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On the Spectrum of the Operator of Inner Waves in a Viscous Compressible Stratified Fluid

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Abstract. We study the structure of the spectrum of differential operators which arise in the problems modelling the inner oscillations of viscous compressible barotropic exponentially stratified three-dimensional fluid. For Dirichlet problem, we prove that the essential spectrum consists of three real points. We find the sector of the complex plane to which all the eigenvalues belong.

1. Viscous Stratified Compressible Fluid

Let us consider a bounded domain $\Omega \subset R^3$ with the boundary $\partial\Omega$ of the class C^∞ and the following system of fluid dynamics

$$(1.1)\qquad \begin{cases} \frac{\partial u_1}{\partial t} - \nu\Delta u_1 - \nu\beta\frac{\partial}{\partial x_1}\left(\operatorname{div}\overrightarrow{u}\right) + \frac{\partial p}{\partial x_1} = 0 \\ \frac{\partial u_2}{\partial t} - \nu\Delta u_2 - \nu\beta\frac{\partial}{\partial x_2}\left(\operatorname{div}\overrightarrow{u}\right) + \frac{\partial p}{\partial x_2} = 0 \\ \frac{\partial u_3}{\partial t} - \nu\Delta u_3 - \nu\beta\frac{\partial}{\partial x_3}\left(\operatorname{div}\overrightarrow{u}\right) + \rho + \frac{\partial p}{\partial x_3} = 0 \\ \frac{\partial \rho}{\partial t} - N^2 u_3 = 0 \\ \alpha^2\frac{\partial p}{\partial t} + \operatorname{div}\overrightarrow{u} = 0 \qquad x \in \Omega,\ t \geq 0. \end{cases}$$

Here $x = (x_1, x_2, x_3)$ is the space variable, $\overrightarrow{u}(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity field, $p(x,t)$ is the scalar field of the dynamic pressure and $\rho(x,t)$ is the dynamic density. In this model, the stationary distribution of density is described by the function e^{-Nx_3} , where N is a positive constant. For the compressibility coefficient α, the kinematic viscosity coefficient ν, and the volume (bulk) viscosity coefficient β we assume $\alpha > 0$, $\nu > 0$, $\beta \geq 0$.

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For inviscid case, the equations 1.1 are deduced in [3], [11] , [17]. For viscous compressible fluid, the system 1.1 is deduced, for example, in [12].

We may observe that, despite an extensive study of stratified flows from the physical point of view (see, for example, [2], [4], [5], [13], [14], [16]), there have been relatively few works considering the mathematical aspect of the problem.

We associate system 1.1 to Dirichlet boundary condition

$$\overrightarrow{u}|_{\partial\Omega} = 0.$$

Let us consider the following problem of normal vibrations

$$\begin{array}{l} \overrightarrow{u}(x,t) = \overrightarrow{v}(x)\, e^{-\lambda t} \\ \rho(x,t) = N v_4(x)\, e^{-\lambda t} \\ p(x,t) = \frac{1}{\alpha} v_5(x)\, e^{-\lambda t}\ ,\ \lambda \in C. \end{array} \tag{1.2}$$

We denote $\widetilde{v} = (\overrightarrow{v}, v_4, v_5)$ and write the system 1.1 in the matrix form

$$L\widetilde{v} = 0 \tag{1.3}$$

where

$$L = M - \lambda I$$

and

$$M = \begin{pmatrix} -\nu\Delta - \nu\beta\frac{\partial^2}{\partial x_1^2} & -\nu\beta\frac{\partial^2}{\partial x_1 \partial x_2} & -\nu\beta\frac{\partial^2}{\partial x_1 \partial x_3} & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_1} \\ -\nu\beta\frac{\partial^2}{\partial x_1 \partial x_2} & -\nu\Delta - \nu\beta\frac{\partial^2}{\partial x_2^2} & -\nu\beta\frac{\partial^2}{\partial x_2 \partial x_3} & 0 & \frac{1}{\alpha}\frac{\partial}{\partial x_2} \\ -\nu\beta\frac{\partial^2}{\partial x_1 \partial x_3} & -\nu\beta\frac{\partial^2}{\partial x_2 \partial x_3} & -\nu\Delta - \nu\beta\frac{\partial^2}{\partial x_3^2} & N & \frac{1}{\alpha}\frac{\partial}{\partial x_3} \\ 0 & 0 & -N & 0 & 0 \\ \frac{1}{\alpha}\frac{\partial}{\partial x_1} & \frac{1}{\alpha}\frac{\partial}{\partial x_2} & \frac{1}{\alpha}\frac{\partial}{\partial x_3} & 0 & 0 \end{pmatrix}. \tag{1.4}$$

