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Holomorphic Curves into the Product Space of the Riemann Spheres

By Yusaku Tiba

Abstract. We prove the second main theorem for the product space of the Riemann spheres by using the meromorphic connection. We deal with the divisors which are totally geodesic with respect to this meromorphic connection.

1. Introduction

The purpose of this paper is to prove the second main theorem for a holomorphic map from the complex plane $\mathbb{C}$ to the product space of the one-dimensional projective spaces $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $[X_0 : X_1]$ and $[Y_0 : Y_1]$ be the homogeneous coordinates in the first and second factors of the product space of the $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let m', n', m'', n'' be positive integers. We define the effective divisors D', D'' on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ by the polynomials $X_0^{m'}Y_0^{n'} - X_1^{m'}Y_1^{n'}$, $X_0^{m''}Y_1^{n''} - X_1^{m''}Y_0^{n''}$. We prove the second main theorem for divisors D' and D''.

We state our main theorem precisely. Let $H_{1,0}$, $H_{1,1}$, $H_{2,0}$ and $H_{2,1}$ be the hyperplanes in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which are defined by the monomials X_0, X_1, Y_0 and Y_1. Put $Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Then there exists the sequence of the blowing-up

$$\begin{gathered}\pi_{1,0} : Z_1 \to Z_0,\\ \pi_{2,1} : Z_2 \to Z_1,\\ \vdots\\ \pi_{k,k-1} : Z_k \to Z_{k-1}.\end{gathered}$$

which satisfies the following condition $(*)$:

Put $\pi_{j,i} = \pi_{i+1,i} \circ \cdots \circ \pi_{j,j-1}$ for $i < j$. Let $\widetilde{D}'$, $\widetilde{D}''$ and $\widetilde{H}_{i,j}$, $1 \le i \le 2, 0 \le j \le 1$ be the proper transform of D', D'', and $H_{i,j}$ under $\pi_{k,0}$. Let

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E_i, $1 \le i \le k$ be the exceptional divisor of the blowing-up $\pi_{i,i-1}$, and let $\widetilde{E}_i$ be the proper transform of E_i under $\pi_{k,i}$. Then

$(*)$ $\widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{2} \sum_{j=0}^{1} \widetilde{H}_{i,j} + \sum_{i=1}^{k} \widetilde{E}_i$ is simple normal crossing in Z_k.

Our goal is the following theorem.

THEOREM 1 (Main Theorem). *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map. Let $\widetilde{f} : \mathbb{C} \to Z_k$ be the lift of f. Assume that*

$$f(\mathbb{C}) \not\subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \quad | \quad C_0 X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\},$$

for all $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and all $(C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, and assume that there exist no holomorphic functions g_1, g_2 on $\mathbb{C}$ and no $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$f = (\exp g_1, \exp g_2),$$

$$a g_1 + b g_2 = (\text{constant}),$$

on $\mathbb{C}$. Then it follows that

$$T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \le N_2(r, \widetilde{f}^* \widetilde{D}') + N_2(r, \widetilde{f}^* \widetilde{D}'') + 2 \sum_{i=1}^{2} \sum_{j=0}^{1} N_1(r, \widetilde{f}^* \widetilde{H}_{i,j}) + 2 \sum_{i=1}^{k} N_1(r, \widetilde{f}^* \widetilde{E}_i) + S_f(r),$$

where $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$. Here "$\|$" means that the inequality holds for all $r \in (0, +\infty)$ possibly except for subset with finite Lebesgue measure.

In Section 4, we prove that $\widetilde{D}' + \widetilde{D}''$ is a big divisor on Z_k. Hence we can compute defects for holomorphic curves $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$.

The second main theorem for hypersurfaces in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ is first obtained in Noguchi [3]. In Section 5 of Noguchi [3], there are some additional conditions for maps from $\mathbb{C}$ to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. In our paper, we do not assume these conditions.

In Siu [7], a meromorphic connection is used to prove the second main theorem. Because $\mathbb{C}^* \times \mathbb{C}^*$ is a Lie group, there exists the canonical connection on $\mathbb{C}^* \times \mathbb{C}^*$. We extend this connection to the meromorphic connection ∇ on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. This meromorphic connection ∇ is used in Section 5 of Noguchi [3]. We also use ∇ to prove our main theorem. This connection does not "vary" under the blowing-up, and plays an important role in our arguments.

Let $i : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the inclusion map where $\mathbb{C}^* = \mathbb{C} \backslash \{0\}$. Then Z_k is the compactification of semi-Abelian variety $\mathbb{C}^* \times \mathbb{C}^*$. If holomorphic map $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ does not intersect $H_{1,0}, H_{1,1}, H_{2,0}, H_{2,1}$, then f is a holomorphic map from $\mathbb{C}$ to semi-Abelian variety $\mathbb{C}^* \times \mathbb{C}^*$. In Noguchi, Winkelmann and Yamanoi [5], [6], the second main theorem for holomorphic map f from $\mathbb{C}$ to a semi-Abelian variety A with D is proved, where D is an effective reduced divisor on A.

Theorem 2 ([6]). *Let $f : \mathbb{C} \to A$ be a holomorphic map such that the image of f is Zariski dense in A. There is the compactification of A such that $\overline{A}$ is smooth, equivalent with respect to the A-action, independent of f, and it follows that*

$$T_f(r, [\overline{D}]) \leq N_1(r, f^*D) + \varepsilon T_f(r, [\overline{D}])\|_\varepsilon,$$

for all $\varepsilon > 0$, where $\overline{D}$ is the closure of D in $\overline{A}$.

If $A = \mathbb{C}^* \times \mathbb{C}^*$, and $D = D' + D''$ in Theorem 2, our main theorem deals with the holomorphic curves into $\overline{A}$.

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2. Notation and Preliminaries

We introduce some functions which play an important role in the Nevanlinna theory. Let E be an effective divisor on $\mathbb{C}$. We write $E = \sum m_j P_j$, where $\{P_j\}$ is a set of discrete points in $\mathbb{C}$ and m_j are positive integers. Put $n_k(r,E) = \sum_{|P_j|<r} \min\{k, m_j\}$. We define the counting function of E by

$$N_k(r,E) = \int_1^r \frac{n_k(t,E)}{t} dt.$$

Let X be a complex projective algebraic manifold, and let D be a divisor on X. Let $[D]$ be the holomorphic line bundle on X which is defined by the divisor D, and let $\operatorname{supp} D$ be the support of D. Let σ be a holomorphic section of $[D]$ such that the zero divisor of σ is D. Let $f : \mathbb{C} \to X$ be a non-constant holomorphic map. We define the proximity function of D by

$$m_f(r,D) = \int_0^{2\pi} \log \frac{1}{\|\sigma(f(re^{i\theta}))\|} \frac{d\theta}{2\pi},$$

where $\|\cdot\|$ is a Hermitian metric in L. Let $R(L, \|\cdot\|)$ be the curvature form of the metrized line bundle $(L, \|\cdot\|)$ representing the first Chern class. Then we define the characteristic function of L by

$$T_f(r,L) = \int_1^r \frac{dt}{t} \int_{\Delta(t)} f^* R(L, \|\cdot\|) + O(1),$$

where $\Delta(t) = \{z \in \mathbb{C} \,|\, |z| < t\}$. We set $T_f(r) = T_f(r,L)$ if L is an ample line bundle on X. The equation

$$T_f(r,L) = N(r, f^*D) + m_f(r,D) + O(1)$$

is called the First Main Theorem (cf. Noguchi and Ochiai [4], Chapter V, §2). If $X = \mathbb{P}^1(\mathbb{C})$, f is a meromorphic function on $\mathbb{C}$. Then we have the lemma on logarithmic derivative (cf. Noguchi and Ochiai [4], Chapter VI, §1)

$$\int_0^{2\pi} \log^+ \left| \frac{f'(re^{i\theta})}{f(re^{i\theta})} \right| d\theta \leq S_f(r),$$

where $\log^+ r = \max\{0, \log r\}$, and $S_f(r) = O(\log^+ T_f(r) + \log^+ r)\|$. Here "$\|$" means that the inequality holds for all $r \in (0, +\infty)$ possibly except for a subset with finite Lebesgue measure.

The following lemma is also fundamental in Nevanlinna theory.

LEMMA 1. *Let $h(r) > 0$ be a monotone increasing function in $r \geq 1$. Then, for arbitrary $\delta > 0$, we have*

$$\frac{dh(r)}{dr} \leq (h(r))^{1+\delta}\|.$$

PROOF. See Noguchi-Ochiai [4], Chapter V, §5. □

Let X be a complex projective algebraic manifold, and let Y be a smooth hypersurface of X. Let s be the holomorphic function on an open subset $U \subset X$ such that $Y \cap U = \{x \in U \mid s(x) = 0\}$. Let ∇ be a holomorphic connection on U. We write

$$\nabla^{(m)} = \overbrace{\nabla \circ \cdots \circ \nabla}^{m-\text{times}}.$$

Then the following lemma holds (see the proof of Lemma 11.13. of J.-P. Demailly [1]).

