabla^3 u_1 - \nabla^3 u_2\|_{h^\gamma} \\ &\le C_{\widetilde{M}}(\|u_1\|_{E_1}^{\omega_1} + \|u_2\|_{E_1}^{\omega_1})\|u_1 - u_2\|_{h^{3+\gamma}}.\end{aligned}$$

We also have to check the sum of powers of E_1-norm in (4.8)–(4.10).

Note that

$$h^{2+\gamma} = (E_0, E_1)^0_{\theta_1,\infty}, \quad h^{3+\gamma} = (E_0, E_1)^0_{\theta_2,\infty},$$

where $\theta_1 = 1/2, \quad \theta_2 = 3/4$.

In (4.8),

$$\omega_1 + \frac{\theta_1 - \alpha}{1-\alpha} = \frac{1+\gamma-\beta}{3+\gamma-\beta} + \frac{4\{1/2 - (1+\beta-\gamma)/4\}}{3+\gamma-\beta} = \frac{2(1+\gamma-\beta)}{3+\gamma-\beta} < 1. \tag{4.11}$$

In (4.9)

$$\omega_2 + \frac{\theta_1 - \alpha}{1-\alpha} = \frac{2+\gamma-\beta}{3+\gamma-\beta} + \frac{4\{1/2 - (1+\beta-\gamma)/4\}}{3+\gamma-\beta} = \frac{3+2(\gamma-\beta)}{3+\gamma-\beta} < 1. \tag{4.12}$$

In (4.10)

$$\omega_1 + \frac{\theta_2 - \alpha}{1-\alpha} = \frac{1+\gamma-\beta}{3+\gamma-\beta} + \frac{4\{3/4 - (1+\beta-\gamma)/4\}}{3+\gamma-\beta} = \frac{3+2(\gamma-\beta)}{3+\gamma-\beta} < 1. \tag{4.13}$$

By (4.11)–(4.13) the sum of powers of E_1-norm for the lower order term is less than one. Thus, the estimate for the lower order term $\mathscr{F}(\cdot,\cdot,\cdot)$ satisfies the assumption of Theorem 1.1. We are now in a position to apply Theorem 1.1 to show the existence and uniqueness of a local-in-time solution of (4.1) with $h^{1+\beta}$-initial data. □

4.2. The surface diffusion flow, the Willmore flow and the anisotropic surface diffusion flow

In this subsection we consider the existence and uniqueness of a family $\{\Gamma(t); t > 0\}$ of smooth hypersurfaces solving the surface diffusion flow

$$\begin{cases} V = -\Delta_\Gamma H, \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{4.14}$$

the anisotropic surface diffusion flow

$$\begin{cases} V = -\Delta_\Gamma H_\mu, \\ \Gamma(0) = \Gamma_0, \end{cases} \tag{4.15}$$

and the Willmore flow

$$\begin{cases} V = -\Delta_\Gamma H - \dfrac{1}{2}H^3 + HR, \\ \Gamma(0) = \Gamma_0. \end{cases} \tag{4.16}$$

Here V is normal velocity of $\Gamma(t)$, while H and R denote the mean curvature and the scalar curvature of $\Gamma(t)$, respectively, and Δ_Γ is the Laplace-Beltrami operator on $\Gamma(t)$. For a smooth function f and a smooth vector field X on Γ, ∇_Γ and $\operatorname{div}_\Gamma$ are defined by

$$\nabla_\Gamma f = \nabla f - \langle \boldsymbol{n}, \nabla f \rangle \boldsymbol{n},$$
$$\operatorname{div}_\Gamma X = \operatorname{Trace}((I - \boldsymbol{n} \otimes \boldsymbol{n}) JX).$$

Here $\boldsymbol{n}$ denotes the unit normal vector on $\Gamma(t)$, $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbf{R}^N$, while JX stands for the Jacobian of X. The Laplace-Beltrami operator Δ_Γ on Γ is defined by

$$\Delta_\Gamma = \operatorname{div}_\Gamma \nabla_\Gamma.$$

Here H_μ denotes an anisotropic mean curvature of $\Gamma(t)$, that is, for a given surface energy density μ_0, the one-homogeneous extension μ of μ_0 is given by

$$\mu(\boldsymbol{p}) := \mu_0(\boldsymbol{p}/|\boldsymbol{p}|)|\boldsymbol{p}|, \quad \boldsymbol{p} \in \mathbf{R}^N.$$

