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Note on the Chen-Lin Result with the Li-Zhang Method

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Abstract. We give a new proof of the Chen-Lin result with the method of moving sphere in a work of Li-Zhang.

Introduction and Results

We set $\Delta = \partial_{11} + \partial_{22}$ the Laplace-Beltrami operator on $\mathbb{R}^2$.

On an open set Ω of $\mathbb{R}^2$ with a smooth boundary we consider the following problem:

$$(P) \quad \begin{cases} -\Delta u = V(x)e^u \text{ in } \Omega, \\ 0 < a \leq V(x) \leq b < +\infty. \end{cases}$$

The previous equation is called the Prescribed Scalar Curvature equation in relation with conformal change of metrics. The function V is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [7, 8, 10, 12, 13, 17, 18, 21, 22, 25]. We can see in [8] different results for the solutions of those type of equations with or without boundary conditions and, with minimal conditions on V, for example we suppose $V \geq 0$ and $V \in L^p(\Omega)$ or $Ve^u \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [8] the following important theorem,

Theorem A (Brezis-Merle [8]). *If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega) \text{ if } \inf_\Omega u_i \geq m.$$

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A simple consequence of this theorem is that, if we assume $u_i = 0$ on $\partial\Omega$, then the sequence $(u_i)_i$ is locally uniformly bounded. We can find in [8] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i}.

If we assume V with more regularity, we can have another type of estimates, $\sup + \inf$. It was proved by Shafrir in [22] that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

$$C\left(\frac{a}{b}\right) \sup_K u_i + \inf_\Omega u_i \leq c = c(a, b, K, \Omega).$$

We can see in [12] an explicit value of $C\left(\frac{a}{b}\right) = \sqrt{\frac{a}{b}}$. In [22] Shafrir has used the Stokes formula and an isoperimetric inequality; see [6]. In [12] Chen and Lin have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a, b, A, K, \Omega)$; see Brezis-Li-Shafrir [7]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin; see [12]. Also, we can see in [17] an extension of the Brezis-Li-Shafrir's result to compact Riemann surface without boundary. We can see in [18] explicit form ($8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [13] and [25] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

On an open set Ω of $\mathbb{R}^2$ we consider the following equation:

$$\begin{cases} -\Delta u_i = V_i e^{u_i} \ \text{ on } \Omega, \\ 0 < a \leq V_i(x) \leq b < +\infty, \ x \in \Omega, \\ |V_i(x) - V_i(y)| \leq A|x-y|^s, \ 0 < s \leq 1, \ x, y \in \Omega. \end{cases}$$

Among other results, we have in [12] the following Harnack type inequality,

THEOREM B (Chen-Lin [12]). *For all compact $K \subset \Omega$ and all $s \in]0, 1]$ there is a constant $c = c(a, b, A, s, K, \Omega)$ such that,*

$$\sup_K u_i + \inf_\Omega u_i \leq c \ \text{ for all } i.$$

Here we try to prove the previous theorem by the moving-plane method and Li-Zhang method; see [19]. The method of moving-plane was developed by Gidas-Ni-Nirenberg; see [14]. We can see in [9] one of the applications of this method and, in particular, the classification of the solutions of some elliptic PDEs.

Note that in our proof we do not need a classification result for some particular elliptic PDEs as showed in [7] and [12].

In a similar way we have in dimension $n \geq 3$, with different methods, some a priori estimates of the type $\sup \times \inf$ for equation of the type:

$$-\Delta u + \frac{n-2}{4(n-1)} R_g(x) u = V(x) u^{(n+2)/(n-2)} \text{ on } M,$$

where R_g is the scalar curvature of a riemannian manifold M, and V is a function. The operator $\Delta = \nabla^i(\nabla_i)$ is the Laplace-Beltrami operator on M.

When $V \equiv 1$ and M compact, the previous equation is the Yamabe equation. T. Aubin and R. Scheon solved the Yamabe problem, see for example [1]. Also, we can have an idea on the Yamabe Problem in [15]. If V is not a constant function, the previous equation is called a prescribing curvature equation, we have many existence results see also [1].

Now, if we look at the problem of a priori bound for the previous equation, we can see in [2], [4], [11], [16], some results concerning the $\sup \times \inf$ type of inequalities when the manifold M is the sphere or more generality a locally conformally flat manifold.

