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A Rigidity Theorem and a Stability Theorem for Two-step Nilpotent Lie Groups

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Abstract. Let G be a Lie group and H a connected Lie subgroup of G. Given any discontinuous subgroup Γ for the homogeneous space $\mathscr{X} = G/H$ and any deformation of Γ, the deformed discrete subgroup may fail to be discontinuous for $\mathscr{X}$. To understand this phenomenon in the case when G is a two-step nilpotent Lie group, we provide a stratification of the deformation space of the action of Γ on $\mathscr{X}$, which depends upon the dimensions of $G-$adjoint orbits. As a direct consequence, a rigidity Theorem is given and a certain sufficient condition for the stability property is derived. We also discuss the Hausdorff property of the associated deformation space.

1. Introduction

We focus attention in this paper on some geometric and topological features of the deformation space of a discontinuous group acting on some nilpotent homogeneous spaces for which, the underlying group in question is two-step nilpotent. The problem of describing explicitly deformations for Clifford-Klein forms for the general non-Riemannian setting was initiated and formalized in [13], *Problem C* by T. Kobayashi, where he formalized this problem from a theoretic point of view. See [9, 12, 13, 14] for further perspectives and basic examples.

As an application of the general theory, T. Kobayashi and S. Nasrin studied in [14] the setup of a properly discontinuous action of a discrete subgroup $\Gamma \simeq \mathbb{Z}^k$ which acts on $\mathbb{R}^{k+1} \simeq G/H$ through a certain two-step nilpotent affine transformation group G of dimension $2k+1$ when the connected subgroup H in question is $\mathbb{R}^k$. In these circumstances, the authors

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gave a complete description of the parameter space

$$\mathscr{R}(\Gamma, G, H) := \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{l} \varphi \text{ is injective, } \varphi(\Gamma) \text{ is discrete and} \\ \text{acts properly and fixed point freely} \\ \text{on } G/H \end{array} \right\} \tag{1}$$

which is introduced in [9] for general contexts. On the basis of this description, they determine explicitly the deformation space $\mathscr{T}(\mathbb{Z}^k, G, H)$ by building up an accurate cross-section of the adjoint orbits of the elements of $\mathscr{R}(\Gamma, G, H)$.

In this paper, we generalize the above context and tackle the setting where the underlying group G is two-step nilpotent. We will provide a stratification of both the parameter and deformation spaces based on the dimensions of the adjoint action of G on the homomorphisms set $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, where $\mathfrak{l}$ stands for the Lie algebra of the syndetic hull of Γ (Theorems 3.3 and 3.6). The algebraic interpretation of these spaces given in Theorem 2.1 appears as a fundamental ingredient in this respect. We also show that the rigidity property fails to hold (Theorem 4.2) and a stability theorem is established (Theorem 5.6). In addition, we prove that whenever the G-orbits in $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ have a common dimension, the deformation space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is a Hausdorff space (Theorem 6.2). This is also the case where for instance the syndetic hull of Γ is abelian and maximal as in Corollary 6.3. A context where $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ fails to be a Hausdorff space is considered as well (Proposition 6.4). This leads to investigate about several kinds of questions of geometric nature related to the structure of the deformation space.

In [3], the authors studied the situation when G stands for the Heisenberg group and showed that the Hausdorff property of the deformation space is equivalent to the fact that $\mathscr{R}(\Gamma, G, H)$ is open in $\mathrm{Hom}(\Gamma, G)$ (which means that the stability property holds). In this context they also provide a description of the spaces $\mathscr{R}(\Gamma, G, H)$ and $\mathscr{T}(\Gamma, G, H)$. Heisenberg groups appear therefore as a special setting, where also the Hausdorff property of the deformation space is equivalent to the fact that the adjoint orbits of the parameter space have a common dimension. We show in Example 5 that this phenomenon fails to hold in general two-step contexts.

The outline of the paper is as follows. The next section is devoted to record some basic properties of Clifford−Klein forms and introduce the parameter and deformation spaces of the action of a discontinuous group on exponential homogeneous spaces for two-step nilpotent Lie groups. A result about the structure of the parameter space is also proved in this section. In section 3, we describe a layering of the parameter and deformation spaces. Section 4 is devoted to prove the local rigidity conjecture 4.1 and section 5 to establish some stability results. We close the paper with a study of the Hausdorff property of the deformation space. We do hope that our study could go farther to encompass other more general nilpotent contexts.

2. Backgrounds and Notations

We begin this section with fixing some notations, terminologies and recording some basic facts about deformations. The readers could consult the references [8, 9, 11, 12, 13] and some references therein for broader information about the subject. Concerning the entire subject, we strongly recommend the papers [8] and [13].

2.1. Proper and fixed point actions

Let $\mathscr{M}$ be a locally compact space and K a locally compact topological group. The continuous action of the group K on $\mathscr{M}$ is said to be:

(1) Proper if, for each compact subset $S \subset \mathscr{M}$ the set $K_S = \{k \in K : k \cdot S \cap S \neq \emptyset\}$ is compact.

(2) Fixed point free (or merely free) if, for each $m \in \mathscr{M}$, the isotropy group $K_m = \{k \in K : \ k \cdot m = m\}$ is trivial.

(3) (CI) if for any $m \in \mathscr{M}$, K_m is compact. (cf. [8]).

(4) Properly discontinuous if, K is discrete and the action of K on $\mathscr{M}$ is proper.

(5) The group K is said to be discontinuous, if it is discrete and it acts on $\mathscr{M}$ properly and fixed point freely.

In the case where $\mathscr{M} = G/H$ is a homogeneous space and K is a closed subgroup of G, then it is well known that the action of K on $\mathscr{M}$ is proper if $SHS^{-1} \cap K$ is compact for any compact set S in G. Likewise the action of K on $\mathscr{M}$ is free if for every $g \in G$, $K \cap gHg^{-1} = \{e\}$.

2.2. Clifford-Klein forms

For any given discontinuous subgroup Γ of a Lie group G for the homogeneous space G/H, the quotient space $\Gamma\backslash G/H$ is said to be a *Clifford-Klein form* for the homogeneous space G/H. The following point was emphasized by Kobayashi. Any Clifford-Klein form is endowed with a smooth manifold structure for which the quotient canonical surjection $\pi : G/H \to \Gamma\backslash G/H$ turns out to be an open covering and particularly a local diffeomorphism. On the other hand, any Clifford-Klein form $\Gamma\backslash G/H$ inherits any G-invariant geometric structure (e.g. complex structure, pseudo-Riemanian structure, conformal structure, symplectic structure,...) on the homogeneous space G/H through the covering map π.

2.3. Parameter and deformation spaces

The material dealt with in this subsection comes from [13]. The reader could also consult the references [12] and [9] for precise definitions. As in the first introductory section, we designate by $\mathrm{Hom}(\Gamma, G)$ the set of group homomorphisms from Γ to G endowed with the point wise convergence topology. The same topology is obtained by taking generators $\gamma_1, \dots, \gamma_k$ of Γ, then using the injective map

$$\mathrm{Hom}(\Gamma, G) \to G \times \cdots \times G, \ \varphi \mapsto (\varphi(\gamma_1), ..., \varphi(\gamma_k))$$

to equip $\mathrm{Hom}(\Gamma, G)$ with the relative topology induced from the direct product $G \times \cdots \times G$. For each $\varphi \in \mathscr{R}(\Gamma, G, H)$, the space $\varphi(\Gamma)\backslash G/H$ is a Clifford-Klein form which is a Hausdorff topological space and even equipped with a structure of a smooth manifold for which, the quotient canonical map is an open covering. Let now $\varphi \in \mathscr{R}(\Gamma, G, H)$ and $g \in G$, we consider the element φ^g of $\mathrm{Hom}(\Gamma, G)$ defined by $\varphi^g(\gamma) = g\varphi(\gamma)g^{-1}, \gamma \in \Gamma$. It is then clear that the element $\varphi^g \in \mathscr{R}(\Gamma, G, H)$ and that the map:

$$\varphi(\Gamma)\backslash G/H \longrightarrow \varphi^g(\Gamma)\backslash G/H, \quad \varphi(\Gamma)xH \mapsto \varphi^g(\Gamma)g^{-1}xH$$

is a natural diffeomorphism. Following ([13], (5.3.1)), we consider then the orbits space

$$\mathscr{T}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, H)/G$$

instead of $\mathscr{R}(\Gamma, G, H)$ in order to avoid the unessential part of deformations arising inner automorphisms and to be quite precise on parameters. The quotient space $\mathscr{T}(\Gamma, G, H)$ is called the deformation space of the discontinuous action of Γ on the homogeneous space G/H.

2.4. Algebraic description of the parameter and deformation spaces

Let $\mathfrak{g}$ be a finite dimensional real exponential solvable Lie algebra and G its associated Lie group. This means that the exponential map $\exp : \mathfrak{g} \to G$ is a global C^∞-diffeomorphism from $\mathfrak{g}$ into G. That is, G is connected and simply connected. Let log denote the inverse map of exp. The Lie algebra $\mathfrak{g}$ acts on $\mathfrak{g}$ by the adjoint representation ad, that is $\mathrm{ad}_T(Y) = [T, Y],\ T,\ Y \in \mathfrak{g}$. The group G acts on $\mathfrak{g}$ by the adjoint representation Ad, defined by $\mathrm{Ad}_g = \exp(\mathrm{ad}_T),\ g = \exp T \in G$. Let $H = \exp \mathfrak{h}$ be a closed connected subgroup of G. Let Γ be an discrete subgroup of G of rank k and define the parameter space $\mathscr{R}(\Gamma, G, H)$ as given in (1). Let L be the syndetic hull of Γ which is the smallest (and hence the unique) connected Lie subgroup of G which contains Γ co-compactly (see [2]). Recall that the Lie subalgebra $\mathfrak{l}$ of L is the real span of the lattice $\log \Gamma$, which is generated by $\{\log \gamma_1, \ldots, \log \gamma_k\}$ where $\{\gamma_1, \ldots, \gamma_k\}$ is a set of generators of Γ. The group G also acts on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ by:

$$g \cdot \psi = \mathrm{Ad}_g \circ \psi. \tag{2}$$

Let $\mathrm{Hom}^{\mathrm{inj}}(\mathfrak{l}, \mathfrak{g})$ be the set of injective homomorphisms from $\mathfrak{l}$ to $\mathfrak{g}$. The following useful result was originated in [14] and obtained in [2].