Define the Cahn-Hoffman vector $\boldsymbol{\nu}$ by $\boldsymbol{\nu} = \nabla\mu$. Then H_μ is defined as

$$H_\mu = -\operatorname{div}_\Gamma \boldsymbol{\nu}(\boldsymbol{n}).$$

In this subsection we consider the case when the hypersurface $\Gamma(t)$ is represented as the graph of a smooth function $u(x,t)$, i.e., $\Gamma(t) = \{x_n = u(x',t); x' = (x_1, x_2, \ldots, x_{n-1}) \in \mathbf{R}^{n-1}\}$. We take the unit outer normal vector $\boldsymbol{n}$ as

$$\boldsymbol{n} = \left(-\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}, \frac{1}{(1+|\nabla u|^2)^{1/2}} \right),$$

where $\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \dots, \partial u/\partial x_{n-1}) = (\partial_1 u, \partial_2 u, \dots, \partial_{n-1} u)$. Then, from [11, Chapter 1], we have

$$V = \frac{\partial_t u}{(1+|\nabla u|^2)^{1/2}}, \tag{4.17}$$

$$H = \operatorname{div}\left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}}\right) = \left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\right)\frac{\partial_i \partial_j u}{(1+|\nabla u|^2)^{1/2}}. \tag{4.18}$$

The Laplace-Beltrami operator Δ_Γ is of the form

$$\begin{aligned}\Delta_\Gamma = &\left(\delta_{kl} - \frac{\partial_k u \partial_l u}{1+|\nabla u|^2}\right)\frac{\partial^2}{\partial x_k \partial x_l} \\ &+ \frac{1}{(1+|\nabla u|^2)^{1/2}}\frac{\partial}{\partial x_k}\left\{(1+|\nabla u|^2)^{1/2}\left(\delta_{kl} - \frac{\partial_k u \partial_l u}{1+|\nabla u|^2}\right)\right\}\frac{\partial}{\partial x_l}.\end{aligned} \tag{4.19}$$

Here the summation runs from 1 to $(n-1)$ for all indices. We now apply the result of Theorem 4.1 to show the unique local solvability of equations (4.14) and (4.16).

THEOREM 4.2. *Let the initial hypersurface Γ_0 be given as the graph of a function $u_0 \in h^{1+\beta}(\mathbf{R}^{n-1})$ with $\beta \in (0,1)$, i.e., $\Gamma_0 = \{x_n = u_0(x'); x' = (x_1, x_2, \dots, x_{n-1}) \in \mathbf{R}^{n-1}\}$. Then for $\gamma \in (0,\beta)$ there exist positive constants $T = T(u_0) > 0$ and $\varepsilon = \varepsilon(u_0) > 0$ such that the problem (4.k) with $k = 14, 16$ has a unique classical solution*

$$u(\cdot, w) \in BUC_{1-\alpha}([0,T], h^{4+\gamma}) \cap BUC^1_{1-\alpha}([0,T], h^\gamma),$$

for any initial value $w \in \overline{\mathbb{B}}_{h^{1+\beta}(\mathbf{R}^{n-1})}(u_0, \varepsilon)$. Here $\alpha = (1+\beta-\gamma)/4$.

REMARK 4.3. The structure of the problem (4.15) is the same as the structure of problems (4.14) and (4.16), and therefore we can state similar theorem for problem (4.15). In order $\boldsymbol{\nu} = \nabla\mu$ to be a continuous function, we must assume μ is of C^2-class away from the origin.

PROOF OF THEOREM 4.2. First we study the surface diffusion flow

(4.14). By (4.17)–(4.19) the surface diffusion flow is

$$(4.20)\quad \frac{\partial u}{\partial t} = -(1+|\nabla u|^2)^{1/2}\left\{\left(\delta_{kl} - \frac{\partial_k u \partial_l u}{1+|\nabla u|^2}\right)\partial_k \partial_l \right.$$
$$\times \left[\left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\right)\frac{\partial_i \partial_j u}{(1+|\nabla u|^2)^{1/2}}\right]$$
$$- \frac{1}{(1+|\nabla u|^2)^{1/2}}\partial_k\left[(1+|\nabla u|^2)^{1/2}\left(\delta_{kl} - \frac{\partial_k u \partial_l u}{1+|\nabla u|^2}\right)\right]$$
$$\left. \times\, \partial_l\left[\left(\delta_{ij} - \frac{\partial_i u \partial_j u}{1+|\nabla u|^2}\right)\frac{\partial_i \partial_j u}{(1+|\nabla u|^2)^{1/2}}\right]\right\}.$$

