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Invariant Differential Operators on the Schrödinger Model for the Minimal Representation of the Conformal Group

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Abstract. We consider the Schrödinger model of the minimal representation for the conformal group $O(2n, 2)\,(n > 1)$ which was constructed by Kobayashi-Ørsted [Adv. Math. 2003], and enriched by a series of papers by Kobayashi-Mano [Memoirs of AMS 2011, etc]. We get the joint spectra of the differential operators on the model for generators of the center of the Lie algebra of $U(k) \times U(n-k) \times U(1)$ for $k = 1, \ldots, n-1$. Further, we obtain the generators of the algebra consisting of all invariant differential operators for two compact subgroups $I_1 \times SO(2n-1) \times SO(2)$ and $SO(2n) \times SO(2)$ of $O(2n, 2)$.

1. Introduction and Statement of Main Results

1.1. Differential operators on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$

We consider three differential operators on $\mathbb{R}^{2n-1}$:

$$D_1 := \left(x_1 + \frac{x_1}{4}\Delta - \frac{1}{2}E\frac{\partial}{\partial x_1} - \frac{n-1}{2}\frac{\partial}{\partial x_1}\right), \tag{1.1}$$

$$D_2 := \sqrt{-1}\sum_{j=1}^{n-1}\left(x_{2j}\frac{\partial}{\partial x_{2j+1}} - x_{2j+1}\frac{\partial}{\partial x_{2j}}\right), \tag{1.2}$$

$$D_3 := |x|\left(\frac{1}{4}\Delta - 1\right), \tag{1.3}$$

where $|x|$ denotes the norm $(\sum_{j=1}^{2n-1} x_j^2)^{1/2}$ for any $x = (x_1, \ldots, x_{2n-1}) \in \mathbb{R}^{2n-1}$, Δ the Laplace operator $\sum_{j=1}^{2n-1} \partial^2/\partial x_j^2$ and E the Euler operator $\sum_{j=1}^{2n-1} x_j\partial/\partial x_j$.

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The following theorem implies that D_1, D_2 and D_3 are *natural* differential operators not on the Hilbert space $L^2(\mathbb{R}^{2n-1})$ with respect to the ordinary Lebesgue measure but on the Hilbert space $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$ of square integrable functions on $\mathbb{R}^{2n-1}$ with respect to the weighted measure $|x|^{-1}dx = |x|^{-1}dx_1 \dots dx_{2n-1}$.

Theorem 1.1. (1) *The differential operators D_1, D_2 and D_3 extend to self-adjoint operators on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$.*

(2) *D_1, D_2 and D_3 mutually commute.*

(3) *D_1, D_2 and D_3 have only discrete spectra on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$, respectively.*

(4) *The set of the joint eigenvalues of (D_1, D_2, D_3) is given as follows:*

$$\left\{(x, y, z) \in \mathbb{Z}^3 : \begin{matrix} x + y + z - n + 1 \equiv 0 \mod 2 \\ |x| + |y| \leq -z - n + 1 \end{matrix}\right\}. \tag{1.4}$$

(5) *D_1, D_2 and D_3 are algebraically independent.*

The properties of D_3 were studied by Kobayashi–Mano [9, Introduction and Section 3.4]. This operator corresponds to the Schrödinger operator on the Schrödinger model of the Weil representation (see Introduction in [9]). Since D_2 is the vector field generated by one-parameter subgroups of a compact group, it is not hard to get the properties of D_2. D_1 is one of the operators which are called *fundamental differential operators* by Kobayashi–Mano [11].

We shall show the more general theorem than Theorem 1.1 in Section 4. The idea of proving this theorem is to use representation theory. For the indefinite orthogonal group $O(2n, 2)$ and its identity component $SO_0(2n, 2)$, it is known in [9, Section 3.1] that a unitary highest weight representation π_+ of $SO_0(2n, 2)$ can be realized on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$ through the Schrödinger model of $O(2n, 2)$ given by Kobayashi–Ørsted [12, Part III]. Then we consider a compact subgroup $U := U(n_1) \times U(n_2) \times U(1)$ ($n_1, n_2 \geq 1$, $n_1 + n_2 = n$) of $SO_0(2n, 2)$. Here U is minimal among the subgroups of the direct product form $U(n_1) \times \cdots \times U(n_k) \times U(1)$ ($n_1, \dots, n_k \geq 1, n_1 + \cdots + n_k = n$) to which the restriction of π_+ is still multiplicity-free (see Introduction and statement of main results in [13] and (2.3) in Section 2.2). Under the restriction $SO_0(2n, 2) \downarrow U$ if $n_1 = 1$, one can see that D_1, D_2 and D_3 are given as the infinitesimal action of the center of the Lie algebra $\mathfrak{u}$ of U

(see Section 2.3). In addition, we use the explicit branching formula of the restriction $\pi_+|_U$ (see (2.3) and Proposition 3.3) to show the theorem.

1.2. Invariant differential operators on $\mathbb{R}^{2n-1} \setminus \{0\}$

By (1) of Theorem 1.1, they extend to self-adjoint operators on $\mathcal{H}$, where we set $\mathcal{H} := L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$. Let $\mathcal{H}^\infty$ denote the space of all C^∞-vectors for $(\pi_+, \mathcal{H})$ of $SO_0(2n, 2)$. Here, we recall that if $v \in \mathcal{H}$ is such that the function $\pi_+(g)v$ is of class C^∞ from $SO_0(2n, 2)$ to $\mathcal{H}$, then v is called a C^∞-vector. We consider the universal enveloping algebra $U(\mathfrak{so}(2n, 2)_\mathbb{C})$ of the complexification of the Lie algebra $\mathfrak{so}(2n, 2)$ of $SO_0(2n, 2)$ and the algebra $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})$ consisting of all differential operators on $\mathbb{R}^{2n-1} \setminus \{0\}$. Then, for the infinitesimal action $d\pi_+$ we have

$$d\pi_+(U(\mathfrak{so}(2n, 2)_\mathbb{C})) \subset \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\}) \cap \mathrm{End}(\mathcal{H}^\infty). \tag{1.5}$$

Here, we note that the origin is the singular point of the measure $|x|^{-1}dx$ on $\mathbb{R}^{2n-1}$.

Now, we briefly review the following algebras of invariant differential operators on representation theory:

(1) (Harmonic analysis) Let (G, K) be a Riemannian symmetric pair. We consider the algebra $\mathcal{D}(G/K)^G$ consisting of all G-invariant differential operators on a Riemannian symmetric space G/K. The spherical functions on G/K are obtained as the joint eigenfunctions for all differential operators in $\mathcal{D}(G/K)^G$. Then $\mathcal{D}(G/K)^G$ is generated by l algebraically independent differential operators, where l is the rank of G/K (see [4, Chapter II, §4] for example).

(2) (Characters) Let G be a connected semisimple Lie group and $\mathcal{D}(G)^G$ the algebra consisting of all left G-invariant differential operators on G. The character of each irreducible representation of G is a joint eigenfunction or eigendistribution of the differential equations defined by all right-invariant differential operators in $\mathcal{D}(G)^G$. This is given by Harish-Chandra (see [5, Chapter X, §4] for example). The right invariants of $\mathcal{D}(G)^G$ is isomorphic to the center of $U(\mathfrak{g}_\mathbb{C})$.

We will consider the invariant differential operators for $(\pi_+, \mathcal{H})$ of $SO_0(2n, 2)$. In the above cases (1) and (2), the actions of G on the function spaces of the left-regular representations are induced by the actions on the base manifolds respectively. In our case, it is remarkable that not all elements in $SO_0(2n, 2)$ act on $\mathbb{R}^{2n-1} \setminus \{0\}$ (see [9]), however, they act on the function space $\mathcal{H}$. For this, the infinitesimal action $d\pi_+$ of $\mathfrak{so}(2n, 2)$ on $\mathcal{H}$ contains not only vector fields but also differential operators of second order, such as D_1 and D_3.

For any subgroup H of $SO_0(2n, 2)$, we define an algebra $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H$ consisting of all invariant differential operators under the action $\pi_+(H)$ by

$$\begin{aligned}(1.6)\quad & \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H := \\ & \qquad \left\{D \in \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\}) : D \circ \pi_+(h) = \pi_+(h) \circ D, \text{ for any } h \in H\right\}.\end{aligned}$$

When $H = SO_0(2n, 2)$, the algebra $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^{SO_0(2n,2)}$ is $\mathbb{C}$ because π_+ is an irreducible representation of $SO_0(2n, 2)$. When $H = U$, by Proposition 3.3 in Section 3.4 one can see that each $D \in \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^U$ can be diagonalized with respect to the same fixed basis. Indeed, by the multiplicity freeness, Schur's lemma implies that D acts as a scalar multiplication on any irreducible summand. Hence, we have

LEMMA 1.2. *The algebra* $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^U$ *is commutative.*

Then we ask:

QUESTION 1.3. Determine the structure of the commutative algebra $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^U$. Namely, find the generators of $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^U$.

We recall an immediate consequence of (2.3) in Section 2.2 and Proposition 3.3.

COROLLARY 1.4. *For any subgroup* H *of* $SO_0(2n, 2)$ *which contains* U*, the restriction* $\pi_+|_H$ *is multiplicity-free.*

In the same way as that to show Lemma 1.2, we have

LEMMA 1.5. *For any subgroup* H *of* $SO_0(2n, 2)$ *which contains* U*, the algebra* $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H$ *is commutative.*

We note that the larger subgroups H tend to be, the smaller the algebras $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H$ become.

Hereafter we set $G := SO_0(2n,2)$ and compact subgroups $K := SO(2n) \times SO(2)$, $H := I_1 \times SO(2n-1) \times SO(2)$, and $R := I_1 \times SO(2n-1) \times I_2 \simeq SO(2n-1)$ of G. Then we have

$$G \supset K \supset H \supset R. \tag{1.7}$$

The restriction of π_+ to H is multiplicity-free (see (2.3) and Proposition 3.1). That of π_+ to R is not multiplicity-free (see Remark 3.2), however, the explicit branching formula of $\pi_+|_R$ is given by Kobayashi–Mano (see Fact 3.5). By this formula, we get the following result:

THEOREM 1.6. *We have the following:*

$$\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H = \mathbb{C}[D_3, D_4], \tag{1.8}$$

where D_4 is the differential operator on $\mathbb{R}^{2n-1} \setminus \{0\}$ defined by

$$D_4 := \sum_{1 \le i < j \le 2n-1} \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right)^2. \tag{1.9}$$

Here, D_4 is nothing but the Casimir operator with respect to $SO(2n-1) \simeq R$, up to a scalar constant.

Theorem 1.6 means that $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H = \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^{SO(2n-1)} \cap (\ker \mathrm{ad} D_3)$ since the action of $SO(2n-1)$ is given by rotation and the action of $SO(2)$ is infinitesimally given by D_3. Theorem 1.6 leads to the following corollary:

COROLLARY 1.7. *We have the following:*

$$\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^K = \mathbb{C}[D_3]. \tag{1.10}$$

By Corollary 1.7, D_3 is nothing but the generator of the K-invariant differential operators. The joint eigenfunctions with respect to $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^K$ are given in [9].

1.3. The background of our theorems and question

Theorem 1.1, Question 1.3, Theorem 1.6 and Corollary 1.7 are based on the philosophy posed by Kobayashi [6] of applying multiplicity-free properties to problems in analysis. The advantage of the irreducible decomposition for a multiplicity-free representation is that such a decomposition diagonalizes any operator commuting with the group action.

This article is organized as follows. In Section 2, we introduce the representation theoretic setting and describe D_1, D_2, D_3 through the action of certain elements in $\mathfrak{so}(2n, 2)$. In Section 3, we review various branching laws for the minimal representation of $O(2n, 2)$ and introduce the action of D_1 (Proposition 3.6). The proof of Proposition 3.6 is given in Appendix because of a lot of technical computation of special functions. The more general theorem than Theorem 1.1 is shown in Section 4. Theorem 1.6 and Corollary 1.7 are shown in Section 5. Theorem 1.6 is the immediate consequence of Proposition 5.1. The proof of this proposition is given in Section 6 since this involves a lot of technical computation.

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2. The Descriptions of D_1, D_2, D_3

In this section, we review the background of our theorems and state the source of differential operators D_1, D_2, D_3, owing to notation of Kobayashi, Ørsted and Mano [8, 9, 12].