LEMMA 2. *Let X and U be as above. Let $f : \mathbb{C} \to X$ be a holomorphic function. Assume that Y is totally geodesic with respect to ∇ on U. Then there exist holomorphic functions $h_0, h_1, \cdots, h_m$ on U such that*

$$\begin{aligned} ds \cdot \nabla_{f'}^{(m)} f'(z) &= h_0(f(z)) s \circ f(z) \\ &\quad + \sum_{i=1}^{m} h_i(f(z)) \frac{d^i(s \circ f)}{dz^i}(z) + \frac{d^{m+1}(s \circ f)}{dz^{m+1}}(z), \end{aligned}$$

for $z \in f^{-1}(U)$.

Let X and $\widetilde{X}$ be n-dimensional complex projective algebraic manifolds. Let $\pi : \widetilde{X} \to X$ be a surjective holomorphic map. Then there exists a proper subvariety S of X such that $\widetilde{X} \setminus \pi^{-1}(S)$ and $X \setminus S$ are locally biholomorphic. Let ∇ be a meromorphic connection on X. Let V be a small neighborhood of $p \in \widetilde{X}$, and let u, v be holomorphic vector fields on a small neighborhood V of p. Then $V \setminus \pi^{-1}(S)$ is locally biholomorphic with $\pi(V) \setminus S$. We define the meromorphic connection $\pi^*\nabla$ on $\widetilde{X} \setminus \pi^{-1}(S)$ by

$$(\pi^*\nabla)_u v|_{V \setminus \pi^{-1}(S)} = (\pi_*|_{V \setminus \pi^{-1} S})^{-1} \nabla_{\pi_* u} \pi_* v.$$

Then the meromorphic vector field $(\pi^*\nabla)_u v$ on $V \setminus \pi^{-1} S$ is uniquely extended to the meromorphic vector field $(\pi^* \nabla_u v)$ on V. In this way, we define the meromorphic connection $\pi^*\nabla$ on $\widetilde{X}$.

3. Meromorphic Connection and Blowing-Up

Let D', D'' be divisors on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ defined by the polynomials $X_0^{m'}Y_0^{n'} - X_1^{m'}Y_1^{n'}$, $X_0^{m''}Y_1^{n''} - X_1^{m''}Y_0^{n''}$. Let $i : \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the inclusion map. Then $\operatorname{supp} i^*D'$ is a subgroup of $\mathbb{C}^* \times \mathbb{C}^*$. Therefore there exists the canonical connection ∇ on $\mathbb{C}^* \times \mathbb{C}^*$ such that $\operatorname{supp} i^*D'$ is totally geodesic with respect to ∇. This connection is extended to the meromorphic connection on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. We also denote this extended connection by ∇. Let $U_{i,j} = \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_i \neq 0, Y_j \neq 0\}$, $0 \leq i, j \leq 1$. Take the canonical local coordinate system (z, w) on $U_{i,j} \simeq \mathbb{C} \times \mathbb{C}$. Then, the meromorphic connection ∇ is written by

$$d + \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix},$$

on $U_{i,j}$. It is easy to see that $\operatorname{supp} i^*D''$ is also totally geodesic with respect to ∇.

The universal covering space of $\mathbb{C}^* \times \mathbb{C}^*$ is $\mathbb{C} \times \mathbb{C}$. The connection on $\mathbb{C} \times \mathbb{C}$ which is induced by ∇ is the flat connection d on $\mathbb{C} \times \mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$ be a non-constant holomorphic map, and let $F : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ be the lift of f. Then $f' \wedge \nabla_{f'} f' \equiv 0$ if and only if F is a translation of a linear map.

Lemma 3. *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map such that $f(\mathbb{C})$ is not contained in the support of $H_{i,j}$, $i = 1, 2$, $j = 0, 1$. Then f satisfies*

$$f' \wedge \nabla_{f'} f' \equiv 0,$$

if and only if f satisfies the following condition (i) or (ii):

(i)

$$f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \,|\, X_0^{r_1} Y_0^{r_2} - C X_1^{r_1} Y_1^{r_2} = 0\},$$

for some $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$ and some $C \in \mathbb{C} \setminus \{0\}$.

(ii)

There exist holomorphic functions g_1, g_2 on $\mathbb{C}$ and $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0, 0)\}$ such that

$$f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),$$

$$ag_1 + bg_2 = (\text{constant}),$$

on $\mathbb{C}$.

PROOF. Without loss of generality, we may assume that $f(0) \in \mathbb{C}^* \times \mathbb{C}^*$. The holomorphic map

$$\big(\exp(2\pi\sqrt{-1}\,\cdot\,), \exp(2\pi\sqrt{-1}\,\cdot\,)\big) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*,$$

is the universal covering of $\mathbb{C}^* \times \mathbb{C}^*$. The induced connection on the covering space $\mathbb{C} \times \mathbb{C}$ by ∇ is the flat connection d. We put $f = (f_1, f_2)$ where f_1 and f_2 are meromorphic functions on $\mathbb{C}$. Let

$$h_i = \frac{1}{2\pi\sqrt{-1}} \log f_i, \qquad i = 1,\, 2.$$

Assume that $f' \wedge \nabla_{f'} f' \equiv 0$. Then there exists a meromorphic function h on $\mathbb{C}$ such that

$$\begin{pmatrix} h_1''(z) \\ h_2''(z) \end{pmatrix} = h(z) \begin{pmatrix} h_1'(z) \\ h_2'(z) \end{pmatrix},$$

on $\mathbb{C}$.

This means that

$$h_i'(z) = h_i'(0) \exp H(z), \qquad i = 1,\, 2,$$

in a simple connected neighborhood U of $0 \in \mathbb{C}$. Here

$$H(z) = \int_0^z h(t)dt.$$

If $h_i'(0) = 0$, it follows that h_i is a constant function. Hence $(h_1'(0), h_2'(0)) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$. It holds that

$$h_i(z) = h_i'(0) \int_0^z \exp H(t)dt + h_i(0), \qquad i = 1,\, 2.$$

It follows that

$$h_2'(0)h_1(z) - h_1'(0)h_2(z) = h_2'(0)h_1(0) - h_1'(0)h_2(0).$$

Conversely, assume that there exists $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$ah_1(z) + bh_2(z) = (\text{constant}),$$

on $\mathbb{C}$. Then $ah_1'(z) + bh_2'(z) = 0, ah_1''(z) + bh_2''(z) = 0$. Hence it follows that $f' \wedge \nabla_{f'} f' \equiv 0$.

Therefore $f' \wedge \nabla_{f'} f' \equiv 0$ if and only if there exists $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$a \log f_1(z) + b \log f_2(z) = (\text{constant}),$$

on $\mathbb{C}$.

Assume that

$$a \log f_1(z) + b \log f_2(z) = c, \tag{1}$$

for some $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, $c \in \mathbb{C}$. Without loss of generality, we may assume that $a = 1$. For $x \in \mathbb{C}$, we put

$$f_i(z) = (z-x)^{r_i} h_i(z), \qquad i = 1, 2,$$

where $r_i \in \mathbb{Z}$, $h_i(z)$ is a holomorphic function on an open neighborhood of x such that $h_i(x) \neq 0$. Then, by (1), we have $r_1 + br_2 = 0$. When $r_2 \neq 0$ for some $x \in \mathbb{C}$, it follows that

$$\log(f_1(z))^{r_2} + \log(f_2(z))^{-r_1} = r_2 c.$$

Then it holds that the meromorphic function $(f_1(z))^{r_2}(f_2(z))^{-r_1}$ is a constant function on $\mathbb{C}$. This means that

$$f(\mathbb{C}) \subset \{([X_0 : X_1], [Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \mid X_1^{r_2} Y_0^{r_1} - C X_0^{r_2} Y_1^{r_1} = 0\},$$

for $r_1 \in \mathbb{Z}$, $r_2 \in \mathbb{Z} \setminus \{0\}$, $C \in \mathbb{C}$. When $r_2 = 0$ for all $x \in \mathbb{C}$, we have $r_1 = 0$ for all $x \in \mathbb{C}$. This means that there exist holomorphic functions g_1, g_2 on $\mathbb{C}$ such that $f_i = \exp g_i$, $i = 1, 2$. Then $g_1 + bg_2 = c$.