We have to verify that the principal parts and the lower order terms of (4.20) satisfy the assumptions (A1), (A2) and (F). The principal symbol of the right-hand side of (4.20) is given by

$$(4.21)\quad \frac{1}{(1+|\boldsymbol{\zeta}|^2)^2}\left[(\delta_{kl}(1+|\boldsymbol{\zeta}|^2) - \zeta_k\zeta_l)(\delta_{ij}(1+|\boldsymbol{\zeta}|^2) - \zeta_i\zeta_j)\right] p_k p_l p_i p_j$$
$$= \frac{1}{(1+|\boldsymbol{\zeta}|^2)^2}\left[(1+|\boldsymbol{\zeta}|^2)|\boldsymbol{p}|^2 - (\boldsymbol{\zeta}\cdot\boldsymbol{p})^2\right]^2,$$

where we denote $\boldsymbol{\zeta} = (\zeta_1, \zeta_2, \ldots, \zeta_{n-1})$ for variable for the first order derivative $\nabla u = (\partial_1 u, \partial_2 u, \ldots, \partial_{n-1} u)$ and $\boldsymbol{p} = (p_1, p_2, \ldots, p_{n-1}) \in \mathbf{R}^{n-1} \setminus \{0\}$. Cauchy's inequality then yields uniform ellipticity: there exists a constant $\mathcal{V} > 0$ such that

$$(4.21) \geq \mathcal{V}|\boldsymbol{p}|^4,$$

for all $\boldsymbol{p} = (p_1, p_2, \ldots, p_{n-1}) \in \mathbf{R}^{n-1} \setminus \{0\}$. Thus for a fixed $\boldsymbol{\zeta}$, the principal part $\mathscr{A}(\boldsymbol{\zeta})$ is an elliptic operator and of C^1-class with respect to $\boldsymbol{\zeta}$ so that (A2) is fulfilled. (See [1, Section 7], [2, Section 3] for various definitions of *uniformly elliptic operator*). It is easy to see that this estimate for (4.21) is locally uniform for $\boldsymbol{\zeta}$ so that $\mathscr{A}(\boldsymbol{\zeta})$ fulfills (A1).

The lower order terms of (4.20) are of the form

$$(4.22)\quad \mathscr{F}_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi}) = -\frac{\zeta_l\zeta_k\zeta_j\eta_{ij}\xi_{ikl}}{(1+|\boldsymbol{\zeta}|^2)^2},$$

$$(4.23)\quad \widetilde{\mathscr{F}}_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi}) = -\frac{\zeta_k\zeta_l\eta_{il}\eta_{jk}\eta_{ij}}{(1+|\boldsymbol{\zeta}|^2)^2}.$$

Here we denote $\boldsymbol{\eta} = \{\eta_{ij}\}$ and $\boldsymbol{\xi} = \{\xi_{ijk}\}$ by the second order derivative and the third order derivative respectively. For example, η_{ij} denotes $\partial_i\partial_j u$ and ξ_{ijk} denotes $\partial_i\partial_j\partial_k u$. Note that the sum of powers of the polynomials $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ of (4.22) and (4.23). In (4.22), the sum of powers of $\boldsymbol{\eta}$ is 1 and that of $\boldsymbol{\xi}$ is also 1, while in (4.23) the sum of powers of $\boldsymbol{\zeta}$ is 3. Thus, we can conclude that $\mathscr{F}_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi})$ and $\widetilde{\mathscr{F}}_1(\boldsymbol{\zeta}, \boldsymbol{\eta}, \boldsymbol{\xi})$ satisfy the assumption (F).

