J. Math. Sci. Univ. Tokyo
18 (2011), 441–463.

Virial Identity and Dispersive Estimates for the n-Dimensional Dirac Equation

By Federico CACCIAFESTA

Abstract. We extend to general dimension $n \geq 1$ the virial identity proved in [3] for the 3D magnetic Dirac equation. As an application we deduce Strichartz estimates for an n-dimensional Dirac equation perturbed with a magnetic potential.

1. Introduction

The *Dirac equation on* $\mathbb{R}^{1+n}$ is a constant coefficient, hyperbolic system of the form

$$iu_t + \mathcal{D}u + m\beta u = 0 \tag{1.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}^n_x \to \mathbb{C}^M$, the *Dirac operator* is defined by

$$\mathcal{D} = i^{-1} \sum_{k=1}^{n} \alpha_k \frac{\partial}{\partial x_k} = i^{-1}(\alpha \cdot \nabla),$$

and the *Dirac matrices* $\alpha_0 \equiv \beta$, $\alpha_1, \ldots, \alpha_n$ are a set of $M \times M$ hermitian matrices satisfying the anti-commutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk}\mathbb{I}_M, \qquad 0 \leq j, k \leq n. \tag{1.2}$$

The quantity $m \geq 0$ is called the *mass* and in the classical 3D model is linked with the mass of a spin 1/2 particle.

REMARK 1.1. For each dimension $n \geq 1$ there exist different choices of M and of matrices α_j satisfying all of the above conditions; the original Dirac equation corresponds to $n = 3$, $M = 4$, in which case the 4 matrices

2010 *Mathematics Subject Classification.* 35Q41.
Key words: Singular integrals, weighted spaces, Schrödinger operator, Schrödinger equation, Strichartz estimates, smoothing estimates.

can be chosen from a well known set of 16 anticommuting matrices (see [22]). A possible way to construct a familiy of matrices satisfying such properties is the following.
For $n = 1$ let

$$\alpha_0^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \alpha_1^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For $n \geq 2$ let

$$\alpha_j^{(n)} = \begin{pmatrix} 0 & \alpha_j^{(n-1)} \\ \alpha_j^{(n-1)} & 0 \end{pmatrix}, \quad j = 0, ..., n-1, \qquad \alpha_n^{(n)} = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}.$$

Notice that in this case $M = 2^n$ (for a more detailed analysis of general Dirac matrices, see [19], [16], [20])

An easy consequence of the anticommutation relations is the identity

$$(1.3) \qquad (i\partial_t - \mathcal{D} - m\beta)(i\partial_t + \mathcal{D} + m\beta) = (\Delta - m^2 - \partial_{tt}^2)\mathbb{I}_M.$$

which reduces the study of (1.1) to a corresponding study of the Klein-Gordon equation, or the wave equation in the massless case $m = 0$. The analysis of the important Maxwell-Dirac and Dirac-Klein-Gordon systems of quantum electrodynamics in [1]- [2] was based on this method; notice however that in the reduction step some essential details of the structure may be lost, as recently pointed out in [9], [8], [10].

From (1.3) one can deduce in a straightforward way the dispersive properties of the Dirac flow from the corresponding properties of the wave-Klein-Gordon flow. Based on this approach, an extensive theory of local and global well posedness for nonlinear perturbations of (1.1) was developed in [11], [12], [19], [18]; see also [5], [6] for a study of the dispersive properties of the Dirac equation perturbed by a magnetic field.

The goal of this paper is to study the dispersive properties of the system (1.1) perturbed by a magnetic field, thus extending to the n-dimensional setting the smoothing and Strichartz estimates proved in [3] for the 3D magnetic Dirac equation. Denoting with

$$A(x) = (A^1(x), ..., A^n(x)) : \mathbb{R}^n \to \mathbb{R}^n$$

a static magnetic potential, the standard way to express its interaction with a particle is by replacing the derivatives ∂_k with their covariant counterpart $\partial_k - iA^k$, thus obtaining the *magnetic Dirac operator*

$$\mathcal{D}_A = i^{-1}\sum_{k=1}^{n}\alpha_k(\partial_k - iA^k) = i^{-1}\alpha\cdot\nabla_A, \qquad \nabla_A = \nabla - iA(x). \tag{1.4}$$

Here and in the following we denote with a dot the scalar product of two vectors of operators:

$$(P_1,\dots,P_m)\cdot(Q_1,\dots,Q_m) = \sum_{j=1}^{m} P_jQ_j.$$

We shall also use the unified notation

$$\mathcal{H} = i^{-1}\alpha\cdot\nabla_A + m\beta = \mathcal{D}_A + m\beta \tag{1.5}$$

to include both the massive and the massless case.

Thus we plan to investigate the dispersive properties of the flow $e^{it\mathcal{H}}f$ defined as the solution to the Cauchy problem

$$iu_t(t,x) + \mathcal{H}u(t,x) = 0, \qquad u(0,x) = f(x). \tag{1.6}$$

It is natural to require that the operator $\mathcal{H}$ be selfadjoint. Several sufficient conditions are known for selfadjointness (see [22]). For greatest generality, we prefer to make an abstract selfadjointness assumtpion; we also include a density condition which allows to approximate rough solutions with smoother ones, locally uniformly in time, and is easily verified in concrete cases. The condition is the following:

Self-Adjointness Assumption (A). *The operator $\mathcal{H}$ is essentially selfadjoint on $C_c^\infty(\mathbb{R}^n)$, and in addition for initial data $f \in C_c^\infty(\mathbb{R}^n)$ the flow $e^{it\mathcal{H}}f$ belongs at least to $C(\mathbb{R}, H^{3/2})$.*

Remark 1.2. It is easy to show, using Fourier transform, the conservation of the mass under the magnetic Dirac flow: being $e^{it\mathcal{H}}$ unitary we have indeed

$$\|e^{it\mathcal{H}}f\|_{L^2} = \|f\|_{L^2}.$$

The main tool used here is the method of Morawetz multipliers, in the version of [7], [3]. This method allows to partially overcome the smallness assumption on the potential which was necessary for the perturbative approach of [6]. An additional advantage is that the assumptions on the potential are expressed in terms of the *magnetic field* B rather than the vector potential A; indeed, B is a physically measurable quantity while A should be thought of as a mathematical abstraction. We recall that in dimension 3 the magnetic field B is defined as

$$B = \mathrm{curlA}.$$

In arbitrary dimension n, a natural generalization of the previous definition is the following

Definition 1.1. Given a magnetic potential $A : \mathbb{R}^n \to \mathbb{R}^n$, the *magnetic field* $B : \mathbb{R}^n \to \mathcal{M}_{n\times n}(\mathbb{R})$ is the matrix valued function

$$B = DA - DA^t, \qquad B^{jk} = \frac{\partial A^j}{\partial x^k} - \frac{\partial A^k}{\partial x^j}$$

and its *tangential component* $B_\tau = \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$B_\tau = \frac{x}{|x|}B.$$

Notice indeed that $B_\tau(x)$ is orthogonal to x for all x.