2.1. L^2-model of the minimal representation

We set $\mathbb{N} := \{0, 1, 2, \dots\}$. Let $(\pi^c, L^2(C))$ be the L^2-model of the minimal representation of $O(2n, 2)$ for $n \in \mathbb{N}$ with $n > 1$ considered by Kobayashi–Ørsted (see [12, III]). Here, $O(2n, 2)$ is the indefinite orthogonal group defined by

$$O(2n, 2) := \{g \in GL(2n + 2, \mathbb{R}) : {}^t g I_{2n,2}\, g = I_{2n,2}\},$$

where $I_{2n,2} := \mathrm{diag}(1, \dots, 1, -1, -1) \in GL(2n+2, \mathbb{R})$. Further, C is the closed cone:

$$C = \{\zeta \in \mathbb{R}^{2n} \setminus \{0\} : Q(\zeta) = 0\},$$

where $Q(\zeta) := \zeta_1^2 + \cdots + \zeta_{2n-1}^2 - \zeta_{2n}^2$. Let $C_\pm$ be the forward and the backward light cone respectively:

$$C_\pm := \left\{(\zeta_1, \dots, \zeta_{2n}) \in \mathbb{R}^{2n} : \pm\zeta_{2n} > 0,\ \zeta_1^2 + \cdots + \zeta_{2n-1}^2 = \zeta_{2n}^2\right\}.$$

Then one can see that C is the disjoint union $C_+ \cup C_-$. The direct sum decomposition

$$L^2(C) = L^2(C_+) \oplus L^2(C_-) \tag{2.1}$$

yields a branching law $\pi^c = \pi_+^c \oplus \pi_-^c$ with respect to the restriction $O(2n, 2) \downarrow G = SO_0(2n, 2)$. The irreducible representations π_+^c and π_-^c of G are contragradient to each other, one is a highest weight module, and the other is a lowest weight module.

2.2. Schrödinger model on the flat space $\mathbb{R}^{2n-1}$

The representation $(\pi_+^c, L^2(C_+))$ of $G = SO_0(2n, 2)$ can be realized on the space of L^2-functions with a weighted measure on $\mathbb{R}^{2n-1}$. In this subsection, we review this fact.

We have used the variables $\zeta = (\zeta_1, \dots, \zeta_{2n})$ for $C_+ \subset \mathbb{R}^{2n}$ and will use $x = (x_1, \dots, x_{2n-1})$ for the coordinate of $\mathbb{R}^{2n-1}$. The projection

$$p : \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}, \quad (\zeta_1, \dots, \zeta_{2n-1}, \zeta_{2n}) \mapsto (\zeta_1, \dots, \zeta_{2n-1}) \tag{2.2}$$

induces a diffeomorphism from C_+ onto $\mathbb{R}^{2n-1} \setminus \{0\}$, and the measure $d\mu$ on C_+ is given by $\delta(Q)$ (see [3]), and therefore is pushed forward to $(2|x|)^{-1}dx$, where $dx = dx_1 \dots dx_{2n-1}$. Thus, we have a unitary isomorphism:

$$\sqrt{2}p^* : L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx) \xrightarrow{\sim} L^2(C_+). \tag{2.3}$$

Through this isomorphism, the representation of G is realized on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$. This model is given by Kobayashi–Mano [9, Section 3.1] and called the *Schrödinger model*. This representation is π_+ in Introduction.

2.3. The center of the Lie algebra of U

Let us denote by $\mathfrak{u}$ the Lie algebra of $U = U(n_1)\times U(n_2)\times U(1)$ $(n_1, n_2 \geq 1, n_1 + n_2 = n)$. In this subsection, we get the infinitesimal action of the center $\mathfrak{z}(\mathfrak{u})$ of $\mathfrak{u} \subset \mathfrak{so}(2n,2)$. Here we use the embedding $\mathfrak{u}(k) \subset \mathfrak{so}(2k)$ corresponding to $z_j = \zeta_{2j-1} + \sqrt{-1}\zeta_{2j}$ $(z_j \in \mathbb{C},\ \zeta_{2j-1}, \zeta_{2j} \in \mathbb{R})$ for $j = 1, \ldots, k$.

Under our inclusion $\mathfrak{u} \subset \mathfrak{so}(2n,2)$, we can identify the Lie algebra $\mathfrak{u}$ with $\mathfrak{u}(n_1) \oplus \mathfrak{u}(n_2) \oplus \mathfrak{u}(1)$. Then one can see that generators of $\mathfrak{z}(\mathfrak{u}) \subset \mathfrak{so}(2n,2)$ are the following:

$$J_1^{(n_1)} = \frac{\overline{N_1} + N_1}{2} + \sum_{j=1}^{n_1-1}(-E_{2j,2j+1} + E_{2j+1,2j}), \tag{2.4}$$

$$J_2^{(n_2)} = \sum_{j=n_1}^{n-1}(-E_{2j,2j+1} + E_{2j+1,2j}), \tag{2.5}$$

$$J_3 = \frac{N_{2n} - \overline{N_{2n}}}{2}. \tag{2.6}$$

Here, E_{ij} $(0 \leq i, j \leq 2n+1)$ is the matrix with entry 1 in the i-th row and the j-column and 0 else. Further we set

$$\varepsilon_j := \begin{cases} 1 & (1 \leq j \leq 2n-1), \\ -1 & (j = 2n) \end{cases}$$

and

$$\begin{aligned} \overline{N}_j &:= E_{j,0} + E_{j,2n+1} - \varepsilon_j E_{0,j} + \varepsilon_j E_{2n+1,j} \qquad (1 \leq j \leq 2n), \\ N_j &:= E_{j,0} - E_{j,2n+1} - \varepsilon_j E_{0,j} - \varepsilon_j E_{2n+1,j} \qquad (1 \leq j \leq 2n). \end{aligned} \tag{2.7}$$

We note that $\overline{N}_j, N_j$ are elements of Lie algebra $\mathfrak{so}(2n,2)$ and $J_1^{(n_1)}, J_2^{(n_2)}, J_3$ are the following block diagonal matrices:

$$J_1^{(n_1)} = \mathrm{diag}(\underbrace{J, \ldots, J}_{n_1}, \underbrace{0, \ldots, 0}_{2n_2}, 0, 0), \tag{2.8}$$

$$J_2^{(n_2)} = \mathrm{diag}(\underbrace{0, \ldots, 0}_{2n_1}, \underbrace{J, \ldots, J}_{n_2}, 0, 0), \tag{2.9}$$

$$J_3 = \mathrm{diag}(\underbrace{0, \ldots, 0}_{2n}, J), \tag{2.10}$$

where we set

$$J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.11}$$

When $n_1 = 1$, we set

$$J_1 := J_1^{(1)} \quad \text{and} \quad J_2 := J_2^{(n-1)}. \tag{2.12}$$

Namely, we have

$$J_1 = \frac{\overline{N_1} + N_1}{2} \quad \text{and} \quad J_2 = \sum_{j=1}^{n-1} (-E_{2j,2j+1} + E_{2j+1,2j}). \tag{2.13}$$

It is easily seen that

$$J_1^{(n_1)} + J_2^{(n_2)} = J_1 + J_2 \tag{2.14}$$

for $n_1 = 1, \ldots, n-1$ and hence

$$d\pi_\varepsilon^c(J_1^{(n_1)}) + d\pi_\varepsilon^c(J_2^{(n_2)}) = d\pi_\varepsilon^c(J_1) + d\pi_\varepsilon^c(J_2) \tag{2.15}$$

for $n_1 = 1, \ldots, n-1$ and $\varepsilon \in \{+, -\}$ (see Section 2.4).

2.4. Infinitesimal action of the minimal representation

In order to describe D_1, D_2, D_3, we use the following linear transformations on the space $\mathcal{S}'(\mathbb{R}^{2n})$ of tempered distributions given in [12, Part III, Lemma 3.2] and [9, Section 2.4]:

$$d\hat{\varpi}(\overline{N}_j) := 2\sqrt{-1}\zeta_j, \tag{2.16}$$

$$d\hat{\varpi}(N_j) := \sqrt{-1}\left(-(n+1)\,\varepsilon_j \frac{\partial}{\partial \zeta_j} - E_\zeta \varepsilon_j \frac{\partial}{\partial \zeta_j} + \frac{1}{2}\zeta_j \Box_\zeta \right) \tag{2.17}$$

for $\overline{N}_j$, N_j $(1 \le j \le 2n) \in \mathfrak{so}(2n, 2)$, where we set

$$\Box_\zeta := \frac{\partial^2}{\partial \zeta_1^2} + \cdots + \frac{\partial^2}{\partial \zeta_{2n-1}^2} - \frac{\partial^2}{\partial \zeta_{2n}^2} \quad \text{and} \quad E_\zeta := \sum_{j=1}^{2n} \zeta_j \frac{\partial}{\partial \zeta_j}.$$

We denote by $L^2(C)_K$ the space of K-finite functions in $L^2(C)$, where K is the maximal compact subgroup $SO(2n) \times SO(2)$ of $G = SO_0(2n, 2)$.

The map $\imath : L^2(C)_K \to \mathcal{S}'(\mathbb{R}^{2n})$ defined by $u(\zeta) \mapsto u(\zeta)\delta(Q)$ is well-defined and injective (see [12, Part III, Section 3.4]). Then we have the following commutative diagram for any $X \in \mathfrak{so}(2n, 2)$:

$$\begin{array}{ccc} L^2(C)_K & \xrightarrow{\imath} & \mathcal{S}'(\mathbb{R}^{2n}) \\ d\pi^c(X) \Big\downarrow & & \Big\downarrow d\hat{\varpi}(X) \\ L^2(C)_K & \xrightarrow{\imath} & \mathcal{S}'(\mathbb{R}^{2n}). \end{array} \tag{2.18}$$

This gives the infinitesimal action $d\pi^c$ of $\mathfrak{so}(2n, 2)$.

2.5. The descriptions of D_1, D_2, D_3

We give the differential operators D_1, D_2, D_3 in Section 1.1 through the explicit descriptions of the infinitesimal actions $d\pi_+(J_1)$, $d\pi_+(J_2)$, $d\pi_+(J_3)$ on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$.

Let Δ be the Laplace operator $\sum_{j=1}^{2n-1} \partial^2/\partial x_j^2$. We recall from [9, Section 3.4] the following fact:

FACT 2.1. *$d\pi_+(J_3)$ is described as the following differential operator:*

$$d\pi_+(J_3) = \sqrt{-1}|x| \left(\frac{1}{4}\Delta - 1 \right). \tag{2.19}$$

Obviously, $\exp tJ_2$ is an element in $I_2 \times U(n-1) \times I_2 \subset R = I_1 \times SO(2n-1) \times I_2$. Hence, by the explicit action of the representation $(\pi^c, L^2(C))$ in [9, (2.2.3)], through (2.2) $\pi_+(\exp tJ_2)$ acts on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$ by the pull-back of the action as the rotation on $\mathbb{R}^{2n-1}$. Thus we get

PROPOSITION 2.2. *$d\pi_+(J_2)$ is described as the following differential operator:*

$$d\pi_+(J_2) = \sum_{j=1}^{n-1} \left(-x_{2j} \frac{\partial}{\partial x_{2j+1}} + x_{2j+1} \frac{\partial}{\partial x_{2j}} \right). \tag{2.20}$$

Let E be the Euler operator $\sum_{j=1}^{2n-1} x_j \partial/\partial x_j$. For $J_1 = \left(\overline{N_1} + N_1\right)/2$, one can describe $d\pi_+(J_1)$ as a differential operator on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$ by equations (2.3.15), (2.3.18) and (2.3.19) in [11].

PROPOSITION 2.3. *$d\pi_+(J_1)$ is described as the following differential operator:*

$$d\pi_+(J_1) = \sqrt{-1}\left(x_1 + \frac{x_1}{4}\Delta - \frac{1}{2}E\frac{\partial}{\partial x_1} - \frac{n-1}{2}\frac{\partial}{\partial x_1}\right). \tag{2.21}$$

Therefore, by Fact 2.1 and Propositions 2.2 and 2.3, (D_1, D_2, D_3) in Theorem 1.1 is nothing but $(d\pi_+(J_1), d\pi_+(J_2), d\pi_+(J_3))/\sqrt{-1}$.

REMARK 2.4. By (2.14) and (2.15), we have

$$\begin{aligned} d\pi_+(J_1^{(n_1)}) = \sqrt{-1}&\left(x_1 + \frac{x_1}{4}\Delta - \frac{1}{2}E\frac{\partial}{\partial x_1} - \frac{n-1}{2}\frac{\partial}{\partial x_1}\right) \\ &+ \sum_{j=1}^{n_1-1}\left(-x_{2j}\frac{\partial}{\partial x_{2j+1}} + x_{2j+1}\frac{\partial}{\partial x_{2j}}\right) \end{aligned} \tag{2.22}$$

and

$$d\pi_+(J_2^{(n_2)}) = \sum_{j=n_1}^{n-1}\left(-x_{2j}\frac{\partial}{\partial x_{2j+1}} + x_{2j+1}\frac{\partial}{\partial x_{2j}}\right). \tag{2.23}$$

3. Branching Laws of $\pi_\pm^c$ and π_+

In this section, we consider the K-type formula of $L^2(C_\pm)$, the restriction of $L^2(C_\pm)$ to U and the explicit irreducible decomposition of $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)_K$ with respect to the restriction $K \downarrow R$ in order to get the joint eigenvalues of (D_1, D_2, D_3).

