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On Non-Sensitive Homeomorphisms of the Boundary of a Proper Cocompact CAT(0) Space

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Abstract. We investigate the homeomorphism $\bar{f}$ of the boundary ∂X of a proper cocompact CAT(0) space X with $|\partial X| > 2$ induced by an isometry f of X, and we study when the induced homeomorphism $\bar{f}$ of the boundary ∂X is non-expansive or non-sensitive.

1. Introduction

In this paper, we study non-expansive homeomorphisms and non-sensitive homeomorphisms of the boundary of a proper cocompact CAT(0) space. Definitions and basic properties of CAT(0) spaces and their boundaries are found in [1]. We introduce some basic of CAT(0) spaces and their boundaries in Section 2. For a proper CAT(0) space X and the boundary ∂X of X, we can define a metric on the boundary ∂X as follows: We first fix a basepoint $x_0 \in X$. Let $\alpha, \beta \in \partial X$. There exist unique geodesic rays $\xi_{x_0,\alpha}$ and $\xi_{x_0,\beta}$ in X with $\xi_{x_0,\alpha}(0) = \xi_{x_0,\beta}(0) = x_0$, $\xi_{x_0,\alpha}(\infty) = \alpha$ and $\xi_{x_0,\beta}(\infty) = \beta$. Then the metric $d_{\partial X}^{x_0}(\alpha, \beta)$ of α and β on ∂X with respect to the basepoint x_0 is defined by

$$d_{\partial X}^{x_0}(\alpha, \beta) = \sum_{i=1}^{\infty} \min\{d(\xi_{x_0,\alpha}(i), \xi_{x_0,\beta}(i)),\ \frac{1}{2^i}\}.$$

The metric $d_{\partial X}^{x_0}$ depends on the basepoint x_0 and the topology of ∂X does not depend on x_0.

An isometry f of a proper CAT(0) space X naturally induces the homeomorphism $\bar{f}$ of the boundary ∂X (cf. [1, p.264, Corollary II.8.9]). The purpose of this paper is to investigate when the homeomorphism $\bar{f}$ of the

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boundary ∂X is non-expansive or non-sensitive. Here, in this paper, non-expansive homeomorphisms and non-sensitive homeomorphisms are defined as follows: A homeomorphism $g : Y \to Y$ of a metric space (Y, d) is said to be *non-expansive* if for any $\epsilon > 0$ there exist $x, y \in Y$ with $x \neq y$ such that $d(g^i(x), g^i(y)) < \epsilon$ for any $i \in \mathbb{Z}$. Also a homeomorphism $g : Y \to Y$ is said to be *non-sensitive* if for any $\epsilon > 0$ there exist a point $x \in Y$ and a neighborhood U of x in Y such that the diameter $\operatorname{diam} g^i(U) < \epsilon$ for any $i \in \mathbb{Z}$. (We note that non-expansiveness and non-sensitiveness of a homeomorphism g of a metric space (Y, d) depends on the topology of Y and does not depend on the metric d of Y.) In dynamical systems and chaos theory, (non-)expansive homeomorphisms and (non-)sensitive homeomorphisms are important concepts. In this paper, we would like to obtain some information of homeomorphisms of boundaries of CAT(0) spaces by using a concept of the dynamical systems and the chaotic theory. We can find some recent research using a concept of the dynamical systems and the chaotic theory on minimality and scrambled sets of boundaries of CAT(0) groups and Coxeter groups in [7], [8], [9], [10], [11] and [13].

We introduce some remarks on isometries of CAT(0) spaces and induced homeomorphisms of boundaries in Section 3, and we show the following theorem in Sections 4–7.