Remark 1.3. The previous definition reduces to the standard one in dimension $n = 3$: indeed the matrix B satisfies for all $v \in \mathbb{R}^3$

$$Bv = \mathrm{curlA} \wedge \mathrm{v}$$

and in this sense B can be identified with curl$A$. Notice also that

$$B_\tau = \frac{x}{|x|} \wedge \mathrm{curl}A.$$

Our first result is the following (formal) virial identity for the n-dimensional magnetic Dirac equation (1.6):

Theorem 1.2 (Virial identity). *Assume that the operator $\mathcal{H}$ defined in* (1.5) *satisfies (A), and let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a real valued function. Then any solution $u(t,x)$ of* (1.6) *satisfies the formal virial identity*

$$
\begin{aligned}
&2\int_{\mathbb{R}^n} \nabla_A u \cdot D^2\phi \cdot \overline{\nabla_A u} - \frac{1}{2}\int_{\mathbb{R}^n} |u|^2 \Delta^2\phi + \\
&+ 2\int_{\mathbb{R}^n} \Im\left(u\nabla\phi \cdot B \cdot \overline{\nabla_A u}\right) + \\
&+ \int_{\mathbb{R}^n} \overline{u} \cdot \sum_{j<k} \alpha_j\alpha_k(\nabla\phi \cdot \nabla B^{jk})u = \\
&= -\frac{d}{dt}\int_{\mathbb{R}^n} \Re\left(u_t(2\nabla\phi \cdot \overline{\nabla_A u} + \overline{u}\Delta\phi)\right).
\end{aligned} \tag{1.7}
$$

Remark 1.4. If $\phi = \phi(|x|)$ is a radial function, as we shall always assume in the following, the virial identity can be considerably simplified. In particular, notice that

$$\sum_{j<k} \alpha_j\alpha_k(\nabla\phi \cdot \nabla B^{jk}) = \phi'(|x|)\sum_{j<k} \alpha_j\alpha_k\partial_r B^{jk}.$$

As a direct consequence of the previous virial identity, we can prove a smoothing estimate for the n-dimensional magnetic Dirac equation (1.6).

In the following we shall denote respectively with $\nabla_A^r u$ and $\nabla_A^\tau u$ the radial and tangential components of the covariant gradient, namely

$$\nabla_A^r u := \frac{x}{|x|} \cdot \nabla_A u, \qquad \nabla_A^\tau u := \nabla_A u - \frac{x}{|x|} \cdot \nabla_A^r u$$

so that

$$|\nabla_A^r u|^2 + |\nabla_A^\tau u|^2 = |\nabla_A u|^2.$$

We shall use the notation

$$[B]_1 = \sum_{j,k=1}^n |B^{jk}|$$

to denote the ℓ^1 norm of a matrix (i.e. the sum of the absolute values of its entries), and we shall measure the size of matrix valued functions using norms like

$$\|B\|_{L^\infty} = \|[B(x)]_1\|_{L^\infty_x}$$

Then we have:

Theorem 1.3 (Smoothing estimates). *Let $n \geq 4$. Let the operator $\mathcal{H}$ defined in (1.5) satisfies assumption (A). Let $B = DA - DA^t = B_1 + B_2$ with $B_2 \in L^\infty$, and assume that*

$$|B_\tau(x)| \leq \frac{C_1}{|x|^2}, \quad \frac{1}{2}[\partial_r B(x)]_1 \leq \frac{C_2}{|x|^3} \tag{1.8}$$

for all $x \in \mathbb{R}^n$ and for some constants C_1, C_2 such that

$$\left(\frac{9}{4}\right) C_1^2 + 3C_2 \leq (n-1)(n-3) \tag{1.9}$$

Assume moreover that

$$C_0 = \||x|^2 B_1\|_{L^\infty(\mathbb{R}^n)} < \frac{(n-2)^2}{4}.$$

Finally, in the massless case restrict the choice to $B_1 = B$, $B_2 = 0$ in the above assumptions.

Then for all $f \in L^2$ the following smoothing estimate holds

$$\sup_{R>0} \frac{1}{R} \int_{-\infty}^{+\infty} \int_{|x|\leq R} |e^{it\mathcal{H}} f|^2 dx dt \lesssim \|f\|_{L^2}^2. \tag{1.10}$$

Remark 1.5. As in [13] and [3], a sharper estimate can be proved if inequality (1.9) is strict, but we won't deal with the details of this aspect here.

The limitation to $n \geq 3$ space dimensions is intrinsic in the multiplier method; low dimensions $n = 1, 2$ require a different approach (see e.g. [4] for a general result in dimension 1). In the present paper we shall only deal with the case $n \geq 4$, the 3-dimensional case being exaustively discussed in [3]. Notice that, as it often occurs, the three dimensional case yields different hypothesis on the potential, being slightly different the multiplicator that one needs to consider.