We define the domain of the operator M as follows.

$$D(M) =$$

$$= \left\{ \overrightarrow{v} \in \left(\overset{0}{W}{}_2^1(\Omega) \right)^3,\ v_4 \in L_2(\Omega),\ v_5 \in L_2(\Omega):\ M\widetilde{v} \in (L_2(\Omega))^5 \right\},$$

where $\overset{0}{W}{}_2^1(\Omega)$ is a closure of the functional space $C_0^\infty(\Omega)$ in the norm

$$\|f\| = \left(\int_\Omega \left[|\nabla f|^2 + f^2\right] dx\right)^{\frac{1}{2}}.$$

In this paper, we will study the spectrum of the operator M.

From the point of view of applications, the separation of variables 1.2 serves as a tool to describe every non-stationary motion modelled by 1.1 as a linear superposition of the stationary modes. The knowledge of the spectrum of normal vibrations may be very useful for studying the stability of the flows. Additionally, the spectrum of operator M plays an important role in the investigation of weakly non-linear flows, since the bifurcation points where the small non-linear solutions arise, belong to the spectrum of linear normal vibrations, i.e., to the spectrum of operator M.

It can be easily seen that the operator M is a closed operator, and its domain is dense in $(L_2(\Omega))^5$.

Let is denote by $\sigma_{ess}(M)$ the essential spectrum of operator M. We recall that the essential spectrum

$$\sigma_{ess}(M) = \{\lambda \in C : \ (M - \lambda I) \text{ is not of Fredholm type}\},$$

is composed of the points belonging to the continuous spectrum, limit points of the point spectrum and the eigenvalues of infinite multiplicity ([10], [15]).

To find the essential spectrum of the operator M, we will use the following property ([9]):

$$\sigma_{ess}(M) = Q \cup S,$$

where

$$Q = \{\lambda \in C : \ (M - \lambda I) \text{ is not elliptic in sense of Douglis-Nirenberg}\}$$

and

$$S = \left\{ \begin{array}{c} \lambda \in C \setminus Q : \text{ the boundary conditions for the operator } (M - \lambda I) \\ \text{do not satisfy Lopatinski conditions} \end{array} \right\}.$$

We recall the following two definitions.

DEFINITION 1. Let us consider a differential matrix operator

$$L = \begin{pmatrix} l_{11} & ... & l_{1N} \\ ... & ... & ... \\ l_{N1} & ... & l_{NN} \end{pmatrix}, \quad l_{ij} = \sum_{|\alpha| \leq n_{ij}} a_{ij}^{(\alpha)} D^{\alpha}, \quad \alpha = (\alpha_1, ..., \alpha_n),$$

$$|\alpha| = \alpha_1 + ... + \alpha_n, \quad D_j = \frac{\partial}{\partial x_j}, \quad D^{\alpha} = D_1^{\alpha_1} ... D_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}.$$

Let $\{s_i\}_{i=1}^{N}$, $\{t_j\}_{j=1}^{N}$ be two sets of integer numbers such that, if $l_{ij} \neq 0$, then $n_{ij} = \deg l_{ij} \leq s_i + t_j$. In case $l_{ij} = 0$, we do not require any condition for the sum $s_i + t_j$. Now, we construct the main symbol of $L(D)$ as follows.