Conversely, if f satisfies the condition (i) or (ii). It is easy to see that there exists $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$a \log f_1(z) + b \log f_2(z) = (\text{constant}),$$

on $\mathbb{C}$. $\square$

REMARK 1. The condition of (b) in Lemma 3 does not mean the algebraical degeneracy of $f(\mathbb{C})$. For example, take

$$f(z) = (\exp z, \exp \sqrt{-1}z) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}).$$

The divisor

$$D' + D'' + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j},$$

is not simple normal crossing at $\{([0:1],[0:1]),([0:1],[1:0]),([1:0],[0:1]),([1:0],[1:0])\} \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Put $Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $\pi_{1,0} : Z_1 \to Z_0$ be the blowing-up of Z_0 at the center $\{([0:1],[0:1]),([0:1],[1:0]),([1:0],[0:1]),([1:0],[1:0])\}$. Let $D'_1, D''_1, H_{i,j,1}$ be the strict transform of $D', D'', H_{i,j}$ under $\pi_{1,0}$, and let E_1 be the exceptional divisor of $\pi_{1,0}$.

If the divisor

$$D'_1 + D''_1 + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,1} + E_1,$$

is not simple normal crossing in Z_1, we blow up Z_1 at the points where this divisor is not simple normal crossing. We repeat this process for several times. We put the l-th blowing-up $\pi_{l,l-1} : Z_l \to Z_{l-1}$. Let E_l be the exceptional divisor of $\pi_{l,l-1}$. We define

$$\pi_{j,i} = \pi_{i+1,i} \circ \pi_{i+2,i+1} \circ \cdots \circ \pi_{j,j-1},$$

for $i \leq j$ (we define $\pi_{i,i} = \mathrm{Id}$). Let D'_l, D''_l, $H_{i,j,l}$ be the strict transform of D', D'', $H_{i,j}$ under $\pi_{l,0}$, and let $E_{i,l}$, $1 \leq i \leq l$, be the strict transform of E_i under $\pi_{l,i}$.

Then there exists a positive integer k such that

$$D'_k + D''_k + \sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,k} + \sum_{i=1}^{k} E_{i,k},$$

is simple normal crossing. We put $\widetilde{D}' = D'_k$, $\widetilde{D}'' = D''_k$, $\widetilde{H}_{i,j} = H_{i,j,k}$, and $\widetilde{E}_i = E_{i,k}$.

Example 1. Let D', D'' be the divisor which are defined by the polynomials

$$X_0^2 Y_0 - X_1^2 Y_1, \qquad X_0^3 Y_1^2 - X_1^3 Y_0^2.$$

Let $\pi_{1,0} : Z_1 \to Z_0$ be the blowing-up as above. Then $D_1' + D_1'' + \sum_{i=1}^2 \sum_{j=0}^1 H_{i,j,1} + E_1$ is not simple normal crossing at four points in Z_1. Then $\pi_{2,1} : Z_2 \to Z_1$ is the blowing-up at these four points. We see that $D_2' + D_2'' + \sum_{i=1}^2 \sum_{j=0}^1 H_{i,j,2} + \sum_{i=1}^2 E_{i,2}$ is not simple normal crossing at two points in Z_2. Then $\pi_{3,2} : Z_3 \to Z_2$ is the blowing-up at these two points. Then $D_2' + D_2'' + \sum_{i=1}^2 \sum_{j=0}^1 H_{i,j,2} + \sum_{i=1}^2 E_{i,2}$ is normal crossing.

Let E_i' and E_i'' be unions of irreducible components of E_i such that $\pi_{i,0}(\operatorname{supp} E_i') \subset \operatorname{supp} D'$ and $\pi_{i,0}(\operatorname{supp} E_i'') \subset \operatorname{supp} D''$. Then $E_1 = E_1' + E_1''$, $E_2 = E_2' + E_2''$ and $E_3 = E_3''$. Let $\widetilde{E}_i'$ and $\widetilde{E}_i''$ be the proper transform of E_i' and E_i''. Then it follows that

$$\pi_{3,0}^* D' = \widetilde{D}' + \widetilde{E}_1' + 2\widetilde{E}_2',$$

and

$$\pi_{3,0}^* D'' = \widetilde{D}'' + 2\widetilde{E}_1'' + 3\widetilde{E}_2'' + 6\widetilde{E}_3''.$$

LEMMA 4. *There exist affine open coverings $\{U_s^l\}_{1 \le s \le N_l}$ of Z_l, for $0 \le l \le k$, such that every U_s^l satisfies the following five conditions:*

(i)

$$U_s^l \simeq \mathbb{C} \times \mathbb{C}.$$

Let (z, w) be the canonical local coordinate system of U_s^l.

(ii)

$$\sum_{i=1}^{2} \sum_{j=0}^{1} H_{i,j,l}|_{U_s^l} + \sum_{1 \le i \le l} E_{i,l}|_{U_s^l} = (z) + (w),$$

on U_s^l.

(iii)

$$D_l'|_{U_s^l} = (z^{p'} - w^{q'}) \qquad (\text{or} \qquad (1 - z^{p'} w^{q'}) \qquad \text{respectively}),$$

on U^l_s, where p' and q' are non-negative integers (p', q' may depend on l and s).

(iv)

$$D''_l|_{U^l_s} = (1 - z^{p''} w^{q''}) \qquad (\text{or} \qquad (z^{p''} - w^{q''}) \qquad \text{respectively}),$$

on U^l_s, where p'' and q'' are non-negative integers (p'', q'' may depend on l and s).

(v)

$$\pi^*_{l,0}\nabla|_{U^l_s} = d + \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix},$$

on U^l_s.

PROOF. We take affine open coverings $\{U^l_s\}_{1\le s\le N_l}$ by induction over l. For $l = 0$, we put $\{U^0_s\}_{1\le s\le 4} = \{U_{i,j}\}_{0\le i,j\le 1}$. Here $U_{i,j} = \{[X_0 : X_1], [Y_0 : Y_1] \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) | X_i \neq 0, Y_j \neq 0\}$. Then $\{U^0_s\}_{1\le s\le 4}$ satisfies above five conditions. Assume that we take the affine open covering $\{U^{l-1}_s\}_{1\le s\le N_{l-1}}$ of Z_{l-1} for $l \le k$ which satisfies the above five conditions. Let $U^{l-1}_t \in \{U^{l-1}_s\}_{1\le s\le N_{l-1}}$. Take the canonical local coordinate system (z, w) of $U^{l-1}_t \simeq \mathbb{C} \times \mathbb{C}$.

If $D'_{l-1}|_{U^{l-1}_t} = (z^{p'} - w^{q'})$ for some positive integers p', q'. Then

$$D''_{l-1}|_{U^{l-1}_t} = (1 - z^{p''} w^{q''}),$$

for some non-negative integers p'', q''. The divisor

$$D'_{l-1} + D''_{l-1} + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j,l-1} + \sum_{1\le i\le l-1} E_{i,l-1},$$

in Z_{l-1}, is not normal crossing at $(0, 0) \in U^{l-1}_t$. Then $(0, 0)$ is contained in the center of the blowing-up $\pi_{l,l-1}$. We have

$$\pi^{-1}_{l,l-1}(U^{l-1}_t) = \{((z, w), [W_0 : W_1]) \in U^{l-1}_t \times \mathbb{P}^1(\mathbb{C}) \,|\, zW_1 = wW_0\}.$$

Let $V_i = \{((z, w), [W_0 : W_1]) \in \pi^{-1}_{l,l-1}(U^{l-1}_t) \,|\, W_i \neq 0\}$, $i = 0, 1$. Then $\{V_0, V_1\}$ is an affine open covering of $\pi^{-1}_{l,l-1}(U^{l-1}_t)$. We show that affine open sets V_0 and V_1 satisfy the five conditions of lemma. Let $u = W_1/W_0$ be the holomorphic function on V_0. Then (z, u) is the local coordinate system of V_0. It is easy to verify that V_0 satisfies (i), (ii), (iii) and (iv). Since

$$\pi_{l,l-1}^* z = z, \qquad \pi_{l,l-1}^* w = zu,$$

we have

$$\pi_{l,l-1\,*}\left(\frac{\partial}{\partial z}\ \frac{\partial}{\partial u}\right) = \left(\frac{\partial}{\partial z}\ \frac{\partial}{\partial w}\right)\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}.$$

Let Γ be the connection form of $\pi_{l,l-1}^*\nabla|_{V_0}$ with respect to the frame $\partial/\partial z$, $\partial/\partial u$. Then it follows that

$$\begin{aligned}\Gamma &= \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} d\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} \\ &\quad + \begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} \pi_{l,l-1}^* \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dw}{w} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}.\end{aligned}$$

Since

$$\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{u}{z} & \frac{1}{z} \end{pmatrix},$$

we have

$$\begin{aligned}\Gamma &= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -\frac{u}{z} & \frac{1}{z} \end{pmatrix}\begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{dz}{z} - \frac{du}{u} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} -\frac{dz}{z} & 0 \\ \frac{u}{z}\frac{dz}{z} & -\frac{1}{z}\left(\frac{dz}{z} + \frac{du}{u}\right) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & z \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ \frac{du}{z} & \frac{dz}{z} \end{pmatrix} + \begin{pmatrix} -\frac{dz}{z} & 0 \\ \frac{u}{z}\frac{dz}{z} - \frac{u}{z}\left(\frac{dz}{z} + \frac{du}{u}\right) & -\frac{dz}{z} - \frac{du}{u} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{dz}{z} & 0 \\ 0 & -\frac{du}{u} \end{pmatrix}.\end{aligned}$$

Hence V_0 satisfies (v). In the same way, we can show that V_1 satisfies the conditions of the lemma.