THEOREM 1.1. *Let X be a proper cocompact CAT(0) space with $|\partial X| > 2$. Suppose that $f : X \to X$ is an isometry and $\bar{f} : \partial X \to \partial X$ is the homeomorphism induced by f.*

(1) *If f is an elliptic isometry, then there exists a point $x_0' \in X$ such that $\bar{f} : (X, d_{\partial X}^{x_0'}) \to (X, d_{\partial X}^{x_0'})$ is an isometry, and hence $\bar{f}$ is a non-expansive and non-sensitive homeomorphism of ∂X with respect to any metric on the boundary ∂X.*

(2) *If the CAT(0) space X is non-hyperbolic, then $\bar{f}$ is a non-expansive homeomorphism of ∂X.*

(3) *If the CAT(0) space X is hyperbolic, then $\bar{f}$ is a non-sensitive homeomorphism of ∂X.*

(4) *$\bar{f}$ is a non-expansive homeomorphism of ∂X.*

Here we note that the boundary ∂X of a proper cocompact CAT(0) space X with $|\partial X| > 2$ has no isolated points (cf. [6]). Hence if $\bar{f}$ is a non-sensitive homeomorphism of the boundary ∂X, then $\bar{f}$ is a non-expansive homeomorphism of ∂X. Thus, in Theorem 1.1, the statements (2) and (3) implies (4).

We introduce sensitiveness of the induced homeomorphisms of the boundary with respect to neighborhoods of a point in Section 8, and we provide some remarks and questions in Section 9.

2. CAT(0) Spaces and Their Boundaries

We say that a metric space (X, d) is a *geodesic space* if for each $x, y \in X$, there exists an isometric embedding $\xi : [0, d(x, y)] \to X$ such that $\xi(0) = x$ and $\xi(d(x, y)) = y$ (such ξ is called a *geodesic*). Also a metric space X is said to be *proper* if every closed metric ball is compact.

Let X be a geodesic space and let T be a geodesic triangle in X. A *comparison triangle* for T is a geodesic triangle $\overline{T}$ in the Euclidean plane $\mathbb{R}^2$ with same edge lengths as T. Choose two points x and y in T. Let $\bar{x}$ and $\bar{y}$ denote the corresponding points in $\overline{T}$. Then the inequality

$$d(x, y) \le d_{\mathbb{R}^2}(\bar{x}, \bar{y})$$

is called the *CAT(0)-inequality*, where $d_{\mathbb{R}^2}$ is the usual metric on $\mathbb{R}^2$. A geodesic space X is called a *CAT(0) space* if the CAT(0)-inequality holds for all geodesic triangles T and for all choices of two points x and y in T.

Let X be a proper CAT(0) space and $x_0 \in X$. The *boundary of X with respect to x_0*, denoted by $\partial_{x_0} X$, is defined as the set of all geodesic rays issuing from x_0. Then we define a topology on $X \cup \partial_{x_0} X$ by the following conditions:

(1) X is an open subspace of $X \cup \partial_{x_0} X$.

(2) For $\alpha \in \partial_{x_0} X$ and $r, \epsilon > 0$, let

$$U_{x_0}(\alpha; r, \epsilon) = \{x \in X \cup \partial_{x_0} X \mid x \notin B(x_0, r),\ d(\alpha(r), \xi_x(r)) < \epsilon\},$$

where $\xi_x : [0, d(x_0, x)] \to X$ is the geodesic from x_0 to x ($\xi_x = x$ if $x \in \partial_{x_0} X$). Then for each $\epsilon_0 > 0$, the set

$$\{U_{x_0}(\alpha; r, \epsilon_0) \mid r > 0\}$$

is a neighborhood basis for α in $X \cup \partial_{x_0} X$.

This topology is called the *cone topology* on $X \cup \partial_{x_0} X$. It is known that $X \cup \partial_{x_0} X$ is a metrizable compactification of X ([1]).

Let X be a proper CAT(0) space. Two geodesic rays $\xi, \zeta : [0, \infty) \to X$ are said to be *asymptotic* if there exists a constant N such that $d(\xi(t), \zeta(t)) \le N$ for any $t \ge 0$. It is known that for each geodesic ray ξ in X and each point $x \in X$, there exists a unique geodesic ray ξ' issuing from x such that ξ and ξ' are asymptotic.

Let x_0 and x_1 be two points of a proper CAT(0) space X. Then there exists a unique bijection $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ such that ξ and $\Phi(\xi)$ are asymptotic for any $\xi \in \partial_{x_0} X$. It is known that $\Phi : \partial_{x_0} X \to \partial_{x_1} X$ is a homeomorphism ([1]).