Theorem 2.1. *Let $G = \exp \mathfrak{g}$ be a completely solvable Lie group, $H = \exp \mathfrak{h}$ a closed connected subgroup of G, Γ a discontinuous subgroup for the homogeneous space G/H and $L = \exp \mathfrak{l}$ its syndetic hull. Then up to a homeomorphism, the parameter space $\mathscr{R}(\Gamma, G, H)$ is given by:*

$$\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \{\psi \in \mathrm{Hom}^{inj}(\mathfrak{l}, \mathfrak{g}) : \exp(\psi(\mathfrak{l})) \textit{ acts properly on } G/H\}.$$

The deformation space $\mathscr{T}(\Gamma, G, H)$ is likewise homeomorphic to the space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})/\mathrm{Ad}$, where the action Ad *of G is given as in (2). Furthermore, when G is completely solvable, the assumption on Γ to be abelian can be removed.*

2.5. On the structure of the parameter space

Definition 2.2. Let G be an exponential solvable Lie group and H a connected and closed subgroup of G. A pair (G, H) is said to have the

Lipsman property if for any connected subgroup L of G, there is equivalence between proper and (CI) for the triple (L, G, H).

Definition 2.3 (cf.[5]).
1. A subset V of $\mathbb{R}^n$ is called semi-algebraic if it admits some representation of the form

$$V = \bigcup_{i=1}^{s} \bigcap_{j=1}^{r_i} \{x \in \mathbb{R}^n : P_{i,j}(x) \quad s_{ij} \quad 0\},$$

where for each $i = 1, ..., s$ and $j = 1, ..., r_i$, $P_{i,j}$ are some polynomials on $\mathbb{R}^n$ and $s_{ij} \in \{>, =, <\}$.
2. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semi-algebraic sets. A map $f : X \to Y$ is called semi-algebraic if its graph is a semi-algebraic set of $\mathbb{R}^{n+m}$.

Any information concerning the spaces $\mathrm{Hom}(\Gamma, G)$ and $R(\Gamma, G, H)$ may help to understand the properties and the structure of the deformation space $\mathcal{T}(\Gamma, G, H)$. The sets $\mathrm{Hom}(\Gamma, G)$ and $R(\Gamma, G, H)$ may have some singularities and there is no clear raison, to say that the parameter space $R(\Gamma, G, H)$ is an analytic or algebraic or smooth manifold. For instance, when the parameter space is a semi-algebraic set, it has certainly a finite number of connected components, which means in turn, that the deformation space itself enjoys this feature. The following proposition, will be of use in the sequel and shows that the parameter space is semi-algebraic whenever the pair (G, H) has the Lipsman property with G a connected simply connected nilpotent Lie group.

Proposition 2.4. *Let (G, H) be a pair having the Lipsman property with G a connected simply connected nilpotent Lie group, Γ a discontinuous subgroup for G/H, and $\mathfrak{l}$ the Lie algebra of the syndetic hull of Γ. Then the parameter space $\mathcal{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is semi-algebraic.*

Proof. Note that the action of $\exp(\varphi(\mathfrak{l}))$ on G/H is free if and only if $\mathrm{Ad}_g \varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\}$ for all $g \in G$. Let $\mathcal{L}(\mathfrak{l}, \mathfrak{g})$ be the set of linear map of $\mathfrak{l}$ into $\mathfrak{g}$. Consider the maps

$$\begin{array}{rcl} i : \mathfrak{g} \times \mathcal{L}(\mathfrak{l}, \mathfrak{g}) & \longrightarrow & \mathcal{L}(\mathfrak{l}, \mathfrak{g}) \\ (X, \varphi) & \longmapsto & \mathrm{Ad}_{\exp(X)} \varphi \end{array}$$

and define the semi-algebraic set

$$S = \left\{ (X, \varphi) \;\middle|\; \begin{array}{l} \varphi \in \mathrm{Hom}^{\mathrm{inj}}(\mathfrak{l}, \mathfrak{g}) \\ \mathrm{Ad}_{\exp(X)}\, \varphi(\mathfrak{l}) \cap \mathfrak{h} \neq \{0\} \end{array} \right\} \subset \mathfrak{g} \times \mathcal{L}(\mathfrak{l}, \mathfrak{g}).$$

The map $P_{r_2} : \mathfrak{g} \times \mathcal{L}(\mathfrak{l}, \mathfrak{g}) \longrightarrow \mathcal{L}(\mathfrak{l}, \mathfrak{g});\ (X, \varphi) \longmapsto \varphi$ is semi-algebraic, then the set

$$P_{r_2}(S) = \left\{ \begin{array}{l} \varphi \in \mathrm{Hom}^0(\mathfrak{l}, \mathfrak{g}) : \text{ there exists } X \in \mathfrak{g} \text{ such that} \\ \mathrm{Ad}_{\exp(X)}\, \varphi(\mathfrak{l}) \cap \mathfrak{h} \neq \{0\} \end{array} \right\}$$

is semi-algebraic and its complement

$$\begin{aligned} \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) &= P_{r_2}(S)^c \\ &= \left\{ \varphi \in \mathrm{Hom}^{\mathrm{inj}}(\mathfrak{l}, \mathfrak{g}) : \ \mathrm{Ad}_{\exp(X)}\, \varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\},\ \forall X \in \mathfrak{g} \right\} \end{aligned}$$

is also semi-algebraic. □

3. The Deformation Space for Two-step Nilpotent Lie Groups

We assume henceforth that $\mathfrak{g}$ is a two-step nilpotent Lie algebra, $\mathfrak{l}$ a subalgebra of $\mathfrak{g}$ and $\mathfrak{z}$ the (non-trivial) center of $\mathfrak{g}$. We consider the decompositions

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{g}' \text{ and } \mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{l}' \tag{3}$$

where $\mathfrak{g}'$ (respectively $\mathfrak{l}'$) designates a subspace of $\mathfrak{g}$ (of $\mathfrak{l}$ respectively). Any $\varphi \in \mathcal{L}(\mathfrak{l}, \mathfrak{g})$ can be written as

$$\varphi = \begin{pmatrix} A_\varphi & B_\varphi \\ C_\varphi & D_\varphi \end{pmatrix}, \tag{4}$$

where $A_\varphi \in \mathcal{L}([\mathfrak{l}, \mathfrak{l}], \mathfrak{z})$, $B_\varphi \in \mathcal{L}(\mathfrak{l}', \mathfrak{z})$, $C_\varphi \in \mathcal{L}([\mathfrak{l}, \mathfrak{l}], \mathfrak{g}')$ and $D_\varphi \in \mathcal{L}(\mathfrak{l}', \mathfrak{g}')$. Let

$$\varphi' = \begin{pmatrix} A_\varphi & 0 \\ 0 & D_\varphi \end{pmatrix}. \tag{5}$$

We first remark the following assertion.

Lemma 3.1. *Any element $\varphi \in \mathcal{L}(\mathfrak{l}, \mathfrak{g})$ is a Lie algebra homomorphism if and only if $C_\varphi = 0$ and $\varphi' \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$.*

PROOF. We point out first that if $\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, then $\varphi([\mathfrak{l}, \mathfrak{l}]) = [\varphi(\mathfrak{l}), \varphi(\mathfrak{l})] \subset \mathfrak{z}$, in particular $C_\varphi = 0$ and $\varphi = \begin{pmatrix} A_\varphi & B_\varphi \\ 0 & D_\varphi \end{pmatrix}$. Let $x = x_1 + y_1$ and $x' = x_2 + y_2 \in \mathfrak{l}$ where $x_i \in [\mathfrak{l}, \mathfrak{l}]$ and $y_i \in \mathfrak{l}'$, then

$$\begin{aligned} \varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) &\Leftrightarrow [\varphi(x), \varphi(x')] = \varphi([x, x']) \\ &\Leftrightarrow [A_\varphi(x_1) + B_\varphi(y_1) + D_\varphi(y_1), A_\varphi(x_2) \\ &\qquad + B_\varphi(y_2) + D_\varphi(y_2)] \\ &\qquad = \varphi([x_1 + y_1, x_2 + y_2]) \\ &\Leftrightarrow [D_\varphi(y_1), D_\varphi(y_2)] = \varphi([y_1, y_2]) = A_\varphi([y_1, y_2]) \\ &\Leftrightarrow [D_\varphi(y_1), D_\varphi(y_2)] = A_\varphi([y_1, y_2]) \\ &\Leftrightarrow \varphi' \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}). \ \square \end{aligned}$$

For any $g \in G$, let as earlier Ad_g designate the adjoint representation which can be written making use the decomposition (3) as

$$\mathrm{Ad}_g = \begin{pmatrix} I_{\mathfrak{z}} & \sigma(g) \\ 0 & I_{\mathfrak{g}'} \end{pmatrix} \tag{6}$$

for some map $\sigma : G \to \mathcal{L}(\mathfrak{g}', \mathfrak{z})$. Here $I_{\mathfrak{z}}$ and $I_{\mathfrak{g}'}$ denote the identity maps of $\mathfrak{z}$ and $\mathfrak{g}'$ respectively.

