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The Cone Conjecture for Abelian Varieties

By Artie Prendergast-Smith

The purpose of this paper is to write down a complete proof of the Morrison–Kawamata cone conjecture for abelian varieties. The conjecture predicts, roughly speaking, that for a large class of varieties (including all smooth varieties with numerically trivial canonical bundle) the automorphism group acts on the nef cone with rational polyhedral fundamental domain. (See Section 1 for a precise statement.) The conjecture has been proved in dimension 2 by Sterk–Looijenga, Namikawa, Kawamata, and Totaro [Ste85, Nam85, Kaw97, Tot10], but in higher dimensions little is known in general.

Abelian varieties provide one setting in which the conjecture is tractable, because in this case the nef cone and the automorphism group can both be viewed as living inside a larger object, namely the real endomorphism algebra. In this paper we combine this fact with known results for arithmetic group actions on convex cones to produce a proof of the conjecture for abelian varieties.

Here is the main result.

Theorem 0.1. *Let X be an abelian variety and $\overline{A(X)}^e$ its effective nef cone. Then there is a rational polyhedral fundamental domain for the action of the automorphism group $Aut(X)$ on $\overline{A(X)}^e$.*

The conclusion of the theorem was already known in some cases. It was proved for abelian surfaces by Kawamata [Kaw97], adapting the proof of Sterk–Looijenga for $K3$ surfaces. In the same paper, Kawamata also proved the conjecture for all self-products of an elliptic curve without complex multiplication. Finally, Bauer [Bau98] showed that for an abelian variety, the nef cone is rational polyhedral if and only if the variety is isogenous to a product of mutually non-isogenous abelian varieties of Picard number 1, which in particular implies the theorem for abelian varieties of this special type.

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1. The Cone Conjecture

We work throughout the paper over an arbitrary algebraically closed field.

Morrison [Mor93] gave the original statement of the cone conjecture for Calabi–Yau threefolds, motivated by considerations from mirror symmetry. The statement was generalised by Kawamata [Kaw97] to families of varieties with numerically trivial canonical bundle, and from there to so-called klt Calabi–Yau pairs [Tot10]. As mentioned in the introduction, the conjecture has been verified in dimension 2, but in general it remains wide open. See [Tot10, Section 1] for history and a summary of the current status.

Here we state the conjecture in a rather simple form applicable to abelian varieties. The symbol $\equiv$ denotes numerical equivalence of divisors, and for a projective variety $N^1(X)$ denotes the real vector space $(\mathrm{Div}(X)/\equiv) \otimes \mathbf{R}$ where $\mathrm{Div}(X)$ is the free abelian group spanned by Cartier divisors. The cones $\overline{A(X)}$ and $\overline{M(X)}$ are the closed cones in $N^1(X)$ spanned by the classes of nef or movable divisors. The cone $\mathrm{Eff}(X)$ is the cone spanned by the classes of all effective divisors, and $\overline{A(X)}^e$ and $\overline{M(X)}^e$ denote the intersections $\overline{A(X)} \cap \mathrm{Eff}(X)$ and $\overline{M(X)} \cap \mathrm{Eff}(X)$. Finally, a *pseudo-automorphism* of X is a birational map $X \dashrightarrow X$ which is an isomorphism outside a subset of codimension 2. Note that a pseudo-automorphism maps a movable or effective divisor to another movable or effective divisor, therefore preserves the cone $\overline{M(X)}^e$.

Conjecture 1.1 (Morrison–Kawamata). *Let X be a smooth projective variety with $K_X \equiv 0$. Then:*

(1) There exists a rational polyhedral cone Π which is a fundamental domain for the action of $Aut(X)$ on $\overline{A(X)}^e$ in the sense that

(a) $\overline{A(X)}^e = Aut(X) \cdot \Pi$,

(b) $Int\,\Pi \cap g^*(Int\,\Pi) = \emptyset$ for $g^* \neq \mathrm{id}$ in $GL(N^1(X))$.

(2) There exists a rational polyhedral cone Π' which is a fundamental domain (in the sense above) for the action of the pseudo-automorphism group $PsAut(X)$ on $\overline{M(X)}^e$.

Part (1) of the conjecture would imply in particular that any variety with numerically trivial canonical bundle has finitely many contractions up to automorphisms. This is because any contraction of a projective variety is determined by a semi-ample line bundle, and two semi-ample line bundles give the same contraction if they belong to the interior of the same face in the nef effective cone.

For abelian varieties part (2) of the conjecture is implied by part (1), because the effective nef cone $\overline{A(X)}^e$ and effective movable cone $\overline{M(X)}^e$ are the same. Indeed, on an abelian variety any effective divisor is semi-ample by the 'theorem of the square' [Mum70, §6, Corollary 4], so $\overline{A(X)}^e = \overline{M(X)}^e = \mathrm{Eff}(X)$. As a consequence Theorem 0.1 implies the full cone conjecture for abelian varieties.

The equality of cones in the last paragraph can be strengthened to give the following result; a reference is [Bau98, Proposition 1.1].

Proposition 1.2. *Let D be a Cartier divisor on an abelian variety X and let $[D]$ denote the class of D in $N^1(X)$. Then $[D] \in \mathit{Eff}(X)$ if and only if $[D] \in \overline{A(X)}$.*

This implies that $\overline{A(X)}^e$ is equal to the *rational hull* $A(X)_+$ of the ample cone $A(X)$, defined as the convex hull of the rational points in the closure $\overline{A(X)}$. This will be important later, because from the point of view of reduction theory for arithmetic groups $A(X)_+$ is a more natural object to consider than $\overline{A(X)}^e$.