Let X be a proper CAT(0) space. The asymptotic relation is an equivalence relation on the set of all geodesic rays in X. The *boundary of* X, denoted by ∂X, is defined as the set of asymptotic equivalence classes of geodesic rays. The equivalence class of a geodesic ray ξ is denoted by $\xi(\infty)$. For each $x_0 \in X$ and each $\alpha \in \partial X$, there exists a unique element $\xi \in \partial_{x_0} X$ with $\xi(\infty) = \alpha$. Thus we may identify ∂X with $\partial_{x_0} X$ for each $x_0 \in X$.

We can define the metric $d_{\partial X}^{x_0}$ on the boundary ∂X as in Section 1. In this paper, we suppose that every CAT(0) space X has a fixed basepoint x_0 and $d_{\partial X}^{x_0}$ is the metric on the boundary ∂X as in Section 1.

Let X be a non-compact proper cocompact CAT(0) space. (Here X is said to be *cocompact* if there exists a compact subset K of X such that $\mathrm{Isom}(X) \cdot K = X$, where $\mathrm{Isom}(X)$ is the isometry group of X.) Then X is *almost geodesically complete* by [5, Corollary 3] (cf. [5] and [12]). Hence by the proof of [6, Theorem 3.1], we can obtain the following proposition.

Proposition 2.1. *Let X be a proper cocompact CAT(0) space with $|\partial X| > 2$. Then every point of ∂X is an accumulation point, i.e., ∂X has no isolated points.*

3. On Homeomorphisms of Boundaries Induced by Isometries of CAT(0) Spaces

Let (X, d) be a metric space and let $f : X \to X$ be an isometry of X. Then the *translation length* of f is defined as $|f| := \inf\{d(x, f(x)) \,|\, x \in X\}$.

We also define the set $\mathrm{Min}(f) := \{x \in X \mid d(x, f(x)) = |f|\}$. An isometry f of a metric space X is said to be *semi-simple* if $\mathrm{Min}(f)$ is non-empty.

DEFINITION 3.1 (cf. [1, p.229]). Let f be an isometry of a metric space X.

(1) f is called *elliptic* if f has a fixed-point (in this case, $|f| = 0$ and $\mathrm{Min}(f)$ is the fixed-points set of f).

(2) f is called *hyperbolic* if f is semi-simple and $|f| > 0$.

(3) f is called *parabolic* if f is not semi-simple, i.e., $\mathrm{Min}(f)$ is empty.

For a hyperbolic isometry of a CAT(0) space, the following remark is well-known (cf. [1, p.231, Theorem II.6.8]).

REMARK. Let f be a hyperbolic isometry of a proper CAT(0) space X. Then there exists a geodesic line $\sigma : \mathbb{R} \to X$ such that $f(\sigma(t)) = \sigma(t + |f|)$ for any $t \in \mathbb{R}$. Such a geodesic line is called an *axis* of f. We note that $\mathrm{Im}\,\sigma \subset \mathrm{Min}(f)$. It is known that the axes of f are parallel to each other and $\mathrm{Min}(f)$ is the union of the all axes. Hence $\mathrm{Min}(f)$ splits as $\mathrm{Min}(f) = Y \times \mathbb{R}$ for some $Y \subset X$.

For an axis σ of f, we define $f^{\infty} := \sigma(\infty)$ and $f^{-\infty} := \sigma(-\infty)$. Here the two points f^{∞} and $f^{-\infty}$ of the boundary ∂X are not dependent on the axis σ. Also we note that for every point $x \in X$, the sequence $\{f^i(x)\}_i$ converges to f^{∞} as $i \to \infty$ in $X \cup \partial X$, and the sequence $\{f^i(x)\}_i$ converges to $f^{-\infty}$ as $i \to -\infty$ in $X \cup \partial X$.

Let f be an isometry of a proper CAT(0) space X. For each geodesic ray ξ in X, the map $f \circ \xi$ is also a geodesic ray in X since f is an isometry of X. We define the map $\bar{f} : \partial X \to \partial X$ by $\bar{f}([\xi]) := [f \circ \xi]$ for $[\xi] \in \partial X$ (where $[\xi]$ is the equivalence class of asymptotic relation of a geodesic ray ξ in X). Then it is known that $\bar{f}$ is a homeomorphism of the boundary ∂X (cf. [1, p.264, Corollary II.8.9]).