LEMMA 3.2. *The map σ is a group homomorphism. In particular the range of σ is a linear subspace of $\mathcal{L}(\mathfrak{g}', \mathfrak{z})$.*

PROOF. Clearly $\mathrm{Ad} : G \to GL(\mathfrak{g})$ is a group homomorphism. Then for g and $g' \in G$

$$\begin{aligned} \mathrm{Ad}_{gg'} = \mathrm{Ad}_g \mathrm{Ad}_{g'} &\Leftrightarrow \begin{pmatrix} I_{\mathfrak{z}} & \sigma(gg') \\ 0 & I_{\mathfrak{g}'} \end{pmatrix} = \begin{pmatrix} I_{\mathfrak{z}} & \sigma(g) \\ 0 & I_{\mathfrak{g}'} \end{pmatrix} \begin{pmatrix} I_{\mathfrak{z}} & \sigma(g') \\ 0 & I_{\mathfrak{g}'} \end{pmatrix} \\ &\Leftrightarrow \begin{pmatrix} I_{\mathfrak{z}} & \sigma(gg') \\ 0 & I_{\mathfrak{g}'} \end{pmatrix} = \begin{pmatrix} I_{\mathfrak{z}} & \sigma(g) + \sigma(g') \\ 0 & I_{\mathfrak{g}'} \end{pmatrix} \\ &\Leftrightarrow \sigma(gg') = \sigma(g) + \sigma(g'). \end{aligned}$$

In particular σ is continuous and G is connected which means therefore that $\mathrm{Im}(\sigma)$ is a connected (linear) subgroup of the linear space $\mathcal{L}(\mathfrak{g}', \mathfrak{z})$. $\square$

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and consider the decompositions

$$\mathfrak{g} = (\mathfrak{z} \cap \mathfrak{h}) \oplus \mathfrak{z}' \oplus \mathfrak{h}' \oplus V \text{ and } \mathfrak{l} = [\mathfrak{l}, \mathfrak{l}] \oplus \mathfrak{l}', \tag{7}$$

where $\mathfrak{z} = (\mathfrak{z} \cap \mathfrak{h}) \oplus \mathfrak{z}'$, $\mathfrak{h} = (\mathfrak{z} \cap \mathfrak{h}) \oplus \mathfrak{h}'$ and V is a linear subspace supplementary to $(\mathfrak{z} \cap \mathfrak{h}) \oplus \mathfrak{z}' \oplus \mathfrak{h}'$ in $\mathfrak{g}$. Then with respect to these decompositions, the adjoint representation Ad_g, $g \in G$ can once again written down as

$$\mathrm{Ad}_g = \begin{pmatrix} I_1 & 0 & \sigma_1(g) & \delta_1(g) \\ 0 & I_2 & \sigma_2(g) & \delta_2(g) \\ 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & I_4 \end{pmatrix},$$

where $\sigma(g) = \begin{pmatrix} \sigma_1(g) & \delta_1(g) \\ \sigma_2(g) & \delta_2(g) \end{pmatrix}$, $\sigma_1(g) \in \mathcal{L}(\mathfrak{h}', \mathfrak{z} \cap \mathfrak{h})$, $\delta_1(g) \in \mathcal{L}(V, \mathfrak{z} \cap \mathfrak{h})$, $\sigma_2(g) \in \mathcal{L}(\mathfrak{h}', \mathfrak{z}')$ and $\delta_2(g) \in \mathcal{L}(V, \mathfrak{z}')$(here I_1, I_2, I_3 and I_4 designate the identity maps of $\mathfrak{z} \cap \mathfrak{h}$, $\mathfrak{z}'$, $\mathfrak{h}'$ and V respectively). This leads to the fact that any element of $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ can be written accordingly, as a matrix

$$\varphi(A, B) = \begin{pmatrix} A_1 & B_1 \\ A_2 & B_2 \\ 0 & B_3 \\ 0 & B_4 \end{pmatrix},$$

where $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ and $B = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \\ B_4 \end{pmatrix}$. Here $A_1 \in \mathcal{L}([\mathfrak{l}, \mathfrak{l}], \mathfrak{z} \cap \mathfrak{h})$, $A_2 \in \mathcal{L}([\mathfrak{l}, \mathfrak{l}], \mathfrak{z}')$, $B_1 \in \mathcal{L}(\mathfrak{l}', \mathfrak{z} \cap \mathfrak{h})$, $B_2 \in \mathcal{L}(\mathfrak{l}', \mathfrak{z}')$, $B_3 \in \mathcal{L}(\mathfrak{l}', \mathfrak{h}')$ and $B_4 \in \mathcal{L}(\mathfrak{l}', V)$. We can now state our first result.

THEOREM 3.3. *Let G be a two-step nilpotent Lie group, H a connected subgroup of G and Γ a discontinuous subgroup for G/H. Then the parameter space $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ splits into the disjoint union $\mathscr{R}_1 \bigsqcup \mathscr{R}_2$, where*

$$\mathscr{R}_1 := \left\{ \varphi(A, B) \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \,\middle|\, \begin{array}{l} \mathrm{rk}(B_4) = \dim(\mathfrak{l}') \\ \textit{and } \mathrm{rk}(A_2) = \dim([\mathfrak{l}, \mathfrak{l}]) \end{array} \right\}$$

and

$$\mathscr{R}_2 := \left\{ \varphi(A,B) \in \operatorname{Hom}(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \operatorname{rk}(B_4) < \dim(\mathfrak{l}') \textit{ and for all } g \in G \\ \operatorname{rk}\begin{pmatrix} A_2 & B_2 + \sigma_2(g)B_3 + \delta_2(g)B_4 \\ 0 & B_4 \end{pmatrix} \\ \quad = \dim(\mathfrak{l}) \end{array} \right\}.$$

Proof. As the pair (G,H) has the Lipsman property ([1] and [16]), Theorem 2.1 enables us to state that

$$\mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) = \left\{ \varphi(A,B) \in \operatorname{Hom}(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \dim \varphi(A,B)(\mathfrak{l}) = \dim(\mathfrak{l}) \\ \operatorname{Ad}_g \varphi(A,B)(\mathfrak{l}) \cap \mathfrak{h} = \{0\} \\ \quad \text{for all } g \in G \end{array} \right\}.$$

Now,

$$\operatorname{Ad}_g \varphi(A,B) = \begin{pmatrix} A_1 & B_1 + \sigma_1(g)B_3 + \delta_1(g)B_4 \\ A_2 & B_2 + \sigma_2(g)B_3 + \delta_2(g)B_4 \\ 0 & B_3 \\ 0 & B_4 \end{pmatrix}, \tag{8}$$

which means that the condition $[\operatorname{Ad}_g \varphi(A,B)](\mathfrak{l}) \cap \mathfrak{h} = \{0\}$ is equivalent to the fact that $\operatorname{rk}\begin{pmatrix} A_2 & B_2 + \sigma_2(g)B_3 + \delta_2(g)B_4 \\ 0 & B_4 \end{pmatrix} = \dim(\mathfrak{l})$, which is in turn equivalent to

$$\operatorname{rk}(B_4) = \dim(\mathfrak{l}') \text{ and } \operatorname{rk}(A_2) = \dim([\mathfrak{l},\mathfrak{l}])$$

or

$$\operatorname{rk}(B_4) < \dim(\mathfrak{l}') \text{ and } \operatorname{rk}\begin{pmatrix} A_2 & B_2 + \sigma_2(g)B_3 + \delta_2(g)B_4 \\ 0 & B_4 \end{pmatrix} = \dim(\mathfrak{l}). \ \square$$

3.1. Description of the deformation space $\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$

We fix in this section our objects and we keep the same notation. Let $\mathfrak{g}$ and $\mathfrak{l}$ be as above and

$$\operatorname{Hom}_1(\mathfrak{l},\mathfrak{g}) := \left\{\varphi' : \ \varphi \in \operatorname{Hom}(\mathfrak{l},\mathfrak{g})\right\},$$

where φ' is as in (5). The group G acts on $\operatorname{Hom}_1(\mathfrak{l},\mathfrak{g}) \times \mathcal{L}(\mathfrak{l},\mathfrak{g})$ through the following law

$$g \cdot (\varphi', B_\varphi) = (\varphi', B_\varphi + \sigma(g)D_\varphi).$$

where B_φ and D_φ are as in formula (4) and σ is as in (6).

Lemma 3.4. *The map*

$$\begin{array}{rcl}\psi : \mathrm{Hom}(\mathfrak{l},\mathfrak{g}) & \longrightarrow & \mathrm{Hom}_1(\mathfrak{l},\mathfrak{g}) \times \mathcal{L}(\mathfrak{l}',\mathfrak{z}) \\ \varphi & \longmapsto & \big(\varphi', B_\varphi\big)\end{array}$$

is a G-equivariant homeomorphism.