Finally it should be emphasised that for abelian varieties, it makes no difference to the conjecture whether Aut denotes the group of automorphisms in the category Var/k of varieties over the ground field k or in the category $GrVar/k$ of group varieties over k. This is because any (Var/k)-automorphism of an abelian variety can be composed with a translation automorphism to give a $(GrVar/k)$-automorphism, and translations act trivially on $N^1(X)$. For the rest of the paper we will use Aut to denote the group of $(GrVar/k)$-automorphisms.

Examples. The following examples illustrate how the nef effective cone and automorphism group of an abelian variety can vary, compatibly with the cone conjecture. We will consider two abelian surfaces of Picard number 2. By standard results on quadratic forms and the Hodge Index Theorem, for any such surface X we can choose a rational basis $\{v_1, v_2\}$ for $N^1(X)$ in which the intersection form has matrix $\operatorname{diag}(a, -b)$, with a and b positive. When X is an abelian surface, Proposition 1.2 says that a Cartier divisor D on X is ample if and only if $D^2 > 0$ and $D \cdot H > 0$ for some fixed ample divisor H. So if we choose a basis $\{v_1, v_2\}$ for $N^1(X)$ as above, the ample cone of X is described as

$$A(X) = \left\{x_1 v_1 + x_2 v_2 \in N^1(X) \,|\, a x_1^2 - b x_2^2 > 0,\, x_1 > 0\right\}.$$

The two extremal rays of $\overline{A(X)}$ are spanned by the vectors $v_1 \pm (\sqrt{a/b}) v_2$, so $\overline{A(X)}^e = A(X)_+$ is a rational polyhedral cone if and only if a/b is a square in $\mathbf{Q}$.

Before giving our examples we mention the following useful fact. To verify the cone conjecture for a variety X, it suffices to find a rational polyhedral cone $\Pi \subset \overline{A(X)}^e$ whose translates by $\operatorname{Aut}(X)$ cover the whole nef effective cone; it is then relatively straightforward to produce a precise fundamental domain. (This step will be explained at the end of Section 5, but we mention it now to keep the exposition clear.)

For the first example we take X to be a product $E_1 \times E_2$ of non-isogenous elliptic curves over $\mathbf{C}$. Then $\operatorname{Aut}(X) = \operatorname{Aut}(E_1) \times \operatorname{Aut}(E_2)$, and $\operatorname{Aut}(E_i)$ is a cyclic group of order 2, 4, or 6. Therefore in this case $\operatorname{Aut}(X)$ is a finite group acting on $\overline{A(X)}^e$. On the other hand, by taking suitable rational linear combinations of the divisors $E_1 \times \{0\}$ and $\{0\} \times E_2$, the intersection form on $N^1(X)$ can be transformed to have matrix $\operatorname{diag}(1, -1)$. By our description of the extremal rays a few paragraphs ago, $\overline{A(X)}^e$ is therefore a rational polyhedral cone, so by the fact in the previous paragraph the cone conjecture is true for X, taking $\Pi = \overline{A(X)}^e$.

For the second example we take X to be an abelian surface with *real multiplication*, by which we mean that the endomorphism algebra $\operatorname{End}^0(X) := \operatorname{End}(X) \otimes \mathbf{Q}$ is isomorphic to the number field $\mathbf{Q}(\sqrt{d})$ for some square-free integer $d > 0$. (The simplest examples of such surfaces are Jacobians of certain genus 2 curves; explicit models can be found in [Wil00].) By Dirichlet's unit theorem, the automorphism group $\operatorname{Aut}(X)$ has rank equal to $r_1 + r_2 - 1$, where r_1 is the number of embeddings $\mathbf{Q}(\sqrt{d}) \hookrightarrow \mathbf{R}$ and r_2 is the number of conjugate pairs of embeddings $\mathbf{Q}(\sqrt{d}) \hookrightarrow \mathbf{C}$ whose image is not contained in $\mathbf{R}$. One checks easily that $r_1 = 2$ and $r_2 = 0$, so $\operatorname{Aut}(X)$ has rank 1. What about the cone $\overline{A(X)}^e$? In this case the matrix of the intersection form di-

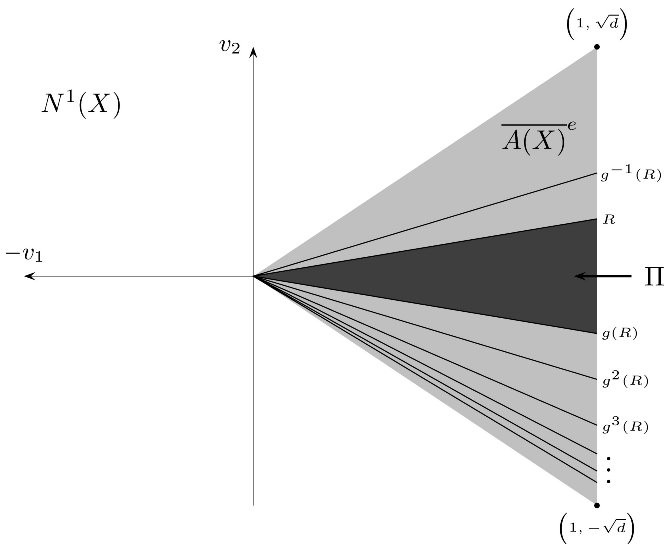

Fig. 1. Nef cone of an abelian surface with real multiplication.

agonalises to $\mathrm{diag}(a, -b)$ with $a/b = d$, so the boundary rays are irrational. To find the required rational polyhedral fundamental domain, we proceed as follows. Choose an arbitrary rational ray $R \subset \overline{A(X)}^e$ and an element g of infinite order in $\mathrm{Aut}(X)$. A little thought shows $\{g^i(R) \,|\, i = 1, 2, 3 \ldots\}$ is a sequence of rational rays, which either converges to one extremal ray, or decomposes into two subsequences, one converging to each extremal ray. Composing g with a torsion element of $\mathrm{Aut}(X)$, we can assume that the first case occurs. Then the cone Π spanned by the rays R and $g(R)$ is a rational polyhedral cone whose translates by $\mathrm{Aut}(X)$ cover the whole nef effective cone (Figure 1). Again by the fact above, this proves the cone conjecture for X.