If

$$D'_{l-1}|_{U_t^{l-1}} = (1 - z^{p'} w^{q'}), \qquad D''_{l-1}|_{U_t^{l-1}} = (z^{p''} - w^{q''}),$$

for some non-negative integers p', q' and some positive integers p'', q''. In the same way as above, we can take the affine open sets in $\pi_{l,l-1}^{-1}(U_t^{l-1})$ which satisfy the five conditions of the lemma.

If

$$D'_{l-1}|_{U_t^{l-1}} = (1 - z^{p'} w^{q'}), \qquad D''_{l-1}|_{U_t^{l-1}} = (1 - z^{p''} w^{q''}),$$

for some non-negative integers p', q', p'', q''. Then $U_t^{l-1} \simeq \pi_{l,l-1}^{-1}(U_t^{l-1})$ because U_t^{l-1} does not contain the center of the blowing-up $\pi_{l,l-1}$. By the assumption of induction, the affine open subset $\pi_{l,l-1}^{-1}(U_t^{l-1})$ satisfies the five conditions of the lemma. This completes the proof. □

4. Proof of the Bigness of $\widetilde{D}' + \widetilde{D}''$

In this section, we show that $\widetilde{D}' + \widetilde{D}''$ is big in Z_k. We note that there exists the proof of the bigness for more general cases in Proposition 3.9. of [6].

To prove the bigness of the line bundle $\widetilde{D}' + \widetilde{D}''$, it is sufficient to show the following lemma (cf. Theorem 2.2.16. of R. Lazarsfeld [2]).

LEMMA 5. *The divisor $\widetilde{D}' + \widetilde{D}''$ is nef and $(\widetilde{D}' + \widetilde{D}'')^2 > 0$.*

PROOF. Because

$$(\widetilde{D}' + \widetilde{D}'')^2 = (\widetilde{D}')^2 + 2\widetilde{D}' \cdot \widetilde{D}'' + (\widetilde{D}'')^2,$$

it is enough to show that $(\widetilde{D}')^2 = (\widetilde{D}'')^2 = 0$ and $\widetilde{D}'$ and $\widetilde{D}''$ are nef. Without loss of generality, we may assume that $m' \leq n'$. Let E'_1 be the reduced divisor on Z_1 such that

$$\pi_{1,0}^* D' = D'_1 + m' E'_1.$$

Let F' be the divisor on Z_k such that

$$\pi_{k,1}^* D'_1 = \widetilde{D}' + F'.$$

It follows that

$$\begin{aligned}(\widetilde{D}')^2 &= (\pi_{k,0}^* D' - F' - m'\pi_{k,1}^* E_1')^2 \\ &= (\pi_{k,0}^* D')^2 + (F')^2 + m'^2(\pi_{k,1}^* E_1')^2 - 2\pi_{k,0}^* D' \cdot F' \\ &\quad + 2m'F' \cdot \pi_{k,1}^* E_1' - 2m'\pi_{k,1}^* E_1' \cdot \pi_{k,0}^* D' \\ &= 2m'n' + (F')^2 - 2m'^2 - 2(\widetilde{D}' + F' + m'\pi_{k,1}^* E_1') \cdot F' \\ &\quad + 2m'F' \cdot \pi_{k,1}^* E_1' - 2m'\pi_{k,1}^* E_1' \cdot (\pi_{k,1}^* D_1' + m'\pi_{k,1}^* E_1') \\ &= 2m'n' - 2m'^2 - (F')^2 - 2\widetilde{D}' \cdot F' - 2m' D_1' \cdot E_1' - 2m'^2 (E_1')^2.\end{aligned}$$

Because $D_1' \cdot E_1' = 2m'$, we have

$$(\widetilde{D}')^2 = 2m'n' - 2m'^2 - (F')^2 - 2\widetilde{D}' \cdot F'. \tag{2}$$

If $m' = n'$, then $\widetilde{D}' = D_1'$, $F' = 0$ and we have $(\widetilde{D}')^2 = 0$.

Now we prove $(\widetilde{D}')^2 = 0$ by the induction over the positive integer $m' + n'$. Let E_i', $i = 2, \cdots k$ be reduced effective divisors on Z_i such that

$$\operatorname{supp}(\pi_{i,i-1}^* D_{i-1}' - D_i') = \operatorname{supp} E_i'.$$

Let $\widetilde{E}_i'$ be the strict transform of E_i' under $\pi_{k,i}$. There exist non-negative integers $a_2, a_3, \cdots, a_k$ such that

$$F = a_2 \widetilde{E}_2' + a_3 \widetilde{E}_3' + \cdots + a_k \widetilde{E}_k'.$$

Now we take another divisor A' on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is defined by the polynomial

$$X_0^{m'} Y_0^{n'-m'} - X_1^{m'} Y_1^{n'-m'}.$$

There is, as in Section 3, the sequence of the blowing-up

$$\begin{aligned}\sigma_{1,0} \ &: \ W_1 \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}), \\ &\ \ \vdots \\ \sigma_{k-1,k-2} \ &: \ W_{k-1} \to W_{k-2},\end{aligned}$$

such that the following condition $(**)$ satisfies:

Let S be the reduced divisor such that

$$\operatorname{supp}\left(\sigma_{k-1,0}^{*}\left(A' + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j}\right)\right) = \operatorname{supp} S,$$

where $\sigma_{k-1,1} = \sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2}$. Then

$(**)$ S is normal crossing in W_{k-1}.

Let B'_i be the exceptional divisor of $\sigma_{i,i-1}$, and let $\widetilde{B}'_i$ be the strict transform of B'_i under $\sigma_{i+1,i} \circ \cdots \circ \sigma_{k-1,k-2}$. Let $\widetilde{A}'$ be the strict transform of A' under $\sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2}$. It follows that

$$(\sigma_{1,0} \circ \cdots \circ \sigma_{k-1,k-2})^{*} A' = \widetilde{A}' + a_2 \widetilde{B}'_1 + a_3 \widetilde{B}'_2 + \cdots + a_k \widetilde{B}'_{k-1},$$

and

$$\widetilde{E}'_i \cdot \widetilde{E}'_j = \widetilde{B}'_{i-1} \cdot \widetilde{B}'_{j-1}, \qquad \widetilde{D}' \cdot \widetilde{E}'_i = \widetilde{A}' \cdot \widetilde{B}'_{i-1},$$

for all $2 \leq i, j \leq k$. Put $G' = a_2 \widetilde{B}'_1 + a_3 \widetilde{B}'_2 + \cdots + a_k \widetilde{B}'_{k-1}$. We have

$$(F')^2 = (G')^2, \qquad \widetilde{D}' \cdot F' = \widetilde{A}' \cdot G'. \tag{3}$$

It follows that

$$\begin{aligned}
(\widetilde{A}')^2 &= (\sigma_{k-1,0}^{*} A' - G')^2 \\
&= 2m'(n' - m') - 2\sigma_{k-1,0}^{*} A' \cdot G' + (G')^2 \\
&= 2m'(n' - m') - 2(\widetilde{A}' + G') \cdot G' + (G')^2 \\
&= 2m'(n' - m') - (G')^2 - 2\widetilde{A}' \cdot G'.
\end{aligned}$$

By the assumption of the induction, we have

$$(G')^2 + 2\widetilde{A}' \cdot G' - 2m'(n' - m') = 0. \tag{4}$$

By (2), (3) and (4), it follows that

$$\begin{aligned}
(\widetilde{D}')^2 &= 2m'n' - 2m'^2 - (F')^2 - 2\widetilde{D}' \cdot F' \\
&= 2m'(n' - m') - (G')^2 - 2\widetilde{A}' \cdot G' = 0.
\end{aligned}$$

Then we complete the induction. By the same way, we can show that $(\widetilde{D}'')^2 = 0$.