3.1. K-type decomposition

In order to recall the K-type formula of $L^2(C_\pm)$, we review the basic facts of spherical harmonics.

For $m \in \mathbb{N}$ with $m \geq 2$, the space of spherical harmonics of degree $a \in \mathbb{N}$ is defined to be

$$\mathcal{H}^a(\mathbb{R}^m) = \{f \in C^\infty(S^{m-1}) : \Delta_{S^{m-1}} f = -a(a+m-2)f\}, \tag{3.1}$$

where $\Delta_{S^{m-1}}$ is the Laplace–Beltrami operator on the unit sphere S^{m-1} endowed with the standard Riemannian metric. Each $\mathcal{H}^a(\mathbb{R}^m)$ is irreducible as an $O(m)$-module, and still irreducible as an $SO(m)$-module ($m > 2$); however,

$$\mathcal{H}^a(\mathbb{R}^2) = \mathbb{C}e^{\sqrt{-1}a\theta} \oplus \mathbb{C}e^{-\sqrt{-1}a\theta} \quad \text{for} \quad a \geq 1 \tag{3.2}$$

as $SO(2)$-modules. In addition, we have an irreducible decomposition of $O(m-1)$-modules:

$$\mathcal{H}^a(\mathbb{R}^m)|_{O(m-1)} \simeq \bigoplus_{0 \leq b \leq a} \mathcal{H}^b(\mathbb{R}^{m-1}). \tag{3.3}$$

This is also an irreducible decomposition of $SO(m-1)$-modules ($m > 3$).

The K-type formula of $(\pi^{c}_{\pm}, L^2(C_\pm))$ is given as follows (see [9, Section 2.3] for example):

$$L^2(C_\pm)_K \simeq \bigoplus_{\substack{a,b\in\mathbb{N} \\ a+n=b+1}} \mathcal{H}^a(\mathbb{R}^{2n}) \boxtimes \mathbb{C}e^{\pm b\sqrt{-1}\theta}, \tag{3.4}$$

where each $\mathcal{H}^a(\mathbb{R}^{2n}) \boxtimes \mathbb{C}e^{\pm b\sqrt{-1}\theta}$ is an outer tensor product representation of K.

Likewise, the representation $(\pi^c, L^2(C))$ of $O(2n, 2)$ decomposes when it is restricted to its maximal compact subgroup as follows (see [12, Part I, Theorem 3.6.1]):

$$L^2(C)_{O(2n)\times O(2)} \simeq \bigoplus_{\substack{a,b\in\mathbb{N} \\ a+n=b+1}} \mathcal{H}^a(\mathbb{R}^{2n}) \boxtimes \mathcal{H}^b(\mathbb{R}^2). \tag{3.5}$$

By (3.2), each $\mathcal{H}^b(\mathbb{R}^2)$ in the right-hand side of (3.4) decomposes into $\mathbb{C}e^{\sqrt{-1}b\theta} \oplus \mathbb{C}e^{-\sqrt{-1}b\theta}$. This decomposition corresponds to (2.1):

$$L^2(C) = L^2(C_+) \oplus L^2(C_-),$$

for which the K-type formula is given by (3.4).

3.2. Branching formulas of the restriction of $\pi^c_\pm$ to $H = I_1 \times SO(2n-1) \times SO(2)$

By the K-type formula (3.4), we get the branching formula of the restriction of $\pi^c_\pm$ to H.

PROPOSITION 3.1. *When $(\pi^c_\pm, L^2(C_\pm))$ is restricted to H, its irreducible decomposition is multiplicity-free. Then we have an isomorphism as H-modules:*

$$L^2(C_\pm)_K \simeq \bigoplus_{\substack{l,a\in\mathbb{N}\\ l\le a}} \mathcal{H}^l(\mathbb{R}^{2n-1}) \boxtimes \mathbb{C}e^{\pm(a+n-1)\sqrt{-1}\theta}. \tag{3.6}$$

PROOF. Applying (3.3) with respect to the restriction $SO(2n) \downarrow SO(2n-1)$ $(n > 1)$ to the right-hand side of the K-type formula (3.4), we have

$$L^2(C_\pm)_K \simeq \bigoplus_{\substack{a,b,l\in\mathbb{N}\\ a+n=b+1\\ 0\le l\le a}} \mathcal{H}^l(\mathbb{R}^{2n-1}) \boxtimes \mathbb{C}e^{\pm b\sqrt{-1}\theta} \tag{3.7}$$

$$\simeq \bigoplus_{\substack{l,a\in\mathbb{N}\\ l\le a}} \mathcal{H}^l(\mathbb{R}^{2n-1}) \boxtimes \mathbb{C}e^{\pm(a+n-1)\sqrt{-1}\theta}. \tag{3.8}$$

Since $\mathcal{H}^l(\mathbb{R}^k)$ $(k \ge 3)$ and $e^{m\sqrt{-1}\theta}$ $(m \in \mathbb{Z})$ are irreducible and mutually inequivalent as $SO(k)$-modules and $SO(2)$-modules respectively, it is clear that the decomposition (3.6) is multiplicity-free. $\square$

REMARK 3.2. When $(\pi^c_\pm, L^2(C_\pm))$ is restricted to $R = I_1 \times SO(2n-1) \times I_2$, its irreducible decomposition is not multiplicity-free and the multiplicity of each irreducible summand is infinite. However, the explicit branching formula of $\pi_+|_R$ is given by Kobayashi–Mano [9] (see Fact 3.5).

3.3. Spherical harmonics on $\mathbb{C}^n$

First, we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by $z_j = \zeta_{2j-1} + \sqrt{-1}\zeta_{2j}$ $(j = 1, \dots, n)$. In order to get the branching formulas of the restrictions $\pi^c_\pm|_U$, we review the facts about the $U(n)$-modules $\mathcal{H}^{p,q}(\mathbb{C}^n)$ by fixing the inclusion $U(n) \subset SO(2n)$ corresponding to the coordinate change $z_j = \zeta_{2j-1} + \sqrt{-1}\zeta_{2j}$ $(j = 1, \dots, n)$.

Considering $\mathbb{C}^n$ as a real vector space, we denote by $\mathcal{P}(\mathbb{C}^n)$ the algebra over $\mathbb{C}$ of polynomials in $z_1, \dots, z_n$ and $\bar{z}_1, \dots, \bar{z}_n$. Then we see that $\mathcal{P}(\mathbb{C}^n) \simeq$

$\mathcal{P}(\mathbb{R}^{2n})$. The Laplace operator is written as

$$\Delta_\zeta = \sum_{j=1}^{2n} \frac{\partial^2}{\partial \zeta_j^2} = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_j \partial \bar{z}_j}. \tag{3.9}$$

Thus we set

$$H(\mathbb{C}^n) := \{p \in \mathcal{P}(\mathbb{C}^n) : \Delta_\zeta p = 0\}, \tag{3.10}$$

and each element in $H(\mathbb{C}^n)$ is called a harmonic polynomial on $\mathbb{C}^n$.

For $l, l' \in \mathbb{N}$, let $\mathcal{P}^{l,l'}(\mathbb{C}^n)$ be the subspace of $\mathcal{P}(\mathbb{C}^n)$ consisting of homogeneous polynomials $p(z) \equiv p(z, \bar{z})$ of degree l in z_j and of degree l' in $\bar{z}_j$:

$$p(cz) = c^l \bar{c}^{l'} p(z), \quad \text{for } c \in \mathbb{C} \setminus \{0\}. \tag{3.11}$$

Then we set

$$H^{l,l'}(\mathbb{C}^n) := H(\mathbb{C}^n) \cap \mathcal{P}^{l,l'}(\mathbb{C}^n). \tag{3.12}$$

Polynomials in $\mathcal{P}^{l,l'}(\mathbb{C}^n)$ are uniquely determined by their values on the unit sphere

$$S^{2n-1} \simeq \{z \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = 1\}. \tag{3.13}$$

Hence a space $\mathcal{H}^{l,l'}(\mathbb{C}^n)$ is defined by restricting functions from $H^{l,l'}(\mathbb{C}^n)$ onto S^{2n-1}. In particular, $\mathcal{H}^{l,l'}(\mathbb{C}^n)$ is regarded as a subspace of the space of spherical harmonics $\mathcal{H}^{l+l'}(\mathbb{R}^{2n})$.

Each space $\mathcal{H}^{l,l'}(\mathbb{C}^n)$ is invariant under the action of $U(n)$ on $\mathcal{H}^{l+l'}(\mathbb{R}^{2n})$, because $U(n) \subset SO(2n)$ commutes with $\tilde{J} = \mathrm{diag}(J, \ldots, J) \in GL(2n, \mathbb{R})$ (J is given in (2.11)). Hence each $\mathcal{H}^{l,l'}(\mathbb{C}^n)$ defines a representation of $U(n)$. We denote by $\mathcal{H}^{l,l'}(\mathbb{C}^n)$ this representation. It is known that the $\mathcal{H}^{l,l'}(\mathbb{C}^n)$ are irreducible and mutually inequivalent $U(n)$ modules (see [15, Section 11.2.2] for example). Further, the group $\mathbb{C}^\times \cap U(n)$ of scalar unitary matrices acts on $\mathcal{H}^{l,l'}(\mathbb{C}^n)$ as

$$c \mapsto c^{-l+l'} \tag{3.14}$$

for any $c = e^{\sqrt{-1}\theta} I_n \in \mathbb{C}^\times \cap U(n)$. We note that $\mathcal{H}^{l,l'}(\mathbb{C}^n)$ is nonzero for any $l, l' \geq 0$ if $n \geq 2$ and only

$$\mathcal{H}^{l,0}(\mathbb{C}) = \mathbb{C}z^l \quad \text{and} \quad \mathcal{H}^{0,l'}(\mathbb{C}) = \mathbb{C}\bar{z}^{l'} \tag{3.15}$$

are nonzero when $n = 1$, that is,

$$\mathcal{H}^{l,l'}(\mathbb{C}) = \{0\} \ \text{ for } l, l' \geq 1. \tag{3.16}$$

3.4. Multiplicity-free decompositions of $\pi^c_\pm$ under the restriction to U

For $U = U(n_1) \times U(n_2) \times U(1) \subset O(2n, 2)$ ($n_1, n_2 \geq 1$, $n_1 + n_2 = n$), by [13, Theorem A'] one can get the following proposition:

Proposition 3.3. *The restriction of $(\pi^c_\pm, L^2(C_\pm))$ to U is discretely decomposable and multiplicity-free. Then we have an isomorphism as U-modules:*

$$L^2(C_\pm)_K \simeq \bigoplus_{M \in S_1} \mathcal{H}^{m_1,m'_1}(\mathbb{C}^{n_1}) \boxtimes \mathcal{H}^{m_2,m'_2}(\mathbb{C}^{n_2}) \boxtimes \mathbb{C}e^{\pm m\sqrt{-1}\theta}, \tag{3.17}$$

where we set

$$S_1 := \left\{ M \in \mathbb{N}^5 : \begin{array}{c} M = (m_1, m'_1, m_2, m'_2, m) \\ f(m_1, m'_1, m_2, m'_2) + m - n + 1 \equiv 0 \mod 2 \\ g(m_1, m'_1, m_2, m'_2) \leq m - n + 1 \end{array} \right\} \tag{3.18}$$

and

$$f(m_1, m'_1, m_2, m'_2) := m_1 - m'_1 + m_2 - m'_2, \tag{3.19}$$

$$g(m_1, m'_1, m_2, m'_2) := m_1 + m'_1 + m_2 + m'_2. \tag{3.20}$$

In particular, Proposition 3.3 when $n_1 = 1$ and $n_1 = n_2 = 1$ is expressed as the following corollary:

Corollary 3.4. *For $U = U(1) \times U(n-1) \times U(1)$ when $n_1 = 1$, the branching formula of the restriction of $(\pi^c_\pm, L^2(C_\pm))$ to U is given as follows:*

$$L^2(C_\pm)_K \simeq \bigoplus_{M \in S_2} \mathbb{C}e^{m_1\sqrt{-1}\psi} \boxtimes \mathcal{H}^{m_2,m'_2}(\mathbb{C}^{n-1}) \boxtimes \mathbb{C}e^{\pm m\sqrt{-1}\theta}, \tag{3.21}$$

where we set

$$(3.22)\quad S_2 := \left\{ M \in \mathbb{Z} \times \mathbb{N}^3 : \begin{array}{c} M = (m_1, m_2, m_2', m) \\ m_1 + m_2 - m_2' + m - n + 1 \equiv 0 \mod 2 \\ |m_1| + m_2 + m_2' \leq m - n + 1 \end{array} \right\}.$$

In particular, for $U = U(1) \times U(1) \times U(1)$ *when* $n = 2$, *the branching formula of the restriction of* $(\pi_\pm^c, L^2(C_\pm))$ *to* U *is given as follows:*

$$(3.23)\qquad L^2(C_\pm)_K \simeq \bigoplus_{M \in S_3} \mathbb{C}e^{m_1\sqrt{-1}\psi} \boxtimes \mathbb{C}e^{m_2\sqrt{-1}\varphi} \boxtimes \mathbb{C}e^{\pm m\sqrt{-1}\theta},$$

where we set

(3.24)
$$S_3 := \left\{ M = (m_1, m_2, m) \in \mathbb{Z}^2 \times \mathbb{N} : \begin{array}{c} m_1 + m_2 + m - 1 \equiv 0 \mod 2 \\ |m_1| + |m_2| \leq m - 1 \end{array} \right\}.$$

3.5. Branching formula of the restriction of $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)_K$ to R and the action of $d\pi_+(J_1)$

We shall recall the explicit irreducible decomposition of $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)_K$ with respect to the restriction $K \downarrow R$ in order to get the explicit action of $d\pi_+(J_1)$.