$$\widetilde{L}(D) = \begin{pmatrix} \widetilde{l}_{11}(D) & ... & \widetilde{l}_{1N}(D) \\ ... & ... & ... \\ \widetilde{l}_{N1}(D) & ... & \widetilde{l}_{NN}(D) \end{pmatrix},$$

$$\widetilde{l}_{ij} = \begin{cases} 0 \ \text{ if } l_{ij}(D) = 0 \text{ or } \deg l_{ij}(D) < s_i + t_j \\ \sum\limits_{|\alpha| = s_i + t_j} a_{ij}^{(\alpha)} D^{\alpha} \text{ if } \deg l_{ij}(D) = s_i + t_j \end{cases}.$$

If there exist the sets s and t which satisfy the above conditions and, additionally, if the following condition holds,

$$\det \widetilde{L}(\xi) \neq 0 \quad \text{for all } \xi \in R^n \setminus \{0\},$$

then the operator $L(D)$ is called *elliptic in sense of Douglis-Nirenberg* (see[1]).

DEFINITION 2. Let us consider $\xi = (\xi_1, \xi_2, \xi_3)$, $\widetilde{\xi} = (\xi_1, \xi_2)$, $\widehat{L}(\xi)$ −the matrix of the algebraic complements of the main symbol matrix $\widetilde{L}(\xi)$, $G(\xi)$ is the main symbol of the matrix $G(D)$ which defines the boundary conditions, $M^{+}\left(\widetilde{\xi}, \tau\right) = \prod \left(\tau - \tau_j\left(\widetilde{\xi}\right)\right)$, $\tau_j\left(\widetilde{\xi}\right)$ are the roots of the equation $\det \widetilde{L}\left(\widetilde{\xi}, \tau\right) = 0$ with positive imaginary part. If the rows of the matrix $G\left(\widetilde{\xi}, \tau\right) \widehat{L}\left(\widetilde{\xi}, \tau\right)$ are linearly independent with respect to the module

$M^+\left(\widetilde{\xi},\tau\right)$ for $\left|\widetilde{\xi}\right| \neq 0$, then we will say that the *conditions of Lopatinski are satisfied* (see[9]).

Now we establish the following main result.

THEOREM 1. *The essential spectrum of operator M is composed of three real points*

$$\sigma_{ess}(M) = \left\{0,\ \frac{1}{\nu\alpha^2(\beta+1)},\ \frac{1}{\nu\alpha^2(\beta+2)}\right\}.$$

PROOF. We observe that the main symbol of the operator $L = M - \lambda I$ is:

$$\widetilde{L}(\xi) = \begin{pmatrix} -\nu|\xi|^2 - \nu\beta\xi_1^2 & -\nu\beta\xi_1\xi_2 & -\nu\beta\xi_1\xi_3 & 0 & \frac{1}{\alpha}\xi_1 \\ -\nu\beta\xi_1\xi_2 & -\nu|\xi|^2 - \nu\beta\xi_2^2 & -\nu\beta\xi_2\xi_3 & 0 & \frac{1}{\alpha}\xi_2 \\ -\nu\beta\xi_1\xi_3 & -\nu\beta\xi_2\xi_3 & -\nu|\xi|^2 - \nu\beta\xi_3^2 & 0 & \frac{1}{\alpha}\xi_3 \\ 0 & 0 & 0 & -\lambda & 0 \\ \frac{1}{\alpha}\xi_1 & \frac{1}{\alpha}\xi_2 & \frac{1}{\alpha}\xi_3 & 0 & -\lambda \end{pmatrix}.$$

We calculate the determinant of the last matrix

$$\det\left(\widetilde{M-\lambda I}\right)(\xi) = \frac{\lambda\nu^2}{\alpha^2}|\xi|^6\left(\nu\lambda\alpha^2(\beta+1)-1\right),$$

and thus we can see that for two points $\lambda = 0$ and $\lambda = \frac{1}{\nu\alpha^2(\beta+1)}$ the operator $L = M - \lambda I$ is not elliptic in sense of Douglis-Nirenberg. We will show, additionally, that for the point $\lambda = \frac{1}{\nu\alpha^2(\beta+2)}$ the condition of Lopatinski is not satisfied.