A natural application of the smoothing estimate (1.10) is to derive Strichartz estimates for the perturbed flow $e^{it\mathcal{H}} f$, both in the massless and massive case. Our concluding result is the following:

THEOREM 1.4 (Strichartz estimates). *Let $n \geq 4$. Assume $\mathcal{H}$, A, B are as in Theorem 1.3, and in addition assume that*

$$\sum_{j\in\mathbb{Z}} 2^j \sup_{|x|\cong 2^j} |A| < \infty. \tag{1.11}$$

Then the perturbed Dirac flow satisfies the Strichartz estimates

$$\| |D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} e^{it\mathcal{H}} f \|_{L^p L^q} \lesssim \|f\|_{L^2} \tag{1.12}$$

where, in the massless case $m = 0$, the couple (p, q) is any wave admissibile, non-endpoint couple i.e. such that

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \qquad 2 < p \leq \infty \qquad \frac{2(n-1)}{n-3} > q \geq 2, \tag{1.13}$$

while in the massive case the same bound holds for all Schrödinger adimmissible couple, non-endpoint (p, q), i.e. such that

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \qquad 2 < p \leq \infty \qquad \frac{2n}{n-2} > q \geq 2. \tag{1.14}$$

The paper is organized as follows: in Section 2 we shall prove Theorem 1.2, deriving it from a classical virial identity for the wave equation (see Theorem 2.1) plus the algebric structure of the Dirac operator. In Section 3 we shall use the multiplicator technique to prove the smoothing estimate (1.10) from Theorem 1.2. Finally in Section 4 we shall derive the Strichartz estimates of Theorem 1.4 by a perturbative argument based on the smoothing estimates. Section 5 is devoted to the proof of a magnetic Hardy inequality for the Dirac operator, needed at several steps in the proof of the previous theorems.

Acknowledgments. The author wishes to thank the referee for many useful suggestions.

2. Proof of the Virial Identity

Let u be a solution to equation (1.1). Using identity

$$0 = (i\partial_t - \mathcal{H})(i\partial_t + \mathcal{H})u = (-\partial_{tt} - \mathcal{H}^2)u,$$

we see that u solves the Cauchy problem for a magnetic wave equation:

$$\begin{cases} u_{tt} + \mathcal{H}^2 u = 0 \\ u(0) = f \\ u_t(0) = i\mathcal{H}f. \end{cases} \tag{2.1}$$

In [3] the following general result was proved for a solution $u(t,x)$ of wave-type equations:

Theorem 2.1 ([3]). *Let L be a selfadjoint operator on $L^2(\mathbb{R}^n)$, and let $u(t,x)$ be a solution of the equation*

$$u_{tt}(t,x) + Lu(t,x) = 0.$$

Let $\phi : \mathbb{R}^n \to \mathbb{R}$ and define the quantity

$$\Theta(t) = (\phi u_t, u_t) + \mathcal{R}((2\phi L - L\phi)u, u). \tag{2.2}$$

Then $u(t,x)$ satisfies the formal virial identities

$$\dot{\Theta}(t) = \mathcal{R}([L,\phi]u, u_t) \tag{2.3}$$

$$\ddot{\Theta}(t) = -\frac{1}{2}([L,[L,\phi]]u, u). \tag{2.4}$$

In order to apply this proposition to our case we thus need to compute explicitly the commutators in (2.3), (2.4) with the choice $L = \mathcal{H}^2$. We begin by expanding the square

$$\mathcal{H}^2 = (\mathcal{H}_0 - \alpha \cdot A)^2 = \mathcal{H}_0^2 - \mathcal{H}_0(\alpha \cdot A) - (\alpha \cdot A)\mathcal{H}_0 + (\alpha \cdot A)(\alpha \cdot A),$$

and we recall that the unperturbed part of the operator

$$\mathcal{H}_0 = \mathcal{D} + m\beta = i^{-1}\alpha \cdot \nabla + m\beta$$

satisfies

$$\mathcal{H}_0^2 = (m^2 - \Delta)\mathbb{I}_M.$$

Since β anticommutes with each α_j we get

$$\mathcal{H}^2 = \mathcal{H}_0^2 - i^{-1}(\alpha \cdot \nabla)(\alpha \cdot A) - i^{-1}(\alpha \cdot A)(\alpha \cdot \nabla) + (\alpha \cdot A)(\alpha \cdot A). \tag{2.5}$$

We need a notation to distinguish the composition of the operators (multiplication by) A_k and ∂_j, which we shall denote with $\partial_j \circ A^k$, i.e.,

$$\partial_j \circ A^k u = \partial_j (A^k u)$$

and the simple derivative $\partial_j A^k$. After a few steps we obtain (we omit for simplicity the factor $\mathbb{I}_M$ in diagonal operators)

$$\mathcal{H}^2 = \mathcal{H}_0^2 + i(\nabla \cdot A) + i(A \cdot \nabla) + |A|^2 + i \sum_{j \neq k}^{n} \alpha_j \alpha_k (\partial_j \circ A^k + A^j \partial_k).$$

or equivalently

$$\mathcal{H}^2 = (m^2 - \Delta_A) + i \sum_{j \neq k}^{n} \alpha_j \alpha_k (\partial_j \circ A^k + A^j \partial_k), \tag{2.6}$$

where

$$\Delta_A = (\nabla - iA)^2 = \nabla_A^2.$$

Now we observe that

$$\begin{aligned}
&\sum_{j \neq k} \alpha_j \alpha_k (\partial_j \circ A^k + A^j \partial_k) \\
&\qquad = \sum_{j<k} \alpha_j \alpha_k [(\partial_j \circ A^k + A^j \partial_k) - (\partial_k \circ A^j + A^k \partial_j)] = \\
&\qquad = \sum_{j<k} \alpha_j \alpha_k (\partial_j A^k - \partial_k A^j) \\
&\qquad = \sum_{j<k} \alpha_j \alpha_k B^{jk} = \\
&\qquad = \frac{1}{4} \sum_{j,k=1}^{n} (\alpha_j \alpha_k - \alpha_k \alpha_j) B^{jk}
\end{aligned}$$

since B is skewsymmetric. If we introduce the matrix $S = [S_{jk}]$ whose entries are the matrices

$$S_{jk} = \frac{1}{4}(\alpha_j \alpha_k - \alpha_k \alpha_j) \equiv \frac{1}{2} \alpha_j \alpha_k$$

and we use the notation

$$[a_{jk}]\cdot[b_{jk}]=\sum_{j,k=1}^{n}a_{jk}b_{jk}$$

for the scalar product of matrices, the above identity can be compactly written in the form

$$\sum_{j\neq k}\alpha_j\alpha_k(\partial_j\circ A^k+A^j\partial_k)=S\cdot B.$$

In conclusion we have proved that

$$\mathcal{H}^2=(m^2-\Delta_A)\mathbb{I}_M+iS\cdot B \tag{2.7}$$

and hence for the massless case

$$\mathcal{D}_A^2=-\Delta_A\mathbb{I}_M+iS\cdot B. \tag{2.8}$$

Thus the commutator with ϕ reduces to

$$[\mathcal{H}^2,\phi]=[m^2,\phi]-[\Delta_A,\phi]+i[S\cdot B,\phi]=-[\Delta_A,\phi].$$