For general manifolds M of dimension $n \geq 3$ we have some Harnack type estimates; see for example [3, 5], [19] and [20], for equation of the type,

$$-\Delta u + h(x) u = V(x) u^{(n+2)/(n-2)} \text{ on } M.$$

Also, there are similar problems defined on complex manifolds for the Complex Monge-Ampere equation; see [23, 24]. They consider, on compact Kahler manifold (M, g), the following equation

$$\begin{cases} (\omega_g + \partial\bar{\partial}\phi)^n = e^{f - t\phi}\omega_g^n, \\ \omega_g + \partial\bar{\partial}\phi > 0 \text{ on } M \end{cases}$$

And, they prove some estimates of type $\sup_M(\phi - \psi) + m \inf_M(\phi - \psi) \leq C(t)$ or $\sup_M(\phi - \psi) + m \inf_M(\phi - \psi) \geq C(t)$ under the positivity of the first Chern class of M.

The function ψ is a C^2 function such that

$$\omega_g + \partial\bar{\partial}\psi \geq 0 \text{ and } \int_M e^{f-t\psi}\omega_g^n = Vol_g(M),$$

New Proof of the Theorem B.

We argue by contradiction and we want to prove that

$$\exists\ R > 0, \text{ such that } 4\log R + \sup_{B_R(0)} u + \inf_{B_{2R}(0)} u \leq c = c(a, b, A),$$

Thus, by contardition we can assume

$$\exists\ (R_i)_i,\ (u_i)_i\ R_i \to 0,\ 4\log R_i + \sup_{B_{R_i}(0)} u_i + \inf_{B_{2R_i}(0)} u_i \to +\infty.$$

Step 1. The blow-up analysis

For $x_0 \in \Omega$ we want to prove the theorem locally around x_0. We use the previous assertion with $x_0 = 0$. The classical blow-up analysis gives the existence of the sequence $(x_i)_i$ and a sequence of functions $(v_i)_i$ satisfying the following properties.

We set

$$\sup_{B_{R_i(0)}} u_i = u_i(\bar{x}_i),$$

$$s_i(x) = 2\log(R_i - |x - \bar{x}_i|) + u_i(x), \text{ and}$$
$$s_i(x_i) = \sup_{B_{R_i(\bar{x}_i)}} s_i,\ \ \sigma_i = \frac{1}{2}(R_i - |x_i - \bar{x}_i|).$$

Also, we set

$$v_i(x) = u_i[x_i + xe^{-u_i(x_i)/2}] - u_i(x_i),\ \ \bar{V}_i(x) = V_i[x_i + xe^{-u_i(x_i)/2}],$$

Then, with this classical selection process, we have

$$2\log M_i = u_i(x_i) \geq u_i(\bar{x}_i)$$

$$u_i(x) \leq C_1 u_i(x_i),\ \forall\ x \in B(x_i, \sigma_i),$$

where C_1 is a constant independant of i.

Also,

$$u_i(x_i) + \min_{\partial B(x_i,R_i)} u_i + 4\log R_i \geq u_i(\bar{x}_i) + \min_{B(0,2R_i)} u_i + 4\log R_i \to +\infty,$$

and

$$\lim_{i\to+\infty} R_i e^{u_i(x_i)/2} = \lim_{i\to+\infty} \sigma_i e^{u_i(x_i)/2} = +\infty.$$

Finally, we have

$$\begin{cases} \Delta v_i + \bar{V}_i e^{v_i} = 0 \text{ for } |y| \leq R_i M_i, \\ v_i(0) = 0, \\ v_i(y) \leq C_1 \text{ for } |y| \leq \sigma_i M_i, \\ \lim_{i\to+\infty} \min_{|y|=2R_iM_i} (v_i(y) + 4\log|y|) = +\infty. \end{cases}$$

Because of the classical elliptic estimates and the classical Harnack inequality, we can prove the uniform convergence on each compact of $\mathbb{R}^2$

$$v_i \to v \text{ when } v \text{ is a solution on } \mathbb{R}^2 \text{ of}$$

$$\begin{cases} \Delta v + V(0)e^v = 0 \text{ in } \mathbb{R}^2, \\ v(0) = 0,\ 0 < v \leq C_1. \end{cases}$$

with $V(0) = \lim_{i\to+\infty} V_i(x_i)$ and $0 < a \leq V(0) \leq b < +\infty$.

Step 2. The moving-plane method

Here we use the Kelvin transform and the Li-Zhang's method.