The Dirichlet boundary condition can be written in a matrix form

$$G\widetilde{v}|_{\partial\Omega} = 0\ , \quad G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

If we denote $\widetilde{\xi} = (\xi_1, \xi_2)$, $\xi_3 = \tau$, then

$$\det\left(\widetilde{M-\lambda I}\right)\left(\widetilde{\xi},\tau\right) = \frac{\lambda\nu^2}{\alpha^2}\left(\left|\widetilde{\xi}\right|^2+\tau^2\right)^3\left(\nu\lambda\alpha^2(\beta+1)-1\right)$$

and thus the equation $\det\left(\widetilde{M-\lambda I}\right)\left(\widetilde{\xi},\tau\right)=0$ for $\lambda\neq 0,\ \frac{1}{\nu\alpha^2(\beta+1)}$ has one root $\tau=i\left|\widetilde{\xi}\right|$ of triple multiplicity in the upper half of the complex plane.

In this way, $M^+\left(\widetilde{\xi},\tau\right)=\left(\tau-i\left|\widetilde{\xi}\right|\right)^3$. Since the elements of the matrices $\widetilde{M-\lambda I}$ and G are homogeneous functions with respect to $\widetilde{\xi},\tau$, then it is sufficient to verify the Lopatinski condition for unitary vectors $\widetilde{\xi}$. Let us choose a local system of coordinates so that $\xi_1=1$, $\xi_2=0$. Then, we have

$$M^+\left(\widetilde{\xi},\tau\right)=(\tau-i)^3$$

and

$$\left(\widetilde{M-\lambda I}\right)(\tau)=$$

$$=\begin{pmatrix} -\nu(\beta+1)-\nu\tau^2 & 0 & -\nu\beta\tau & 0 & \frac{1}{\alpha} \\ 0 & -\nu\left(1+\tau^2\right) & 0 & 0 & 0 \\ -\nu\beta\tau & 0 & -\nu-\nu(\beta+1)\tau^2 & 0 & \frac{\tau}{\alpha} \\ 0 & 0 & 0 & -\lambda & 0 \\ \frac{1}{\alpha} & 0 & \frac{\tau}{\alpha} & 0 & -\lambda \end{pmatrix}.$$

For the matrix $\left(\widetilde{M-\lambda I}\right)$ we construct first the adjoint matrix $\left(\widehat{M-\lambda I}\right)$ (which is composed of algebraic complements of the original matrix), then we multiply $\left(\widehat{M-\lambda I}\right)$ by the boundary conditions matrix G, after which we divide $G\left(\widehat{M-\lambda I}\right)$ by the polynomial $(\tau-i)^3$, and, finally, we consider the matrix M_1 which is composed of the residues of that division. After some elementary transformations of the rows of M_1 and making the notation $\eta=\beta-\frac{1}{\nu\lambda\alpha^2}$, we obtain the following matrix M_2 :

$$M_2=-\nu^2\lambda(\tau-i)\times$$

$$\times\begin{pmatrix} (4+3\eta)i-(5\eta+4)\tau & 0 & -\eta(1+3\tau i) & 0 & 0 \\ 0 & -4(1+\eta)(\tau-i) & 0 & 0 & 0 \\ -\eta(1+3\tau i) & 0 & (4+\eta)i+(\eta-4)\tau & 0 & 0 \end{pmatrix}.$$

If we define as M_3 the 3^{rd} order square minor of M_2 without the last two columns, then the linear independence of the rows of the matrices M_2, M_1 can be verified by calculating the determinant of the matrix M_3.

We observe that

$$\det M_3 = 16\left(\eta+1\right)\left(\eta+2\right)^2\left(\nu^2\lambda\right)^3\left(\tau-i\right)^6 =$$
$$= 16\left(\left(\beta+1\right)-\frac{1}{\nu\alpha^2\lambda}\right)\left(\left(\beta+2\right)-\frac{1}{\nu\alpha^2\lambda}\right)^2\left(\nu^2\lambda\right)^3\left(\tau-i\right)^6.$$