The purpose of this paper is to investigate the homeomorphism $\bar{f}$ of the boundary ∂X induced by an isometry f of X.

4. On Homeomorphisms of Boundaries Induced by Elliptic Isometries of CAT(0) Spaces

In this section, we consider the homeomorphism $\bar{f}$ of the boundary ∂X induced by an *elliptic* isometry f of a proper cocompact CAT(0) space X.

We show the following theorem.

Theorem 4.1. *Let X be a proper cocompact CAT(0) space with $|\partial X| > 2$ and let $f : X \to X$ be an elliptic isometry. Then there exists a point $x_0' \in X$ such that $\bar{f}$ is an isometry of the metric space $(\partial X, d_{\partial X}^{x_0'})$. Hence $\bar{f}$ is a non-expansive and non-sensitive homeomorphism of the boundary ∂X with respect to any metric on the boundary ∂X.*

Proof. Since f is an elliptic isometry, there exists a fixed-point $x_0' \in X$ of f. Let $\alpha, \beta \in \partial X$ and let ξ and ζ be the geodesic rays in X such that $\xi(0) = \zeta(0) = x_0'$, $\xi(\infty) = \alpha$ and $\zeta(\infty) = \beta$. Then $f(x_0') = x_0'$, and $f \circ \xi$ and $f \circ \zeta$ are the geodesic rays issuing from x_0' such that $f \circ \xi(\infty) = \bar{f}(\alpha)$ and $f \circ \zeta(\infty) = \bar{f}(\beta)$.

Now $d(f \circ \xi(t), f \circ \zeta(t)) = d(\xi(t), \zeta(t))$ for any $t \geq 0$ because f is an isometry. Hence

$$\begin{aligned} d_{\partial X}^{x_0'}(\bar{f}(\alpha), \bar{f}(\beta)) &= \sum_{i=1}^{\infty} \min\{d(f \circ \xi(i), f \circ \zeta(i)),\ \frac{1}{2^i}\} \\ &= \sum_{i=1}^{\infty} \min\{d(\xi(i), \zeta(i)),\ \frac{1}{2^i}\} \\ &= d_{\partial X}^{x_0'}(\alpha, \beta), \end{aligned}$$

that is, $\bar{f}$ is an isometry of $(\partial X, d_{\partial X}^{x_0'})$.

For any $\epsilon > 0$, we take a point $\alpha \in \partial X$ and $\epsilon/4$-neighborhood U of α in $(\partial X, d_{\partial X}^{x_0'})$. Then

$$\operatorname{diam} \bar{f}^i(U) = \operatorname{diam} U < \epsilon$$

for any $i \in \mathbb{Z}$ because $\bar{f}$ is an isometry of $(\partial X, d_{\partial X}^{x_0'})$. Hence $\bar{f}$ is a non-sensitive homeomorphism of ∂X. Here the non-sensitiveness of $\bar{f}$ is not dependent on the metric $d_{\partial X}^{x_0'}$. In particular, it is independent of the point x_0'.

Since X is a proper cocompact CAT(0) space with $|\partial X| > 2$, every point of the boundary ∂X is an accumulation point and ∂X has no isolated points by Proposition 2.1. Thus $\bar{f}$ is also a non-expansive homeomorphism of ∂X. $\square$

5. On Hyperbolic Spaces

In this section, we introduce *hyperbolic* CAT(0) spaces.

We first introduce a definition of hyperbolic spaces. A geodesic space X is called a *hyperbolic space*, if there exists a number $\delta \geq 0$ such that every geodesic triangle in X is "δ-thin". Here "δ-thin" is defined as follows: Let $x, y, z \in X$ and let $\triangle := \triangle xyz$ be a geodesic triangle in X. There exist unique non-negative numbers a, b, c such that

$$d(x, y) = a + b,\ d(y, z) = b + c,\ d(z, x) = c + a.$$

Then we can consider the metric tree $T_\triangle$ that has three vertices of valence one, one vertex of valence three, and edges of length a, b and c. Let o be the vertex of valence three in $T_\triangle$ and let v_x, v_y, v_z be the vertices of $T_\triangle$ such that $d(o, v_x) = a$, $d(o, v_y) = b$ and $d(o, v_z) = c$. Then the map $\{x, y, z\} \to \{v_x, v_y, v_z\}$ extends uniquely to a map $f : \triangle \to T_\triangle$ whose restriction to each side of $\triangle$ is an isometry. For some $\delta \geq 0$, the geodesic triangle $\triangle$ is said to be *δ-thin* if $d(p, q) \leq \delta$ for each points $p, q \in \triangle$ with $f(p) = f(q)$.