Now we show that $\widetilde{D}'$ is nef. Let $m' = dp$, $n' = dq$, where d is the greatest common divisor of m' and n'. Then it follows that

$$X_0^{m'}Y_0^{n'} - X_1^{m'}Y_1^{n'} = \prod_{i=0}^{d-1}\left(X_0^pY_0^q - (\varepsilon_d)^iX_1^pY_1^q\right),$$

where $\varepsilon_d = \exp((2\pi\sqrt{-1})/d)$. Let C_i be the irreducible divisor on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which is defined by the polynomial $X_0^pY_0^q - (\varepsilon_d)^iX_1^pY_1^q$, and let $\widetilde{C}_i$ be the strict transform of C_i under $\pi_{k,0}$. By the above arguments, we have $(\widetilde{C}_0)^2 = 0$. Because $\widetilde{C}_0$ and $\widetilde{C}_i$, $1 \le i \le d-1$, are linearly equivalent, we have

$$\widetilde{C}_i \cdot \widetilde{D}' = (\widetilde{C}_i)^2 = (\widetilde{C}_0)^2 = 0.$$

Therefore $\widetilde{D}'$ is nef. By the same way, we can show that $\widetilde{D}''$ is nef. $\square$

5. Proof of the Main Theorem

Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be the holomorphic map, let $\widetilde{f} : \mathbb{C} \to Z_k$ be the lift of f, and let $\widetilde{\nabla} = \pi_{k,0}^*\nabla$.

$$\begin{tikzcd}
Z_0 & Z_1 \arrow[l, "\pi_{1,0}"'] & Z_2 \arrow[l, "\pi_{2,1}"'] & \cdots \arrow[l, "\pi_{3,2}"'] & Z_k \arrow[l, "\pi_{k,k-1}"'] \\
\mathbb{C} \arrow[u, "f"] \arrow[urrrr, dashed, "\widetilde{f}"'] & & & &
\end{tikzcd}$$

Let $\widetilde{\sigma}' \in \Gamma(Z_k, [\widetilde{D}'])$, $\widetilde{\sigma}'' \in \Gamma(Z_k, [\widetilde{D}''])$, $\widetilde{h}_{i,j} \in \Gamma(Z_k, [\widetilde{H}_{i,j}])$, $\widetilde{e}_i \in \Gamma(Z_k, \widetilde{E}_i)$ be the holomorphic section such that

$$(\widetilde{\sigma}') = \widetilde{D}', \qquad (\widetilde{\sigma}'') = \widetilde{D}'', \qquad (\widetilde{h}_{i,j}) = \widetilde{H}_{i,j}, \qquad (\widetilde{e}_i) = \widetilde{E}_i.$$

Lemma 6. *Assume that*

$$f(\mathbb{C}) \not\subset \operatorname{supp}\left(D' + D'' + \sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j}\right),$$

and assume that

$$f' \wedge \nabla_{f'} f' \not\equiv 0.$$

Then it follows that

$$\int_{|z|=r} \log^+ \frac{\|\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}'(z)\|_{\wedge^2 TZ_k}}{\|\widetilde{\sigma}'(\widetilde{f})\|_{[D']} \|\widetilde{\sigma}''(\widetilde{f})\|_{[D'']} \prod_{i=1}^{2} \prod_{j=0}^{1} \|\widetilde{h}_{i,j}(\widetilde{f})\|_{[H_{i,j}]} \prod_{i=1}^{k} \|\widetilde{e}_i(\widetilde{f})\|_{[E_i]}} \frac{d\theta}{2\pi}$$
$$\leqq S_f(r).$$

Proof. For the convenience of the notation, we assume that D' and D'' are irreducible. Put

$$A = \widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{2} \sum_{j=0}^{1} \widetilde{H}_{i,j} + \sum_{i=1}^{k} \widetilde{E}_i,$$

and put

$$\xi(z) = \frac{\|\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}'(z)\|_{\wedge^2 TZ_k}}{\|\widetilde{\sigma}'(\widetilde{f})\|_{[\widetilde{D}']} \|\widetilde{\sigma}''(\widetilde{f})\|_{[\widetilde{D}'']} \prod_{i=1}^{2} \prod_{j=0}^{1} \|\widetilde{h}_{i,j}(\widetilde{f})\|_{[\widetilde{H}_{i,j}]} \prod_{i=1}^{k} \|\widetilde{e}_i(\widetilde{f})\|_{[\widetilde{E}_i]}}.$$

Note that A is simple normal crossing in Z_k.

Let

$$x \in \bigcup_{i=1}^{2} \bigcup_{j=0}^{1} \operatorname{supp} \widetilde{H}_{i,j} \cap \bigcup_{i=1}^{k} \operatorname{supp} \widetilde{E}_l.$$

By Lemma 4, there exists an affine open neighborhood U_x of x and local coordinate system z_x, w_x on U_x which satisfy the five conditions of Lemma 4. We put

$$V_x = U_x \setminus \operatorname{supp}(\widetilde{D}' + \widetilde{D}''),$$

and put

$$\widetilde{f}_1 = z_x \circ \widetilde{f}, \qquad \widetilde{f}_2 = w_x \circ \widetilde{f},$$

on $\widetilde{f}^{-1}(V_x)$. It follows that

$$\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'}\widetilde{f}' = \left(\widetilde{f}_1'\widetilde{f}_2'' - \widetilde{f}_1''\widetilde{f}_2' + \widetilde{f}_1'\widetilde{f}_2'\frac{\widetilde{f}_1'}{\widetilde{f}_1} - \widetilde{f}_1'\widetilde{f}_2'\frac{\widetilde{f}_2'}{\widetilde{f}_2}\right)\frac{\partial}{\partial z_x} \wedge \frac{\partial}{\partial w_x}.$$

on $\widetilde{f}^{-1}(V_x)$. Then it follows that

$$\xi(z) = \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1}\frac{\widetilde{f}_2''}{\widetilde{f}_2} - \frac{\widetilde{f}_1''}{\widetilde{f}_1}\frac{\widetilde{f}_2'}{\widetilde{f}_2} + \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1}\right)^2\frac{\widetilde{f}_2'}{\widetilde{f}_2} - \frac{\widetilde{f}_1'}{\widetilde{f}_1}\left(\frac{\widetilde{f}_2'}{\widetilde{f}_2}\right)^2\right)\Phi_x(f(z)), \tag{5}$$

on $\widetilde{f}^{-1}(V_x)$, where Φ_x is a smooth function on V_x.

Let

$$x \in \operatorname{supp}\widetilde{D}' \cap \left(\bigcup_{i=1}^{2}\bigcup_{j=0}^{1}\operatorname{supp}\widetilde{H}_{i,j} \cup \bigcup_{i=1}^{k}\operatorname{supp}\widetilde{E}_i\right)$$

$$\left(\text{or} \qquad x \in \operatorname{supp}\widetilde{D}'' \cap \left(\bigcup_{i=1}^{2}\bigcup_{j=0}^{1}\operatorname{supp}\widetilde{H}_{i,j} \cup \bigcup_{i=1}^{k}\operatorname{supp}\widetilde{E}_i\right),\ \text{respectively}\right)$$

By Lemma 4, there exists an affine open neighborhood U_x of x and local coordinate system z_x, w_x on U_x which satisfy the five condition of Lemma 4. Because D' and D'' are irreducible, it follows that

$$D'|_{U_x} = (z_x - 1) \qquad (\text{or } D''|_{U_x} = (z_x - 1),\ \text{respectively}),$$

and $z_x(x) = 1, w_x(x) = 0$. We take $z_x' = z_x - 1$. Let V_x be an affine open subset of U_x such that

$$A|_{V_x} = (z_x') + (w_x),$$

and

$$\nabla|_{V_x} = d + \begin{pmatrix} -(dz_x')/(z_x'+1) & 0 \\ 0 & -(dw_x)/w_x \end{pmatrix}.$$

We note that $z_x'(x) + 1 \neq 0$ on $\widetilde{f}^{-1}(V_x)$. We put

$$\widetilde{f}_1 = z_x' \circ \widetilde{f}, \qquad \widetilde{f}_2 = w_x \circ \widetilde{f},$$

on $\widetilde{f}^{-1}(V_x)$. It follows that

$$\xi(z) = \left(\frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2''}{\widetilde{f}_2} - \frac{\widetilde{f}_1''}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2} - \frac{\widetilde{f}_1'}{\widetilde{f}_1} \left(\frac{\widetilde{f}_2'}{\widetilde{f}_2} \right)^2 \right) \Phi_x(f(z)) + \widetilde{f}_1' \frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2} \Psi_x(f(z)), \tag{6}$$

on $\widetilde{f}^{-1}(V_x)$, where Φ_x and Ψ_x are smooth functions on V_x.