As we can see, for $\lambda \neq 0,\ \frac{1}{\nu\alpha^2(\beta+1)},\ \frac{1}{\nu\alpha^2(\beta+2)}$ the rows of the matrix M_1 are linearly independent. We have already proved that the first two points belong to the essential spectrum of the operator M. In this way, due to the verification of Lopatinski condition, we have that the point $\lambda = \frac{1}{\nu\alpha^2(\beta+2)}$ also belongs to the essential spectrum of the operator M and thus

$$\sigma_{ess}\left(M\right) = \left\{0,\ \frac{1}{\nu\alpha^2\left(\beta+1\right)},\ \frac{1}{\nu\alpha^2\left(\beta+2\right)}\right\},$$

which concludes the proof of the Theorem. □

THEOREM 2. *The spectrum of operator M is symmetrical with respect to the real axis, and all the eigenvalues of operator M are in the following sector of the complex plane:*

$$Z = \left\{\lambda \in C:\ \ \mathrm{Re}\,\lambda \geq 0,\ \ \left|\mathrm{Im}\,\lambda\right| \leq N + \frac{\left(\mathrm{Re}\,\lambda\right)}{\nu\alpha^2\beta N}\right\}.$$

PROOF. Let us denote $v^* = (v_1, v_2, v_3, v_4)$ and $Sv^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & N \\ 0 & 0 & -N & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$.

Then, the system $\left(M-\lambda I\right)\{v^*, v_5\} = 0$ can be written in the form

$$\left\{\begin{array}{l} -\lambda v^* + Sv^* - \nu\Delta\overrightarrow{v} - \nu\beta\,\mathrm{div}\,\overrightarrow{v} + \frac{1}{\alpha}\nabla v_5 = 0 \\ -\lambda v_5 + \frac{1}{\alpha}\,\mathrm{div}\,\overrightarrow{v} = 0 \end{array}\right. .$$

Now we multiply this system by $\overline{\{v^*, v_5\}}$ and then integrate by parts in Ω. In this way, we obtain the following equations:

$$\begin{array}{l} -\lambda\left\|v^*\right\|^2 + \left(Sv^*, v^*\right) + \nu\sum\limits_{k=1}^{3}\left\|\nabla v_k\right\|^2 + \nu\beta\left\|\mathrm{div}\,\overrightarrow{v}\right\|^2 - \frac{1}{\alpha}\left(v_5, \mathrm{div}\,\overrightarrow{v}\right) = 0 \\ -\lambda\left\|v_5\right\|^2 + \frac{1}{\alpha}\left(\mathrm{div}\,\overrightarrow{v}, v_5\right) = 0 \end{array}.$$

We sum up these two equations

$$-\lambda\left(\|v^*\|^2+\|v_5\|^2\right)+(Sv^*,v^*)+\nu\sum_{k=1}^{3}\|\nabla v_k\|^2+\nu\beta\left\|\operatorname{div}\overrightarrow{v}\right\|^2+$$
$$+\frac{1}{\alpha}\left[(\operatorname{div}\overrightarrow{v},v_5)-(v_5,\operatorname{div}\overrightarrow{v})\right]=0$$

and then separate the real and the imaginary parts, keeping in mind the fact that, since S is skew-symmetric matrix, then the expression (Sv^*, v^*) is imaginary.

$$\operatorname{Re}\lambda=\frac{\nu\sum\limits_{k=1}^{3}\|\nabla v_k\|^2+\nu\beta\left\|\operatorname{div}\overrightarrow{v}\right\|^2}{\|v^*\|^2+\|v_5\|^2}\geq 0,$$

$$|\operatorname{Im}\lambda|=-i\frac{(Sv^*,v^*)+\frac{1}{\alpha}\left[(\operatorname{div}\overrightarrow{v},v_5)-(v_5,\operatorname{div}\overrightarrow{v})\right]}{\|v^*\|^2+\|v_5\|^2}.$$

Using the inequalities

$$(f,g)_{L_2}\leq\|f\|_{L_2}\|g\|_{L_2}\ ,\quad 2\frac{a}{\sqrt{N}}b\sqrt{N}\leq\frac{a^2}{N}+b^2N,$$

we estimate

$$|\operatorname{Im}\lambda|\leq\frac{N\|v^*\|^2+\frac{2}{\alpha}\left\|\operatorname{div}\overrightarrow{v}\right\|\|v_5\|}{\|v^*\|^2+\|v_5\|^2}\leq\frac{N\|v^*\|^2+N\|v_5\|^2+\frac{\|\operatorname{div}\overrightarrow{v}\|^2}{\alpha^2N}}{\|v^*\|^2+\|v_5\|^2}.$$