2. Homogeneous Self-Dual Cones and Reduction Theory

In this section we give the results we need about reduction theory for arithmetic group actions on homogeneous self-dual convex cones. It should be mentioned that this theory has a rich history that we touch on here only briefly; see [AMRT] for historical discussion and references.

From now on V will always denote a finite-dimensional real vector space. By a *cone* in V we always mean a convex cone $C \subset V$ which is *non-degenerate* (meaning that its closure $\bar{C}$ contains no nonzero subspaces of V). The *dual cone* $C^* \subset V^*$ is defined to be the interior of the cone $\overline{C^*}$ consisting of linear forms on V which are nonnegative on C.

Now suppose C is an open cone in a vector space V. We define the *automorphism group* $G(C)$ of C to be the subgroup of $GL(V)$ consisting of linear transformations such that $g(C) = C$. The cone C is said to be *homogeneous* if $G(C)$ acts transitively on C. Suppose further that V carries an inner product, giving an identification of V with V^*. We say C is *self-dual* if this identification takes C to its dual C^*. (This condition depends on the choice of inner product, but the dependence will not matter in what follows.) The basic theorem about the automorphism group of a homogeneous self-dual cone is the following, due to Vinberg [Vin65].

Theorem 2.1 (Vinberg). *Let $C \subset V$ be a homogeneous self-dual convex cone. Then the automorphism group $G(C)$ is the group of real points of a reductive algebraic group $\mathcal{G}(C)$.*

It is an amazing fact that homogeneous self-dual convex cones can be completely classified into a small number of cases. More precisely, define the *direct sum* of cones C_1 and C_2 in vector spaces V_1 and V_2 to be the cone $C_1 \oplus C_2 := \{v_1 + v_2 \in V_1 \oplus V_2 | v_i \in C_i\}$ and call a cone *indecomposable* if it is not the direct sum of two nontrivial cones. The classification theorem, due to Koecher and Vinberg, is the following [Vin63, I, §1, Proposition 2], [Vin65, p. 71].

Theorem 2.2 (Koecher–Vinberg). *Any convex cone C can be written as a direct sum $\oplus_i C_i$ of indecomposable cones. The product $\prod G(C_i)$ is a finite-index subgroup of $G(C)$. The cones C_i are homogeneous and self-dual if and only if C is. Any indecomposable homogeneous self-dual cone is isomorphic to one of the following:*

1. *the cone $P_r(\mathbf{R})$ of positive-definite matrices in the space $\mathcal{H}_r(\mathbf{R})$ of $r \times r$ real symmetric matrices;*
2. *the cone $P_r(\mathbf{C})$ of positive-definite matrices in the space $\mathcal{H}_r(\mathbf{C})$ of $r \times r$ complex Hermitian matrices;*

3. *the cone $P_r(\mathbf{H})$ of positive-definite matrices in the space $\mathcal{H}_r(\mathbf{H})$ of $r \times r$ quaternionic Hermitian matrices;*

4. *the spherical cone* $\left\{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} \mid x_0 > \sqrt{x_1^2 + \cdots x_n^2}\right\}$*;*

5. *the 27-dimensional cone of positive-definite* 3×3 *octonionic Hermitian matrices.*

The inner product for which the cone is self-dual is $\langle x, y \rangle = \mathrm{Tr}(xy^*)$ in all cases except 4, and the usual inner product on $\mathbf{R}^{n+1}$ in case 4.

The proof uses the surprising correspondence between self-dual homogeneous convex cones and formally real Jordan algebras; see [AMRT, Chapter II, §2] for details.

Vinberg [Vin65] computed the automorphism groups of all the cones in the list of Theorem 2.2. Here we state the part of his result which will be relevant to abelian varieties.

THEOREM 2.3 (Vinberg). *Let C be one of the cones $P_r(k)$ in the previous theorem: that is, the cone of positive-definite matrices in the vector space $\mathcal{H}_r(k)$ of $r \times r$ symmetric or Hermitian matrices over k, where $k = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$. Then the identity component $G(C)^0$ of the automorphism group of C consists of all $\mathbf{R}$-linear transformations of $\mathcal{H}_r(k)$ of the form $D \mapsto M^*DM$ for some $M \in GL(r, k)$.*

Now we come to reduction theory. For a $\mathbf{Q}$-algebraic group $\mathcal{G}$, a subgroup $\Gamma \subset \mathcal{G}(\mathbf{Q})$ is called an *arithmetic subgroup* of $\mathcal{G}$ if Γ and the group $\mathcal{G}(\mathbf{Z})$ of integer points of $\mathcal{G}$ are commensurable (meaning their intersection inside $\mathcal{G}(\mathbf{Q})$ has finite index in each). The basic problem of reduction theory for convex cones is the following: given a convex cone C and an arithmetic subgroup Γ of the automorphism group $\mathcal{G}(C)$, can we find a rational polyhedral fundamental domain for the action of Γ on C? The first results of this kind go back to Hermite and Minkowski, who found rational polyhedral fundamental domains for the adjoint action of $SL(r, \mathbf{Z})$ on the cone $P_r(\mathbf{R})$ of positive-definite real symmetric matrices. That is, there is a finite set of integral linear inequalities such that any quadratic form can be reduced by an integral change of basis to a form whose coefficients satisfy those inequalities. This explains the name 'reduction theory'.

The main result of the theory we will use is the following, due to Ash [AMRT, Chapter II].