We now apply Theorem 4.1 to obtain the unique local classical solution of the surface diffusion flow with $h^{1+\beta}$-initial data. For the Willmore flow (4.16) and the anisotropic surface diffusion flow (4.15), the structure of the principal part and the lower order terms is the same as the structure of the surface diffusion flow (4.14). For this reason, we leave the details to the reader. □

4.3. The evolution equations for the height of a crystal

In this subsection we study the evolution equations for the height of a crystal which is derived in [15] as a limit of microscopic models

$$\begin{cases} \dfrac{\partial u}{\partial t} = -B\nabla\cdot\left\{\Lambda\cdot\nabla\left[\nabla\cdot\left(\dfrac{\nabla u}{|\nabla u|}\right) + \dfrac{g_3}{g_1}\nabla\cdot(|\nabla u|\nabla u)\right]\right\}, \\ \\ u(0) = u_0. \end{cases} \tag{4.24}$$

Here

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} \dfrac{1}{1+q|\nabla u|}\dfrac{(\partial_x u)^2}{|\nabla u|^2} + \dfrac{(\partial_y u)^2}{|\nabla u|^2} & -\dfrac{q|\nabla u|}{1+q|\nabla u|}\dfrac{(\partial_x u)(\partial_y u)}{|\nabla u|^2} \\ -\dfrac{q|\nabla u|}{1+q|\nabla u|}\dfrac{(\partial_x u)(\partial_y u)}{|\nabla u|^2} & \dfrac{1}{1+q|\nabla u|}\dfrac{(\partial_y u)^2}{|\nabla u|^2} + \dfrac{(\partial_x u)^2}{|\nabla u|^2} \end{pmatrix}.$$

The quantities B, q, g_1 and g_3 are given positive constants. We will try to solve the initial value problem for (4.24). Unfortunately, the equation (4.24) is degenerate (not parabolic) and singular at $\nabla u = 0$. Our aim in this subsection is to construct a unique classical solution for regularized and relaxed problems which are parabolic. We introduce $|\nabla h|_\varepsilon := (\varepsilon^2 + |\nabla h|^2)^{1/2}$

for (small) $\varepsilon > 0$ and relax and regularize the original equation (4.24) to get

$$\begin{cases} \dfrac{\partial u}{\partial t} = -B\nabla \cdot \left\{ \Lambda_\varepsilon \cdot \nabla \left[\nabla \cdot \left(\dfrac{\nabla u}{|\nabla u|_\varepsilon} \right) + \dfrac{g_3}{g_1} \nabla \cdot (|\nabla u|_\varepsilon \nabla u) \right] \right\}, \\ \\ u(0) = u_0, \end{cases} \tag{4.25}$$

where

$$\Lambda_\varepsilon = \begin{pmatrix} \dfrac{1}{1+q|\nabla u|_\varepsilon} \dfrac{(\partial_x u)^2}{|\nabla u|_\varepsilon^2} + \dfrac{(\partial_y u)^2}{|\nabla u|_\varepsilon^2} & -\dfrac{q|\nabla u|_\varepsilon}{1+q|\nabla u|_\varepsilon} \dfrac{(\partial_x u)(\partial_y u)}{|\nabla u|_\varepsilon^2} \\ -\dfrac{q|\nabla u|_\varepsilon}{1+q|\nabla u|_\varepsilon} \dfrac{(\partial_x u)(\partial_y u)}{|\nabla u|_\varepsilon^2} & \dfrac{1}{1+q|\nabla u|_\varepsilon} \dfrac{(\partial_y u)^2}{|\nabla u|_\varepsilon^2} + \dfrac{(\partial_x u)^2}{|\nabla u|_\varepsilon^2} \end{pmatrix}.$$

Theorem 4.4. *Assume $\varepsilon > 0$. For $\beta \in (0,1)$ let $u_0 \in h^{1+\beta}(\mathbf{R}^2)$. Then for each $\gamma \in (0,\beta)$ there exist positive constants $T = T(u_0) > 0$ and $\varepsilon = \varepsilon(u_0) > 0$ such that the problem of (4.25) admits a unique classical solution*

$$u(\cdot, w) \in BUC_{1-\alpha}([0,T], h^{4+\gamma}) \cap BUC^1_{1-\alpha}([0,T], h^\gamma),$$

for any initial value $w \in \overline{\mathbb{B}}_{h^{1+\beta}(\mathbf{R}^2)}(u_0, \varepsilon)$. Here $\alpha = (1+\beta-\gamma)/4$.