Using the Leibnitz rule

$$\nabla_A(fg)=g\nabla_A f+f\nabla g,$$

we arrive at the explicit formula

$$[\mathcal{H}^2,\phi]=-[\Delta_A,\phi]=-2\nabla\phi\cdot\nabla_A-(\Delta\phi). \tag{2.9}$$

Recalling (2.2) and (2.3) we thus obtain

$$\dot{\Theta}(t)=-\Re\int_{\mathbb{R}^n}u_t(2\nabla\phi\cdot\overline{\nabla_A u}+\overline{u}\Delta\phi). \tag{2.10}$$

We now turn to the second commutator. By formulas (2.7) and (2.9) we have

$$[\mathcal{H}^2,[\mathcal{H}^2,\phi]]=[\Delta_A,[\Delta_A,\phi]]-i[S\cdot B,[\Delta_A,\phi]]. \tag{2.11}$$

The first commutator is well known and was computed e.g. in [7]; taking formula (2.19) there (with $V \equiv 0$) we obtain

$$(u, [\Delta_A, [\Delta_A, \phi]]u) = 4\int_{\mathbb{R}^n} \nabla_A u D^2\phi\overline{\nabla_A u} - \int_{\mathbb{R}^n} |u|^2\Delta^2\phi + + 4\Im\int_{\mathbb{R}^n} u\nabla\phi B_\tau \cdot \overline{\nabla_A u}. \tag{2.12}$$

By (2.9) the last term in (2.11) becomes

$$\begin{aligned}
[S\cdot B, [\Delta_A, \phi]] &= 2[S\cdot B, \nabla\phi\cdot\nabla_A] = \\
&= 2(S\cdot B\nabla\phi\cdot\nabla_A - \nabla\phi\cdot\nabla_A S\cdot B) = \\
&= \sum_{j<k}\alpha_j\alpha_k B^{jk}\nabla\phi\cdot\nabla_A - \nabla\phi\cdot\nabla_A\sum_{j<k}\alpha_j\alpha_k B^{jk} = \\
&= \sum_{j<k}\alpha_j\alpha_k[B^{jk}, \nabla\phi\cdot\nabla_A] = \\
&= -\sum_{j<k}\alpha_j\alpha_k(\nabla\phi\cdot\nabla B^{jk}).
\end{aligned} \tag{2.13}$$

Identity (1.7) then follows from (2.4), (2.10), (2.11), (2.12) and (2.13).

3. Smoothing Estimates

We shall use the following radial multiplier (for a detailed description see [13], [3]):

$$\tilde{\phi}_R(x) = \phi(x) + \varphi_R(x) \tag{3.1}$$

where

$$\phi(x) = |x|$$

for which we have

$$\phi'(r) = 1, \qquad \phi''(r) = 0, \qquad \Delta^2\phi(r) = -\frac{(n-1)(n-3)}{r^3}$$

with the notation $r = |x|$, and φ_R is the rescaled $\varphi_R(r) = R\varphi_0(\frac{r}{R})$, of the multiplier

$$\varphi_0(r) = \int_0^r \varphi'(s)ds \tag{3.2}$$

where

$$\varphi_0'(r) = \begin{cases} \frac{n-1}{2n} r, & r \le 1 \\ \frac{1}{2} - \frac{1}{2nr^{n-1}}, & r > 1 \end{cases} \tag{3.3}$$

and so

$$\varphi_0''(r) = \begin{cases} \frac{n-1}{2n}, & r \le 1 \\ \frac{n-1}{2nr^n}, & r > 1. \end{cases}$$

Thus we have

$$\varphi_R'(r) = \begin{cases} \frac{(n-1)r}{2nR}, & r \le R \\ \frac{1}{2} - \frac{R^{n-1}}{2nr^{n-1}}, & r > R \end{cases} \tag{3.4}$$

$$\varphi_R''(r) = \begin{cases} \frac{1}{R}\frac{n-1}{2n}, & r \le R \\ \frac{1}{R}\frac{R^n(n-1)}{2nr^n}, & r > R \end{cases}. \tag{3.5}$$

$$\Delta^2 \varphi_R = -\frac{n-1}{2R^2}\delta_{|x|=R} - \frac{(n-1)(n-3)}{2r^3}\chi_{[R,+\infty)}. \tag{3.6}$$

Notice that φ_R', φ_R'', $\Delta\varphi_R \ge 0$ and moreover $\sup_{r\ge 0} \varphi'(r) \le \dfrac{1}{2}$.

Thus it's easy to show the bounds for the derivatives of the perturbed multiplier

$$\sup_{r\ge 0} \tilde{\phi}_R' \le \frac{3}{2}, \qquad \Delta\tilde{\phi}_R \le \frac{n}{r}. \tag{3.7}$$

We separate the estimates of the LHS and the RHS of (1.7)

Estimate of the RHS of (1.7)