We use the polar coordinate

$$(3.25)\quad \mathbb{R}_+ \times (0, \pi) \times S^{2n-3} \ni (r, \theta, \eta) \mapsto \begin{pmatrix} r\cos\theta \\ r\sin\theta\ \eta \end{pmatrix} \in \mathbb{R} \times (\mathbb{R}^{2n-2} \backslash \{0\}).$$

Obviously, $\mathbb{R} \times (\mathbb{R}^{2n-2} \backslash \{0\})$ is a open set of $\mathbb{R}^{2n-1}$. We write η of the polar coordinates (3.25) as:

$$(3.26)\qquad \eta = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 \\ \cdots \\ \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{2n-4} \cos\theta_{2n-3} \\ \sin\theta_1 \sin\theta_2 \ldots \sin\theta_{2n-4} \sin\theta_{2n-3} \end{pmatrix} \in S^{2n-3}.$$

We recall from Section 2.2 that $L^2(C_+) \simeq L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$. Here, we use the following explicit decomposition of the K-type formula (3.4) when it is restricted further to the subgroup $R = I_1 \times SO(2n-1) \times I_2$.

FACT 3.5 ([9, Section 3.2]). *Let $L_n^\alpha(x)$ be a Laguerre polynomial of degree n. Then we have the explicit irreducible decomposition with respect to the restriction $K \downarrow R$:*

$$L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)_K = \bigoplus_{\substack{a,l\in\mathbb{N}\\ l\leq a}} W_{a,l} \tag{3.27}$$

where we set

$$W_{a,l} := f_{a,l}\cdot \mathcal{H}^l(\mathbb{R}^{2n-1}), \tag{3.28}$$

and $f_{a,l}$ are defined by $f_{a,l}(r) := L_{a-l}^{2n-3+2l}(4r)\, r^l e^{-2r}$ on $\mathbb{R}^+$.

By Fact 7.1, $\mathcal{H}^l(\mathbb{R}^{2n-1})$ is spanned over $\mathbb{C}$ by the following basis:

$$\begin{aligned}\big\{C_l^{\nu_1,k}(\cos\theta)C_k^{\nu_2,k_1}(\cos\theta_1)\cdots \\ \cdots C_{k_{2n-5}}^{\nu_{2n-3},k_{2n-4}}(\cos\theta_{2n-4})e^{-\sqrt{-1}k_{2n-4}\theta_{2n-3}} : \\ k,k_1,\dots,k_{2n-3}\in\mathbb{Z} \text{ with } l\geq k\geq k_1\geq\cdots\geq k_{2n-5}\geq |k_{2n-4}|\big\},\end{aligned} \tag{3.29}$$

where $\nu_j := (2n-2-j)/2$ and $C_l^{\nu,m}(x)$ are associated Gegenbauer polynomials (see Section 7.1). Then, we have

$$\mathcal{H}^l(\mathbb{R}^{2n-1}) = \bigoplus_{0\leq k\leq l} C_l^{\nu_1,k}(\cos\theta)\,\mathcal{H}^k(\mathbb{R}^{2n-2}). \tag{3.30}$$

As is well-known, (3.30) corresponds to the irreducible decomposition under the restriction $SO(2n-1)\downarrow SO(2n-2)$. Hence we have

$$W_{a,l} = \bigoplus_{0\leq k\leq l} f_{a,l}(r)\, C_l^{\nu_1,k}(\cos\theta)\,\mathcal{H}^k(\mathbb{R}^{2n-2}). \tag{3.31}$$

For any $a,l,k\in\mathbb{N}$ with $a\geq l\geq k$, we define an element $u_{a,l}^k$ by

$$u_{a,l}^k := f_{a,l}(r)\, C_l^{\nu_1,k}(\cos\theta). \tag{3.32}$$

Here, we set $u_{a,l}^k = 0$ unless $a\geq l\geq k$. We note that each $u\in W_{a,l}$ can be expressed as a linear combination by $u_{a,l}^k\,\varphi_k$, where $\varphi_k\in\mathcal{H}^k(\mathbb{R}^{2n-2})$. By (2.21), one can get the action of $d\pi_+(J_1)$.

PROPOSITION 3.6. *For any $a, l, k \in \mathbb{N}$ with $a \geq l \geq k$, we have*

$$\begin{aligned}(3.33)\qquad & d\pi_+(J_1)\left(u^k_{a,l}\,\varphi_k\right) \\ &= -\frac{4(l+1-k)}{2n-3+2l}\sqrt{-1}u^k_{a,l+1}\,\varphi_k \\ &\quad -\frac{(2n-4+l+k)(2n-3+a+l)(a-l+1)}{4(2n-3+2l)}\sqrt{-1}u^k_{a,l-1}\,\varphi_k.\end{aligned}$$

We shall show this proposition in Appendix.

4. Proof of Theorem 1.1

In this section, we shall show the more general theorem than Theorem 1.1. Then we have Theorem 1.1 by this theorem, (2.3) and the explicit descriptions of $d\pi_+(J_1), d\pi_+(J_2)$ and $d\pi_+(J_3)$ as the differential operators on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$ (see Section 2.5).

THEOREM 4.1. *Suppose that $\varepsilon \in \{+,-\}$, and $n_1, n_2 \in \mathbb{N}$ with $n_1, n_2 \geq 1$ and $n_1 + n_2 = n$.*

(1) *The infinitesimal actions $d\pi^c_\varepsilon(J_1^{(n_1)}), d\pi^c_\varepsilon(J_2^{(n_2)})$ and $d\pi^c_\varepsilon(J_3)$ extend to self-adjoint operators on $L^2(C_\varepsilon)$.*

(2) *$d\pi^c_\varepsilon(J_1^{(n_1)}), d\pi^c_\varepsilon(J_2^{(n_2)})$ and $d\pi^c_\varepsilon(J_3)$ mutually commute on $L^2(C_\varepsilon)$.*

(3) *$d\pi^c_\varepsilon(J_1^{(n_1)}), d\pi^c_\varepsilon(J_2^{(n_2)})$ and $d\pi^c_\varepsilon(J_3)$ have only discrete spectra on $L^2(C_\varepsilon)$ respectively.*

(4) *The set of the joint eigenvalues of $(d\pi^c_\varepsilon(J_1^{(n_1)}), d\pi^c_\varepsilon(J_2^{(n_2)}), d\pi^c_\varepsilon(J_3))/\sqrt{-1}$ is given as follows:*

$$(4.1)\qquad \sigma := \left\{(x,y,z) \in \mathbb{Z}^3 : \begin{array}{c} x+y+z-n+1 \equiv 0 \mod 2 \\ |x|+|y| \leq -\varepsilon z - n + 1 \end{array}\right\}.$$

(5) *$d\pi^c_\varepsilon(J_1^{(n_1)}), d\pi^c_\varepsilon(J_2^{(n_2)})$ and $d\pi^c_\varepsilon(J_3)$ are algebraically independent.*

We note that the properties of $d\pi^c_+(J_3/\sqrt{-1})$ are given in [9, Sections 3.4] and $d\pi^c_-(J_3/\sqrt{-1}) = -d\pi^c_+(J_3/\sqrt{-1})$.

PROOF. (1) Since $(\pi^c_\varepsilon, L^2(C_\varepsilon))$ is a unitary representation of $G = SO_0(2n,2)$, the infinitesimal action $\sqrt{-1}d\pi^c_\varepsilon(X)$ for any $X \in \mathfrak{so}(2n,2)$ extends to a self-adjoint operator on $L^2(C_\varepsilon)$. Hence, (1) holds.

(2) This assertion follows from the fact that $J_1^{(n_1)}, J_2^{(n_2)}$ and J_3 are the generators of the center $\mathfrak{z}(\mathfrak{u})$.

(3) and (4) Hereafter, we identify each U-module

$$\mathcal{H}^{m_1,m_1'}(\mathbb{C}^{n_1}) \boxtimes \mathcal{H}^{m_2,m_2'}(\mathbb{C}^{n_2}) \boxtimes \mathbb{C}e^{\varepsilon m\sqrt{-1}\theta} \tag{4.2}$$

appearing in the summation of the right-hand side of (3.17) with the corresponding subspace of $L^2(C_\varepsilon)_K$.

By the definitions, we see that $\exp tJ_1^{(n_1)} \in U(n_1)$, $\exp tJ_2^{(n_2)} \in U(n_2)$ and $\exp tJ_3 \in U(1)$. Hence, by (3.14) and (3.17) we see that each infinitesimal action $d\pi_\varepsilon^c(J_1^{(n_1)})$, $d\pi_\varepsilon^c(J_2^{(n_2)})$ and $d\pi_\varepsilon^c(J_3)$ acts on the space (4.2) as a scalar multiplication as follows:

$$\begin{aligned}(d\pi_\varepsilon^c(J_1^{(n_1)}), d\pi_\varepsilon^c(J_2^{(n_2)}), d\pi_\varepsilon^c(J_3))& \\ = (-m_1+m_1', -m_2+m_2', -\varepsilon m)\sqrt{-1}.&\end{aligned} \tag{4.3}$$

Thus, (3) follows from (4.3).

Further, by the commutativity (2), the infinitesimal actions (4.3) and S_1 (see (3.18)), we see that the map $S_1 \to \sigma$ defined by

$$(m_1, m_1', m_2, m_2', m) \mapsto (-m_1+m_1', -m_2+m_2', -\varepsilon m)$$

is surjective. Hence, we have shown the assertion (4).

When $U = U(1)\times U(n-1)\times U(1)$ and $U(1)\times U(1)\times U(1)$, we also get the same σ.

(5) Suppose that there exists a non-zero polynomial $P \in \mathbb{C}[X_1, X_2, X_3]$ such that

$$P(d\pi_\varepsilon^c(J_1^{(n_1)}), d\pi_\varepsilon^c(J_2^{(n_2)}), d\pi_\varepsilon^c(J_3)) \equiv 0 \tag{4.4}$$

on $L^2(C_\varepsilon)_K$. For any $(x,y,z) \in \sigma$, there exists a non-zero joint eigenfunction $h \in L^2(C_\varepsilon)_K$ with respect to an eigenvalue $(x,y,z)\sqrt{-1}$ of $(d\pi_\varepsilon^c(J_1^{(n_1)})$, $d\pi_\varepsilon^c(J_2^{(n_2)})$, $d\pi_\varepsilon^c(J_3))$. Then we have

$$P(d\pi_\varepsilon^c(J_1^{(n_1)}), d\pi_\varepsilon^c(J_2^{(n_2)}), d\pi_\varepsilon^c(J_3))h = P(x\sqrt{-1}, y\sqrt{-1}, z\sqrt{-1})h \tag{4.5}$$

and by assumption

$$P(x\sqrt{-1}, y\sqrt{-1}, z\sqrt{-1})h = 0. \tag{4.6}$$

Hence, we see that $P(x\sqrt{-1}, y\sqrt{-1}, z\sqrt{-1}) = 0$ for any $(x, y, z) \in \sigma$.