Proof. The proof is the same as the proof of Theorem 4.1. From now on we denote $a^\varepsilon_{ij}(\nabla u)$ by the (i,j)-th component of the matrix Λ_ε. We also observe that

$$\nabla \cdot \left(\frac{\nabla u}{|\nabla u|_\varepsilon} \right) = \sum_{k,l} q^\varepsilon_{kl}(\nabla u) \frac{\partial^2}{\partial x_k \partial x_l} u,$$

with

$$q^\varepsilon_{kl}(\nabla u) = \frac{1}{|\nabla u|_\varepsilon} \left(\delta_{kl} - \frac{\partial_{x_k} u \partial_{x_l} u}{|\nabla u|_\varepsilon^2} \right).$$

Then the highest order term of $\nabla \cdot \Lambda_\varepsilon \cdot \nabla[\nabla \cdot (\nabla u/|\nabla u|_\varepsilon)]$ is calculated by

$$\sum_{i,j,k,l} a^\varepsilon_{ij} q^\varepsilon_{kl} \frac{\partial^4}{\partial x_i \partial x_j \partial x_k \partial x_l} u, \tag{4.26}$$

where the indices run from 1 to 2. Thus the principal symbol is

$$\sum_{i,j,k,l} a^{\varepsilon}_{ij}(\boldsymbol{\zeta}) q^{\varepsilon}_{kl}(\boldsymbol{\zeta}) p_i p_j p_k p_l = \left(\sum_{i,j} a^{\varepsilon}_{ij}(\boldsymbol{\zeta}) p_i p_j \right) \left(\sum_{k,l} q^{\varepsilon}_{kl}(\boldsymbol{\zeta}) p_k p_l \right), \tag{4.27}$$

for $\boldsymbol{p} = (p_1, p_2) \in \mathbf{R}^2 \setminus \{0\}, \boldsymbol{\zeta} = (\zeta_1, \zeta_2), |\boldsymbol{\zeta}| \le M$. We now calculate

$$\begin{aligned} \sum_{k,l} q^{\varepsilon}_{k,l}(\boldsymbol{\zeta}) p_k p_l &= \frac{1}{|\boldsymbol{\zeta}|_\varepsilon} \left(|\boldsymbol{p}|^2 - \frac{\zeta_k \zeta_l p_k p_l}{|\boldsymbol{\zeta}|^2_\varepsilon} \right) \\ &= \frac{1}{|\boldsymbol{\zeta}|^3_\varepsilon} (|\boldsymbol{p}|^2 |\boldsymbol{\zeta}|^2_\varepsilon - (\boldsymbol{p} \cdot \boldsymbol{\zeta})^2) \\ &\ge \frac{\varepsilon^2}{|\boldsymbol{\zeta}|^3_\varepsilon} |\boldsymbol{p}|^2, \end{aligned} \tag{4.28}$$

$$\begin{aligned} \sum_{i,j} a^{\varepsilon}_{ij}(\boldsymbol{\zeta}) p_i p_j &= \begin{pmatrix} p_1 & p_2 \end{pmatrix} \Lambda_\varepsilon \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} \\ &= \left(\frac{1}{1 + q|\boldsymbol{\zeta}|_\varepsilon} \frac{\zeta_1^2}{|\boldsymbol{\zeta}|^2_\varepsilon} + \frac{\zeta_2^2}{|\boldsymbol{\zeta}|^2_\varepsilon} \right) p_1^2 \\ &\quad + 2 \left(- \frac{q|\boldsymbol{\zeta}|_\varepsilon}{1 + q|\boldsymbol{\zeta}|_\varepsilon} \frac{\zeta_1 \zeta_2}{|\boldsymbol{\zeta}|^2_\varepsilon} \right) p_1 p_2 \\ &\quad + \left(\frac{1}{1 + q|\boldsymbol{\zeta}|_\varepsilon} \frac{\zeta_2^2}{|\boldsymbol{\zeta}|^2_\varepsilon} + \frac{\zeta_1^2}{|\boldsymbol{\zeta}|^2_\varepsilon} \right) p_2^2 \\ &= \frac{1}{1 + q|\boldsymbol{\zeta}|_\varepsilon} \frac{1}{|\boldsymbol{\zeta}|^2_\varepsilon} \Big[(\zeta_1^2 + (1 + q|\boldsymbol{\zeta}|_\varepsilon)\zeta_2^2) p_1^2 \\ &\quad - 2q|\boldsymbol{\zeta}|_\varepsilon \zeta_1 \zeta_2 p_1 p_2 + (\zeta_2^2 + (1 + q|\boldsymbol{\zeta}|_\varepsilon)\zeta_1^2) p_2^2 \Big]. \end{aligned} \tag{4.29}$$