Consider the expression

$$\int_{\mathbb{R}^n} u_t(2\nabla\phi \cdot \overline{\nabla_A u} + u\Delta\phi) = (u_t, 2\nabla\phi \cdot \nabla_A u + \overline{u}\Delta\phi)_{L^2}$$

appearing at the right hand side of (1.7). Since u solves the equation we can replace u_t with

$$u_t = -i\mathcal{H}u = -im\beta u - i\mathcal{D}_A u.$$

By the selfadjointess of β it is easy to check that

$$\Re[-im(\beta u, 2\nabla\phi \cdot \nabla_A u) - im(\beta u, \Delta\phi u)] = 0$$

so that

$$\Re[(u_t, 2\nabla\phi \cdot \nabla_A u + u\Delta\phi) = 2\mathcal{I}(\mathcal{D}_A u, \nabla\phi \cdot \nabla_A u)] + \mathcal{I}(\mathcal{D}_A u, \Delta\phi u)$$

and by Young inequality we obtain

$$\left|\Re\left(\int_{\mathbb{R}^n} u_t(2\nabla\phi \cdot \overline{\nabla_A u} + u\Delta\phi)\right)\right| \leq \frac{3}{2}\|\mathcal{D}_A u\|_{L^2}^2 + \|\nabla\phi \cdot \nabla_A u\|_{L^2}^2 + + \frac{1}{2}\|u\Delta\phi\|_{L^2}^2. \tag{3.8}$$

Now we put in (3.8) the multiplicator $\tilde{\phi}$ defined in (3.1). From the boundedness of φ and the magnetic Hardy inequality (5.2) we have, with the choice $\varepsilon = (n-2)^2 - 4C_0$ which is positive in virtue of the assumption $C_0 < (n-2)^2/4$,

$$\|\nabla\tilde{\phi} \cdot \nabla_A u\|_{L^2}^2 \leq \frac{3}{2}\frac{1}{(n-2)^2 - 4C_0}\|\mathcal{D}_A u\|_{L^2}^2. \tag{3.9}$$

The third term in (3.8) can be estimated again using Hardy inequality with

$$\|u\Delta\tilde{\phi}\|_{L^2}^2 \leq \frac{4n}{(n-2)^2 - 4C_0}\|\mathcal{D}_A u\|_{L^2}^2. \tag{3.10}$$

Summing up, by (3.8), (3.9) and (3.10) we can conclude

$$\left|\Re\left(\int_{\mathbb{R}^n} u_t(2\nabla\phi \cdot \overline{\nabla_A u} + \overline{u}\Delta\phi)\right)\right| \leq c(n)\|\mathcal{D}_A u\|_{L^2}^2. \tag{3.11}$$

Estimate of the LHS of (1.7)

We shall make use of the following identity, that holds in every dimension:

$$\nabla_A u D^2\phi\overline{\nabla_A u} = \frac{\phi'(r)}{r}|\nabla_A^\tau u|^2 + \phi''(r)|\nabla_A^r u|^2. \tag{3.12}$$

For the seek of simplicity, we divide this part in two steps, first considering just the multiplier $\phi(r) = r$, for which the calculations turn out fairly straightforward, and then perturbating it to $\tilde{\phi}$.

Step 1.

With the choice $\phi(r) = r$, by (3.12) we can rewrite the LHS of (1.7) as follows:

$$2\int_{\mathbb{R}^n} \frac{|\nabla_A^\tau u|^2}{|x|}dx + \frac{(n-1)(n-3)}{2}\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3}dx + \\ +2\int_{\mathbb{R}^n} \Im(uB_\tau \cdot \overline{\nabla_A u})dx + \int_{\mathbb{R}^n} \overline{u} \cdot \sum_{j<k} \alpha_j\alpha_k\partial_r B^{jk}u. \tag{3.13}$$

The first thing to be done is to prove this quantity to be positive. For what concerns the perturbative term, assuming that

$$|B_\tau| \leq \frac{C_1}{|x|^2}$$

we have

$$\begin{aligned} -&\left|2\int_{\mathbb{R}^n} \Im(uB_\tau \cdot \overline{\nabla_A u})dx\right| \\ &\geq -2\left(\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3}dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} |x|^3|B_\tau|^2|\nabla_A^\tau u|^2dx\right)^{\frac{1}{2}} \\ &\geq -2C_1K_1K_2, \end{aligned} \tag{3.14}$$

where

$$K_1 = \left(\int_{\mathbb{R}^n} \frac{|u|^2}{|x|^3}dx\right)^{\frac{1}{2}}$$

$$K_2 = \left(\int_{\mathbb{R}^n} \frac{|\nabla_A^\tau u|^2}{|x|}dx\right)^{\frac{1}{2}}.$$

Analogously, assuming

$$\Big\|\sum_{j<k} \alpha_j\alpha_k\partial_r B^{jk}(x)\Big\|_{M\times M} \leq \frac{1}{2}[\partial_r B(x)]_1 \leq \frac{C_2}{|x|^3}$$

(recall that here $\|\cdot\|_{M\times M}$ denotes the operator norm of $M\times M$ matrices and $[\cdot]_1$ denotes the sum of absolute values of the entries of a matrix) we have

$$(3.15)\quad -\left|\int_{\mathbb{R}^n}\overline{u}\cdot\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}u\,dx\right| \geq -\int |u|^2\Big\|\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}\Big\|_{M\times M}dx \geq -C_2K_1^2$$

where K_1 is as before. Thus we have reached the following estimate

$$(3.16)\quad 2\int_{\mathbb{R}^n}\frac{|\nabla_A^\tau u|^2}{|x|}dx+\frac{(n-1)(n-3)}{2}\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^3}dx+ +2\int_{\mathbb{R}^n}\Im(uB_\tau\cdot\overline{\nabla_A u})dx+\int_{\mathbb{R}^n}\overline{u}\cdot\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}u \geq$$
$$\geq 2K_2^2-2C_1K_1K_2-C_2K_1^2+\frac{(n-1)(n-3)}{2}K_1^2=:C(C_1,C_2,K_1,K_2).$$

As usual, we want to optimize the condition on the constants C_1, C_2 under which the quantity C is positive for all K_1, K_2. Fixing $K_2=1$ and requiring that

$$\left(\frac{(n-1)(n-3)}{2}-C_2\right)K_1^2-2C_1K_1+2\geq 0$$

we can easily conclude that the resulting condition on the constants is given by

$$(3.17)\quad C_1^2+2C_2\leq (n-1)(n-3).$$

Thus, if condition (3.17) is satisfied, we have that the quantity in (3.13) is positive.

Step 2.