Proof. The fact that ψ is a well defined homeomorphism comes directly from Lemma 3.1. Let $g \in G$ and $\varphi \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g})$, then

$$\begin{array}{rcl}\psi\,(g \cdot \varphi) & = & \psi\,(\mathrm{Ad}_g\,\varphi) \\ & = & \big(\varphi', B_\varphi + \sigma(g) D_\varphi\big) \\ & = & g \cdot \psi\,(\varphi)\,,\end{array}$$

which proves the lemma. □

3.2. Decomposition of $\mathrm{Hom}_1(\mathfrak{l},\mathfrak{g})$.

As in Lemma 3.2, the group $\sigma(G)$ is a linear space. For any $\varphi \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g})$, let l_φ be the linear map defined by

$$\begin{array}{rcl}l_\varphi : \sigma(G) & \longrightarrow & \mathcal{L}(\mathfrak{l}',\mathfrak{z}) \\ \sigma(g) & \longmapsto & \sigma(g) D_\varphi.\end{array}$$

The range of l_φ is a linear subspace of $\mathcal{L}(\mathfrak{l}',\mathfrak{z})$ and the orbit $G \cdot (\varphi', B_\varphi) = (\varphi', B_\varphi + \mathrm{Im}(l_\varphi))$. Let $m = \dim(\mathcal{L}(\mathfrak{l}',\mathfrak{z}))$ and $q = \dim(\sigma(G))$. For $t = 0, 1, \ldots, q$, we define the sets

$$\mathrm{Hom}_1^t(\mathfrak{l},\mathfrak{g}) := \big\{\varphi' \in \mathrm{Hom}_1(\mathfrak{l},\mathfrak{g}) :\ \mathrm{rk}(l_\varphi) = t\big\}\,.$$

Then clearly,

$$\mathrm{Hom}_1(\mathfrak{l},\mathfrak{g}) = \bigsqcup_{t=0}^{q} \mathrm{Hom}_1^t(\mathfrak{l},\mathfrak{g}).$$

We fix a basis $\{e_1, \ldots, e_m\}$ of $\mathcal{L}(\mathfrak{l}',\mathfrak{z})$ and let

$$I(m, m-t) = \{(i_1, \ldots, i_{m-t}); 1 \le i_1 < \cdots < i_{m-t} \le m\}.$$

For $\beta = (i_1, \dots, i_{m-t}) \in I(m, m-t)$, we consider the subspace $V_\beta := \oplus_{j=1}^{m-t} \mathbb{R} e_{i_j}$.

Proposition 3.5. *For any $\varphi \in \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g})$, let $P_\varphi : \mathcal{L}(\mathfrak{l}', \mathfrak{z}) \to \mathcal{L}(\mathfrak{l}', \mathfrak{z}) / \mathrm{Im}(l_\varphi)$ and $\mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g}) := \{\varphi \in \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g}) : \det(P_\varphi(e_{i_1}), \dots, P_\varphi(e_{i_{m-t}})) \neq 0\}$. Then*

$$\mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g}) = \bigcup_{\beta \in I(m,m-t)} \mathrm{Hom}_{1,\beta}^t(\mathfrak{l}, \mathfrak{g})$$

as a union of open subsets.

Proof. We know that for all $\varphi \in \mathrm{Hom}_1^t(\mathfrak{l}, \mathfrak{g})$, the set $\mathrm{Im}(l_\varphi)$ is a linear subspace of $\mathcal{L}(\mathfrak{l}', \mathfrak{z})$ of dimension t. There exists therefore $(i_1, \dots, i_{m-t}) \in I(m, m-t)$ such that the family $\{P_\varphi(e_{i_1}), \dots, P_\varphi(e_{i_{m-t}})\}$ forms a basis of $\mathcal{L}(\mathfrak{l}', \mathfrak{z}) / \mathrm{Im}(l_\varphi)$ and consequently

$$\det(P_\varphi(e_{i_1}), \dots, P_\varphi(e_{i_{m-t}})) \neq 0. \ \square$$

We are now ready to prove our main result in this section.

Theorem 3.6. *Let G, H and $\mathfrak{l}$ be as before. The deformation space reads*

$$\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \bigsqcup_{i=1}^{2} \bigsqcup_{t=0}^{q} \bigcup_{\beta \in I(m,m-t)} \mathscr{T}_{t,\beta,i},$$

where for $\beta = (i_1, \dots, i_{m-t})$, the set $\mathscr{T}_{t,\beta,1}$ is homeomorphic to the semi-algebraic set

$$\mathcal{T}_{t,\beta,1} := \left\{ \varphi(A,B) \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in V_\beta,\ \mathrm{rk}(l_{\varphi(A,B)}) = t \\ \det(P_{\varphi(A,B)}(e_{i_1}), \dots, P_{\varphi(A,B)}(e_{i_{m-t}})) \neq 0 \\ \mathrm{rk} \begin{pmatrix} A_2 & 0 \\ 0 & B_4 \end{pmatrix} = \dim(\mathfrak{l}) \end{array} \right\}$$

and $\mathcal{T}_{t,\beta,2}$ *is homeomorphic to*

$$\mathcal{T}_{t,\beta,2} := \left\{ \varphi(A,B) \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g}) \;\middle|\; \begin{array}{l} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in V_\beta,\ \mathrm{rk}(l_{\varphi(A,B)}) = t \\ \det(P_{\varphi(A,B)}(e_{i_1}), \ldots, P_{\varphi(A,B)}(e_{i_{m-t}})) \neq 0 \\ \mathrm{rk}(B_4) < \dim(\mathfrak{l}') \\ \mathrm{rk}\begin{pmatrix} A_2 & B_2 + \sigma_2(g)B_3 + \delta_2(g)B_4 \\ 0 & B_4 \end{pmatrix} \\ \quad = \dim(\mathfrak{l}) \textit{ for all } g \in G \end{array} \right\},$$

which is also semi-algebraic.

Proof. It is easy to see that $D_\varphi = D_{g\cdot\varphi}$, which means that $l_\varphi = l_{g\cdot\varphi}$ and $P_{\varphi'} = P_{g\cdot\varphi'}$ for all $\varphi \in \mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ and $g \in G$, then for all $\beta \in I(m, m-t)$ and $0 \leq t \leq q$, the set $\mathrm{Hom}^t_{1,\beta}(\mathfrak{l},\mathfrak{g}) \times \mathcal{L}(\mathfrak{l}',\mathfrak{z})$ is G-stable. Let $\mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g}) = \psi^{-1}(\mathrm{Hom}^t_{1,\beta}(\mathfrak{l},\mathfrak{g}) \times \mathcal{L}(\mathfrak{l}',\mathfrak{z}))$, then

$$\mathrm{Hom}(\mathfrak{l},\mathfrak{g}) = \bigsqcup_{t=0}^{q} \mathrm{Hom}^t(\mathfrak{l},\mathfrak{g}),$$

where

$$\mathrm{Hom}^t(\mathfrak{l},\mathfrak{g}) := \bigcup_{\beta\in I(m,m-t)} \mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g}). \tag{9}$$

Since ψ is G-equivariant, it is clear that for all $t \in \{0, \ldots, q\}$ and $\beta \in I(m, m-t)$, the set $\mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})$ is G-stable and by (9), we get:

$$\mathrm{Hom}^t(\mathfrak{l},\mathfrak{g})/G = \bigcup_{\beta\in I(m,m-t)} \mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})/G, \tag{10}$$

and then

$$\mathrm{Hom}(\mathfrak{l},\mathfrak{g})/G = \bigsqcup_{t=0}^{q} \bigcup_{\beta\in I(m,m-t)} \mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})/G.$$

Whence

$$\mathcal{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) = \bigsqcup_{i=1}^{2} \bigsqcup_{t=0}^{q} \bigcup_{\beta\in I(m,m-t)} \mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g}) \cap \mathcal{R}_i.$$

We now consider the maps

$$\begin{array}{rcl}\overline{\psi}:\mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})/G & \to & \left(\mathrm{Hom}^t_{1,\beta}(\mathfrak{l},\mathfrak{g})\times\mathcal{L}(\mathfrak{l}',\mathfrak{z})\right)/G\\ G\cdot\varphi & \mapsto & G\cdot\psi(\varphi),\end{array}$$

$$\begin{array}{rcl}\pi^t_\beta:\left(\mathrm{Hom}^t_{1,\beta}(\mathfrak{l},\mathfrak{g})\times\mathcal{L}(\mathfrak{l}',\mathfrak{z})\right)/G & \to & \mathrm{Hom}^t_{1,\beta}(\mathfrak{l},\mathfrak{g})\times V_\beta\\ \left(\varphi',B_\varphi+\mathrm{Im}(l_\varphi)\right) & \mapsto & \left(\varphi',P^{-1}_{\varphi_{|V_\beta}}(B_\varphi+\mathrm{Im}(l_\varphi))\right)\end{array}$$

and $\varepsilon^t_\beta=\psi^{-1}\circ\pi^t_\beta\circ\overline{\psi}$.