Theorem 2.4 (Ash). *Let C be a self-dual homogeneous convex cone in a real vector space V with $\mathbf{Q}$-structure. Let $G(C)$ be the automorphism group of C and $\mathcal{G}(C)$ the associated reductive algebraic group (as in Theorem 2.1). Assume $\mathcal{G}(C)^0$ is defined over $\mathbf{Q}$. Then for any arithmetic subgroup Γ of $\mathcal{G}(C)^0$, there exists a rational polyhedral cone $\Pi \subset C_+$ such that $(\Gamma \cdot \Pi) \cap C = C$.*

Here as before C_+ is the rational hull of C, meaning the convex hull of the rational points in $\overline{C}$.

Applied in the case of abelian varieties, this theorem will provide us with a rational polyhedral cone whose translates by automorphisms cover the ample cone. To prove the cone conjecture, we also need to say something about the rational boundary faces of the nef cone.

By a *face* F of the closed cone $\overline{C}$ we mean a subcone of $\overline{C}$ relatively open in its linear span, and maximal with that property. A face is *rational* if its linear span is a rational subspace of the ambient space. We will see in Section 5 that if C is the ample cone of an abelian variety, then C_+ is the union of the rational faces of $\overline{C}$. To deal with the boundary faces, we need the following consequence of Ash's theorem.

Corollary 2.5. *With the same assumptions as in Theorem 2.4, suppose also that C isomorphic to a product of cones of type $P_r(k)$ (that is, cones of positive-definite Hermitian real, complex, or quaternionic matrices). Then for each rational boundary face F of $\overline{C}$ and arithmetic subgroup $\Gamma \subset \mathcal{G}(C)^0$ there exists a rational polyhedral cone $\Pi_F \subset F_+$ such that $(\Gamma \cdot \Pi_F) \cap F = F$.*

Proof of Corollary. Any face F of a the closure $\overline{C}$ of a homogeneous self-adjoint cone is again a homogeneous self-adjoint cone; the main point of the proof is then to show that $\Gamma \cap \operatorname{Stab} F$, the subgroup of Γ consisting of elements which stabilise the face F, is an arithmetic subgroup of the automorphism group of F.

In more detail, since C is a product of cones of the form $P_r(k)$, any boundary face F of $\overline{C}$ can be transformed by an automorphism g of C to a product of faces $P_{s_i}(k)$ with $\sum s_i < r$ [HW87, Theorem 3.6]. (Here we regard $P_s(k)$ as a subcone of $\overline{P_r(k)}$ via a block embedding of $s \times s$ into $r \times r$ matrices.) By Theorem 2.3 any automorphism of $P_s(k)$ in the connected component of the identity is given by conjugation by an element M of $GL(s,k)$. Therefore any automorphism of the face $P_{s_1}(k) \times \cdots \times P_{s_n}(k)$, given by matrices $M_i \in GL(s_i,k)$ say, extends to the automorphism of $P_r(k)$ given by conjugation by $M_1 \oplus \cdots \oplus M_n \oplus \mathrm{Id} \in GL(r,k)$. Extending the automorphism of F in this way, and composing suitably with the automorphisms g and g^{-1}, we conclude that any automorphism of any face F of $\overline{C}$ extends to an automorphism of $\overline{C}$, or equivalently of C. In other words, there is a homomorphism $G(F) \to G(C)$, clearly injective, with image $G(C) \cap \operatorname{Stab} F$ (here $\operatorname{Stab} F$ denotes the group of linear transformations of the ambient space preserving the face F).

Now suppose that F is a rational boundary face of C. Then the group $\operatorname{Stab} F$ is defined over $\mathbf{Q}$. By assumption the group $G(C)^0$ is also defined over $\mathbf{Q}$, so $H := G(C)^0 \cap \operatorname{Stab} F$ is a subgroup of $G(F)$ defined over $\mathbf{Q}$. The subgroup H must have $G(F)^0$ as identity component; the group $G(F)^0$ is therefore the identity component of an algebraic group defined over $\mathbf{Q}$, hence is itself defined over $\mathbf{Q}$.

Next, the homomorphism $\mathcal{G}(F)^0 \to \mathcal{G}(C)^0$ is an injective homomorphism of $\mathbf{Q}$-algebraic groups. It is then a standard fact (a reference is [Ser79]) that for any arithmetic subgroup $\Gamma \subset \mathcal{G}(C)^0$, the intersection $\Gamma \cap \mathcal{G}(F)^0$ is an arithmetic subgroup of $\mathcal{G}(F)^0$.

Applying Theorem 2.4 with F in place of C and $\Gamma \cap \mathcal{G}(F)^0$ in place of Γ, we obtain a rational polyhedral cone $\Pi_F \subset F_+$ with the property that $\big((\Gamma \cap \mathcal{G}(F)^0) \cdot \Pi_F\big) \cap F = F$. In particular, this implies that $(\Gamma \cdot \Pi_F) \cap F = F$, as required. $\square$

3. The Endomorphism Algebra of an Abelian Variety

In this section we describe the endomorphism algebra of an abelian variety as a product of certain matrix algebras. In particular this gives a description of the automorphism group as an arithmetic group, connecting the cone conjecture for abelian varieties to the reduction theory of the previous section. For a full exposition of the structure theory of the endomorphism algebra, read Chapter 4 of Mumford's book [Mum70].

The first result we need is a form of the Poincaré complete reducibility theorem [Mum70, §19, Theorem 1, Corollary 1].

Theorem 3.1. *Let X be an abelian variety. Then X is isogenous to a product $X_1^{n_1} \times \cdots \times X_k^{n_k}$ where the X_i are simple abelian varieties, not isogeneous for $i \neq j$. The isogeny type of the X_i and the natural numbers n_i are uniquely determined by X.*

Here a *simple* abelian variety is one that has no proper abelian subvarieties. An *isogeny* $X \to Y$ of abelian varieties is a surjective homomorphism with finite kernel, and X and Y are *isogenous* if there exists an isogeny $X \to Y$. In fact given an isogeny $f : X \to Y$ there exists an isogeny $g : Y \to X$ such that $gf = n_X \in \mathrm{End}(X)$ for some natural number n; in particular, the relation 'is isogenous to' is indeed an equivalence relation.