It is known that a geodesic space X is hyperbolic if and only if there exists a number $\delta \geq 0$ such that every geodesic triangle in X is "δ-slim". Here a geodesic triangle is said to be *δ-slim* if each of its sides is contained in the δ-neighborhood of the union of the other two sides.

For a proper hyperbolic space X, we can define the *boundary* ∂X of X, and if the space X is hyperbolic and CAT(0), then these "boundaries" coincide.

Details and basic properties of hyperbolic spaces and their boundaries are found in [1], [2], [3] and [4].

It is known when a proper cocompact CAT(0) space is hyperbolic.

THEOREM 5.1 ([1, p.400, Theorem III.H.1.5]). *A proper cocompact CAT(0) space X is hyperbolic if and only if it does not contain a subspace which is isometric to the flat plane $\mathbb{R}^2$.*

6. On Non-Hyperbolic CAT(0) Spaces

In this section, we consider the homeomorphism $\bar{f}$ of the boundary ∂X induced by an isometry f of a proper cocompact *non-hyperbolic* CAT(0) space X.

We obtain the following theorem from Theorem 5.1 and the proof of [11, Theorem 4.3].

THEOREM 6.1. *Let X be a proper cocompact non-hyperbolic CAT(0) space with $|\partial X| > 2$ and let $f : X \to X$ be an isometry of X (need not to be semi-simple). Then the induced homeomorphism $\bar{f} : \partial X \to \partial X$ is non-expansive.*

PROOF. Since X is not hyperbolic, X contains some subspace Z which is isometric to the flat plane $\mathbb{R}^2$ by Theorem 5.1. To prove that the homeomorphism $\bar{f}$ of the boundary ∂X is non-expansive, we show that for any $\epsilon > 0$, there exist $\alpha, \beta \in \partial Z \subset \partial X$ with $\alpha \neq \beta$ such that

$$d_{\partial X}^{x_0}(\bar{f}^i(\alpha), \bar{f}^i(\beta)) < \epsilon$$

for any $i \in \mathbb{Z}$. Here the proof of [11, Theorem 4.3] implies that for any $\epsilon > 0$, we can take $\alpha, \beta \in \partial Z$ with $\alpha \neq \beta$ as the angle $\angle(\alpha, \beta)$ is small enough in Z and

$$d_{\partial X}^{x_0}(\bar{g}(\alpha), \bar{g}(\beta)) < \epsilon$$

for any isometry g of X and the induced homeomorphism $\bar{g}$ of ∂X. Therefore $\bar{f}$ is a non-expansive homeomorphism of the boundary ∂X. □

7. On Hyperbolic CAT(0) Spaces

In this section, we investigate the homeomorphism $\bar{f}$ of the boundary ∂X induced by an isometry f of a proper cocompact *hyperbolic* CAT(0) space X.

For a parabolic isometry of a hyperbolic space, the following remark is known.

REMARK. Let f be a parabolic isometry of a proper hyperbolic space X. Then f induces a homeomorphism $\bar{f}$ of the boundary ∂X, and there exists a unique fixed-point α_0 of $\bar{f}$ on ∂X. Here, in this paper, we define

$f^{\infty} := \alpha_0$ and $f^{-\infty} := \alpha_0$. We note that for every point $x \in X$, the sequence $\{f^i(x)\}_i$ converges to $f^{\infty} = \alpha_0$ as $i \to \infty$ in $X \cup \partial X$, and the sequence $\{f^i(x)\}_i$ converges to $f^{-\infty} = \alpha_0$ as $i \to -\infty$ in $X \cup \partial X$.