Let $x \in \operatorname{supp} \widetilde{D}' \cap \operatorname{supp} \widetilde{D}''$. There exists an affine open neighborhood V_x of x and holomorphic functions z_x, w_x on V_x such that

$$\widetilde{D}'|_{V_x} = (z_x), \qquad \widetilde{D}''|_{V_x} = (w_x),$$

$$A|_{V_x} = (z_x) + (w_x),$$

on V_x. It follows that dz_x and dw_x are linearly independent on V_x. We put

$$\widetilde{f}_1 = z_x \circ \widetilde{f}, \qquad \widetilde{f}_2 = w_x \circ \widetilde{f}.$$

By Lemma 2, there exist holomorphic functions g_0, g_1, h_0, h_1 on V_x such that

$$dz_x \cdot \nabla_{\widetilde{f}'} \widetilde{f}'(\gamma) = g_0(\widetilde{f}(\gamma)) \widetilde{f}_1(\gamma) + g_1(\widetilde{f}(\gamma)) \widetilde{f}_1'(\gamma) + \widetilde{f}_1''(\gamma),$$

for all $\gamma \in \widetilde{f}^{-1}(V_x)$, and

$$dw_x \cdot \nabla_{\widetilde{f}'} \widetilde{f}'(\gamma) = h_0(\widetilde{f}(\gamma)) \widetilde{f}_2(\gamma) + h_1(\widetilde{f}(\gamma)) \widetilde{f}_2'(\gamma) + \widetilde{f}_2''(\gamma),$$

for all $\gamma \in \widetilde{f}^{-1}(V_x)$. It follows that

$$\widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'} \widetilde{f}' = \Big[\widetilde{f}_1' \left(h_0(\widetilde{f}) \widetilde{f}_2 + h_1(\widetilde{f}) \widetilde{f}_2' + \widetilde{f}_2'' \right) - \widetilde{f}_2' \left(g_0(\widetilde{f}) \widetilde{f}_1 + g_1(\widetilde{f}) \widetilde{f}_1' + \widetilde{f}_2'' \right) \Big] \frac{\partial}{\partial z_x} \wedge \frac{\partial}{\partial w_x}.$$

Then it follows that

$$\xi(z) = \Phi_{x,1}(\widetilde{f}) \frac{\widetilde{f}_1'}{\widetilde{f}_1} + \Phi_{x,2}(\widetilde{f}) \frac{\widetilde{f}_2'}{\widetilde{f}_2} + \Phi_{x,3}(\widetilde{f}) \frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2} + \Phi_{x,4}(\widetilde{f}) \frac{\widetilde{f}_1'}{\widetilde{f}_1} \frac{\widetilde{f}_2''}{\widetilde{f}_2} + \Phi_{x,5}(\widetilde{f}) \frac{\widetilde{f}_1''}{\widetilde{f}_1} \frac{\widetilde{f}_2'}{\widetilde{f}_2}, \tag{7}$$

on $\widetilde{f}^{-1}(V_x)$, where $\Phi_{x,1},\dots,\Phi_{x,5}$ are smooth functions on V_x.

Let $R=\{x\in Z_k|\ x$ is contained in two irreducible components of A $\}$. Note that R is a finite subset of Z_k. For $x\in R$, we take an affine open subset V_x and holomorphic functions z_x, w_x as above arguments. Then $\{V_x\}_{x\in R}$ is an open covering of Z_k. We take an open covering $\{V'_x\}_{x\in R}$ of Z_k such that $V'_x\subset V_x$ and V'_x is relatively compact in V_x. We take a partition of unity $\{\phi_x\}_{x\in R}$ which is subordinate to the covering $\{V'_x\}_{x\in R}$. Fix $x\in R$. Let $\widetilde{f}_1=z_x\circ\widetilde{f}$, $\widetilde{f}_2=w_x\circ\widetilde{f}$ be a holomorphic function on $\widetilde{f}^{-1}(V_x)$. Then $\widetilde{f}_1$ and $\widetilde{f}_2$ are extended to meromorphic functions on $\mathbb{C}$. By (5), (6) and (7), we have

$$\begin{aligned}
&\int_{|z|=r}\phi_i(\widetilde{f}(z))\log^+\xi(z)\frac{d\theta}{2\pi}\\
&\le\int_{|z|=r}\Gamma(\widetilde{f}(z))\frac{d\theta}{2\pi}+4\sum_{i=1}^{2}\int_{|z|=r}\log^+\frac{|\widetilde{f}_i'(z)|}{|\widetilde{f}_i(z)|}\frac{d\theta}{2\pi}\\
&+\sum_{i=1}^{2}\int_{|z|=r}\log^+\frac{|\widetilde{f}_i''(z)|}{|\widetilde{f}_i(z)|}\frac{d\theta}{2\pi}+\int_{|z|=r}\phi_i(\widetilde{f}(z))\log^+|\widetilde{f}_1'(z)|\frac{d\theta}{2\pi},
\end{aligned}$$

where Γ is a bounded smooth function on Z_k. By using the lemma on logarithmic derivative, it follows that

$$\int_{|\gamma|=r}\log^+\frac{|\widetilde{f}_i'(\gamma)|}{|\widetilde{f}_i(\gamma)|}\frac{d\theta}{2\pi}\le S_{\widetilde{f}}(r).$$

It holds that

$$\begin{aligned}
\int_{|z|=r}\phi_i(\widetilde{f}(z))\log^+|\widetilde{f}_1'(z)|\frac{d\theta}{2\pi}&=\frac{1}{2}\int_{|z|=r}\phi_i(\widetilde{f}(z))\log^+|\widetilde{f}_1'(z)|^2\frac{d\theta}{2\pi}\\
&\le\frac{1}{2}\int_{|z|=r}\log^+\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{d\theta}{2\pi}+O(1),
\end{aligned}$$

where $\|\cdot\|_{TZ_k}$ is a hermitian metric of TZ_k. By Lemma 1 and the concavity of log, we have that

$$
\begin{aligned}
&\frac{1}{2}\int_{|z|=r} \log^+ \|\widetilde{f}'(z)\|^2_{TZ_k}\frac{d\theta}{2\pi} \\
&\leq \frac{1}{2}\int_{|z|=r} \log\{\|\widetilde{f}'(z)\|^2_{TZ_k}+1\}\frac{d\theta}{2\pi} \\
&\leq \frac{1}{2}\log\left(1+\int_{|z|=r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{d\theta}{2\pi}\right)+O(1) \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2\pi r}\frac{d}{dr}\int_{|z|\leq r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)+O(1) \\
&\leq \frac{1}{2}\log\left(1+\frac{1}{2\pi r}\left(\int_{|z|\leq r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{1+\delta}\right)+O(1)\| \\
&= \frac{1}{2}\log\left(1+\frac{r^\delta}{2\pi}\left(\frac{d}{dr}\int_1^r\frac{dt}{t}\int_{|z|\leq r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{1+\delta}\right)+O(1)\| \\
&\leq \frac{1}{2}\log\left(1+\frac{r^\delta}{2\pi}\left(\int_1^r\frac{dt}{t}\int_{|z|\leq r}\|\widetilde{f}'(z)\|^2_{TZ_k}\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}\right)^{(1+\delta)^2}\right)+O(1)\| \\
&\leq S_f(r),
\end{aligned}
$$

where δ is any positive number.

Because $\sum_{x\in R}\phi_x(\widetilde{f})=1$ on $\mathbb{C}$, it follows that

$$\int_{|z|=r}\log^+\xi(z)\frac{d\theta}{2\pi}=\sum_{x\in R}\int_{|z|=r}\phi_x(\widetilde{f}(z))\log^+\xi(z)\frac{d\theta}{2\pi}\leq S_f(r). \ \Box$$

The following lemma is useful.

Lemma 7. *It follows that*

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi^*_{k,0}H_{i,j}=\sum_{i=1}^{2}\sum_{j=0}^{1}\widetilde{H}_{i,j}+\sum_{i=1}^{k}\pi^*_{k,i}E_i+\sum_{i=1}^{k}\widetilde{E}_i.$$

Proof. Let the divisor $H_{i,j,l}$ on Z_l be the strict transform of $H_{i,j}$ under $\pi_{l,0}$, and let $E_{i,l}$, $i\leq l$, be the strict transform of E_i under $\pi_{l,i}$, where $E_{l,l}=E_l$.