We fix $x_0, y_0 \in \mathbb{Z}$. Let $\tilde{P}$ be a polynomial of one variable defined by $\tilde{P}(Z) := P(x_0\sqrt{-1}, y_0\sqrt{-1}, Z)$ and $V_{\tilde{P}}$ the set of the roots of $\tilde{P}$. It is well known that $\sharp V_{\tilde{P}} \leq \deg \tilde{P}$, where $\sharp V_{\tilde{P}}$ is the cardinality of $V_{\tilde{P}}$ and $\deg \tilde{P}$ is the degree of $\tilde{P}$. On the other hand, let σ_{x_0,y_0} be the subset of σ for (x_0, y_0) defined by

$$\sigma_{x_0,y_0} = \left\{ (x_0, y_0, z) \in \mathbb{Z}^3 : \begin{matrix} x_0 + y_0 + z - n + 1 \equiv 0 \mod 2 \\ |x_0| + |y_0| \leq -\varepsilon z - n + 1 \end{matrix} \right\}. \tag{4.7}$$

Then $\sharp\sigma_{x_0,y_0}$ is infinite and $P(x_0\sqrt{-1}, y_0\sqrt{-1}, z\sqrt{-1}) = 0$ for any $(x_0, y_0, z) \in \sigma_{x_0,y_0}$ because $\sigma_{x_0,y_0} \subset \sigma$. However, those contradict the fact that $\sharp V_{\tilde{P}} \leq \deg \tilde{P}$. Hence, we see that $\tilde{P} = 0$. Moreover, we get the equations $P_j|_{\sqrt{-1}\mathbb{Z}^2} = 0$ for $0 \leq j \leq \deg \tilde{P}$, where P_j are the polynomials of two variables defined by

$$P(X_1, X_2, X_3) = \sum_{0 \leq j \leq \deg \tilde{P}} P_j(X_1, X_2) X_3^j. \tag{4.8}$$

Since $P_j|_{\sqrt{-1}\mathbb{Z}^2} = 0$, one can see that $P_j = 0$. Thus, we have $P = 0$.

Hence, for any non-zero polynomial $P \in \mathbb{C}[X_1, X_2, X_3]$, the differential operator $P(d\pi^c_\varepsilon(J_1^{(n_1)}), d\pi^c_\varepsilon(J_2^{(n_2)}), d\pi^c_\varepsilon(J_3))$ must be non-zero. Namely, $d\pi^c_\varepsilon(J_1^{(n_1)})$, $d\pi^c_\varepsilon(J_2^{(n_2)})$, $d\pi^c_\varepsilon(J_3)$ are algebraically independent. $\square$

Remark 4.2. Through (2.3), Theorem 1.1 is given by the results of Fact 2.1, Propositions 2.2, 2.3 and Theorem 4.1 if $\varepsilon = +$.

5. The Commutative Algebras of Invariant Differential Operators

In this section, we prove Theorem 1.6 and Corollary 1.7. However, Proposition 5.4 used in the proof is shown in next section.

5.1. Proofs of Theorem 1.6 and Corollary 1.7

In order to show Theorem 1.6, we consider the algebra $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^R$ (see the definition in Section 1.2), where $R = I_1 \times SO(2n-1) \times I_2$. Then we have the inclusive relation:

$$\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^R \supset \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H, \tag{5.1}$$

where $H = I_1 \times SO(2n-1) \times SO(2)$. Since D_3 is given by the infinitesimal action of $I_{2n} \times U(1) \simeq I_{2n} \times SO(2)$ (see Section 2.5), D_3 commutes with the action of H. Hence, we get

$$\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H = \left\{ D \in \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^R : [D, D_3] = 0 \right\}. \tag{5.2}$$

Since D_4 is the Casimir operator with respect to R (see Section 1.2), it also commutes with the action of H. In particular, D_4 commutes with D_3. The following proposition leads to Theorem 1.6.

PROPOSITION 5.1. *If a differential operator $D \in \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^R$ satisfies $[D, D_3] = 0$, then D belongs to $\mathbb{C}[D_3, D_4]$.*

We show this proposition in the next subsection after proving Theorem 1.6.

PROOF OF THEOREM 1.6. By the equation (5.2) and Proposition 5.1, we have

$$\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^H \subset \mathbb{C}[D_3, D_4]. \tag{5.3}$$

The reverse inclusive relation of the above clearly holds. Therefore, we have shown Theorem 1.6. $\square$

Next, we shall show Corollary 1.7.

PROOF OF COROLLARY 1.7. Let D be an element in $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^K$. By Theorem 1.6 (or Proposition 5.1), there exists a polynomial $Q \in \mathbb{C}[x, y]$ such that $D = Q(D_3, D_4)$. On the other hand, by Proposition 3.6 we have

$$D_1(v_{a,l}) = (\sqrt{-1})^{-1} d\pi_+(J_1)(v_{a,l}) = c_1(l) v_{a,l+1} + c_2(a,l) v_{a,l-1} \tag{5.4}$$

for $a \geq l \geq 0$, where we set

$$v_{a,l} := u^0_{a,l} \varphi_0, \tag{5.5}$$

$$c_1(l) := -\frac{4(l+1)}{2n-3+2l}, \tag{5.6}$$

$$c_2(a,l) := -\frac{(2n-4+l)(2n-3+a+l)(a-l+1)}{4(2n-3+2l)}. \tag{5.7}$$

The restrictions of the differential operators D_3 and D_4 to $W_{a,l}$ are scalar operators since D_3 and D_4 commute with the action of R (see Fact 3.5). In fact, they act by $\chi_3(a) := -a-n+1$ and $\chi_4(l) := -l(l+2\nu)$ respectively, where we set $2\nu = 2\nu_1 = 2n-3$ (see Section 7.3).

Since $[D, D_1] = 0$, for $a \geq l \geq 0$, we have

$$\begin{aligned}
0 &= [D, D_1](v_{a,l}) \\
&= DD_1(v_{a,l}) - D_1 D(v_{a,l}) \\
&= c_1(l)\,(Q(\chi_3(a), \chi_4(l+1)) - Q(\chi_3(a), \chi_4(l)))\, v_{a,l+1} \\
&\quad + c_2(a,l)\,(Q(\chi_3(a), \chi_4(l-1)) - Q(\chi_3(a), \chi_4(l)))\, v_{a,l-1}.
\end{aligned} \tag{5.8}$$

Here, (5.8) if $a = l = 1$ equals

$$0 = c_2(1,1)\,(Q(\chi_3(1), \chi_4(0)) - Q(\chi_3(1), \chi_4(1)))\, v_{1,0} \tag{5.9}$$

because $v_{a,l+1} = 0$. Since $c_2(1,1) \neq 0$ and $v_{1,0} \neq 0$, we have $Q(\chi_3(1), \chi_4(0)) = Q(\chi_3(1), \chi_4(1))$. If $a \geq 2$ and $1 \leq l \leq a-1$, then one can see that

$$Q(\chi_3(a), \chi_4(l+1)) = Q(\chi_3(a), \chi_4(l)) = Q(\chi_3(a), \chi_4(l-1)) \tag{5.10}$$

because $\{v_{a,l+1}, v_{a,l-1}\}$ is linearly independent and $c_1(l) \neq 0$, $c_2(a,l) \neq 0$. Hence, we obtain

$$Q(\chi_3(a), \chi_4(l)) = Q(\chi_3(a), \chi_4(0)) = Q(\chi_3(a), 0) \tag{5.11}$$

for $a \geq l \geq 0$. Here we note that (5.11) if $a = 0$ clearly holds.

We set $F(x,y) := Q(x,y) - Q(x,0)$ and $\Gamma := \{(-a-n+1, -l(l+2\nu)) \in \mathbb{Z}^2 : a, l \in \mathbb{N},\ a \geq l\}$. Then (5.11) means that $F|_\Gamma = 0$.

We fix an $l \geq 0$. The polynomial $F(x, -l(l+2\nu))$ of one variable x vanishes on the countable set $\{m \in \mathbb{Z} : m \leq -n-l+1\}$. Hence, $F(x, -l(l+2\nu))$ is zero.

We represent $F(x,y)$ as

$$F(x,y) = \sum_{0 \leq j \leq k} F_j(y) x^j \tag{5.12}$$

by polynomials $F_j(y)$ of one variable y. Then each $F_j(y)$ vanishes on the countable set $\{-l(l+2\nu) : l \in \mathbb{N}\}$ and $F_j(y)$ is zero.

Thus $F(x,y)$ is zero. Therefore we see that

$$D = Q(D_3, D_4) = Q(D_3, 0), \tag{5.13}$$

and have proved Corollary 1.7. □

5.2. Proof of Proposition 5.1

We identify $\mathbb{R}^{2n-1} \setminus \{0\}$ with $\mathbb{R}^+ \times S^{2n-2}$ by polar coordinates:

$$\mathbb{R}^+ \times S^{2n-2} \ni (r, \omega) \mapsto r\omega \in \mathbb{R}^{2n-1} \setminus \{0\}. \tag{5.14}$$

We remark that $S^{2n-2} = SO(2n-1)/SO(2n-2)$ is a rank 1 Riemannian symmetric space and hence the invariant differential operators on it are generated by the Laplacian. Since $\pi_+(R)$ acts on $L^2(\mathbb{R}^{2n-1}, |x|^{-1}dx)$ as the action induced by the natural action of $SO(2n-1)$ on $\mathbb{R}^{2n-1}$, we have the following lemma.

Lemma 5.2. *Under the polar coordinate, we have*

$$\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^R \simeq \mathcal{D}(\mathbb{R}^+) \otimes \mathbb{C}[\Delta_S], \tag{5.15}$$

where $\mathcal{D}(\mathbb{R}^+)$ is the algebra of the differential operators on $\mathbb{R}^+$ and Δ_S is the abbreviated notation instead of $\Delta_{S^{2n-2}}$.

Thus we can regard each element in $\mathcal{D}(\mathbb{R}^+) \otimes \mathbb{C}[\Delta_S]$ as a differential operator in $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^R$ under the identification (5.15). In particular, $1 \otimes \Delta_S$ is identified with D_4. For the sake of simplicity, we will omit $\otimes$ for each element in $\mathcal{D}(\mathbb{R}^+) \otimes \mathbb{C}[\Delta_S]$.

By the polar coordinates (5.14), the Euler operator E is described as $r\partial/\partial r$. We set $\vartheta := r\, d/dr$ in $\mathcal{D}(\mathbb{R}^+)$. Then ϑ is regarded as E under the identification (5.15). Simple computation shows that

$$\left(\frac{d}{dr}\right)^j = r^{-j}\vartheta(\vartheta - 1)\cdots(\vartheta - j + 1), \quad \text{for} \quad j \in \mathbb{N} \setminus \{0\}. \tag{5.16}$$

Hence we have the following lemma:

Lemma 5.3. *$\mathcal{D}(\mathbb{R}^+)$ is generated by ϑ and 1 over $C^\infty(\mathbb{R}^+)$. Namely,*

$$\mathcal{D}(\mathbb{R}^+) = \left\{ \sum_{j=0}^{m} f_j \vartheta^j : m \in \mathbb{N}, f_j \in C^\infty(\mathbb{R}^+) \right\}. \tag{5.17}$$

For any a, $b \in \mathbb{R}$, we set $A := r^{-1}\vartheta^2 + ar^{-1}\vartheta + br$ and $B := r^{-1}$. When $a = 2n-3$ and $b = -4$, we have

$$\begin{aligned} D_3 &= A + B\Delta_S \\ &= r^{-1}\vartheta^2 + ar^{-1}\vartheta + br + r^{-1}\Delta_S. \end{aligned} \tag{5.18}$$

Let $\operatorname{ad} A$ be the linear mapping $\operatorname{ad} A : \mathcal{D}(\mathbb{R}^+) \to \mathcal{D}(\mathbb{R}^+)$ defined by $\operatorname{ad} A(D) := [A, D] = AD - DA$. It is clear that $\operatorname{Ker} \operatorname{ad} A \supset \mathbb{C}[A]$.

PROPOSITION 5.4. *If $b \neq 0$, then we have* $\operatorname{Ker} \operatorname{ad} A = \mathbb{C}[A]$.

Proposition 5.4 is shown in Section 6.

REMARK 5.5. Suppose that $b = 0$ and $a = \pm 1/2$. Simple computation shows that $\operatorname{Ker} \operatorname{ad} A = \mathbb{C}[D_\pm] \supsetneq \mathbb{C}[A]$, where

$$D_\pm = \begin{cases} r^{-\frac{1}{2}}\vartheta + \frac{1}{2}r^{-\frac{1}{2}} & \left(a = \frac{1}{2}\right) \\ r^{-\frac{1}{2}}\vartheta & \left(a = -\frac{1}{2}\right). \end{cases} \tag{5.19}$$

Further, we see that $(D_\pm)^2 = A$.