We now perturb the multiplier to complete the proof. We thus put the multiplier $\widetilde{\phi}_R$ as defined in (3.1) in the LHS of (1.7), and repeat exactly the same calculations as in Step 1. Notice that multiplier φ_R with properties

(3.4)-(3.6) yield the estimate, through (3.12),

$$2\int_{\mathbb{R}^n}\nabla_A u D^2\varphi_R\overline{\nabla_A u}-\frac{1}{2}\int_{\mathbb{R}^n}|u|^2\Delta^2\varphi_R\geq \tag{3.18}$$
$$\geq C(n)\left(\frac{1}{R}\int_{|x|\leq R}|\nabla_A u|^2dx+2\int\frac{|\nabla_A^\tau u|^2}{|x|}\right)+$$
$$+\frac{n-1}{4R^2}\int_{|x|=R}|u|^2d\sigma(x)+\frac{(n-1)(n-3)}{4}\int\frac{|u|^2}{|x|^3}$$

for some positive constant $C(n)$. Using now the complete multiplier $\tilde{\phi}_R$ we notice that estimates (3.14) and (3.15) still hold with the rescaled constants $\widetilde{C}_1=\frac{3}{2}C_1$, $\widetilde{C}_2=\frac{3}{2}C_2$, so that we can rewrite (3.16) as follows

$$\frac{1}{R}\int_{|x|\leq R}|\nabla_A u|^2dx+2\int_{\mathbb{R}^n}\frac{|\nabla_A^\tau u|^2}{|x|}dx+ \tag{3.19}$$
$$+\frac{(n-1)(n-3)}{2}\int_{\mathbb{R}^n}\frac{|u|^2}{|x|^3}dx+$$
$$+2\int_{\mathbb{R}^n}\Im(uB_\tau\cdot\overline{\nabla_A u})dx+\int_{\mathbb{R}^n}\overline{u}\cdot\sum_{j<k}\alpha_j\alpha_k\partial_r B^{jk}u\geq$$
$$\geq\frac{1}{R}\int_{|x|\leq R}|\nabla_A u|^2dx+C(\widetilde{C}_1,\widetilde{C}_2,K_1,K_2).$$

Conditions (1.8)-(1.9) on the potential ensure the positivity of $C(\widetilde{C}_1,\widetilde{C}_2,K_1,K_2)$

Thus putting all together , taking the supremum over $R>0$, integrating in time and dropping the corresponding nonnegative terms we have reached the estimate

$$2\int_{-T}^{T}dt\int_{\mathbb{R}^n}\nabla_A u D^2\phi\overline{\nabla_A u}-\frac{1}{2}\int_{-T}^{T}dt\int_{\mathbb{R}^n}|u|^2\Delta^2\phi+ \tag{3.20}$$
$$2\mathcal{I}\int_{-T}^{T}dt\int_{\mathbb{R}^n}u\phi' B_\tau\cdot\overline{\nabla_A u}+\int_{-T}^{T}dt\int_{\mathbb{R}^n}|u|^2\sum_{j<k}\alpha_j\alpha_k(\nabla\phi\cdot\nabla B^{jk})\geq$$
$$\geq\sup_{R>0}\frac{1}{R}\int_{-T}^{T}dt\int_{|x|\leq R}|\nabla_A u|^2dx\geq$$
$$\geq\sup_{R>0}\frac{1}{R}\int_{-T}^{T}dt\int_{|x|\leq R}|\mathcal{D}_A u|^2dx$$

where in the last step we have used the pointwise inequality $|\mathcal{D}_A u| \leq |\nabla_A u|$. We now integrate in time the virial idetity on $[-T, T]$, and using (3.20) and (3.11) we obtain

$$\sup_{R>0} \frac{1}{R} \int_{-T}^{T} dt \int_{|x|\leq R} |\mathcal{D}_A u|^2 dx \lesssim \|\mathcal{D}_A u(T)\|_{L^2}^2 + \|\mathcal{D}_A u(-T)\|_{L^2}^2. \tag{3.21}$$

Let us now consider the range of D_A: from proposition (5.1) we have that for $C_0 < (n-2)^2/4$ $0 \notin \ker(D_A)$, so $\mathrm{ran(D_A)}$ is either L^2 or it is dense in L^2. Fix now an arbitrary $g \in \mathrm{ran(D_A)}$, there exists $f \in D(D_A) = D(\mathcal{H})$ such that $D_A f = g$. We then consider the solution $u(t, x)$ to the problem

$$\begin{cases} iu_t = -m\beta u + \mathcal{D}_A u \\ u(0, x) = f(x) \end{cases}$$

with opposite mass, and notice that u satisifies (3.21) since no hypothesis on the sign of the mass m have been used for it. If we thus apply to this equation the operator $\mathcal{D}_A$ we obtain, by the anticommutation rules,

$$\begin{cases} i(\mathcal{D}_A u)_t = \beta m(\mathcal{D}_A u) + \mathcal{D}_A(\mathcal{D}_A u) \\ \mathcal{D}_A u(0, x) = \mathcal{D}_A f(x) \end{cases}$$

or, in other words, the function $v = \mathcal{D}_A u$ solves the problem

$$\begin{cases} iv_t = \mathcal{H} v \\ v(0, x) = g \end{cases}$$

so that $v = e^{it\mathcal{H}} g$. Substituting in (3.21) and letting $T \to \infty$ we conclude that, in view of Remark (1.2),

$$\sup_{R>0} \frac{1}{R} \int_{-\infty}^{+\infty} \int_{|x|\leq R} |e^{it\mathcal{H}} g|^2 \lesssim \|g\|_{L^2}^2$$

that is exactly (1.10) for $g \in \mathrm{ran(D_A)}$, which is as we have noticed dense in L^2. Density arguments conclude the proof.

4. Proof of the Strichartz Estimates

We begin by recalling the Strichartz estimates for the free Dirac flow, both in the massless and in the massive case. They are a direct consequence of the corresponding estimates for the wave and Klein-Gordon equations:

Proposition 4.1. *Let $n \geq 3$. Then the following Strichartz estimates hold:*

(i) *in the massless case, for any wave admissible couple (p, q) (see (1.13))*

$$\| |D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} e^{it\mathcal{D}} f\|_{L^p L^q} \lesssim \|f\|_{L^2}; \tag{4.1}$$

(ii) *in the massive case, for any Schrödinger admissible couple (p, q) (see (1.14))*

$$\| |D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} e^{it(\mathcal{D}+\beta)} f\|_{L^p L^q} \lesssim \|f\|_{L^2}. \tag{4.2}$$

Proof. We restrict the proof to the case $n \geq 4$, refering to [3] for an exaustive proof of the 3-dimensional case.