For an abelian variety X we write $\mathrm{End}^0(X)$ to denote the tensor product $\mathrm{End}(X) \otimes \mathbf{Q}$. Note that if X and Y are isogenous, then $\mathrm{End}^0(X) \cong \mathrm{End}^0(Y)$ via pullback by the isogenies in either direction. Therefore as a corollary of Theorem 3.1 we get the following [Mum70, §19, Corollary 2].

Corollary 3.2. *Let X be an abelian variety. If X is simple, then $End^0(X)$ is a division algebra over $\mathbf{Q}$. For any abelian variety, if X is isogenous to $X_1^{n_1} \times \cdots \times X_k^{n_k}$ a product of mutually non-isogenous simple abelian varieties, then*

$$End^0(X) = M_{n_1}(D_1) \times \cdots \times M_{n_k}(D_k)$$

where D_i is the division algebra $End^0(X_i)$ and $M_n(D_i)$ is the ring of $n \times n$ matrices over D_i.

So reduction to the case of simple abelian varieties is straightforward. It is a much deeper problem to determine the $\mathbf{Q}$-division algebras which can appear as $\mathrm{End}^0(X)$ for X a simple abelian varieties. The key point is that the endomorphism algebra has finite rank and is equipped with a positive involution [Mum70, §19 Corollary 3, §21 Theorem 1]:

Proposition 3.3. *Let X be a simple abelian variety. Then $D = End^0(X)$ is a finite-rank $\mathbf{Q}$-division algebra. Moreover D carries an involution $x \mapsto x'$ called the Rosati involution. This involution is positive-definite in the sense that if $x \in D$ is any nonzero element, then $Tr_{\mathbf{Q}}(xx') > 0$, where $Tr_{\mathbf{Q}}$ is the reduced trace over $\mathbf{Q}$ of the division algebra D.*

Now there is a classification (due to Albert) of all finite-rank $\mathbf{Q}$-division algebras with a positive involution, and together with some extra geometric restrictions this gives a complete list of possibilities for $\mathrm{End}^0(X)$ when X is a simple abelian variety. Chapter 4 of Mumford's book gives an exposition of the classification; here we only state a weak form of the result.

THEOREM 3.4. *Let X be a simple abelian variety and $D = End^0(X)$ its endomorphism algebra. Then $D \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic as an $\mathbf{R}$-algebra with involution to one of the following algebras:*

- $\mathbf{R} \times \cdots \times \mathbf{R}$*, with $x \mapsto x'$ the trivial involution;*
- $\mathbf{H} \times \cdots \times \mathbf{H}$ *where $\mathbf{H}$ is the algebra of quaternions, and $x' = \overline{x}$, the usual conjugate;*
- $M_2(\mathbf{R}) \times \cdots \times M_2(\mathbf{R})$ *where $M_2(\mathbf{R})$ is the algebra of 2×2 real matrices, and $x' = x^t$, the transpose;*
- $M_2(\mathbf{C}) \times \cdots \times M_2(\mathbf{C})$*, where $M_2(\mathbf{C})$ is the algebra of 2×2 complex matrices, and $x' = x^*$, the conjugate transpose.*

Combining this result with Corollary 3.2 one can deduce the following description of $\mathrm{End}^0(X) \otimes \mathbf{R}$ for an arbitrary abelian variety X.

COROLLARY 3.5. *Let X be an abelian variety. Then $End^0(X) \otimes \mathbf{R}$ is isomorphic as an algebra with involution to a product*

$$\prod_i M_{r_i}(\mathbf{R}) \times \prod_j M_{s_j}(\mathbf{C}) \times \prod_k M_{t_k}(\mathbf{H})$$

with involution given by conjugate transpose on each factor. The bilinear pairing $\langle x, y \rangle = \mathrm{Tr}(xy^*)$ defines an inner product on $End^0(X) \otimes \mathbf{R}$.

Finally we must explain how the automorphism group $\mathrm{Aut}(X)$ sits inside the algebra $\mathrm{End}^0(X) \otimes \mathbf{R}$. If X is an abelian variety, $\mathrm{Aut}(X)$ is the group of units $\mathrm{End}(X)^\times$ in the endomorphism ring $\mathrm{End}(X)$. Furthermore $\mathrm{End}(X)$ is a lattice in the vector space $\mathrm{End}^0(X) \otimes \mathbf{R}$, therefore induces a $\mathbf{Q}$-structure on $\mathrm{End}^0(X) \otimes \mathbf{R}$, and this $\mathbf{Q}$-structure determines $\mathrm{End}(X)$ as a subring of the $\mathbf{R}$-algebra $\mathrm{End}^0(X) \otimes \mathbf{R}$ up to finite index. So from the previous corollary we have the following:

Corollary 3.6. *Let X be an abelian variety. Then $(End^0(X)\otimes\mathbf{R})^\times$ is an algebraic group defined over $\mathbf{Q}$, and $Aut(X)$ is an arithmetic subgroup.*

4. The Néron–Severi Space of an Abelian Variety

In this section, we explain how the Néron–Severi space of an abelian variety can be identified with a subspace of the space $\mathrm{End}^0(X)\otimes\mathbf{R}$. This identification allows us to describe the action of the automorphism group on the ample cone in terms of matrices.