For $0 < \lambda < \lambda_1$ we define

$$\Sigma_\lambda = B(0, R_i M_i) - B(0, \lambda).$$

First, we set

$$\bar{v}_i^\lambda = v_i^\lambda - 4\log|x| + 4\log\lambda = v_i\left(\frac{\lambda^2 x}{|x|^2}\right) + 4\log\frac{\lambda}{|x|},$$

$$x^\lambda = \frac{\lambda^2 x}{|x|^2} \text{ and } \bar{V}_i^\lambda = \bar{V}_i\left(\frac{\lambda^2 x}{|x|^2}\right),$$

$$M_i = e^{u_i(x_i)/2}.$$

We want to compare v_i and $\bar{v}_i^\lambda$, we set

$$w_\lambda = v_i - \bar{v}_i^\lambda.$$

Then

$$-\Delta \bar{v}_i^\lambda = \bar{V}_i^\lambda e^{\bar{v}_i^\lambda},$$

$$-\Delta(v_i - \bar{v}_i^\lambda) = \bar{V}_i(e^{v_i} - e^{\bar{v}_i^\lambda}) + (\bar{V}_i - \bar{V}_i^\lambda)e^{\bar{v}_i^\lambda},$$

We have the following estimate

$$|\bar{V}_i - \bar{V}_i^\lambda| \leq A M_i^{-s}|x|^s|1 - \frac{\lambda^2}{|x|^2}|^s. \square$$

The auxiliary function:

We take an auxiliary function h_λ.

Because $v_i(x^\lambda) \leq C(\lambda_1) < +\infty$, we have

$$h_\lambda = C_1 M_i^{-s}\lambda^2(\log(\lambda/|x|)) + C_2 M_i^{-s}\lambda^{2+s}[1 - (\frac{\lambda}{|x|})^{2-s}], \;\; |x| > \lambda,$$

with $C_1, C_2 = C_1, C_2(s, \lambda_1) > 0$

$$h_\lambda = M_i^{-s}\lambda^2(1 - \lambda/|x|)(C_1\frac{\log(\lambda/|x|)}{1 - \lambda/|x|} + C_2'),$$

with $C_2' = C_2'(s, \lambda_1) > 0$. We can choose C_1 big enough to have $h_\lambda < 0$.

Lemma 1. *There is an $\lambda_{i,0} > 0$ small enough, such that, for $0 < \lambda < \lambda_{i,0}$, we have*

$$w_\lambda + h_\lambda > 0.$$

Proof of the Lemma 1.

We set

$$f(r, \theta) = v_i(r\theta) + 2\log r,$$

then

$$\frac{\partial f}{\partial r}(r, \theta) = < \nabla v_i(r\theta)|\theta > + \frac{2}{r},$$

According to the blow-up analysis,

$$\exists\ r_0 > 0,\ C > 0,\ |\nabla v_i(r\theta)|\theta >| \leq C,\ \text{for } 0 \leq r < r_0.$$

Then

$$\exists\ r_0 > 0,\ C' > 0,\ \frac{\partial f}{\partial r}(r, \theta) > \frac{C'}{r},\ 0 < r < r_0.$$

Case 1. If $0 < \lambda < |y| < r_0$

$$w_\lambda(y) + h_\lambda(y) = v_i(y) - v_i^\lambda(y) + h_\lambda(y) > C(\log|y| - \log|y^\lambda|) + h_\lambda(y),$$

by the definition of h_λ, we have, for $C, C_0 > 0$ and $0 < \lambda \leq |y| < r_0$,

$$w_\lambda(y) + h_\lambda(y) > (|y| - \lambda)[C\frac{\log|y| - \log|y^\lambda|}{|y| - \lambda} - \lambda^{1+s}C_0 M_i^s],$$

but

$$|y| - |y^\lambda| > |y| - \lambda > 0,\ \text{and } |y^\lambda| = \frac{\lambda^2}{|y|},$$

thus,

$$w_\lambda(y) + h_\lambda(y) > 0 \text{ if } \lambda < \lambda_0^i, \lambda_0^i \text{ (small)},\ \text{and } 0 < \lambda < |y| < r_0.$$

Case 2. If $r_0 \leq |y| \leq R_i M_i$

$$v_i \geq \min v_i = C_i^1,\ v_i^\lambda(y) \leq C_1(\lambda_1, r_0),\ \text{if } r_0 \leq |y| \leq R_i M_i.$$