For a hyperbolic or parabolic isometry f of a proper hyperbolic space X, we define $\mathrm{Fix}(\bar{f})$ as the fixed-point set of the induced homeomorphism $\bar{f}$ of the boundary ∂X.

We obtain the following lemma from [3, Theorems 8.16 and 8.17] and [4, 8.1.F and 8.1.G].

LEMMA 7.1. *Let X be a proper hyperbolic CAT(0) space and let $f : X \to X$ be a hyperbolic isometry or a parabolic isometry.*

(1) *For any $\alpha \in \partial X \setminus \mathrm{Fix}(\bar{f})$, the sequence $\{\bar{f}^i(\alpha)\}_i$ converges to f^{∞} as $i \to \infty$ and converges to $f^{-\infty}$ as $i \to -\infty$ in ∂X.*

(2) *For any compact subset K of $\partial X \setminus \mathrm{Fix}(\bar{f})$ and any neighborhood U^+ (resp. U^-) of f^{∞} (resp. $f^{-\infty}$), there exists a number $n \in \mathbb{N}$ such that $\bar{f}^n(K) \subset U^+$ (resp. $\bar{f}^{-n}(K) \subset U^-$).*

Using Lemma 7.1, we show the following theorem.

THEOREM 7.2. *Let X be a proper cocompact hyperbolic CAT(0) space with $|\partial X| > 2$ and let $f : X \to X$ be an isometry of X. Then the induced homeomorphism $\bar{f} : \partial X \to \partial X$ is non-sensitive.*

PROOF. The isometry f is either elliptic, hyperbolic or parabolic. If f is an elliptic isometry of X, then the induced homeomorphism $\bar{f}$ of ∂X is non-sensitive by Theorem 4.1. We suppose that f is a hyperbolic isometry or a parabolic isometry of X.

Let $\epsilon > 0$ and let $\alpha \in \partial X \setminus \mathrm{Fix}(\bar{f})$. Then we can take a sufficiently small closed neighborhood U_0 of α in ∂X such that

$$U_0 \cap \mathrm{Fix}(\bar{f}) = \emptyset \text{ and } \operatorname{diam} U_0 < \epsilon.$$

Here, by Lemma 7.1 (2), we obtain that

$$\operatorname{diam} \bar{f}^i(U_0) \to 0 \text{ as } i \to \infty \text{ and}$$
$$\operatorname{diam} \bar{f}^i(U_0) \to 0 \text{ as } i \to -\infty.$$

Hence the set

$$A_0 = \{i \in \mathbb{Z} \mid \operatorname{diam} \bar{f}^i(U_0) \geq \epsilon\}$$

is finite.

If A_0 is empty, then $\operatorname{diam} \bar{f}^i(U_0) < \epsilon$ for any $i \in \mathbb{Z}$, i.e., $\bar{f}$ is non-sensitive.

We suppose that A_0 is non-empty. Let $i_0 \in A_0$. Then $\operatorname{diam} \bar{f}^{i_0}(U_0) \geq \epsilon$. Here we note that $\bar{f}^{i_0}(U_0)$ is a neighborhood of $\bar{f}^{i_0}(\alpha)$. Then we can take a small closed neighborhood V_1 of $\bar{f}^{i_0}(\alpha)$ such that $V_1 \subset \bar{f}^{i_0}(U_0)$ and $\operatorname{diam} V_1 < \epsilon$. Let $U_1 := \bar{f}^{-i_0}(V_1)$. Then U_1 is a closed neighborhood of α, $U_1 \subsetneqq U_0$ and $\operatorname{diam} U_1 \leq \operatorname{diam} U_0 < \epsilon$. Here we consider the set

$$A_1 = \{i \in \mathbb{Z} \mid \operatorname{diam} \bar{f}^i(U_1) \geq \epsilon\}.$$

We note that $A_1 \subsetneqq A_0$ because $U_1 \subsetneqq U_0$ and $i_0 \in A_0 \setminus A_1$.

If A_1 is empty, then $\operatorname{diam} \bar{f}^i(U_1) < \epsilon$ for any $i \in \mathbb{Z}$, i.e., $\bar{f}$ is non-sensitive.