Recalling identity (1.3) we immediately have that $u(t, x) = e^{it\mathcal{D}} f$ and $v(t, x) = e^{it(D+\beta)}$ satisfy the two Cauchy problems

$$\begin{cases} u_{tt} - \Delta u = 0 \\ u(0, x) = f(x) \\ u_t(0, x) = i\mathcal{D} f, \end{cases} \tag{4.3}$$

$$\begin{cases} v_{tt} - \Delta v + mv = 0 \\ v(0, x) = f(x) \\ v_t(0, x) = i(\mathcal{D} + \beta) f, \end{cases} \tag{4.4}$$

and so each component of the M-dimensional vectors u and v satisfy the same Strichartz estimates as for the n-dimensional wave equation and Klein-Gordon equation respectively. Thus case (i) follows from the standard estimates proved in [14] and [17], while case (ii) follows from similar techniques (the details can be found e.g. in the Appendix of [6]). □

We turn now to the perturbed flow. In the massless case, from the Duhamel formula we can write

$$u(t,x) \equiv e^{it\mathcal{D}_A} f = e^{it\mathcal{D}} f + \int_0^t e^{i(t-s)\mathcal{D}} \alpha \cdot Au(s) ds. \tag{4.5}$$

The term $e^{it\mathcal{D}} f$ can be directly estimated with (4.1). For the perturbative term we follow the Keel-Tao method [17]: by a standard application of the Christ-Kiselev Lemma, since we only aim at the non-endpoint case, it is sufficient to estimate the untruncated integral

$$\int e^{i(t-s)\mathcal{D}} \alpha \cdot Au(s) ds = e^{it\mathcal{D}} \int e^{-is\mathcal{D}} \alpha \cdot Au(s) ds.$$

Using again (4.1) we have

$$\begin{aligned} \left\| |D|^{\frac{1}{q}-\frac{1}{p}-\frac{1}{2}} e^{it\mathcal{D}} \int e^{-is\mathcal{D}} \alpha \cdot Au(s) ds \right\|_{L^p L^q} &\lesssim \\ \lesssim \left\| \int e^{-is\mathcal{D}} \alpha \cdot Au(s) ds \right\|_{L^2}&. \end{aligned} \tag{4.6}$$

Now we use the dual form of the smoothing estimate (1.10), i.e.

$$\left\| \int e^{-is\mathcal{D}} \alpha \cdot Au(s) ds \right\|_{L^2} \leq \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \| |A| \cdot |u| \|_{L^2_t L^2(|x| \cong 2^j)}, \tag{4.7}$$

where we have used the dual of the Morrey-Campanato norm as in [21]. Hence by Hölder inequality, hypothesis (1.11) and estimate (1.10) we have

$$\begin{aligned} &\sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \| |A| \cdot |u| \|_{L^2_t L^2(|x| \cong 2^j)} \leq \\ &\leq \sum_{j \in \mathbb{Z}} 2^j \sup_{|x| \cong 2^j} |A| \cdot \sup_{j \in \mathbb{Z}} 2^{-\frac{j}{2}} \|u\|_{L^2_t L^2(|x| \cong 2^j)} \lesssim \|f\|_{L^2} \end{aligned} \tag{4.8}$$

which proves (1.12). The proof in the massive case is exactly the same.

Remark 4.1. The endpoint estimates can also be recovered, both in the massless and massive case, adapting the proof of Lemma 13 in [15], but we will not go into details of this aspect.

5. Magnetic Hardy Inequality

This section is devoted to the proof of a version of Hardy's inqeuality adapted to the perturbed Dirac operator

$$\mathcal{H} = \mathcal{D}_A + m\beta, \qquad \mathcal{D}_A = i^{-1}\alpha \cdot \nabla_A \equiv i^{-1}\alpha \cdot (\nabla - iA).$$

The proof is simple but we include it for the sake of completeness.

PROPOSITION 5.1. *Let $B = DA - DA^t = B_1 + B_2$ and assume that*

$$\||x|^2 B_1\|_{L^\infty(\mathbb{R}^n)} < \infty, \quad \|B_2\|_{L^\infty(\mathbb{R}^n)} < \infty. \tag{5.1}$$

Then for every $f : \mathbb{R}^n \to \mathbb{C}^M$ such that $\mathcal{H}f \in L^2$ and any $\varepsilon < 1$ the following inequality holds when $m \neq 0$:

$$m^2 \int_{\mathbb{R}^n} |f|^2 + \left((1-\varepsilon)\frac{(n-2)^2}{4} - \frac{1}{2}\||x|^2 B_1\|_{L^\infty}\right) \int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2} + \tag{5.2}$$
$$+\varepsilon \int_{\mathbb{R}^n} |\nabla_A f|^2 \leq \left(1 + \frac{\|B_2\|_{L^\infty}}{2m^2}\right) \int_{\mathbb{R}^n} |\mathcal{H}f|^2.$$

When $m = 0$, the inequality is also true provided we choose $B_1 = B$, $B_2 = 0$ and we interpret the right hand side of (5.2) *simply as $\int |\mathcal{H}f|^2$.*

PROOF. Denote with $(\cdot,\cdot)$ the inner product in $L^2(\mathbb{R}^n, \mathbb{C}^M)$ and with $\|\cdot\|$ the associated norm. Recalling (2.7), we can write

$$\|\mathcal{H}f\|^2 = m^2\|f\|^2 + \|\nabla_A f\|^2 + i(S \cdot Bf, f)$$

where the matrix $S \cdot B = [S_{jk}] \cdot [B^{jk}]$ is skew symmetric since

$$S_{jk} = \frac{1}{2}\alpha_j\alpha_k, \qquad B^{jk} = \partial_j A^k - \partial_k A^j.$$