We now show that for all $t\in\{0,\dots,q\}$, the collection $\left(\varepsilon^t_\beta,\mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})/G\right)_{\beta\in I(m,m-t)}$ is a family of local sections of the canonical surjection $\pi^t:\mathrm{Hom}^t(\mathfrak{l},\mathfrak{g})\longrightarrow\mathrm{Hom}^t(\mathfrak{l},\mathfrak{g})/G$. Indeed, we have to show that $\pi^t\circ\varepsilon^t_\beta=Id_{\mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})/G}$. Let $\varphi\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ such that the orbit $\begin{pmatrix}A_\varphi & B_\varphi+\mathrm{Im}(l_\varphi)\\ 0 & D_\varphi\end{pmatrix}\in\mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})/G$, then

$$\begin{aligned}&\pi^t\circ\varepsilon^t_\beta\begin{pmatrix}A_\varphi & B_\varphi+\mathrm{Im}(l_\varphi)\\ 0 & D_\varphi\end{pmatrix}\\ &\quad=\pi^t\begin{pmatrix}A_\varphi & P^{-1}_{\varphi'|V_\beta}(B_\varphi+\mathrm{Im}(l_\varphi))\\ 0 & D_\varphi\end{pmatrix}\\ &\quad=\begin{pmatrix}A_\varphi & P^{-1}_{\varphi'|V_\beta}(B_\varphi+\mathrm{Im}(l_\varphi))+\mathrm{Im}(l_\varphi)\\ 0 & D_\varphi\end{pmatrix}\\ &\quad=\begin{pmatrix}A_\varphi & B_\varphi+\mathrm{Im}(l_\varphi)\\ 0 & D_\varphi\end{pmatrix}.\end{aligned}$$

In particular

$$\varepsilon^t_\beta(\mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})/G)=\left\{\varphi\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g})\;\middle|\;\begin{array}{l}\varphi'\in\mathrm{Hom}^t_{1,\beta}(\mathfrak{l},\mathfrak{g})\\ B_\varphi\in V_\beta\end{array}\right\}. \tag{11}$$

Now

$$\begin{array}{rcl}\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) & = & \{[\varphi]\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g})/G:\exp(\varphi(\mathfrak{l}))\text{ acts properly on }G/H\}\\ & = & \displaystyle\bigsqcup_{t=0}^{q}\bigcup_{\beta\in I(m,m-t)}\left\{[\varphi]\in\mathrm{Hom}^t_\beta(\mathfrak{l},\mathfrak{g})/G\right\}\cap\mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h}).\ \square\end{array}$$

4. A Proof of the Local Rigidity Conjecture of Two-step Nilpotent Lie Groups

4.1. Local geometrical proprieties of deformations

We keep the same notation and hypotheses. A. Weil [17] introduced the notions of local rigidity of homomorphisms in the case where the subgroup H is compact. T. Kobayashi [9] generalized it in the case where H is not compact. For non-Riemannian setting G/H with H non-compact, the local rigidity does not hold in general. In the reductive case, Kobayashi first proved in [12] that local rigidity may fail even for irreducible symmetric space of high dimensions. For non-compact setting, the local rigidity does not hold in general in the non-Riemannian case and has been studied in [9, 13, 14]. We briefly recall here some details. For a comprehensible information, we refer the readers to the references [7, 8, 9, 11, 12, 13, 14]. For $\varphi \in \mathscr{R}(\Gamma, G, H)$, the discontinuous subgroup $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* ([9]) as a discontinuous group of G/H if the orbit of φ through the inner conjugation is open in $\mathscr{R}(\Gamma, G, H)$. This means equivalently that any point sufficiently close to φ should be conjugate to φ under an inner automorphism of G. So, the homomorphisms which are locally rigid are those which correspond to those which are isolated points in the deformation space $\mathscr{T}(\Gamma, G, H)$. When every point in $\mathscr{R}(\Gamma, G, H)$ is locally rigid, the deformation space turns out to be discrete and then we say that the Clifford-Klein form $\Gamma\backslash G/H$ can not deform continuously through the deformation of Γ in G. If a given $\varphi \in \mathscr{R}(\Gamma, G, H)$ is not locally rigid, we say that it admits a *continuous deformation* and that the related Clifford-Klein form is continuously deformable.

As a direct consequence from Theorem 3.6, we prove that the rigidity property fails to hold. This gives therefore a positive answer to the following conjecture substantiated in [4]:

Conjecture 4.1. *Let G be a connected simply connected nilpotent Lie group, H a connected subgroup of G and Γ a non-trivial discontinuous subgroup for G/H. Then, the local rigidity fails to hold.*

We will prove the following:

Theorem 4.2. *Let G be a connected and simply connected two-step nilpotent Lie group. Then conjecture 4.1 holds.*

Proof. Note first that $\mathbb{R}_+^\times$ acts on $\mathcal{T}_{t,\beta,i}$, $i=1,2$, by left multiplication

$$\begin{array}{rcl} \mathbb{R}_+^\times \times \mathcal{T}_{t,\beta,i} & \longrightarrow & \mathcal{T}_{t,\beta,i} \\ (\lambda, \varphi(A,B)) & \longmapsto & \lambda \cdot \varphi = \varphi(\lambda^2 A, \lambda B). \end{array}$$

Then this action is well defined. Indeed, we define an $\mathbb{R}_+^\star$-action ρ on $\mathfrak{l} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{l}'$ by

$$\rho(\lambda)(X) := \begin{cases} \lambda^2 X \text{ for } X \in [\mathfrak{l},\mathfrak{l}] \\ \lambda X \text{ for } X \in \mathfrak{l}'. \end{cases}$$

Since $\mathfrak{l}$ is 2-step nilpotent, this action preserves the Lie algebra structure of $\mathfrak{l}$. Therefore, this induces an $\mathbb{R}_+^\star$-action on $\mathcal{L}(\mathfrak{l},\mathfrak{g})$. Its restriction is nothing but the $\mathbb{R}_+^\star$-action on $\mathcal{T}_{t,\beta,i}$. Suppose that there is a local rigid homomorphism $\varphi(A,B) \in \mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$, then its class $[\varphi(A,B)]$ is an open point in $\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ and there exists β and t such that $[\varphi(A,B)] \in \mathcal{T}_{t,\beta,i}$ for some $i \in \{1,2\}$. It follows that the image of $[\varphi(A,B)]$ denoted also by $[\varphi(A,B)]$, in the semi-algebraic set $\mathcal{T}_{t,\beta,i}$ is an isolated point. Therefore, any continuous action of $\mathbb{R}_+^\times$ on $\mathcal{T}_{t,\beta,i}$ fixes $[\varphi(A,B)]$. This means that for any $\lambda \in \mathbb{R}_+^\times$, there is $g(\lambda) \in G$ such that

$$\lambda \cdot \varphi(A,B) = g(\lambda) \cdot \varphi(A,B) = \mathrm{Ad}_{g(\lambda)}\, \varphi(A,B),$$

for some $g(\lambda) \in G$ where $\mathrm{Ad}_{g(\lambda)}$ is defined as in (8). This is equivalent to

$$\lambda \cdot \varphi(A,B) = \begin{pmatrix} A_1 & B_1 + \sigma_1(g(\lambda))B_3 + \delta_1(g(\lambda))B_4 \\ A_2 & B_2 + \sigma_2(g(\lambda))B_3 + \delta_2(g(\lambda))B_4 \\ 0 & B_3 \\ 0 & B_4 \end{pmatrix}$$

for any $\lambda \in \mathbb{R}_+^\times$, and this implies that $\begin{pmatrix} B_3 \\ B_4 \end{pmatrix} = 0$, $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0$ and $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0$, which is a contradiction with the injectivity of $\varphi(A,B)$. $\square$

5. A Stability Theorem

5.1. The notion of stability

Let (G,H,Γ) be as in the second section. The homomorphism φ is said to be *topologically stable* or merely *stable* in the sense of Kobayashi-Nasrin ([14]), if there is an open set in $\mathrm{Hom}(\Gamma,G)$ which contains φ and is contained in $\mathscr{R}(\Gamma, G, H)$. When the set $\mathscr{R}(\Gamma, G, H)$ is an open subset of $\mathrm{Hom}(\Gamma, G)$, then obviously each of its elements is stable which is the case for any irreducible Riemannian symmetric spaces with the assumption that Γ is torsion free uniform lattice of G ([14] and [17]). Furthermore, we point out in this setting that the concept of stability may be one fundamental genesis to understand the local structure of the deformation space.

Coming back to our setting where G is nilpotent and two-step. From the fact that the pair (G, H) have the Lipsman property, if $\mathfrak{l}$ is an ideal of $\mathfrak{g}$ then $\exp(\mathfrak{l})$ acts properly on G/H if and only if $\mathfrak{l} \cap \mathfrak{h} = \{0\}$.

DEFINITION 5.1. Let $\mathfrak{g}$ be a Lie algebra. A maximal abelian subalgebra of $\mathfrak{g}$ is an abelian subalgebra of $\mathfrak{g}$ of maximal dimension. Maximal subalgebras are not unique and contain obviously the center of $\mathfrak{g}$.

As a consequence of this observation, we get the following result concerning the case where $\mathfrak{l}$ is a maximal subalgebra of $\mathfrak{g}$.

PROPOSITION 5.2. *If $\mathfrak{l}$ is a maximal abelian subalgebra of $\mathfrak{g}$, then the stability property holds.*

PROOF. As $\mathfrak{l}$ is maximal and abelian, then $\varphi(\mathfrak{l})$ is also maximal and abelian for any $\varphi \in \mathrm{Hom}^{\mathrm{inj}}(\mathfrak{l}, \mathfrak{g})$. It follows that $\mathfrak{z} \subset \varphi(\mathfrak{l})$ for all $\varphi \in \mathrm{Hom}^{\mathrm{inj}}(\mathfrak{l}, \mathfrak{g})$ and therefore

$$\begin{aligned} \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) &= \left\{\varphi \in \mathrm{Hom}^{\mathrm{inj}}(\mathfrak{l}, \mathfrak{g}) : \ \mathrm{Ad}_g \, \varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\}\right\} \\ &= \left\{\varphi \in \mathrm{Hom}^{\mathrm{inj}}(\mathfrak{l}, \mathfrak{g}) : \ \varphi(\mathfrak{l}) \cap \mathfrak{h} = \{0\}\right\}, \end{aligned}$$

which is an open set of $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$. $\square$

As a direct consequence of Theorem 3.3, we get the following corollary.