If A_1 is non-empty, then we take $i_1 \in A_1$ and by the same argument as above, we obtain a small closed neighborhood V_2 of $\bar{f}^{i_1}(\alpha)$ and $U_2 = \bar{f}^{-i_1}(V_1)$ as U_2 is a closed neighborhood of α, $U_2 \subsetneqq U_1$ and $\operatorname{diam} U_2 \leq \operatorname{diam} U_1 \leq \operatorname{diam} U_0 < \epsilon$. Also we consider the set

$$A_2 = \{i \in \mathbb{Z} \mid \operatorname{diam} \bar{f}^i(U_2) \geq \epsilon\}.$$

Here $A_2 \subsetneqq A_1 \subsetneqq A_0$.

By iterating this argument, we obtain a sequence

$$A_k \subsetneqq \cdots \subsetneqq A_2 \subsetneqq A_1 \subsetneqq A_0.$$

Here there exists a number k such that A_k is empty since A_0 is a finite set. Then $\operatorname{diam} \bar{f}^i(U_k) < \epsilon$ for any $i \in \mathbb{Z}$.

Therefore $\bar{f}$ is a non-sensitive homeomorphism of the boundary ∂X. $\square$

8. On Sensitiveness of the Induced Homeomorphisms with Respect to Neighborhoods of a Point of the Boundary

In this section, we investigate sensitiveness of the homeomorphisms of the boundary induced by an isometry of a proper cocompact CAT(0) space *with respect to neighborhoods of a point* of the boundary.

In this paper, a homeomorphism $g : Y \to Y$ is said to be *sensitive with respect to neighborhoods of a point* y of Y if there exists a number $\epsilon > 0$ such that for any neighborhood U of y in Y, the diameter $\operatorname{diam} g^i(U) \geq \epsilon$ for some $i \in \mathbb{Z}$. Also a homeomorphism $g : Y \to Y$ is said to be *non-sensitive with respect to neighborhoods of a point* y of Y if for any $\epsilon > 0$ there exist a neighborhood U of y in Y such that $\operatorname{diam} g^i(U) < \epsilon$ for any $i \in \mathbb{Z}$.

We obtain the following theorem from the arguments in Sections 4–7.

THEOREM 8.1. *Let X be a proper cocompact CAT(0) space with $|\partial X| > 2$. Suppose that $f : X \to X$ is an isometry and $\bar{f} : \partial X \to \partial X$ is the homeomorphism induced by f.*

(1) *If f is an elliptic isometry, then $\bar{f}$ is non-sensitive with respect to neighborhoods of any point of the boundary ∂X.*

(2) *If the CAT(0) space X is hyperbolic and f is a hyperbolic isometry or a parabolic isometry, then $\bar{f}$ is non-sensitive with respect to neighborhoods of any point of $\partial X \setminus \operatorname{Fix}(\bar{f})$.*

(3) *If the CAT(0) space X is hyperbolic and f is a hyperbolic isometry or a parabolic isometry, then $\bar{f}$ is sensitive with respect to neighborhoods of the points f^{∞} and $f^{-\infty}$.*

PROOF. Theorem 4.1 implies that (1) holds and the proof of Theorem 7.2 implies that (2) holds.

We show that (3) holds. We suppose that X is hyperbolic and f is a hyperbolic isometry. For any neighborhood U of $f^{-\infty}$ in the boundary ∂X, there exists $\alpha \in U$ with $\alpha \neq f^{-\infty}$ since ∂X has no isolated points. Then the sequence $\{\bar{f}^i(\alpha)\}_i$ converges to f^{∞} as $i \to \infty$ by Lemma 7.1 (1). Also $\bar{f}^i(f^{-\infty}) = f^{-\infty}$ for any $i \in \mathbb{Z}$. Hence

$$\operatorname{diam} \bar{f}^i(U) \geq d_{\partial X}^{x_0}(\bar{f}^i(f^{-\infty}), \bar{f}^i(\alpha)) = d_{\partial X}^{x_0}(f^{-\infty}, \bar{f}^i(\alpha)),$$

where $d_{\partial X}^{x_0}(f^{-\infty}, \bar{f}^i(\alpha))$ converges to $d_{\partial X}^{x_0}(f^{-\infty}, f^{\infty})$ as $i \to \infty$. Therefore $\bar{f}$ is sensitive with respect to neighborhoods of the point $f^{-\infty}$. We also obtain that $\bar{f}$ is sensitive with respect to neighborhoods of the point f^{∞} by the same argument.