Let $\mathbb{C}[\Delta_S]^m$ be the set of all differential operators $Q(\Delta_S)$, where $Q(x)$ is a polynomial and its degree is less than or equal to m for $m \in \mathbb{N}$. Let $O(P)$ be the order of the differential operator P. The following lemma immediately shows Proposition 5.1.

LEMMA 5.6. *If P is an element of $\mathcal{D}(\mathbb{R}^+) \otimes \mathbb{C}[\Delta_S]$ satisfying $[D_3, P] = 0$, then there exists a polynomial Q such that $P = Q(D_3, \Delta_S)$.*

For the proof of Lemma 5.6, we show an inequality (5.24) for any element of $\mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^R$. By Lemma 5.2, each $P \in \mathcal{D}(\mathbb{R}^{2n-1} \setminus \{0\})^R$ is described as

$$P = \sum_{j=0}^{m} P_j (\Delta_S)^j \tag{5.20}$$

for some $m \in \mathbb{N}$ and some $P_j \in \mathcal{D}(\mathbb{R}^+)$. If $[D_3, P] = 0$, then we have

$$[A, P_0] = 0, \tag{5.21}$$

$$[A, P_j] = [P_{j-1}, B], \quad (1 \leq j \leq m) \tag{5.22}$$

$$0 = [P_m, B]. \tag{5.23}$$

Since $B(= r^{-1})$ is not a constant function, we can easily see by induction that $O(P_j) \le 2(m-j)$ for $0 \le j \le m$. In particular, we have

$$O(P_0) \le 2m. \tag{5.24}$$

Proof of Lemma 5.6. We write $\deg(Q)$ for the degree of a one-variable polynomial Q. We prove this lemma by induction on m. When $m = 0$, we take a differential operator $P \in \mathcal{D}(\mathbb{R}^+)$ satisfying $[D_3, P] = 0$. By (5.24), we have $O(P) = 0$. Moreover, by $[A, P] = 0$ and Proposition 5.4, there exists a polynomial $\hat{Q}$ such that $P = \hat{Q}(A)$. Thus P is a constant. Hence we have shown the lemma when $m = 0$.

Assume that the lemma holds for $l \in \mathbb{N}$ with $0 \le l < m$. Let P be an element in $\mathcal{D}(\mathbb{R}^+) \otimes \mathbb{C}[\Delta_S]^m$ satisfying $[D_3, P] = 0$. Then P is described as (5.20) and (5.21) holds. Hence, by Proposition 5.4 there exists a polynomial Q such that $P_0 = Q(A)$. By (5.24), we have $O(P_0) \le 2m$. Further, we see that $\deg(Q) \le m$ because $O(A) = 2$ and $O(P_0) = O(Q(A)) = 2\deg(Q)$. Next we consider the differential operator $Q(D_3)$. Since $D_3 = A + B\Delta_S$, there exist differential operators ν_j $(1 \le j \le q)$ on $\mathbb{R}^+$ such that

$$Q(D_3) = Q(A) + \nu_1 \Delta_S + \cdots + \nu_q \Delta_S^q,$$

where $q = \deg(Q)$. Now we set

$$\tilde{P} := \sum_{j=1}^{q} (P_j - \nu_j)(\Delta_S)^{j-1} + \sum_{j=q+1}^{m} P_j (\Delta_S)^{j-1}.$$

Then we can easily see that $P - Q(D_3) = \Delta_S \tilde{P}$ and $[D_3, \tilde{P}] = 0$. By the assumption, there exists a polynomial $\tilde{Q}$ such that $\tilde{P} = \tilde{Q}(D_3, \Delta_S)$. Hence we have $P = Q(D_3) + \Delta_S \tilde{Q}(D_3, \Delta_S)$ and proved the lemma. □

Proposition 5.1 immediately follows from Lemma 5.6.

6. Proof of Proposition 5.4

For any $a, b \in \mathbb{R}$ with $b \ne 0$, we consider $A = r^{-1}\vartheta^2 + ar^{-1}\vartheta + br$. Each $D \in \ker \operatorname{ad} A$ such that $O(D) = m$ is described as

$$D = \sum_{0 \le j \le m} f_j \vartheta^j \tag{6.1}$$

by $f_j \in C^\infty(\mathbb{R}^+)$. Since the coefficient of each ϑ^j in $[D, A] = 0$ must be zero, one can get the following equations:

LEMMA 6.1. *We have*

$$(2\vartheta + m)(f_m) = 0, \tag{6.2}$$

$$\begin{aligned}(2\vartheta + \nu - 1)(f_{\nu-1}) \\ = -\left\{\vartheta^2 + a\vartheta + a\binom{\nu}{\nu-1} - \binom{\nu}{\nu-2}\right\}(f_\nu) \\ - \sum_{\nu+1\le j\le m}\left\{a\binom{j}{\nu-1} - \binom{j}{\nu-2}\right\}(-1)^{j-\nu} f_j \\ + br^2 \sum_{\nu+1\le j\le m}\binom{j}{\nu} f_j,\end{aligned} \tag{6.3}$$

where $\nu \in \mathbb{N}$ *with* $1 \le \nu \le m$, *and*

$$\vartheta(\vartheta + a)(f_0) = br^2 \sum_{1\le j\le m} f_j. \tag{6.4}$$

Here, $\binom{p}{q}$ *is the binomial coefficient for* $p, q \in \mathbb{N}$ *with* $p \ge q$.

Then the following proposition holds.

PROPOSITION 6.2. *Let D be an element in* $\operatorname{Ker}\operatorname{ad} A$. *Then $O(D)$ is even.*

Namely, there is no operator $D \in \mathcal{D}(\mathbb{R}^+)$ such that $[D, A] = 0$ and $O(D)$ is odd. We shall show Proposition 6.2 after the proof of Proposition 5.4.

PROOF OF PROPOSITION 5.4. It is sufficient to prove that $\operatorname{Ker}\operatorname{ad} A \subset \mathbb{C}[A]$. By Proposition 6.2, the order $O(D) = m$ of each $D \in \operatorname{Ker}\operatorname{ad} A$ is even. D is described as (6.1). By (6.2), there is $c_m \in \mathbb{C}$ such that $f_m = c_m r^{-m/2}$.

We use induction on m. If $m = 0$, then (6.2) leads to the equation $D = c_0$ for some $c_0 \in \mathbb{C}$. Thus, $D \in \mathbb{C}[A]$. We assume that for any $Q \in \operatorname{Ker}\operatorname{ad} A$ with $O(Q) < m$, Q belongs to $\mathbb{C}[A]$. Here, we consider $D - c_m A^{m/2}$. Then we see that $O(D-c_mA^{m/2}) < m$ and $[A, D-c_mA^{m/2}] = 0$. By the assumption, $D - c_mA^{m/2}$ belongs to $\mathbb{C}[A]$. Hence we have shown the proposition. □

It is difficult to get the explicit solutions f_j of the system of the linear differential equations (6.2), (6.3) and (6.4). However, we can prove Proposition 6.2 by using differential operators which annihilate f_j. We use the following notation.

Definition 6.3. For $\alpha, \beta \in \mathbb{Z}$ with $\alpha \leq \beta$, we define the following differential operators in $\mathcal{D}(\mathbb{R}^+)$:

$$[\alpha \sim \beta] := (2\vartheta + \alpha)(2\vartheta + \alpha + 1) \cdots (2\vartheta + \beta), \tag{6.5}$$

$$[\alpha] := [\alpha \sim \alpha] = (2\vartheta + \alpha), \tag{6.6}$$

where we set $[\alpha \sim \beta] := \mathrm{id}$ when $\alpha > \beta$.

Simple computation shows the following equation:

Lemma 6.4. *For $\lambda \in \mathbb{Z}$, we have*

$$[\lambda] \circ r^2 = r^2 \circ [\lambda + 4]. \tag{6.7}$$

By the assumption of Proposition 6.2, in this subsection we put $m = 2l + 1$ for some $l \in \mathbb{N}$. For any $\mu \in \mathbb{N}$ with $0 \leq \mu \leq l$, we set

$$T_{2\mu+1} := \prod_{0 \leq k \leq [\frac{l-\mu}{2}]} [4\mu + 4k - 2l + 1][4\mu + 4k - 2l + 3 \sim 2l - 4k + 1], \tag{6.8}$$

$$T_{2\mu} := \prod_{0 \leq k \leq [\frac{l-\mu}{2}]} [4\mu + 4k - 2l \sim 2l - 4k + 1], \tag{6.9}$$

where $[(l - \mu)/2]$ is the greatest integer not greater than $(l - \mu)/2$.

Then, by the definitions we have

$$T_{2l} = T_{2l+1} \circ [2l] \tag{6.10}$$

and

$$T_{2\mu} = T_{2\mu+1} \circ \prod_{0 \le k \le [\frac{l-\mu}{2}]} [4\mu + 4k - 2l][4\mu + 4k - 2l + 2], \tag{6.11}$$

$$T_{2\mu+1} = T_{2\mu+2} \circ \begin{cases} \displaystyle\prod_{0 \le k \le [\frac{l-\mu}{2}]} [4\mu + 4k - 2l + 1][4\mu + 4k - 2l + 3], & (l - \mu : \text{odd}) \\ \displaystyle [2\mu + 1] \prod_{0 \le k \le [\frac{l-\mu}{2}]-1} [4\mu + 4k - 2l + 1][4\mu + 4k - 2l + 3], & \\ & (l - \mu : \text{even}) \end{cases} \tag{6.12}$$

for any μ with $0 \le \mu \le l - 1$.

The right-hand sides of (6.10), (6.11) and (6.12) are the products of pairwise commuting differential operators. By (6.10), (6.11) and (6.12), we get the following lemma:

Lemma 6.5. *For $\mu, \nu \in \mathbb{N}$ with $0 \le \nu \le \mu \le 2l + 1$, if $T_\mu(f) = 0$ for some $f \in C^\infty(\mathbb{R}^+)$, then $T_\nu(f) = 0$.*

Let f_j be the solutions of the system of the differential equations in Lemma 6.1. By Lemma 6.5 we have the following lemma:

Lemma 6.6. *For any $\mu \in \mathbb{N}$ with $0 \le \mu \le 2l + 1$, we have*

$$T_\mu(f_\mu) = 0. \tag{6.13}$$

Proof. We show this lemma by induction on μ decreasing. When $\mu = 2l + 1$, we see that

$$T_{2l+1}(f_{2l+1}) = [2l + 1](f_{2l+1}) = 0 \tag{6.14}$$

by (6.2). Further, by (6.2) and (6.3) we have

$$\begin{aligned} T_{2l}(f_{2l}) &= [2l + 1][2l](f_{2l}) \\ &= -\left\{ \vartheta^2 + a\vartheta + a \binom{2l+1}{2l} - \binom{2l+1}{2l-1} \right\} [2l + 1](f_{2l+1}) \\ &= 0. \end{aligned} \tag{6.15}$$

Assume that $T_\nu(f_\nu) = 0$ for any $\nu \in \mathbb{N}$ with $2\mu \leq \nu \leq 2l+1$. We shall show that $T_{2\mu-1}(f_{2\mu-1}) = T_{2\mu-2}(f_{2\mu-2}) = 0$. By (6.12), we have

$$(6.16)\quad T_{2\mu-1} = T_{2\mu} \circ \begin{cases} \displaystyle\prod_{0\leq k\leq[\frac{l-\mu+1}{2}]} [4\mu+4k-2l-3][4\mu+4k-2l-1], & (l-\mu : \text{even}) \\ [2\mu-1] \displaystyle\prod_{0\leq k\leq[\frac{l-\mu+1}{2}]-1} [4\mu+4k-2l-3][4\mu+4k-2l-1] & (l-\mu : \text{odd}) \end{cases}$$
$$= U_{2\mu-1} \circ T_{2\mu} \circ [2\mu-1],$$

where we set

$$(6.17)\quad U_{2\mu-1} := \begin{cases} [2\mu-3] \displaystyle\prod_{0\leq k\leq[\frac{l-\mu}{2}]-1} [4\mu+4k-2l-3][4\mu+4k-2l-1], & (l-\mu : \text{even}) \\ \displaystyle\prod_{0\leq k\leq[\frac{l-\mu}{2}]} [4\mu+4k-2l-3][4\mu+4k-2l-1]. & (l-\mu : \text{odd}) \end{cases}$$

Hence, by the assumption $T_\nu(f_\nu) = 0$ $(\nu \geq 2\mu)$ and Lemma 6.5, $T_{2\mu}$ in the right-hand side of (6.16) annihilates the above terms in the right-hand side of the following equation:

$$(6.18)\qquad \begin{aligned}[2\mu-1](f_{2\mu-1}) &= C_{2\mu}(f_{2\mu}) + C_{2\mu+1}f_{2\mu+1} + \cdots + C_{2l+1}f_{2l+1} \\ &\quad + r^2(C'_{2\mu+1}f_{2\mu+1} + \cdots + C'_{2l+1}f_{2l+1}),\end{aligned}$$

where each C_j (resp. C'_j) is a differential operator (resp. a constant) given by (6.3) and commutes with all $[\alpha]$.