COROLLARY 5.3. *If $\dim \mathfrak{z}' = \dim[\mathfrak{l}, \mathfrak{l}]$, then the stability property holds.*

PROOF. Put $s = \dim([\mathfrak{l}, \mathfrak{l}])$, then we have $A_2 \in M_s(\mathbb{R})$. Thus the inequation $\mathrm{rk}(B_4) < \dim \mathfrak{l}'$ implies

$$\mathrm{rk}\begin{pmatrix} A_2 & \star \\ 0 & B_4 \end{pmatrix} \leq s + \mathrm{rk}(B4) < \dim \mathfrak{l}.$$

This means $\mathscr{R}_2$ is empty and $\mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) = \mathscr{R}_1$, which is open in $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$. $\square$

Remark 5.4. Note that when $\mathfrak{l}$ is abelian, the hypothesis of Corollary 5.3 holds if and only if $\mathfrak{z} \subset \mathfrak{h}$, which means in particular that $\mathfrak{h}$ is an ideal of $\mathfrak{g}$. So the corollary is a consequence of a more general result: If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, then the parameter space is open in $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$.

5.2. A stability Theorem

In the rest of this section, we give a new sufficient criterion of stability, which is useful when Γ is abelian as we show in some examples.

Definition 5.5. Let $\mathfrak{l},\mathfrak{g}$ and $\mathfrak{h}$ be as above, the subalgebra $\mathfrak{l}$ is said to satisfy $(\star)$ for $\mathfrak{g}/\mathfrak{h}$ if there is a decomposition of $\mathfrak{h} = (\mathfrak{z}\cap\mathfrak{h}) \oplus \mathfrak{h}' \oplus \mathfrak{h}''$ and $\mathfrak{g} = (\mathfrak{z}\cap\mathfrak{h}) \oplus \mathfrak{z}' \oplus \mathfrak{h}' \oplus \mathfrak{h}'' \oplus V = \mathfrak{h} \oplus \mathfrak{z}' \oplus V$. Here $\mathfrak{h}''$ and V are some subspaces of $\mathfrak{g}$ such that:

$(\star 1)$ $(\mathfrak{z}\cap\mathfrak{h}) \oplus \mathfrak{h}'$ is an ideal of $\mathfrak{g}$.

$(\star 2)$ $\mathrm{rk}\begin{pmatrix} B_3'' \\ B_4 \end{pmatrix} = \mathrm{rk}(B_4)$ or $B_4 = 0$, where $B_3 = \begin{pmatrix} B_3' \\ B_3'' \end{pmatrix}$.

Example 1. Let $\mathfrak{g} = \mathbb{R}\text{-span}\{X, Y_1, \dots, Y_n, Z_1, \dots, Z_n\}$ such that $[X, Y_i] = Z_i$ for $1 \le i \le n$. For all $r > 0$ and $t \ge 0$, let $\mathfrak{h} = \mathfrak{h}_{r,t} = \mathbb{R}$-span$\{X, Y_1, \dots, Y_{r-t-1}, Z_1, \dots, Z_{r-1}\}$ and $\mathfrak{l} = \mathfrak{l}_{r,t,q} = \mathbb{R}\text{-span}\{Y_{r+t+q}, \dots, Y_n, Z_r, \dots, Z_n\}$ with $0 \le q \le n-r-t$. Then $\mathfrak{h}_{r,t}$ and $\mathfrak{l}_{r,t,q}$ are subalgebras of $\mathfrak{g}$ and the subalgebras $(\mathfrak{z}\cap\mathfrak{h}) \oplus \mathfrak{h}' = \mathbb{R}\text{-span}\{Y_1, \dots, Y_{r+t-1}, Z_1, \dots, Z_{r-1}\}$ of $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{h}'' = \mathbb{R}\text{-span}\{X\}$.

$$B_1 = (u_{i,j})_{\substack{1\le i\le r-1\\ 1\le j\le k}},\ B_2 = (u_{i,j})_{\substack{r\le i\le n\\ 1\le j\le k}},\ B_3' = (v_{i,j})_{\substack{1\le i\le r+t-1\\ 1\le j\le k}},$$
$$B_3'' = (x_1, \dots, x_k) \text{ and } B_4 = (v_{i,j})_{\substack{r+t\le i\le n\\ 1\le j\le k}}.$$

Then,

$$
\begin{aligned}
&\mathrm{Hom}(\mathfrak{l},\mathfrak{g})\\
&= \left\{ \left(\begin{array}{c} (u_{i,j})_{1\leq i\leq n, 1\leq j\leq k} \\ (v_{i,j})_{1\leq i\leq r+t-1, 1\leq j\leq k} \\ (x_j)_{1\leq j\leq k} \\ (v_{i,j})_{r+t\leq i\leq n, 1\leq j\leq k} \end{array} \right) \middle| \begin{array}{l} x_i v_{l,j} - x_j v_{l,i} = 0,\ 1\leq l,i\leq n \\ \text{and } 1\leq j\leq k \end{array} \right\}\\
&= \left\{ \varphi(B)\in M_{2n+1,k}(\mathbb{R}) : B_3^{''} = 0 \right\} \bigcup \\
&\quad \left\{ \varphi(B)\in M_{2n+1,k}(\mathbb{R}) :\ \mathrm{rk} \begin{pmatrix} B_3^{''} \\ B_4 \end{pmatrix} \leq 1 \right\},
\end{aligned}
$$

and $\mathfrak{l}$ satisfies $(\star)$ for $\mathfrak{g}/\mathfrak{h}$.

Example 2. Let $\mathfrak{g} = \mathbb{R}\text{-span}\{X_1,\dots,X_n,(Z_{i,j})_{1\leq i<j\leq n}\}$ such that $[X_i,X_j] = Z_{i,j}$ for $i\neq j$. Let $\mathfrak{h} = \mathbb{R}\text{-span}\{X_1,\dots,X_q,(Z_{i,j})_{1\leq i<j\leq q}\}$, $\mathfrak{h}^{''} = \mathbb{R}\text{-span}\{X_1,\dots,X_q\}$, $\mathfrak{h}' = \{0\}$, $\mathfrak{z}\cap\mathfrak{h} = \mathbb{R}\text{-span}\{(Z_{i,j})_{1\leq i<j\leq q}\}$ of dimension r, $\mathfrak{l} = \mathfrak{z}'\oplus\mathbb{R}X_n$ an abelian subalgebra of $\mathfrak{g}$ with $\mathfrak{z}' = \mathbb{R}\text{-span}\{(Z_{i,j})_{q\leq i<j\leq n}\}$ and $V = \mathbb{R}\text{-span}\{X_{q+1},\dots,X_n\}$. Then we have

$$
\mathrm{Hom}(\mathfrak{l},\mathfrak{g}) = \left\{ \varphi(B)\in M_{r+m'+n,k}(\mathbb{R}) : \mathrm{rk} \begin{pmatrix} B_3^{''} \\ B_4 \end{pmatrix} \leq 1 \right\},
$$

and $\mathfrak{l}$ satisfies $(\star)$ for $\mathfrak{g}/\mathfrak{h}$.

Example 3. Let $\mathfrak{g}$ be a two-step nilpotent Lie group, $\mathfrak{l}$ an subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ an ideal of $\mathfrak{g}$, then we can write $\mathfrak{h} = (\mathfrak{z}\cap\mathfrak{h})\oplus\mathfrak{h}'$ and therefore $\mathfrak{h}'' = \{0\}$, so as in Definition 5.5, for all $\varphi(A,B)\in\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$ we have B_3'' is a trivial matrix, then $\mathfrak{l}$ satisfies the property $(\star)$ for $\mathfrak{g}/\mathfrak{h}$.

Example 4. Let $\mathfrak{g} = \mathbb{R}\text{-span}\{X,T,Y_1,\dots,Y_4,Z_1,\dots,Z_4,U_0,U_3,U_4\}$ such that $[X,Y_i] = Z_i$, $i = 1,\dots,4$, $[X,T] = U_0$, $[T,Y_3] = U_3$ and $[T,Y_4] = U_4$. Take $\mathfrak{h} = \mathbb{R}\text{-span}\{X,Y_4,Z_4,U_4\}$ and $\mathfrak{l} = \mathbb{R}\text{-span}\{Y_1,Y_2,Y_3,Z_1,Z_2,Z_3\}$. Let $\mathcal{B} = \{Z_4,U_0,U_3,U_4,Z_1,Z_2,Z_3,Y_4,X,T,Y_1,Y_2,Y_3\}$ be a basis of $\mathfrak{g}$ and

$$
\varphi = \begin{pmatrix} B_1 \\ B_2 \\ B_3' \\ B_3'' \\ B_4 \end{pmatrix}
$$

with $B_1 = \begin{pmatrix} z_{4,1} & \cdots & z_{4,6} \\ u_{0,1} & \cdots & u_{0,6} \\ u_{3,1} & \cdots & u_{3,6} \\ u_{4,1} & \cdots & u_{4,6} \end{pmatrix}$, $B_2 = \begin{pmatrix} z_{1,1} & \cdots & z_{1,6} \\ z_{2,1} & \cdots & z_{2,6} \\ z_{3,1} & \cdots & z_{3,6} \end{pmatrix}$, $B'_3 = (y_{4,1}, \dots, y_{4,6})$, $B''_3 = (x_1, \dots, x_6)$ and $B_4 = \begin{pmatrix} t_1 & \cdots & t_6 \\ y_{1,1} & \cdots & y_{1,6} \\ y_{2,1} & \cdots & y_{2,6} \\ y_{3,1} & \cdots & y_{3,6} \end{pmatrix}$.