The selfadjoint matrices α_j have norm less than 1 (recall $\alpha_j^2 = \mathbb{I}$), so that

$$|(S \cdot Bf, f)| \leq \frac{1}{2}([B]_1 f, f)$$

where we denote by $[B]_1$ the ℓ^1 matrix norm

$$[B(x)]_1 = \sum_{j,k} |B^{jk}(x)|.$$

Now recalling assumption (5.1) we can write

$$|(S \cdot Bf, f)| \leq \frac{1}{2}\||x|^2 B_1\|_{L^\infty} \left\|\frac{f}{|x|}\right\|^2 + \frac{1}{2}\|B_2\|_{L^\infty}\|f\|^2$$

and in conclusion

$$\|\mathcal{H}f\|^2 \geq m^2\|f\|^2 + \|\nabla_A f\|^2 - \frac{1}{2}\||x|^2 B_1\|_{L^\infty} \left\|\frac{f}{|x|}\right\|^2 - \frac{1}{2}\|B_2\|_{L^\infty}\|f\|^2.$$

We now recall the magnetic Hardy inequality proved in [13]:

$$\frac{(n-2)^2}{4}\int_{\mathbb{R}^n} \frac{|f|^2}{|x|^2} \leq \int_{\mathbb{R}^n} |\nabla_A f|^2. \tag{5.3}$$

Observing now that

$$\|\mathcal{H}f\|^2 = (\mathcal{H}^2 f, f) = m^2\|f\|^2 + \|\mathcal{D}_A f\|^2$$

and that

$$(1-\varepsilon)\frac{(n-2)^2}{4}\left\|\frac{f}{|x|}\right\|^2 + \varepsilon\|\nabla_A f\|^2 \leq \|\nabla_A f\|^2,$$

the proof is complete. □

References

[1] Bournaveas, N., Local existence for the Maxwell-Dirac equations in three space dimensions, Comm. Partial Differential Equations **21**(5–6) (1996), 693–720.

[2] Bournaveas, N., Local existence of energy class solutions for the Dirac-Klein-Gordon equations, Comm. Partial Differential Equations **24**(7–8) (1999), 1167–1193.

[3] Boussaid, N., D'Ancona, P. and L. Fanelli, Virial identity and weak dispersion for the magnetic dirac equation, Journal de Mathématiques Pures et Appliquées **95** (2011), 137–150.

[4] D'Ancona, P. and L. Fanelli, L^p-boundedness of the wave operator for the one dimensional Schrödinger operator, Comm. Math. Phys. **268**(2) (2006), 415–438.

[5] D'Ancona, P. and L. Fanelli, Decay estimates for the wave and Dirac equations with a magnetic potential, Comm. Pure Appl. Math. **60**(3) (2007), 357–392.

[6] D'Ancona, P. and L. Fanelli, Strichartz and smoothing estimates of dispersive equations with magnetic potentials, Comm. Partial Differential Equations **33**(4–6) (2008), 1082–1112.
[7] D'Ancona, P., Fanelli, L., Vega, L. and N. Visciglia, Endpoint Strichartz estimates for the magnetic Schrödinger equation, J. Funct. Anal. **258**(10) (2010), 3227–3240.
[8] D'Ancona, P., Foschi, D. and S. Selberg, Local well-posedness below the charge norm for the Dirac-Klein-Gordon system in two space dimensions, J. Hyperbolic Differ. Equ. **4**(2) (2007), 295–330.
[9] D'Ancona, P., Foschi, D. and S. Selberg, Null structure and almost optimal local regularity for the Dirac-Klein-Gordon system, J. Eur. Math. Soc. (JEMS) **9**(4) (2007), 877–899.
[10] D'Ancona, P., Foschi, D. and S. Selberg, Null structure and almost optimal local well-posedness of the Maxwell-Dirac system, Amer. J. Math. **132**(3) (2010), 771–839.
[11] Dias, J.-P. and M. Figueira, Global existence of solutions with small initial data in H^s for the massive nonlinear Dirac equations in three space dimensions, Boll. Un. Mat. Ital. B (7) **1**(3) (1987), 861–874.
[12] Escobedo, M. and L. Vega, A semilinear Dirac equation in $H^s(\mathbf{R}^3)$ for $s > 1$, SIAM J. Math. Anal. **28**(2) (1997), 338–362.
[13] Fanelli, L. and L. Vega, Magnetic virial identities, weak dispersion and Strichartz inequalities, Math. Ann. **344**(2) (2009), 249–278.
[14] Ginibre, J. and G. Velo, Generalized Strichartz inequalities for the wave equation, In *Partial differential operators and mathematical physics (Holzhau, 1994)*, volume 78 of Oper. Theory Adv. Appl., pages 153–160, Birkhäuser, Basel, 1995.
[15] Ionescu, A. D. and C. Kenig, Well-posedness and local smoothing of solutions of Schrödinger equations, Math. Res. Letters **12** (2005), 193–205.
[16] Kalf, H. and O. Yamada, Essential self-adjointness of n-dimensional Dirac operators with a variable mass term, J. Math. Phys. **42**(6) (2001), 2667–2676.
[17] Keel, M. and T. Tao, Endpoint Strichartz estimates, Amer. J. Math. **120**(5) (1998), 955–980.
[18] Machihara, S., Nakamura, M., Nakanishi, K. and T. Ozawa, Endpoint Strichartz estimates and global solutions for the nonlinear Dirac equation, J. Funct. Anal. **219**(1) (2005), 1–20.
[19] Machihara, S., Nakamura, M. and T. Ozawa, Small global solutions for nonlinear Dirac equations, Differential Integral Equations **17**(5–6) (2004), 623–636.
[20] Ozawa, T. and K. Yamauchi, Structure of Dirac matrices and invariants for nonlinear Dirac equations, Differential Integral Equations **17**(9–10) (2004), 971–982.
[21] Perthame, B. and L. Vega, Morrey-Campanato estimates for Helmholtz equations, J. Funct. Anal. **164**(2) (1999), 340–355.

[22] Thaller, B., *The Dirac equation*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1992.

(Received April 4, 2011)
(Revised November 25, 2011)

SAPIENZA — Università di Roma
Dipartimento di Matematica
Piazzale A. Moro 2
I-00185 Roma, Italy
E-mail: cacciafe@mat.uniroma1.it