Next, by the definition we can get

$$(6.19)\qquad U_{2\mu-1} \circ T_{2\mu} = V_{2\mu-1} \circ (r^2 T_{2\mu+1} r^{-2}),$$

where $V_{2\mu-1}$ is the following differential operator:

$$
(6.20)\qquad \begin{cases} \displaystyle\prod_{0\le k\le[\frac{l-\mu}{2}]} [2l-4k-2 \sim 2l-4k+1], & (l-\mu : \text{odd}) \\ [2\mu-2][2\mu \sim 2\mu+1] \\ \qquad \times \displaystyle\prod_{0\le k\le[\frac{l-\mu}{2}]-1} [2l-4k-2 \sim 2l-4k+1]. & (l-\mu : \text{even}) \end{cases}
$$

Then we have

$$
(6.21)\qquad T_{2\mu-1} = V_{2\mu-1} \circ (r^2 T_{2\mu+1} r^{-2}) \circ [2\mu-1].
$$

Hence, by (6.16), (6.18) and Lemma 6.4 we have

$$
(6.22)\qquad T_{2\mu-1}(f_{2\mu-1}) = V_{2\mu-1} \circ r^2 T_{2\mu+1}(C'_{2\mu+1} f_{2\mu+1} + \cdots + C'_{2l+1} f_{2l+1}).
$$

By the assumption and Lemma 6.5, it holds that $T_{2\mu+1}(f_\nu) = 0$ $(\nu \ge 2\mu+1)$. Hence we see that $T_{2\mu-1}(f_{2\mu-1}) = 0$.

Next, by (6.11) we can get

$$
(6.23)\qquad T_{2\mu-2} = U'_{2\mu-2} \circ T_{2\mu-1} \circ [2\mu-2],
$$

where we set

$$
(6.24)\qquad U'_{2\mu-2} := \begin{cases} \displaystyle\prod_{0\le k\le[\frac{l-\mu}{2}]} [4\mu+4k-2l-4] \\ \qquad \times \displaystyle\prod_{0\le k\le[\frac{l-\mu+1}{2}]} [4\mu+4k-2l-2], & (l-\mu : \text{odd}) \\ \displaystyle\prod_{0\le k\le[\frac{l-\mu+1}{2}]} [4\mu+4k-2l-4] \\ \qquad \times \displaystyle\prod_{0\le k\le[\frac{l-\mu}{2}]-1} [4\mu+4k-2l-2]. & (l-\mu : \text{even}) \end{cases}
$$

Hence, by the assumption $T_\nu(f_\nu) = 0$ $(\nu \ge 2\mu)$, Lemma 6.5 and $T_{2\mu-1}(f_{2\mu-1}) = 0$ shown above, $T_{2\mu-1}$ in the right-hand side of (6.23) annihilates the above terms in the right-hand side of the following equation:

$$
(6.25)\qquad \begin{aligned} [2\mu-2](f_{2\mu-2}) = {} & C_{2\mu-1}(f_{2\mu-1}) + C_{2\mu} f_{2\mu} + \cdots + C_{2l+1} f_{2l+1} \\ & + r^2(C'_{2\mu} f_{2\mu} + \cdots + C'_{2l+1} f_{2l+1}), \end{aligned}
$$

where each C_j (resp. C'_j) is a differential operator (resp. a constant) given by (6.3) and commutes with all $[\alpha]$.

Next, by the definition we can get

$$U'_{2\mu-2} \circ T_{2\mu-1} = V'_{2\mu-2} \circ (r^2 T_{2\mu} r^{-2}), \tag{6.26}$$

where we set

$$V'_{2\mu-2} := \begin{cases} [2\mu-1] \displaystyle\prod_{0\le k\le [\frac{l-\mu}{2}]} [2l-4k-2 \sim 2l-4k+1], & (l-\mu : \text{odd}) \\ [2\mu-1 \sim 2\mu+1] \\ \qquad \times \displaystyle\prod_{0\le k\le [\frac{l-\mu}{2}]-1} [2l-4k-2 \sim 2l-4k+1]. & (l-\mu : \text{even}) \end{cases} \tag{6.27}$$

Then we have

$$T_{2\mu-2} = V'_{2\mu-2} \circ (r^2 T_{2\mu} r^{-2}) \circ [2\mu-2]. \tag{6.28}$$

Hence, by (6.25), (6.28) and Lemma 6.4 we have

$$T_{2\mu-2}(f_{2\mu-2}) = r^2 T_{2\mu} \circ C'(C'_{2\mu} f_{2\mu} + \cdots + C'_{2l+1} f_{2l+1}), \tag{6.29}$$

where C' is a differential operator given by the below term in the right-hand side of (6.28) and commutes with $T_{2\mu}$. By the assumption, Lemma 6.5 and $T_{2\mu-1}(f_{2\mu-1}) = 0$, it holds that $T_{2\mu}(f_\nu) = 0$ ($\nu \ge 2\mu$). Hence we see that $T_{2\mu-2}(f_{2\mu-2}) = 0$. Therefore we have shown the lemma. □

Let us prove Proposition 6.2.

Proof of Proposition 6.2. We consider the differential operator

$$\prod_{1\le k\le l} [2k-2l-3] \circ T_0, \tag{6.30}$$

where $\prod_{1\le k\le l}[2k-2l-3] :=$ id when $l = 0$. Applying (6.30) to the both sides of (6.4), by Lemma 6.6 when $\mu = 0$ we have

$$
\begin{aligned}
0 &= \prod_{1\le k\le l} [2k-2l-3] \circ T_0 \circ \vartheta(\vartheta+a)(f_0) \\
&= \prod_{1\le k\le l} [2k-2l-3] \circ T_0 \left(br^2 \sum_{1\le j\le m} f_j \right) \\
&= br^2 \prod_{1\le k\le l} [2k-2l+1] \circ W_2 \left(\sum_{1\le j\le m} f_j \right),
\end{aligned} \tag{6.31}
$$

where we set

$$
W_2 := \prod_{0\le k\le \left[\frac{l}{2}\right]} [4k-2l+4 \sim 2l-4k+5] \tag{6.32}
$$

and the last equation is given by Lemma 6.4. Since there exists a differential operator S_2 such that

$$
S_2 \circ T_2 = W_2, \tag{6.33}
$$

by Lemmas 6.5 and 6.6 the equation (6.31) equals

$$
\begin{aligned}
0 &= br^2 \prod_{1\le k\le l} [2k-2l+1] \circ W_2(f_1) \\
&= br^2 \prod_{1\le k\le l-1} [2k-2l+1] \circ W_2 \circ [1](f_1).
\end{aligned} \tag{6.34}
$$

Applying (6.3) when $\nu = 2$, Lemmas 6.5 and 6.6, and the condition (6.33) to (6.34), we get

$$
0 = br^2 \prod_{1\le k\le l-1} [2k-2l+1] \circ W_2 \left(br^2 \sum_{3\le j\le 2l+1} \binom{j}{2} f_j \right)
$$

and by Lemma 6.4

$$
0 = b^2 r^4 \prod_{1\le k\le l-1} [2k-2l+5] \circ W_4 \left(\sum_{3\le j\le 2l+1} \binom{j}{2} f_j \right), \tag{6.35}
$$

where we set

$$W_4 := \prod_{0 \le k \le [\frac{l}{2}]} [4k - 2l + 8 \sim 2l - 4k + 9]. \tag{6.36}$$

Since there exists a differential operator S_4 such that

$$S_4 \circ T_4 = W_4, \tag{6.37}$$

by Lemmas 6.5 and 6.6 the equation (6.35) equals

$$\begin{aligned} 0 &= \binom{3}{2} b^2 r^4 \prod_{1 \le k \le l-1} [2k - 2l + 5] \circ W_4 (f_3) \\ &= \binom{3}{2} b^2 r^4 \prod_{1 \le k \le l-2} [2k - 2l + 5] \circ W_4 \circ [3] (f_3). \end{aligned} \tag{6.38}$$

By iteration we get

$$\begin{aligned} 0 = \binom{3}{2} \cdots \binom{2l+1}{2l} b^{l+1} r^{2l+2} \\ \times \prod_{0 \le k \le [\frac{l}{2}]} [4k + 2l + 4 \sim 6l - 4k + 5] (f_{2l+1}). \end{aligned} \tag{6.39}$$

Since f_{2l+1} is an eigenfunction of the differential operator 2ϑ with its eigenvalue $-(2l+1)$ by (6.2), the equation (6.39) equals

$$\begin{aligned} 0 = \binom{3}{2} \cdots \binom{2l+1}{2l} b^{l+1} r^{2l+2} \\ \times \prod_{0 \le k \le [\frac{l}{2}]} (4k+3) \cdots (4l - 4k + 4) f_{2l+1}. \end{aligned} \tag{6.40}$$

Since $b \neq 0$, by (6.40) we have $f_{2l+1} = 0$. Hence we have shown the proposition. □

7. Appendix

In this section, we shall show Proposition 3.6 by using the results of spherical harmonics and special functions.

7.1. A basis of the space of spherical harmonics

We use the polar coordinate:

$$\mathbb{R}_+ \times S^{m-1} \to \mathbb{R}^m \setminus \{0\},\ (r,\omega) \mapsto r\omega, \tag{7.1}$$

where

$$\omega = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 \\ \cdots \\ \sin\theta_1 \sin\theta_2 \dots \sin\theta_{m-2}\cos\theta_{m-1} \\ \sin\theta_1 \sin\theta_2 \dots \sin\theta_{m-2}\sin\theta_{m-1} \end{pmatrix}. \tag{7.2}$$

In [14], for any $l, m, 2\nu \in \mathbb{Z}$ with $l \geq m \geq 0$, associated Gegenbauer polynomials $C_l^{\nu,m}$ are defined by

$$C_l^{\nu,m}(x) := (1-x^2)^{m/2}\frac{d^m}{dx^m}C_l^{\nu}(x) \quad \text{on } |x| < 1, \tag{7.3}$$

where $C_l^{\nu}(x)$ is the classical Gegenbauer polynomial. It is obvious that $C_l^{\nu,0}(x) = C_l^{\nu}(x)$ and $C_l^{1/2,m}(x) = P_l^m(x)$, where $P_l^m(x)$ is the associated Legendre polynomial. Then we have a basis of the space $\mathcal{H}^l(\mathbb{R}^m)$ of spherical harmonics on S^{m-1} of degree l as follows:

FACT 7.1 ([14]). *For $l_1 \in \mathbb{N}$, $\mathcal{H}^{l_1}(\mathbb{R}^m)$ is spanned over $\mathbb{C}$ by the following basis:*

$$\begin{aligned} \big\{ C_{l_1}^{\nu_1,l_2}(\cos\theta_1) C_{l_2}^{\nu_2,l_3}(\cos\theta_2) \cdots C_{l_{m-2}}^{\nu_{m-2},l_{m-1}}(\cos\theta_{m-2}) e^{-\sqrt{-1}l_{m-1}\theta_{m-1}} : \\ l_2, \dots, l_{m-2}, l_{m-1} \in \mathbb{Z} \textit{ with } l_1 \geq l_2 \geq \cdots \geq l_{m-2} \geq |l_{m-1}| \big\}, \end{aligned} \tag{7.4}$$

where $\nu_j := (m-1-j)/2$.

7.2. Laguerre polynomials and Gegenbauer polynomials

We use the equations of Laguerre polynomials and associated Gegenbauer polynomials in Lemmas 7.2 and 7.3 to prove Proposition 3.6.

Lemma 7.2. *Laguerre polynomials satisfy the following equations:*

$$\left(\frac{d^2}{dx^2} - \frac{d}{dx}\right) L_n^\alpha(x) = L_{n-1}^{\alpha+2}(x), \tag{7.5}$$

$$\left(x^2 \frac{d^2}{dx^2} + (2\alpha - x)x\frac{d}{dx} - \alpha x + \alpha(\alpha-1)\right) L_n^\alpha(x) = (n+1)(n+\alpha)L_{n+1}^{\alpha-2}(x). \tag{7.6}$$

Here, we note that the equation (7.5) if $n = 0$ is the following:

$$\left(\frac{d^2}{dx^2} - \frac{d}{dx}\right) L_0^\alpha(x) = 0 \tag{7.7}$$

because $L_0^\alpha(x) \equiv 1$.