First we define the *Néron–Severi space* of an abelian variety X to be the finite-dimensional real vector space $N^1(X) := (\mathrm{Pic}(X)/\mathrm{Pic}^0(X))\otimes\mathbf{R}$. Our first task is to identify $N^1(X)$ with a subspace of $\mathrm{End}^0X\otimes\mathbf{R}$. By linearity, to make such an identification it suffices to identify a Cartier divisor with an element of $\mathrm{End}^0(X)$, which we do in the following way:

$$\begin{array}{ccccc}
\mathrm{Pic}(X) & \xrightarrow{\phi} & \mathrm{Hom}(X,\widehat{X})\otimes\mathbf{Q} & \xrightarrow{\psi} & \mathrm{End}^0(X) \\
\\
D & \longmapsto & \phi_D & \longmapsto & \phi_L^{-1}\phi_D
\end{array}$$

The notation of the diagram is as follows. The variety $\widehat{X}$ is the *dual abelian variety* of X, which can be identified with $\mathrm{Pic}^0(X)$. The homomorphism $\phi_D : X\to\widehat{X}$ is defined by $\phi_D(x)=T_x^*(D)\otimes D^{-1}$, where T_x is translation by the point $x\in X$. Finally, L is any (fixed) ample line bundle on X; ampleness implies that ϕ_L is an isogeny $X\to\widehat{X}$, and therefore has an inverse $\phi_L^{-1}\in\mathrm{Hom}(\widehat{X},X)\otimes\mathbf{Q}$. One checks that the kernel of ϕ is exactly the subgroup $\mathrm{Pic}^0(X)$ of numerically trivial line bundles on X, and that ψ is an isomorphism. Therefore tensoring with $\mathbf{R}$ gives the claimed embedding $N^1(X)\hookrightarrow\mathrm{End}^0X\otimes\mathbf{R}$. (See [Mum70] for proofs of all the assertions here.) Note that by construction this embedding is compatible with the natural $\mathbf{Q}$-structures on the source and target.

The automorphism group $\mathrm{Aut}(X)$ acts naturally on $N^1(X)$ via pull-back of divisors: an automorphism f maps a divisor D to f^*D. Using the diagram above, we can extend this to an action of $\mathrm{Aut}(X)$ on the whole algebra $\mathrm{End}^0(X)\otimes\mathbf{R}$. To compute the action we need the following formula [Mum70, §15, proof of Theorem 1]:

Lemma 4.1. *Let $f : X \to Y$ be an isogeny of abelian varieties, with dual isogeny $\widehat{f} : \widehat{Y} \to \widehat{X}$. Let D be a line bundle on Y. Then $\phi_{f^*D} = \widehat{f} \circ \phi_D \circ f$.*

Via the diagram we can then work out the extension to an action of $\mathrm{Aut}(X)$ on $\mathrm{End}^0(X) \otimes \mathbf{R}$: one computes that $f \in \mathrm{Aut}(X)$ acts by the formula

$$f \cdot x = \phi_L^{-1} \circ \widehat{f} \circ \phi_L \circ x \circ f$$

for any $x \in \mathrm{End}^0(X) \otimes \mathbf{R}$. Moreover the same formula defines an action of the whole group of units $(\mathrm{End}^0(X) \otimes \mathbf{R})^\times$ on $\mathrm{End}(X)^0 \otimes \mathbf{R}$, and this action fixes the subspace $N^1(X)$.

To complete the picture, we observe [Mum70, p.189] that the map $x \mapsto \phi_L^{-1} \circ \widehat{x} \circ \phi_L$ is by definition exactly the Rosati involution on $\mathrm{End}^0(X) \otimes \mathbf{R}$ mentioned in Proposition 3.3. (The involution therefore depends on the choice of an ample line bundle on X, but the dependence does not matter for our purposes.) For an element $e \in \mathrm{End}^0(X) \otimes \mathbf{R}$ let us denote $\phi_L^{-1} \circ \widehat{e} \circ \phi_L$ by e', so that the action of $(\mathrm{End}^0(X) \otimes \mathbf{R})^\times$ on $\mathrm{End}(X)^0 \otimes \mathbf{R}$ is now given by the formula $x \mapsto f' \circ x \circ f$. By Theorem 3.4 there is an isomorphism of $\mathrm{End}(X)^0 \otimes \mathbf{R}$ with a product of matrix algebras which takes the Rosati involution $e \mapsto e'$ to the conjugate transpose involution $x \mapsto x^*$. Using this isomorphism to translate the action above into matrix terms, we get the following theorem.

Theorem 4.2. *Let X be an abelian variety. Then the $\mathbf{Q}$-algebraic group $(End^0(X) \otimes \mathbf{R})^\times$ acts on $N^1(X)$ by the formula $F : D \to F^*DF$, and this extends the action of $Aut(X)$ on $N^1(X)$ by pullback of line bundles.*

At this point we have an explicit description in terms of matrices of the action of the automorphism group on the Néron–Severi space. To apply this to the cone conjecture, we need to identify the ample cone in the same terms (i.e. as a cone in a space of matrices).

The first step is to identify the image of our embedding $N^1(X) \hookrightarrow \mathrm{End}^0(X) \otimes \mathbf{R}$, viewing the target as a product of matrix algebras as in Corollary 3.5. Again the key here is the Rosati involution: as we have just seen, in matrix terms this is simply conjugate-transposition $x \mapsto x^*$.

THEOREM 4.3. *Let X be an abelian variety. Then $N^1(X) \subset End^0(X) \otimes \mathbf{R}$ is exactly the fixed subspace of the Rosati involution. If $End^0(X) \otimes \mathbf{R}$ is isomorphic to a product of matrix algebras*

$$\prod_i M_{r_i}(\mathbf{R}) \times \prod_j M_{s_j}(\mathbf{C}) \times \prod_k M_{t_k}(\mathbf{H})$$

then $N^1(X)$ is isomorphic to the subspace

$$\bigoplus_i \mathcal{H}_{r_i}(\mathbf{R}) \oplus \bigoplus_j \mathcal{H}_{s_j}(\mathbf{C}) \oplus \bigoplus_k \mathcal{H}_k(\mathbf{H})$$

where $\mathcal{H}_r$ denotes the space of $r \times r$ symmetric or Hermitian matrices. Moroever, the ample cone $A(X)$ is the direct sum of the positive-definite cones $P_r(k)$ in each of the direct summands $\mathcal{H}_r(k)$ of $N^1(X)$.