Thus, in $r_0 \leq |y| \leq R_i M_i$ and $\lambda \leq \lambda_1$, we have,

$$w_\lambda + h_\lambda \geq C_i - 4\log\lambda + 4\log r_0 - C'{\lambda_1}^{2+s}$$

then, if $\lambda \to 0$, $-\log\lambda \to +\infty$, and

$$w_\lambda + h_\lambda > 0,\ \text{ if } \lambda < \lambda_1^i, \lambda_1^i \text{ (small)},\ \text{and } r_0 < |y| \leq R_i M_i. \ \square$$

As in Li-Zhang paper, see [19], by the maximum principle and the Hopf boundary lemma, we have

LEMMA 2. *Let $\tilde{\lambda}_i$ be a positive number such that*

$$\tilde{\lambda}_i = \sup\{\lambda < \lambda_1, \ w_\lambda + h_\lambda > 0 \ \text{ in } \ \Sigma_\lambda\}.$$

Then

$$\tilde{\lambda}_i = \lambda_1.$$

PROOF OF THE LEMMA 2.

The blow-up analysis gives the following inequality for the boundary condition.

For $y = |y|\theta = R_i M_i \theta$ we have

$$w_{\lambda^i}(|y| = R_i M_i) + h_{\lambda^i}(|y| = R_i M_i) =$$
$$= u_i(x_i + R_i\theta) - u_i(x_i) - v_i(R_i M_i) - 4\log\lambda + 4\log(R_i M_i) +$$
$$+ C(s,\lambda_1) M_i^{-s} \lambda^{2+s}[1 - (\frac{\lambda}{R_i M_i})^{2-s}],$$

because

$$4\log R_i + u_i(x_i) + \inf_{B_{2R_i}(0)} u_i \to +\infty,$$

which we can write

$$w_{\lambda^i}(|y| = R_i M_i) + h_{\lambda^i}(|y| = R_i M_i) \geq$$
$$\geq \min_{B_{2R_i}(0)} u_i + u_i(x_i) + 4\log R_i - C(s,\lambda_1) \to +\infty,$$

because, $0 < \lambda \leq \lambda_1$.

Finally, we have

$$w_{\tilde{\lambda}_i}(y) + h_{\tilde{\lambda}_i}(y) > 0 \ \ \forall \ |y| = R_i M_i,$$

Now, we have

$$\Delta w_\lambda + \xi V_i w_\lambda = E_\lambda \ \text{in} \ \ \Sigma_\lambda,$$

where ξ stays between v_i and v_i^λ, and

$$E_\lambda = -(V_i - V_i^\lambda) e^{\bar{v}_i^\lambda}.$$

Thus to prove that

$$(\Delta + \xi V_i)(w_\lambda + h_\lambda) \leq 0 \ \text{in} \ \ \Sigma_\lambda,$$

it sufficies to prove that

$$\Delta h_\lambda + (\xi V_i) h_\lambda + E_\lambda \le 0 \text{ in } \Sigma_\lambda.$$

But we have

$$h_\lambda < 0,$$
$$|E_\lambda| \le C_1 \lambda^4 M_i^{-s} |y|^{-4+s} \text{ in } \Sigma_\lambda,$$

and

$$\Delta h_\lambda = -C_1 \lambda^4 M_i^{-s} |y|^{-4+s} \text{ in } \Sigma_\lambda.$$

We can use the maximum principle and the Hopf lemma to have

$$w_{\tilde{\lambda}_i} + h_{\tilde{\lambda}_i} > 0, \text{ in } \Sigma_\lambda,$$

and

$$\frac{\partial}{\partial \nu}(w_{\tilde{\lambda}_i} + h_{\tilde{\lambda}_i}) > 0, \text{ in } \partial B(0, \tilde{\lambda}_i).$$

From above we conclude that $\tilde{\lambda}_i = \lambda_1$ and lemma 2 is proved. □

Conclusion

As in [19], we have

$$\forall\ \lambda_1 > 0,\ v(y) \ge v^\lambda(y),\ \forall |y| \ge \lambda,\ \forall\ 0 < \lambda < \lambda_1.$$

And the same argument may be used to have

$$\forall\ \lambda_1 > 0,\ v(y) \ge v^{\lambda,x}(y),\ \forall\ x, y\ |y-x| \ge \lambda,\ \forall\ 0 < \lambda < \lambda_1,$$

where

$$v^{\lambda,x}(y) = v_i\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) + 4\log\frac{\lambda}{|y-x|}.$$

This implies that v is a constant, and because $v(0) = 0$, $v \equiv 0$ contradicting the fact that

$$-\Delta v = V(0)e^v.$$

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