In order to show that the bilinear form (4.29) is positive definite, we shall calculate the determinant of the coefficient matrix of the bilinear form. The determinant is

$$\begin{aligned} &\left(\frac{1}{1 + q|\boldsymbol{\zeta}|_\varepsilon} \frac{1}{|\boldsymbol{\zeta}|^2_\varepsilon} \right)^2 \Big[\zeta_1^2 \zeta_2^2 + (1 + q|\boldsymbol{\zeta}|_\varepsilon)\zeta_1^4 + (1 + q|\boldsymbol{\zeta}|_\varepsilon)\zeta_2^4 \\ &\quad + (1 + q|\boldsymbol{\zeta}|_\varepsilon)^2 \zeta_1^2 \zeta_2^2 - q^2 |\boldsymbol{\zeta}|^2_\varepsilon \zeta_1^2 \zeta_2^2 \Big] \end{aligned} \tag{4.30}$$

$$= \left(\frac{1}{1+q|\boldsymbol{\zeta}|_\varepsilon}\frac{1}{|\boldsymbol{\zeta}|_\varepsilon^2}\right)^2 \Big[\zeta_1^2\zeta_2^2 + (1+q|\boldsymbol{\zeta}|_\varepsilon)\zeta_1^4 + (1+q|\boldsymbol{\zeta}|_\varepsilon)\zeta_2^4 + (1+2q|\boldsymbol{\zeta}|_\varepsilon)\zeta_1^2\zeta_2^2\Big]$$

$$= \left(\frac{1}{1+q|\boldsymbol{\zeta}|_\varepsilon}\frac{1}{|\boldsymbol{\zeta}|_\varepsilon^2}\right)^2 \Big[(1+q|\boldsymbol{\zeta}|_\varepsilon)|\boldsymbol{\zeta}|^4\Big]$$

$$= \frac{1}{1+q|\boldsymbol{\zeta}|_\varepsilon}\frac{1}{|\boldsymbol{\zeta}|_\varepsilon^4}|\boldsymbol{\zeta}|^4 > 0.$$

The trace of the coefficient matrix is obviously positive. Now we can conclude that there exists $\mathcal{V} = \mathcal{V}_M > 0$ such that

$$\sum_{i,j} a_{ij}^\varepsilon(\boldsymbol{\zeta})p_ip_j \geq \mathcal{V}|\boldsymbol{p}|^2, \tag{4.31}$$

for all $|\boldsymbol{\zeta}| \leq M$. From (4.27), (4.28) and (4.31), we finally obtain the ellipticity of the principal part so that the principal part of the equation fulfills (A1) and (A2).

The typical terms of the lower order terms of (4.25) are

$$\mathscr{F}_1(\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}) = -Bq\frac{\zeta_1^4\zeta_2\eta_{11}\xi_{112}}{(1+q|\boldsymbol{\zeta}|_\varepsilon)^2|\boldsymbol{\zeta}|_\varepsilon^6}, \tag{4.32}$$

$$\widetilde{\mathscr{F}}_1(\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi}) = \frac{2B}{1+q|\boldsymbol{\zeta}|_\varepsilon}\cdot\frac{\zeta_1^2\zeta_2^2\eta_{11}\eta_{12}^2}{|\boldsymbol{\zeta}|_\varepsilon^7}, \tag{4.33}$$

where we denote $\boldsymbol{\eta} = (\eta_{11}, \eta_{12}, \eta_{21}, \eta_{22})$ and $\boldsymbol{\xi} = (\xi_{111}, \xi_{112}, \ldots, \xi_{222})$ by the second order and the third order derivatives respectively. For example η_{11} denotes h_{xx} and ξ_{112} denotes h_{xxy}. Note that the sum of powers of the polynomials $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ of (4.32) and (4.33). The sum of powers of $\boldsymbol{\eta}$ in (4.32) is 1 and that of $\boldsymbol{\xi}$ in (4.32) is also 1. On the other hand, the sum of powers of $\boldsymbol{\eta}$ in (4.33) is 3. Thus, we can conclude that $\mathscr{F}_1(\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi})$ and $\widetilde{\mathscr{F}}_1(\boldsymbol{\zeta},\boldsymbol{\eta},\boldsymbol{\xi})$ satisfy the assumption (F). The other terms can be handled in a similar way. We are now in a position to apply Theorem 4.1 to obtain the unique local classical solution of (4.25) with $h^{1+\beta}$-initial data. □

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