Then

$$\begin{aligned}
\operatorname{Hom}(\mathfrak{l}, \mathfrak{g}) &= \left\{ \begin{pmatrix} B_1 \\ B_2 \\ B'_3 \\ B''_3 \\ B_4 \end{pmatrix} \in M_{13,6}(\mathbb{R}) \;\middle|\; \begin{array}{l} x_j t_k - x_k t_j = 0 \\ x_j y_{i,k} - x_k y_{i,j} = 0,\ i = 1, \dots, 4 \\ t_j y_{i,k} - t_k y_{i,j} = 0,\ i = 3, 4 \\ 1 \leq j, k \leq 6 \end{array} \right\} \\
&= \left\{ \begin{pmatrix} B_1 \\ B_2 \\ B'_3 \\ B''_3 \\ B_4 \end{pmatrix} \in M_{13,6}(\mathbb{R}) \;\middle|\; \operatorname{rk} \begin{pmatrix} B''_3 \\ B_4 \end{pmatrix} \leq 1 \right\} \\
&\cup \left\{ \begin{pmatrix} B_1 \\ B_2 \\ B'_3 \\ 0 \\ B_4 \end{pmatrix} \in M_{13,6}(\mathbb{R}) \;\middle|\; \begin{array}{l} t_j y_{i,k} - t_k y_{i,j} = 0,\ i = 3, 4 \\ 1 \leq j, k \leq 6 \end{array} \right\}.
\end{aligned}$$

Therefore $\mathfrak{l}$ satisfies $(\star)$ for $\mathfrak{g}/\mathfrak{h}$.

We now state the main result of this section.

THEOREM 5.6. *Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra, $\mathfrak{h}$ and $\mathfrak{l}$ some subalgebras of $\mathfrak{g}$. Assume that $\mathfrak{l}$ satisfies the property $(\star)$ for $\mathfrak{g}/\mathfrak{h}$ and that $\dim(\mathfrak{z}') < \dim(\mathfrak{l})$. Then the stability property holds. More precisely:*

$$\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \left\{ \varphi(A, B) \in \operatorname{Hom}(\mathfrak{l}, \mathfrak{g}) : \operatorname{rk} \begin{pmatrix} A_2 & B_2 \\ 0 & B_4 \end{pmatrix} = \dim(\mathfrak{l}) \right\},$$

where $\varphi(A, B)$ is as in definition (5.5).

PROOF. For a $n \times k$ matrix X, we have

$$\operatorname{rk} X = k \Longleftrightarrow \ker X = \{0\}.$$

Thus, it is sufficient to show

$$\ker \begin{pmatrix} A_2 & B_2 \\ 0 & B_4 \end{pmatrix} = \{0\} \Longleftrightarrow (\forall g \in G) \ \ker \begin{pmatrix} A_2 & B_2 + \sigma'(g)B' \\ 0 & B_4 \end{pmatrix} = \{0\}.$$

Thus, it is enough to show

$$\ker \begin{pmatrix} A_2 & B_2 + \sigma'(g)B' \\ 0 & B_4 \end{pmatrix} = \ker \begin{pmatrix} A_2 & B_2 \\ 0 & B_4 \end{pmatrix}$$

for any $g \in G$. Here, fix $g \in G$. Then this equation follows from

$$\ker B_4 \subset \ker \sigma'(g)B'. \tag{12}$$

Here, we decompose

$$\sigma'(g) = \begin{pmatrix} \sigma_2'(g) & \sigma_2''(g) & \delta_2(g) \end{pmatrix}, \quad B' = \begin{pmatrix} B_3' \\ B_3'' \\ B_4 \end{pmatrix}$$

To show the inclusion (12), we use two claims.

CLAIM 1. $\sigma_2'(g) = 0$.

PROOF OF CLAIM 1. By the assumption that $I := (\mathfrak{z} \cap \mathfrak{h}) \oplus \mathfrak{h}'$ is an ideal of $\mathfrak{g}$, we obtain

$$[\mathfrak{g}, I] \subset \mathfrak{z} \cap \mathfrak{h} \tag{13}$$

because we have

$$[\mathfrak{g}, I] \subset [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}, \text{ and } [\mathfrak{g}, I] \subset I \subset \mathfrak{h}.$$

The inclusion (13) means that $\operatorname{ad}(X)Y \in \mathfrak{z} \cap \mathfrak{h}$ for any $X \in \mathfrak{g}$ and any $Y \in I$. Thus, $\sigma_2'(g) = 0$.

CLAIM 2. $\ker B_4 \subset \ker B_3''$.

PROOF OF CLAIM 2. By the assumption $\dim(\mathfrak{z}') < \dim(\mathfrak{l})$, the condition

$$\mathrm{rk}\begin{pmatrix} A_2 & B_2 + \sigma(g)B' \\ 0 & B_4 \end{pmatrix} = k$$

implies $B_4 \neq 0$. Thus, as $\mathfrak{l}$ *satisfies the property* ($\star$) for $\mathfrak{g}/\mathfrak{h}$, we obtain

$$\mathrm{rk}\begin{pmatrix} B_3'' \\ B_4 \end{pmatrix} = \mathrm{rk}(B_4).$$

This means:

$$\ker B_4 \subset \ker B_3''.$$

Thus, Claim 1 and Claim 2 imply the inclusion (12). □

6. On the Hausdorff Property

This section aims to study the Hausdorffness of the deformation space $\mathcal{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ in the setting where $\mathfrak{g}$ is two-step nilpotent. We first prove the following:

LEMMA 6.1. *For all* $\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, *we have*

$$\dim(G \cdot \varphi) = \dim(\mathfrak{g}) - \dim(\varphi(\mathfrak{l})^{\perp}),$$

where $\varphi(\mathfrak{l})^{\perp} = \{X \in \mathfrak{g} : \ [X, Y] = 0, \ \forall\ Y \in \varphi(\mathfrak{l})\}$.

PROOF. Recall that the map $\sigma : G \to \mathcal{L}(\mathfrak{l}', \mathfrak{z})$ is a group homomorphism and it is clear that $\ker(\sigma) = Z(G)$ where $Z(G)$ is the center of G. Then σ factors via the projection map $G \to G/Z(G)$ to obtain an injective homomorphism $\tilde{\sigma} : G/Z(G) \to \mathcal{L}(\mathfrak{l}', \mathfrak{z})$ such that $\tilde{\sigma}(\overline{G}) = \sigma(G)$.

Let now $\varphi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, then $\ker(l_\varphi) = \{\tilde{\sigma}(\overline{e^X}) : \ [X, Y] = 0 \ \forall\ Y \in \varphi(\mathfrak{l})\}$. Thus,

$$\begin{aligned} \dim(G \cdot \varphi) &= \dim(\sigma(G)) - \dim(\overline{e^{\varphi(\mathfrak{l})^{\perp}}}) \\ &= \dim(\mathfrak{g}) - \dim(\mathfrak{z}) - (\dim(\varphi(\mathfrak{l})^{\perp}) - \dim(\mathfrak{z})) \\ &= \dim(\mathfrak{g}) - \dim(\varphi(\mathfrak{l})^{\perp}). \ \square \end{aligned}$$

We now prove the main result of this section.

Theorem 6.2. *Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra, if all G-orbits in $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ have a common dimension, then the deformation space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is a Hausdorff space.*

Proof. In such a situation, there is $t \in \{0, \dots, q\}$ such that $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) \subset \mathrm{Hom}^t(\mathfrak{l}, \mathfrak{g})$, where $\mathrm{Hom}^t(\mathfrak{l}, \mathfrak{g})$ is as in equation (9) and $q = \dim \sigma(G)$ as in section 3. Indeed, let $\varphi \in \mathrm{Hom}^t(\mathfrak{l}, \mathfrak{g})$ then $\mathrm{rk}\, l_\varphi = t$ and so

$$\begin{aligned} \dim G \cdot \varphi &= \dim \psi(G \cdot \varphi) \\ &= \dim G \cdot \psi(\varphi) \\ &= \dim(\varphi', B_\varphi + \sigma(G) D_\varphi) \\ &= \dim(\varphi', B_\varphi + \mathrm{Im}\, l_\varphi) \\ &= \mathrm{rk}\, l_\varphi = t. \end{aligned}$$

The deformation space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is therefore contained in $\mathrm{Hom}^t(\mathfrak{l}, \mathfrak{g})/G$, and it is sufficient to show that $\mathrm{Hom}^t(\mathfrak{l}, \mathfrak{g})/G$ is a Hausdorff space. Let $[\varphi] \neq [\xi]$ be some points in $\mathrm{Hom}^t(\mathfrak{l}, \mathfrak{g})/G$. Suppose that $[\varphi]$ and $[\xi]$ are not separated, then there exist $(\varphi_n)_n \subset \mathrm{Hom}^t(\mathfrak{l}, \mathfrak{g})$ and $g_n \in G$ such that φ_n converges to φ and $g_n \cdot \varphi_n$ converges to ξ in $\mathrm{Hom}^t(\mathfrak{l}, \mathfrak{g})$. Using the map ψ defined in Lemma 3.4, we can see that the sequence $(\varphi'_n, B_{\varphi_n})_n$ converges to (φ', B_φ), $(\varphi'_n, B_{\varphi_n} + \sigma(g_n) D_{\varphi_n})_n$ converges to (φ', B_φ) and $(\varphi'_n, B_{\varphi_n} + \sigma(g_n) D_{\varphi_n})_n$ converges to (ξ', B_ξ). This means that $\varphi' = \xi'$ and in particular $D_\varphi = D_\xi$ and $P_\varphi = P_\xi$. Finally $[\varphi]$ and $[\xi]$ belong to the open set $\mathrm{Hom}^t_\beta(\mathfrak{l}, \mathfrak{g})/G$ for some $\beta \in I(m, m-t)$. From (11), $\mathrm{Hom}^t_\beta(\mathfrak{l}, \mathfrak{g})/G$ is homeomorphic to a semi-algebraic set. Therefore $\mathrm{Hom}^t_\beta(\mathfrak{l}, \mathfrak{g})/G$ is a Hausdorff space. This leads thus to a contradiction. □