Since

$$\frac{\operatorname{Re}\lambda}{\nu\alpha^2\beta N}=\frac{\frac{1}{\alpha^2\beta N}\sum\limits_{k=1}^{3}\|\nabla v_k\|^2}{\|v^*\|^2+\|v_5\|^2}+\frac{\frac{1}{\alpha^2N}\left\|\operatorname{div}\overrightarrow{v}\right\|^2}{\|v^*\|^2+\|v_5\|^2}$$

and

$$|\operatorname{Im}\lambda|\leq N+\frac{\frac{1}{\alpha^2N}\left\|\operatorname{div}\overrightarrow{v}\right\|^2}{\|v^*\|^2+\|v_5\|^2},$$

then we finally have

$$|\mathrm{Im}\,\lambda| \leq N + \frac{(\mathrm{Re}\,\lambda)}{\nu\alpha^2\beta N}.$$

It remains to prove that the spectrum is symmetrical with respect to the real axis.

For that purpose, we apply the complex-conjugation to the original system of $(M - \lambda I)\{v^*, v_5\} = 0$:

$$\begin{cases} -\overline{\lambda v^*} + S\overline{v^*} - \nu\Delta\overline{\overrightarrow{v}} - \nu\beta\,\mathrm{div}\,\overline{\overrightarrow{v}} + \frac{1}{\alpha}\nabla\overline{v_5} = 0 \\ -\overline{\lambda v_5} + \frac{1}{\alpha}\,\mathrm{div}\,\overline{\overrightarrow{v}} = 0 \end{cases}$$

from which we can see that, if λ is an eigenvalue of M, then $\overline{\lambda}$ is also an eigenvalue of operator M, and thus the theorem is proved. □

2. Inviscid Stratified Fluid

For $\nu = 0$ and $\beta = 0$, for the case of compressible stratified fluid ($\alpha > 0$), in [8] it was proved that the essential spectrum of operator M is the interval of the imaginary axis $[-iN,\ iN]$.

For non-compressible inviscid stratified fluid ($\alpha = \beta = \nu = 0$), in [6], [7] it was proved that the essential spectrum of the operator M is the same interval of the imaginary axis $[-iN,\ iN]$, outside of which there can be only the eigenvalues of finite multiplicity.

3. Conclusions

Let us discuss briefly the results obtained in Theorems 1 and 2, comparing them with the corresponding results of the inviscid stratified fluid.

Just as in case of the explicit representations of solutions of Cauchy problems, where, in most cases, the inviscid solution cannot be obtained from viscous solutions as a mere limit for vanishing viscosity parameter; the essential spectrum of normal vibrations for inviscid stratified fluid cannot be obtained from the essential spectrum for viscous stratified fluid by putting the viscosity parameters ν and β equal to zero.

Therefore, the considered problems are remarkable and interesting due to the special property that, for the viscous fluid, the two points of the essential spectrum

$$\frac{1}{\nu\alpha^2(\beta+1)},\ \frac{1}{\nu\alpha^2(\beta+2)}$$

move to infinity for ν, $\beta \to 0$; while the essential spectrum of the inviscid fluid contains an interval of the imaginary axis.

Additionally, as we can observe, the previous results obtained for the inviscid fluid ($[-iN,\ iN]$), correspond, in a certain way, to the statement of Theorem 2, if we formally put $\operatorname{Re}\lambda = 0$:

$$\left(\operatorname{Re}\lambda \geq 0,\ |\operatorname{Im}\lambda| \leq N + \frac{(\operatorname{Re}\lambda)}{\nu\alpha^2\beta N}\right),\quad \text{for } \operatorname{Re}\lambda = 0 \text{ we have } |\operatorname{Im}\lambda| \leq N.$$

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