Proof of Lemma 7.2. With respect to Laguerre polynomials, we recall from [2, II, pp. 188–190 (24)(8)(23)(15)(12)(10)] that

$$L_n^{\alpha-1}(x) = L_n^\alpha(x) - L_{n-1}^\alpha(x), \tag{7.8}$$

$$(n+1)L_{n+1}^\alpha(x) + (x - \alpha - 2n - 1)L_n^\alpha(x) + (n+\alpha)L_{n-1}^\alpha(x) = 0, \tag{7.9}$$

$$xL_n^{\alpha+1}(x) = (n+\alpha+1)L_n^\alpha(x) - (n+1)L_{n+1}^\alpha(x) \tag{7.10}$$

$$\frac{d}{dx}L_n^\alpha(x) = -L_{n-1}^{\alpha+1}(x), \tag{7.11}$$

$$x\frac{d}{dx}L_n^\alpha(x) = nL_n^\alpha(x) - (n+\alpha)L_{n-1}^\alpha(x), \tag{7.12}$$

$$x\frac{d^2}{dx^2}L_n^\alpha(x) + (\alpha + 1 - x)\frac{d}{dx}L_n^\alpha(x) + nL_n^\alpha(x) = 0, \tag{7.13}$$

where $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$.

It is easily seen that (7.5) follows from (7.8) and (7.11).

In order to show (7.6), we apply (7.8) to the right-hand side of (7.12) and have

$$x\frac{d}{dx}L_n^\alpha(x) = nL_n^{\alpha-1}(x) - \alpha L_{n-1}^\alpha(x). \tag{7.14}$$

By (7.8), (7.10) and (7.14), we obtain

$$\left((\alpha-1)\,x\frac{d}{dx} - (\alpha+n)x + \alpha(\alpha-1)\right) L_n^\alpha(x) = (n+1)(n+\alpha)L_{n+1}^{\alpha-2}(x). \tag{7.15}$$

Multiplying both sides of (7.13) by a variable x, one can get (7.6) by (7.15). □

Lemma 7.3. *Associated Gegenbauer polynomials satisfy the following equations:*

$$(l+1-m)\,C_{l+1}^{\nu,m}(x) = 2(l+\nu)\,x\,C_l^{\nu,m}(x) - (l-1+2\nu+m)\,C_{l-1}^{\nu,m}(x), \tag{7.16}$$

$$(1-x^2)\frac{d}{dx}C_l^{\nu,m}(x) = -\,l\,x\,C_l^{\nu,m}(x) + (l-1+2\nu+m)\,C_{l-1}^{\nu,m}(x). \tag{7.17}$$

Proof. With respect to Gegenbauer polynomials, we recall from [2, II, pp.175 f. (13)(15)(23)] that

$$(l+1)C_{l+1}^{\nu}(x) = 2(l+\nu)xC_l^{\nu}(x) - (l+2\nu-1)C_{l-1}^{\nu}(x), \tag{7.18}$$

$$(1-x^2)\frac{d}{dx}C_l^{\nu}(x) = -lxC_l^{\nu}(x) + (l+2\nu-1)C_{l-1}^{\nu}(x), \tag{7.19}$$

$$\frac{d^m}{dx^m}C_l^{\nu}(x) = 2^m(\nu)_m\,C_{l-m}^{\nu+m}(x), \tag{7.20}$$

where $\nu \in \mathbb{C}$, $l \in \mathbb{N}$ and $(\nu)_m = \Gamma(\nu+m)/\Gamma(\nu)$.

It follows from (7.20) that

$$C_l^{\nu,m}(x) = (1-x^2)^{m/2}\,2^m(\nu)_m C_{l-m}^{\nu+m}(x). \tag{7.21}$$

First, we show (7.16). Replacing ν and l with $\nu+m$ and $l-m$ respectively in (7.18), one can see that

$$(l-m+1)C_{l+1-m}^{\nu+m}(x) = 2(l+\nu)xC_{l-m}^{\nu+m}(x) - (l-1+2\nu+m)C_{l-m-1}^{\nu+m}(x). \tag{7.22}$$

We multiply the both sides by $(1-x^2)^{m/2}\,2^m(\nu)_m$, and then get (7.16) by (7.21).

Next, we show (7.17). By the definition, we have

$$\begin{aligned}\frac{d}{dx}C_l^{\nu,m}(x) &= (1-x^2)^{-1}(-mx)(1-x^2)^{m/2}\frac{d^m}{dx^m}C_l^{\nu}(x) \\ &\quad + (1-x^2)^{m/2}\frac{d}{dx}\left(\frac{d^m}{dx^m}C_l^{\nu}(x)\right) \\ &= (1-x^2)^{-1}(-mx)C_l^{\nu,m}(x) \\ &\quad + (1-x^2)^{m/2}\frac{d}{dx}\left(\frac{d^m}{dx^m}C_l^{\nu}(x)\right).\end{aligned} \tag{7.23}$$

Here, by (7.20) we get

$$\begin{aligned}&\frac{d}{dx}\left(\frac{d^m}{dx^m}C_l^{\nu}(x)\right) \\ &= \frac{d}{dx}\left\{2^m(\nu)_m C_{l-m}^{\nu+m}(x)\right\} \\ &= (1-x^2)^{-1}2^m(\nu)_m \\ &\quad \times\left\{-(l-m)xC_{l-m}^{\nu+m}(x) + (l-m-1+2\nu+2m)C_{l-m-1}^{\nu+m}(x)\right\} \qquad (7.24) \\ &= (1-x^2)^{-1-m/2} \\ &\quad \times\left\{-(l-m)xC_l^{\nu,m}(x) + (l-1+2\nu+m)C_{l-1}^{\nu,m}(x)\right\}, \qquad (7.25)\end{aligned}$$

where (7.24) is given by (7.19) and (7.25) by (7.21). Substituting the above equation into (7.23), we obtain

$$\begin{aligned}\frac{d}{dx}C_l^{\nu,m}(x) &= (1-x^2)^{-1}(-mx)C_l^{\nu,m}(x) \\ &\quad + (1-x^2)^{-1}(m-l)xC_l^{\nu,m}(x) \\ &\quad + (1-x^2)^{-1}(l-1+2\nu+m)C_{l-1}^{\nu,m}(x).\end{aligned} \tag{7.26}$$

Hence we have shown (7.17). □

7.3. Proof of Proposition 3.6

Proof of Proposition 3.6. Under the polar coordinate (3.25), $d\pi_+(J_1)$ of (2.21) takes the form:

$$\begin{aligned}d\pi_+(J_1) = \sqrt{-1}\biggl(&r\cos\theta - \frac{1}{4}r\cos\theta\frac{\partial^2}{\partial r^2} + \frac{1}{4}\frac{\cos\theta}{r}\Delta_{S^{2n-2}} \\ &+ \frac{1}{2}\sin\theta\frac{\partial^2}{\partial r\partial\theta} + \frac{n-2}{2}\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\biggr).\end{aligned} \tag{7.27}$$

Hereafter, we write ν as $\nu_1 = (2n-3)/2$. Since $C_l^{\nu,k}(\cos\theta)\,\varphi_k \in \mathcal{H}^l(\mathbb{R}^{2n-1})$, it is obvious (see Section 7.1) that

$$\Delta_{S^{2n-2}}\left(C_l^{\nu,k}(\cos\theta)\,\varphi_k\right) = -l(l+2\nu)\,C_l^{\nu,k}(\cos\theta)\,\varphi_k. \tag{7.28}$$

By (7.10) and (7.28), we have

$$\begin{aligned}
&\sqrt{-1}\left(r\cos\theta - \frac{1}{4}r\cos\theta\frac{\partial^2}{\partial r^2} + \frac{1}{4}\frac{\cos\theta}{r}\Delta_{S^{2n-2}}\right)\left(u_{a,l}^k\,\varphi_k\right) \\
&= -\frac{\sqrt{-1}}{4}\left(r\frac{\partial^2}{\partial r^2} + l(l+2\nu)r^{-1} - 4r\right)(f_{a,l})(r) \\
&\quad\times\left\{\frac{l-k+1}{2(l+\nu)}C_{l+1}^{\nu,k}(\cos\theta) + \frac{l+k+2\nu-1}{2(l+\nu)}C_{l-1}^{\nu,k}(\cos\theta)\right\}\varphi_k.
\end{aligned} \tag{7.29}$$

Next, by (7.11) and (7.10), we have

$$\begin{aligned}
&\sqrt{-1}\left(\frac{1}{2}\sin\theta\frac{\partial^2}{\partial r\partial\theta} + \frac{n-2}{2}r^{-1}\sin\theta\frac{\partial}{\partial\theta}\right)\left(u_{a,l}^k\,\varphi_k\right) \\
&= \frac{\sqrt{-1}}{2}\left(\frac{\partial}{\partial r} + (n-2)r^{-1}\right)(f_{a,l})(r) \\
&\quad\times\left\{\frac{l(l-k+1)}{2(l+\nu)}C_{l+1}^{\nu,k}(\cos\theta)\right. \\
&\qquad\left. - \frac{(l+2\nu)(l+k+2\nu-1)}{2(l+\nu)}C_{l-1}^{\nu,k}(\cos\theta)\right\}\varphi_k.
\end{aligned} \tag{7.30}$$

Thus, we get

$$\begin{aligned}
&d\pi_+(J_1)\left(u_{a,l}^k\,\varphi_k\right) \\
&= -\frac{l-k+1}{8(l+\nu)}\sqrt{-1}\,A_l(f_{a,l})(r)\,C_{l+1}^{\nu,k}(\cos\theta)\,\varphi_k \\
&\quad - \frac{l+k+2\nu-1}{8(l+\nu)}\sqrt{-1}\,B_l(f_{a,l})(r)\,C_{l-1}^{\nu,k}(\cos\theta)\,\varphi_k,
\end{aligned} \tag{7.31}$$

where we set

$$A_l := r\frac{\partial^2}{\partial r^2} - 2l\frac{\partial}{\partial r} + l(l+1)r^{-1} - 4r \tag{7.32}$$

$$B_l := r\frac{\partial^2}{\partial r^2} + 2(l+2\nu)\frac{\partial}{\partial r} + (l+2\nu)(l+2\nu-1)r^{-1} - 4r. \tag{7.33}$$

Here, direct computation shows the following:

$$\frac{\partial}{\partial r} f_{a,l}(r) = \left(\frac{\partial}{\partial r} + (l-2r)r^{-1}\right)\left(L_{a-l}^{2\nu+2l}(4r)\right) r^l e^{-2r}$$

$$r\frac{\partial^2}{\partial r^2} f_{a,l}(r) = \left(r\frac{\partial^2}{\partial r^2} + 2(l-2r)\frac{\partial}{\partial r} + r^{-1}\left((l-2r)^2 - l\right)\right)\left(L_{a-l}^{2\nu+2l}(4r)\right) r^l e^{-2r}.$$

By these equations, we get

$$A_l(f_{a,l})(r) = \left(r\frac{\partial^2}{\partial r^2} - 4r\frac{\partial}{\partial r}\right)\left(L_{a-l}^{2\nu+2l}(4r)\right) r^l e^{-2r} \tag{7.34}$$

$$B_l(f_{a,l})(r) = \left(r\frac{\partial^2}{\partial r^2} + 4(l+\nu-r)\frac{\partial}{\partial r} - 8(l+\nu) + 2(l+\nu)(2l+2\nu-1)r^{-1}\right)\left(L_{a-l}^{2\nu+2l}(4r)\right) r^l e^{-2r}. \tag{7.35}$$

With respect to (7.34), by (7.15) we obtain

$$\left(r\frac{\partial^2}{\partial r^2} - 4r\frac{\partial}{\partial r}\right)\left(L_{a-l}^{2\nu+2l}(4r)\right) = \begin{cases} 16\, r\, L_{a-l-1}^{2\nu+2l+2}(4r) & (a > l) \\ 0 & (a = l). \end{cases} \tag{7.36}$$

Hence, we have

$$A_l(f_{a,l})(r) = \begin{cases} 16\, f_{a,l+1}(r) & (a > l) \\ 0 & (a = l). \end{cases} \tag{7.37}$$

On the other hand, one can see that by (7.6) the right-hand side of (7.35) equals $(a+l+2\nu)(a-l+1)L_{a-l+1}^{2\nu+2l-2}(4r)\, r^{-1}$. Hence, we have

$$B_l(f_{a,l})(r) = (a+l+2\nu)(a-l+1)f_{a,l-1}(r). \tag{7.38}$$

Therefore we have shown Proposition 3.6. □

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