Corollary 6.3. *Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra. If $\mathfrak{l}$ is a maximal abelian subalgebra of $\mathfrak{g}$, then the deformation space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is a Hausdorff space.*

Proof. If $\mathfrak{l}$ is a maximal abelian subalgebra then so is $\varphi(\mathfrak{l})$ and we have $\varphi(\mathfrak{l})^\perp = \varphi(\mathfrak{l})$. Hence from Lemma 6.1

$$\dim(G \cdot \varphi) = \dim(\mathfrak{g}) - \dim(\varphi(\mathfrak{l})) = \dim(\mathfrak{g}) - \dim(\mathfrak{l}),$$

which is constant. This achieves the proof. □

Proposition 6.4. *Let $\mathfrak{l}$ and $\mathfrak{h}$ be two subalgebras of $\mathfrak{g}$ and suppose that*

(1) *There is decomposition* $\mathfrak{g} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{z}_1 \oplus \mathfrak{z}_2 \oplus \mathfrak{z}_3 \oplus \mathfrak{h}' \oplus \mathfrak{l}' \oplus V$, *where* $\mathfrak{z} \cap \mathfrak{l} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{z}_1$, $\mathfrak{l} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{z}_1 \oplus \mathfrak{l}'$, $\mathfrak{z} \cap \mathfrak{h} = \mathfrak{z}_2$ $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{z}_2$ *and* $\mathfrak{z} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{z}_1 \oplus \mathfrak{z}_2 \oplus \mathfrak{z}_3$.

(2) *There is a codimension one subalgebra* $\mathfrak{l}_1$ *of* $\mathfrak{l}$ *such that* $[\mathfrak{l}_1^{\perp},\mathfrak{l}] \nsubseteq \mathfrak{l} \oplus \mathfrak{h}$ *and* $\mathfrak{z}(\mathfrak{l}) + \mathfrak{l}_1 = \mathfrak{l}$.

Then the deformation space $\mathscr{T}(\mathfrak{l},\mathfrak{g},\mathfrak{h})$ *fails to be a Hausdorff space.*

Proof. Recall that $\dim([\mathfrak{l},\mathfrak{l}]) = s$ and set $\dim(\mathfrak{z}_1) = q$, $\dim(\mathfrak{z}_2) = p$, $\dim(\mathfrak{z}_3) = r$ and $\dim(\mathfrak{z}) = m$. The Lie bracket of $\mathfrak{g}$ is given by

$$[X,Y] = \sum_{i=1}^{m} b_i(X,Y)Z_i$$

where $\{Z_1,\dots,Z_m\}$ is a basis of $\mathfrak{z}$ passing through the decomposition $\mathfrak{z} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{z}_1 \oplus \mathfrak{z}_2 \oplus \mathfrak{z}_3$. There a basis $\{L_1,\dots,L_{k-q-s}\}$ of $\mathfrak{l}'$ such that $\{Z_1,\dots,Z_{s+q},L_2,\dots,L_{k-q-s}\}$ is a basis of the subalgebra $\mathfrak{l}_1$ satisfying assumption (2). The vector $L_1 \in \mathfrak{z}(\mathfrak{l})$ and there is $X_0 \in \mathfrak{l}_1^{\perp}$ such that the bracket $[X_0,L_1] \notin \mathfrak{l} \oplus \mathfrak{h}$. In particular there is $p+q+s < i_0 \le m$ such that $b_{i_0}(X_0,L_1) = \alpha \neq 0$. If we complete the vectors $Z_1,\dots,Z_m,L_1,\dots,L_{k-q-s}$ to a basis of $\mathfrak{g}$ passing through the decomposition $\mathfrak{g} = [\mathfrak{l},\mathfrak{l}] \oplus \mathfrak{z}_1 \oplus \mathfrak{z}_2 \oplus \mathfrak{z}_3 \oplus \mathfrak{l}' \oplus \mathfrak{h}' \oplus V$, then the matrix

$$M_1 = \begin{pmatrix} I_s & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_1 \\ 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{R}(\mathfrak{l},\mathfrak{g},\mathfrak{h}) \text{ with}$$

$$D = \begin{pmatrix} 0 & 0 \\ 0 & I_{k-q-s-1} \end{pmatrix} \in M_{k-q-s}(\mathbb{R}),$$

$C_1 = (u_1\ 0) \in M_{r,k-q-s}(\mathbb{R})$, where $u_1 = {}^t(0,\dots,0,x_1,0,\dots,0) \in \mathbb{R}^r$ and x_1 is the i_0 coordinate. Let $C_s = (u_s\ 0) \in M_{s,k-q-s}(\mathbb{R})$ with $u_s = {}^t(a_1,\dots,a_s) \in \mathbb{R}^s$,

$$C_q = (u_q\ 0) \in M_{q,k-q-s}(\mathbb{R}) \text{ with } u_q = {}^t(a_{s+1},\dots,a_{s+q}) \in \mathbb{R}^q$$

and

$$C_p = (u_p\ 0) \in M_{p,k-q-s}(\mathbb{R}) \text{ with } u_p = {}^t(a_{q+s+1}, \dots, a_{q+s+p}) \in \mathbb{R}^p.$$

Let $C_2 = (u_2\ 0) \in M_{m-q-s-p,k-q-s}(\mathbb{R})$, $u_2 = {}^t(a_{q+s+p+1}, \dots, a_{i_0-1}, x_2, a_{i_0+1}, \dots, a_m) \in \mathbb{R}^{m-q-s-p}$ such that $x_2 \neq x_1$ and $a_i = \frac{x_2-x_1}{\alpha} b_i(X_0, L_1)$. Then

$$M_2 = \begin{pmatrix} I_s & 0 & C_s \\ 0 & I_q & C_q \\ 0 & 0 & C_p \\ 0 & 0 & C_2 \\ 0 & 0 & D \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$$

and $G \cdot M_1 \neq G \cdot M_2$.

Let $\mathcal{V}_i$ be a neighborhood of M_i for $i = 1, 2$. For $\varepsilon > 0$ small enough, the elements

$$M_{1,\varepsilon} = \begin{pmatrix} I_s & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & 0 \\ 0 & 0 & C_1 \\ 0 & 0 & D_\varepsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{V}_1 \text{ and } M_{2,\varepsilon} = \begin{pmatrix} I_s & 0 & C_s \\ 0 & I_q & C_q \\ 0 & 0 & C_p \\ 0 & 0 & C_2 \\ 0 & 0 & D_\varepsilon \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{V}_2,$$

where $D_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & I_{k-q-s-1} \end{pmatrix}$ and $\mathrm{Ad}_{\exp \frac{x_2-x_1}{\alpha\varepsilon} X_0} M_{1,\varepsilon} = M_{2,\varepsilon}$. This means that $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is not a Hausdorff space. $\square$

Remark 6.5. In the situation of the Heisenberg group, it is shown in [3] that the Hausdorff property of the deformation space is equivalent to the fact that the stability property holds. The following example shows that such a phenomen may fail for general two-step nilpotent Lie groups.

Example 5. Let us resume Example 4 above. In this situation, we have $\mathfrak{z}' = \mathbb{R}\text{-span}\{Z_1, Z_2, Z_3, U_0, U_3\}$ and $\dim(\mathfrak{z}') < \dim(\mathfrak{l})$, so the hypothesis of Theorem 5.6 is satisfied. This shows that $\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is open in $\mathrm{Hom}(\mathfrak{l},\mathfrak{g})$. We now write $\mathfrak{g} = \mathfrak{z}_1 \oplus \mathfrak{z}_2 \oplus \mathfrak{z}_3 \oplus \mathfrak{h}' \oplus \mathfrak{l}' \oplus V$, where $\mathfrak{z} \cap \mathfrak{l} = \mathbb{R}$-span$\{Z_1, Z_2, Z_3\} = \mathfrak{z}_1$, $\mathfrak{l}' = \mathbb{R}$-span$\{Y_1, Y_2, Y_3\}$, $\mathfrak{z} \cap \mathfrak{h} = \mathfrak{z}_2 = \mathbb{R}$-span$\{U_4, Z_4\}$, $\mathfrak{h}' = \mathbb{R}$-span$\{X, Y_4\}$, $\mathfrak{z}_3 = \mathbb{R}$-span$\{U_0, U_3\}$ and $V = \mathbb{R}$-span$\{T\}$. If we choose $\mathfrak{l}_1 = \mathbb{R}$-span$\{Y_1, Y_2, Z_1, Z_2, Z_3\}$ as a subalgebra of $\mathfrak{l}$ of codimension one, the we obtain $\mathfrak{l}_1^{\perp} = \mathbb{R}$-span$\{T, Y_1, Y_2, Y_3, Y_4, Z_1, Z_2, Z_3, Z_4, U_0, U_3, U_4\}$, then $[\mathfrak{l}_1^{\perp}, \mathfrak{l}] \ni U_3 \notin \mathfrak{h} \oplus \mathfrak{l}$ and $\mathfrak{z}(\mathfrak{l}) = \mathfrak{l}$, which means that $\mathfrak{l}_1 + \mathfrak{z}(\mathfrak{l}) = \mathfrak{l}$. So by Proposition 6.4, $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ fails to be a Hausdorff space.

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