We show

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l,0}^{*}H_{i,j}=\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,l}+\sum_{i=1}^{l}\pi_{l,i}^{*}E_i+\sum_{i=1}^{l}E_{i,l},$$

by induction over l. If $l=1$, we have

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{1,0}^{*}H_{i,j}=\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,1}+2E_i.$$

Therefore the statement of the induction holds for $l=1$. Assume that the statement holds for $l-1$, $1<l\leq k$. Let C_i $i=1,2,\dots,r$ be irreducible divisor on Z_{l-1} such that

$$\operatorname{supp}\left(\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l-1,0}^{*}H_{i,j}\right)=\bigcup_{i=1}^{r}\operatorname{supp}C_i.$$

There exist positive integers $a_1,a_2,\dots,a_r$ such that

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l-1,0}^{*}H_{i,j}=\sum_{i=1}^{r}a_iC_i.$$

By the assumption of the induction, we have

$$\sum_{i=1}^{l}\pi_{l,i}^{*}E_i=\sum_{i=1}^{r}(a_i-1)C_i.$$

Let $x\in Z_{l-1}$ be one of the points of the center of $\pi_{l,l-1}$, and let F_l be the irreducible component of E_l such that $\pi_{l,l-1}(\operatorname{supp}F_l)=x$. Assume that $x\in\operatorname{supp}C_p\cap\operatorname{supp}C_q$ for $1\leq p<q\leq r$. Then the coefficients of F_l in $\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l,0}^{*}H_{i,j}$ is a_p+a_q, and the coefficient of F_l in $\sum_{i=1}^{l-1}\pi_{l,i}^{*}E_i$ is a_p+a_q-2. Therefore we have

$$\sum_{i=1}^{2}\sum_{j=0}^{1}\pi_{l,0}^{*}H_{i,j}-\sum_{i=1}^{l}\pi_{l,i}^{*}E_i=\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,l}+\sum_{i=1}E_{i,l}.$$

This complete the induction, and the lemma follows. $\square$

Proof of the Main Theorem. We put $W_{\widetilde{\nabla}}(\widetilde{f}) = \widetilde{f}' \wedge \widetilde{\nabla}_{\widetilde{f}'}\widetilde{f}'$. We denote by $\operatorname{ord}_z g$ the order of zero of g at z, where g is a holomorphic section of a line bundle on a neighborhood of z. By (5), (6) and (7) in Lemma 6, it follows that

$$\begin{aligned}
&\operatorname{ord}_z \left(\widetilde{\sigma}'(\widetilde{f})\widetilde{\sigma}''(\widetilde{f}) \prod_{i=1}^{2}\prod_{j=0}^{1} \widetilde{h}_{i,j}(\widetilde{f}) \prod_{i=1}^{k} \widetilde{e}_i(\widetilde{f}) \right) - \operatorname{ord}_z \left(W_{\widetilde{\nabla}}(\widetilde{f}) \right) \\
&\leq \min\{\operatorname{ord}_z \widetilde{\sigma}'(\widetilde{f}), 2\} + \min\{\operatorname{ord}_z \widetilde{\sigma}''(\widetilde{f}), 2\} \\
&+ 2\sum_{i=1}^{2}\sum_{j=0}^{1} \min\{\operatorname{ord}_z \widetilde{h}_{i,j}(\widetilde{f}), 1\} + 2\sum_{i=1}^{k} \min\{\operatorname{ord}_z \widetilde{e}_i(\widetilde{f}), 1\}.
\end{aligned}$$

Therefore it follows that

$$\begin{aligned}
(8)\qquad & T_{\widetilde{f}}(r, K_{Z_k}) + T_{\widetilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) + \sum_{i=1}^{2}\sum_{j=0}^{1} T_{\widetilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \widetilde{E}_i) \\
&\leq N_2(r, \widetilde{f}^*\widetilde{D}') + N_2(r, \widetilde{f}^*\widetilde{D}'') + 2\sum_{i=1}^{2}\sum_{j=0}^{1} N_1(r, \widetilde{f}^*\widetilde{H}_{i,j}) \\
&\quad + 2\sum_{1\leq i\leq k} N_1(r, \widetilde{f}^*\widetilde{E}_i) + S_f(r),
\end{aligned}$$

where K_{Z_k} is the canonical line bundle of Z_k. The canonical line bundle of Z_k is equal to

$$\pi_{k,0}^* K_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})} + \pi_{k,1}^* E_1 + \pi_{k,2}^* E_2 + \cdots + E_k.$$

By Lemma 7, it follows that

$$\begin{aligned}
(9)\qquad -T_f(r, K_{\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})}) &= T_f(r, \mathcal{O}(2,2)) = \sum_{i=1}^{2}\sum_{j=0}^{1} T_{\widetilde{f}}(r, \pi_{k,0}^* H_{i,j}) \\
&= \sum_{i=1}^{2}\sum_{j=0}^{1} T_{\widetilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \pi_{k,i}^* E_i) \\
&\quad + \sum_{i=1}^{k} T_{\widetilde{f}}(r, \widetilde{E}_i)
\end{aligned}$$

By (8), (9), it follows that

$$\begin{aligned}T_{\tilde{f}}(r,[\widetilde{D}'+\widetilde{D}'']) \leq\ & N_2(r,\widetilde{f}^*\widetilde{D}')+N_2(r,\widetilde{f}^*\widetilde{D}'')\\ &+2\sum_{i=1}^{2}\sum_{j=0}^{1}N_1(r,\widetilde{f}^*\widetilde{H}_{i,j})+2\sum_{i=1}^{k}N_1(r,\widetilde{f}^*\widetilde{E}_i)+S_f(r).\end{aligned}$$

By Lemma 3 and Lemma 5, our main theorem follows. □

Corollary 1. *Let $f:\mathbb{C}\to\mathbb{C}^*\times\mathbb{C}^*\subset\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})$ be a non-constant map. Assume that*

$$\begin{aligned}f(\mathbb{C})\not\subset\{([X_0:X_1],[Y_0:Y_1])\\ \in\mathbb{P}^1(\mathbb{C})\times\mathbb{P}^1(\mathbb{C})\,|\,C_0X_0^{r_1}Y_0^{r_2}-C_1X_1^{r_1}Y_1^{r_2}=0\},\end{aligned}$$

for all $(r_1,r_2)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}$ and all $(C_0,C_1)\in\mathbb{C}\times\mathbb{C}\setminus\{(0,0)\}$, and assume that there exists no $(a,b)\in\mathbb{C}\times\mathbb{C}\setminus\{(0,0)\}$ such that

$$a\log f_1+b\log f_2=(\text{constant}),$$

on $\mathbb{C}$. Then it follows that

$$T_{\tilde{f}}(r,[\widetilde{D}])\leq N_2(r,f^*D')+N_2(r,f^*D'')+S_f(r).$$

Proof. Because $N_2(r,\widetilde{f}^*\widetilde{H}_{i,j})=0$ and $N_2(r,\widetilde{f}^*\widetilde{E}_i)=0$, we have the corollary. □

Example 2. Let D',D'' be the divisor which are defined by the polynomials

$$X_0Y_0-X_1Y_1,\qquad X_0Y_1-X_1Y_0.$$

Then

$$D_1'+D_1''+\sum_{i=1}^{2}\sum_{j=0}^{1}H_{i,j,1}+E_1,$$

is normal crossing in Z_1. Therefore $\widetilde{D}'=D_1'$, $\widetilde{D}''=D_1''$. Let $E_{(0,0)},E_{(0,\infty)}$, $E_{(\infty,0)},E_{(\infty,\infty)}$ be irreducible components of E_1 such that

$$\pi_{1,0}(\operatorname{supp}E_{(0,0)})=([0:1],[0:1]),\qquad \pi_{1,0}(\operatorname{supp}E_{(0,\infty)})=([0:1],[1:0]),$$

$$\pi_{1,0}(\operatorname{supp} E_{(\infty,0)}) = ([1:0],[0:1]), \qquad \pi_{1,0}(\operatorname{supp} E_{(\infty,\infty)}) = ([1:0],[1:0]).$$

Let $f = (f_1, f_2) : \mathbb{C} \to Z_0$ be a non-constant holomorphic map, and let $\widetilde{f} : \mathbb{C} \to Z_1$ be the lift of f. It follows that

$$T_{\widetilde{f}}(r, [\widetilde{D}']) = T_{\widetilde{f}}(r, [\pi_{1,0}^* D']) - T_{\widetilde{f}}(r, [E_{(0,\infty)}]) - T_{\widetilde{f}}(r, [E_{(\infty,0)}]),$$

and

$$T_{\widetilde{f}}(r, [\pi_{1,0}^* D']) = T_f(r, \mathcal{O}(1,1)) = T(r, f_1) + T(r, f_2),$$

where

$$T(r, f_i) = \int_{|z|=r} \log^+ |f_i| \frac{d\theta}{2\pi} + N(r, (f_i)_\infty),$$

for $i = 1, 2$. By the first main theorem, we have

$$T_{\widetilde{f}}(r, E_{(0,\infty)}) = N(r, \widetilde{f}^* E_{(0,\infty)}) + m_{\widetilde{f}}(r, E_{(0,\infty)}),$$

$$T_{\widetilde{f}}(r, E_{(\infty,0)}) = N(r, \widetilde{f}^* E_{(\infty,0)}) + m_{\widetilde{f}}(r, E_{(\infty,0)}).$$

It holds that

$$m_{\widetilde{f}}(r, E_{(0,\infty)}) = \int_{|z|=r} \log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} \frac{d\theta}{2\pi},$$

and

$$m_{\widetilde{f}}(r, E_{(\infty,0)}) = \int_{|z|=r} \log^+ \frac{1}{\sqrt{|f_1^{-1}|^2 + |f_2|^2}} \frac{d\theta}{2\pi}.$$