We use additive notation for $N^1(X)$ to emphasise the point that it need not be a subalgebra of $\mathrm{End}^0(X) \otimes \mathbf{R}$: for divisors D_1 and D_2 fixed by the Rosati involution, their product $D_1 D_2 \in \mathrm{End}^0(X) \otimes \mathbf{R}$ need not be fixed, as one can check using suitable matrices.

Theorem 4.3 shows in particular that the ample cone of an abelian variety is a self-dual homogeneous cone, since it is a direct sum of cones on the list of Theorem 2.2. Indeed since it is a sum of cones of the form $P_r(k)$, Theorem 2.3 tells us that $G(A(X))^0$ acts transitively on $A(X)$. Moreover from Theorem 4.3 together with the description of the automorphism group in Theorem 2.3 we can also deduce the following:

COROLLARY 4.4. *Let X be an abelian variety. Then the homomorphism*

$$\begin{aligned}\left(End^0(X) \otimes \mathbf{R}\right)^\times &\to G(A(X))^0 \\ M &\mapsto (D \mapsto M^* D M)\end{aligned}$$

given by the action of $(End^0(X) \otimes \mathbf{R})^\times$ on $N^1(X)$ is surjective.

PROOF. Suppose for simplicity that $N^1(X)$ has a single direct summand, say $N^1(X) = \mathcal{H}_r(\mathbf{R})$. By Theorem 2.3 the identity component $G(A(X))^0$ of the automorphism group of the ample cone is exactly the group of linear transformations of $N^1(X)$ of the form $D \mapsto M^* D M$ with

$M \in GL(r, \mathbf{R})$. Such a linear transformation is the image under the homomorphism of $M \in GL(r, \mathbf{R}) = (\mathrm{End}^0(X) \otimes \mathbf{R})^\times$, which proves surjectivity. The proof in the case of more than one direct summand works in the same way, since by Theorem 2.2 the identity component $G(A(X))^0$ is the direct product of the identity components of the automorphism groups of the direct summand cones. $\square$

5. Boundary Faces of the Nef Cone

In this section, we put together the results of the previous sections to complete the proof of the cone conjecture. Using the results of Section 4, Theorem 2.4 will provide us with a rational polyhedral cone Π° whose translates cover the interior of $A(X)_+$. Corollary 2.5 will say something about the rational boundary faces, but to complete the proof we need to know in addition that the cone $A(X)_+$ has only finitely many faces up to the action of $\mathrm{Aut}(X)$. The key point is to identify these faces with abelian subvarieties of X, and then apply a finiteness result of Lenstra–Oort–Zarhin.

Lemma 5.1. *Let X be an abelian variety. Then there is an $Aut(X)$-equivariant bijection between the set of faces of $A(X)_+$ and the set of abelian subvarieties of X.*

Proof. We can identify the set of faces of $A(X)_+$ with the set of contractions of X, or more precisely the set of equivalence classes of contractions, where two morphisms are identified if they contract the same curves. (In one direction, pulling back an ample line bundle via a contraction gives a nef line bundle, and this line bundle belongs to the relative interior of a unique face of $A(X)_+$. In the other direction, given a face F of $A(X)_+$, choose a line bundle whose class lies in the relative interior of F; as shown in Section 1, such a line bundle is semi-ample, hence defines a contraction of X.) To prove the lemma, we will give a bijection between the set of abelian subvarieties of X and the set of contractions of X.

First suppose $Z \subset X$ is an abelian subvariety. I claim there is a contraction morphism $X \to S$ whose fibres are exactly the translates of Z. To see this, first note [Mum70, §19 Theorem 1] there is an abelian subvariety $Y \subset X$ such that the addition morphism $+ : Y \times Z \to X$ is an isogeny. Moreover [Mum70, p.169], there is also an isogeny $f : X \to Y \times Z$ such that $+\circ f : X \to X$ is exactly n_X, multiplication by n on X, for some integer n. Now consider the projection $\pi : Y \times Z \to Y$ and the composition $\pi \circ f : X \to Y$. Taking the Stein factorisation of this morphism we get a commuting diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \times Z \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{h} & Y \end{array}$$

where g has connected fibres and h is a finite morphism. I claim that the fibres of the morphism g are exactly the translates of Z in X. To see this, first note that by Nori's "Remarks on Effective Divisors" [Mum70, p.88], there is a unique abelian subvariety $Z' \subset X$ such that the fibres of g are exactly the translates of Z' in X. Next, note that the fibres of g must have the same dimension as those of π, since f and h are both finite, so $\dim Z' = \dim Z$. Moreover, the subset $f^{-1}(\{0\} \times Z) \subset X$ is contracted to a point by $\pi \circ f$, so by finiteness of h, the set $g\left((f^{-1}(\{0\} \times Z)\right)$ must have dimension zero. Therefore it suffices to show the inclusion $Z \subset f^{-1}(\{0\} \times Z)$, since then $g(Z)$ must have dimension zero, hence by connectedness must be a point, therefore contained in the fibre Z', and by the equality of dimensions above must in fact be all of Z'.