We suppose that X is hyperbolic and f is a parabolic isometry. Let $\alpha \in \partial X \setminus \{f^{\infty}\}$ and let $\epsilon_0 = d_{\partial X}^{x_0}(\alpha, f^{\infty})$. Then for any neighborhood U of

$f^\infty = f^{-\infty}$ in the boundary ∂X, there exists a number $i_0 \in \mathbb{N}$ such that $\bar{f}^{i_0}(\alpha) \in U$ by Lemma 7.1 (1). Hence $\alpha \in \bar{f}^{-i_0}(U)$ and

$$\operatorname{diam} \bar{f}^{-i_0}(U) \geq d_{\partial X}^{x_0}(\alpha, f^\infty) = \epsilon_0.$$

Therefore $\bar{f}$ is sensitive with respect to neighborhoods of the point $f^\infty = f^{-\infty}$. □

9. Remarks

We introduce an example of an isometry of a proper cocompact CAT(0) space which is not hyperbolic.

Example 9.1. Let $G = (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$ and let X be a proper CAT(0) space on which G acts properly and cocompactly by isometries. Here we denote $G = \langle \{a, b, c\} \mid ab = ba \rangle$, i.e., $G = (\langle a \rangle \times \langle b \rangle) * \langle c \rangle$. Also, for example, we can suppose that X is the CAT(0) complex whose 1-skeleton is the Cayley graph of G with respect to the generating set $\{a, b, c\}$. Then we consider the hyperbolic isometry $f := a$ of X.

We first note that if Z is the flat plane in X on which $\langle a \rangle \times \langle b \rangle$ acts, then $\bar{f}(\alpha) = \alpha$ for any $\alpha \in \partial Z$. In particular, $\bar{f}(b^\infty) = b^\infty$.

Next, we note that the sequence $\{\bar{f}^i(c^\infty)\}_i$ converges to a^∞ as $i \to \infty$ and converges to $a^{-\infty}$ as $i \to -\infty$. Also, in fact, for any $\alpha \in \partial X \setminus \partial Z$, the sequence $\{\bar{f}^i(\alpha)\}_i$ converges to a^∞ as $i \to \infty$ and converges to $a^{-\infty}$ as $i \to -\infty$.

For any neighborhood U of b^∞ in ∂X, there exists $\alpha \in U \setminus \partial Z$ and the sequence $\{\bar{f}^i(\alpha)\}_i$ converges to a^∞ as $i \to \infty$. Here $\bar{f}^i(b^\infty) = b^\infty$ for any $i \in \mathbb{Z}$. Hence we obtain that $\bar{f}$ is sensitive with respect to neighborhoods of the point b^∞.

On the other hand, for any small neighborhood U of c^∞ in ∂X with $U \cap \partial Z = \emptyset$,

$$\operatorname{diam} \bar{f}^i(U) \to 0 \text{ as } i \to \infty \text{ and}$$
$$\operatorname{diam} \bar{f}^i(U) \to 0 \text{ as } i \to -\infty.$$

Hence we obtain that $\bar{f}$ is non-sensitive with respect to neighborhoods of the point c^∞.

Thus there exist points $\beta, \gamma \in \partial X$ such that $\bar{f}$ is sensitive with respect to neighborhoods of the point β and $\bar{f}$ is non-sensitive with respect to neighborhoods of the point γ.

On a hyperbolic isometry of a proper cocompact CAT(0) space which is not hyperbolic, Theorem 6.1 implies that the induced homeomorphism of the boundary is non-expansive. On the other hand, we do not know whether the induced homeomorphism of the boundary is non-sensitive.

The author has the following question.

Question 9.2. Let X be a proper cocompact non-hyperbolic CAT(0) space with $|\partial X| > 2$ and let $f : X \to X$ be a hyperbolic isometry or a parabolic isometry of X. Then is it the case that the induced homeomorphism $\bar{f} : \partial X \to \partial X$ is non-sensitive?

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