The required inclusion can be written as $f(Z) \subset \{0\} \times Z$. Recall that we have a diagram

$$X \xrightarrow{f} Y \times Z \xrightarrow{+} X$$

where $+\circ f = n_X$, multiplication by n on X. So $+(f(Z)) = n_X(Z) = Z$, that is to say $f(Z) \subset +^{-1}(Z)$. Now a point $(y, z) \in Y \times Z$ belongs to $+^{-1}(Z)$ if and only if $y \in Y \cap Z$: that is, $+^{-1}(Z) = (Y \cap Z) \times Z \subset Y \times Z$. So $f(Z) \subset (Y \cap Z) \times Z$, a disjoint union of finitely many translates of $\{0\} \times Z$. By connectedness, we conclude that $f(Z) \subset \{0\} \times Z$, the required inclusion.

We have shown that given an abelian subvariety $Z \subset X$, there is a contraction morphism $g : X \to S$ whose fibres are exactly the translates of Z. Conversely, given a contraction $g : X \to S$, the result of Nori mentioned above says that there is an abelian subvariety $Z \subset X$ such that the fibres of g are exactly the translates of Z in X. So there is a bijection between the set of contraction morphisms on X and the set of abelian subvarieties of X. Finally, if $\mathrm{Aut}(X)$ acts on the set of contractions by pre-composition, the bijection is immediately seen to be $\mathrm{Aut}(X)$-equivariant, completing the proof. □

COROLLARY 5.2. *The cone $A(X)_+$ has finitely many faces modulo the action of Aut(X).*

PROOF. By Lenstra–Oort–Zarhin [LOZ] the set of abelian subvarieties of X is finite up to the action of $\mathrm{Aut}(X)$, so by Lemma 5.1 we get the conclusion. □

Finally we need the following lemma:

LEMMA 5.3. *Any face of $A(X)_+$ is a rational face of $\overline{A(X)}$.*

PROOF. Given a face F of $A(X)_+$, choose a a line bundle L whose class lies in the relative interior of F. This gives a contraction $f : X \to Y$ with the property that $f^*(\overline{A(Y)})$ is a face of $\overline{A(X)}$ contained in F. But $f^*(\overline{A(Y)}$ contains L, so in fact $F = f^*(\overline{A(Y)}) = \overline{A(X)} \cap f^*(N^1(Y))$ [Kaw97, Definition 1.8]. Since $N^1(Y)$ is a rational subspace of $N^1(X)$, this shows that F is a rational face. □

We can now complete the proof of Theorem 0.1, putting together the results of previous sections. The inclusion $N^1(X) \subset \mathrm{End}^0(X) \otimes \mathbf{R}$ described in Section 4 is compatible with $\mathbf{Q}$-structures, so by Corollary 4.4 the connected component $\mathcal{G}(A(X))^0$ is a $\mathbf{Q}$-algebraic subgroup of $GL(N^1(X))$. By Corollary 3.6 the automorphism group $\mathrm{Aut}(X)$ is an arithmetic subgroup of $(\mathrm{End}^0(X) \otimes \mathbf{R})^\times$, so the image Γ of $\mathrm{Aut}(X)$ in $GL(N^1(X))$ is an arithmetic subgroup of $\mathcal{G}(A(X))^0$. Therefore by Theorem 2.4 there is a rational polyhedral cone $\Pi^\circ \subset A(X)_+$ such that $(\Gamma \cdot \Pi^\circ) \cap A(X) = A(X)$.

To deal with the boundary of $A(X)_+$, recall that by Theorem 4.3 the ample cone $A(X)$ is a homogeneous self-adjoint cone which is a product of cones of the form $P_r(k)$. Therefore by Corollary 2.5 for every face F of $A(X)_+$ we can find a rational polyhedral cone $\Pi_F \subset F$ with the property that $(\mathrm{Aut}(X) \cdot \Pi_F) \cap F = F$. Now let $\{F_1, \cdots, F_n\}$ be a complete set of representatives for the (finitely many) $\mathrm{Aut}(X)$-orbits of boundary faces of $A(X)_+$ and choose corresponding rational polyhedral cones $\Pi_{F_1}, \ldots, \Pi_{F_n}$.

Define Π to be the convex hull of Π^o and all the cones Π_{F_i}. Since there are finitely many F_i, the cone Π is again rational polyhedral. Now each cone Π_{F_i} has the property that $F_i \subset \mathrm{Aut}(X) \cdot \Pi_{F_i}$, so $\mathrm{Aut}(X) \cdot \Pi$ covers each of the faces F_i. But the F_i are a complete set of representatives for the $\mathrm{Aut}(X)$-orbits of boundary faces of $A(X)_+$, so in fact $\mathrm{Aut}(X) \cdot \Pi$ covers every boundary face of $A(X)_+$. Also, we know that $\mathrm{Aut}(X) \cdot \Pi^\circ$ covers the interior of $A(X)_+$, so we conclude that $\mathrm{Aut}(X) \cdot \Pi = A(X)_+$.

Finally, given such a rational polyhedral cone Π, one can then produce a precise fundamental domain as explained in [AMRT, p. 116]. In a little more detail, the result there gives a rational polyhedral cone $\Pi_{exact} \subset A(X)_+$ whose translates by $\mathrm{Aut}(X)$ cover the cone $A(X)$ (with disjoint interiors). Now the interior of the rational polyhedral cone Π must be covered by a finite collection of translates of Π_{exact}, by the Siegel property, implying that Π itself is covered by those finitely many translates (since it is rational polyhedral). This shows that $\mathrm{Aut}(X) \cdot \Pi_{exact}$ contains $Aut(X) \cdot \Pi = A(X)_+$, so Π_{exact} is indeed a rational polyhedral fundamental domain for the action of $\mathrm{Aut}(X)$ on $A(X)_+$. □

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Institut fuer Algebraische Geometrie
Leibniz Universitaet Hannover
Welfengarten 1
Hannover 30167, Germany

Current address:
University of Illinois at Chicago
Department of Mathematics,
Statistics, and Computer
Science, Chicago, IL 60607, U.S.A.
E-mail: artie@math.uic.edu