By these equations, we have

$$\begin{aligned} T_{\widetilde{f}}(r, \widetilde{D}') = {} & N(r, (f_1)_\infty) + N(r, (f_2)_\infty) - N(r, \widetilde{f}^* E_{(0,\infty)}) - N(r, \widetilde{f}^* E_{(\infty,0)}) \\ & + \int_{|z|=r} \left(\log^+ |f_1| + \log^+ |f_2|\right) \frac{d\theta}{2\pi} \\ & - \int_{|z|=r} \left(\log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} + \log^+ \frac{1}{\sqrt{|f_1^{-1}|^2 + |f_2|^2}} \right) \frac{d\theta}{2\pi} \end{aligned}$$

Let $f_1 = P(z), f_2 = \exp z$, where $P(z)$ is a polynomial of degree p on $\mathbb{C}$. Then $T(r, f_1) = p \log r + O(1)$, and $T(r, f_2) = |r| + O(1)$. Because

$$\log^+ \frac{1}{\sqrt{|f_1|^2 + |f_2^{-1}|^2}} \leq \log^+ \frac{1}{|f_1|},$$

it follows that

$$m_{\tilde{f}}(r, E_{(0,\infty)}) \leq T(r, f_1^{-1}) = T(r, f_1) + O(1) = p \log |r| + O(1).$$

Therefore we have

$$m_{\tilde{f}}(r, E_{(0,\infty)}) = o(r).$$

By the same arguments, we have

$$m_{\tilde{f}}(r, E_{(\infty,0)}) = o(r).$$

Then it holds that

$$T_{\tilde{f}}(r, \widetilde{D}') = r + o(r).$$

Let D' and D'' be divisors on $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ which are defined by the polynomials

$$X_0^m Y_0^n - X_1^m Y_1^n, \qquad X_0^n Y_1^m - X_1^n Y_0^m.$$

(i,e,. $m = m' = n''$ and $n = n' = m''$.) We have the following theorem.

THEOREM 3. *Let $f : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ be a non-constant holomorphic map. Let $\widetilde{f} : \mathbb{C} \to Z_k$ be the lift of f. Assume that*

$$\begin{aligned} f(\mathbb{C}) \not\subset \{([X_0 : X_1], [Y_0 : Y_1]) \\ \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \,|\, C_0 X_0^{r_1} Y_0^{r_2} - C_1 X_1^{r_1} Y_1^{r_2} = 0\}, \end{aligned}$$

for all $(r_1, r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and all $(C_0, C_1) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$, and assume that there exist no holomorphic functions g_1, g_2 on $\mathbb{C}$ and no $(a, b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),$$

$$ag_1 + bg_2 = (\text{constant}),$$

on $\mathbb{C}$. *Then it follows that*

$$\left(1 - \frac{4}{m+n}\right) T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \leq N_2(r, \tilde{f}^* \widetilde{D}') + N_2(r, \tilde{f}^* \widetilde{D}'') + S_f(r).$$

Proof. Let $a_1 = \min\{m, n\}$. It follows that

$$\pi_{1,0}^*(D' + D'') = D_1' + D_1'' + a_1 E_1,$$

on Z_1, where D_1' and D_1'' are proper transforms of D' and D'' under $\pi_{1,0}$. Let $a_2 = \min\{\max\{m, n\} - a_1, a_1\} \leq a_1$. It follows that

$$\pi_{2,0}^*(D' + D'') = D_2' + D_2'' + a_2 E_2 + a_1 \pi_{2,1}^* E_1,$$

on Z_2, where D_2' and D_2'' are proper transforms of D' and D'' under $\pi_{2,0}$. Repeating this process, there exist positive integers $a_3 \; \cdots , a_k$ such that

$$\pi_{k,0}^*(D' + D'') = \widetilde{D}' + \widetilde{D}'' + \sum_{i=1}^{k} a_i \pi_{k,i}^* E_i.$$

Without loss of generality, we may assume that $m \leq n$. Then it holds that $m \geq a_1 \geq a_2 \geq \cdots \geq a_k$. It follows that

$$T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \geq T_{\tilde{f}}(r, \pi_{k,0}^* \mathcal{O}(m+n, m+n)) - m \sum_{i=1}^{k} T_{\tilde{f}}(r, \pi_{k,i}^* E_i).$$

By Lemma 7, we have

$$T_{\tilde{f}}(r, \pi_{k,0}^* \mathcal{O}(2,2)) = \sum_{i=1}^{2} \sum_{j=0}^{1} T_{\tilde{f}}(r, \widetilde{H}_{i,j}) + \sum_{i=1}^{k} T_{\tilde{f}}(r, \pi_{k,i}^* E_i) + \sum_{i=1}^{k} T_{\tilde{f}}(r, \widetilde{E}_i)$$

Then we have

$$\begin{aligned} & T_{\tilde{f}}(r, [\widetilde{D}' + \widetilde{D}'']) \\ & \geq \frac{m+n}{2} \left(T_{\tilde{f}}(r, \pi_{k,0}^* \mathcal{O}(2,2)) - \sum_{i=1}^{k} T_{\tilde{f}}(r, \pi_{k,i}^* E_i) \right) \end{aligned}$$

$$+\left(\frac{m+n}{2}-m\right)\sum_{i=1}^{k} T_{\tilde{f}}(r,\pi_{k,i}^{*}E_i)$$
$$\geq \frac{m+n}{2}\left(\sum_{i=1}^{2}\sum_{j=0}^{1} T_{\tilde{f}}(r,\widetilde{H}_{i,j})+\sum_{i=1}^{k} T_{\tilde{f}}(r,\widetilde{E}_i)\right).$$

By Theorem 1, it follows that

$$T_{\tilde{f}}(r,[\widetilde{D}'+\widetilde{D}'']) \leq N_2(r,\tilde{f}^{*}\widetilde{D}')+N_2(r,\tilde{f}^{*}\widetilde{D}''))$$
$$+\frac{4}{m+n}T_{\tilde{f}}(r,[\widetilde{D}'+\widetilde{D}''])+S_f(r).$$

Then the theorem follows. □

Corollary 2. *Assume the hypothesis of Theorem 3, and assume that*

$$f(\mathbb{C}) \subset \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \setminus \operatorname{supp}(D'+D'').$$

If $m+n \geq 5$, then it follows that $f(\mathbb{C}) \subset \operatorname{supp} H_{i,j}$ for $i=1,2$ and $j=0,1$.

Proof. Assume that $f(\mathbb{C})$ is not contained in the support of $\sum_{i=1}^{2}\sum_{j=0}^{1} H_{i,j}$. By Theorem 3, f satisfies the following condition (i) or condition (ii):

(i) $$f(\mathbb{C}) \subset \{([X_0 : X_1],[Y_0 : Y_1]) \in \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}) \,|\, X_0^{r_1}Y_0^{r_2}-C_1X_1^{r_1}Y_1^{r_2}=0\},$$

for some $(r_1,r_2) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$ and some $C_1 \in \mathbb{C} \setminus \{(0)\}$.

(ii) There exist holomorphic functions g_1, g_2 on $\mathbb{C}$ and $(a,b) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$ such that

$$f = (\exp g_1, \exp g_2) : \mathbb{C} \to \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C}),$$

$$ag_1 + bg_2 = (\text{constant}),$$

on $\mathbb{C}$.

If f satisfies condition (i), without loss of generality, we may assume that $r_1 > 0, r_2 \geq 0$. Assume that $r_2 > 0$. Let R be an irreducible component of $\{X_0^{r_1}Y_0^{r_2} - CX_1^{r_1}Y_1^{r_2} = 0\}$. Then $([0:1],[1:0]),([1:0],[0:1]) \in \operatorname{supp} R \cap \operatorname{supp} D'$, and $\operatorname{supp} R \cap \operatorname{supp} D''$ contains at least one point which is not $([0:1],[1:0])$ nor $([1:0],[0:1])$. Therefore the holomorphic map

$$f : \mathbb{C} \to \operatorname{supp} R \setminus \operatorname{supp}(D' + D'')$$

is a constant map.
Assume that $r_2 = 0$. We have

$$f(\mathbb{C}) \subset \{([X_0 : X_1],[Y_0 : Y_1]) \in \mathbb{P}(\mathbb{C}) \times \mathbb{P}(\mathbb{C}) \mid X_0^{r_1} - CX_1^{r_1} = 0\}.$$

Let S be an irreducible component of $\{X_0^{r_1} - CX_1^{r_1} = 0\}$. Because $m+n \geq 5$, m or n is more than 2, it follows that $\operatorname{supp} S \cap \operatorname{supp} D'$ or $\operatorname{supp} S \cap \operatorname{supp} D''$ contains at least three points. Then f is a constant map.

If f satisfies condition (ii), it is easy to see that f is a constant map. $\square$

Remark 2. Let $x_{1,0} = ([0:1],[1:1]), x_{1,1} = ([1:0],[1:1]), x_{2,0} = ([1:1],[0:1]), x_{2,1} = ([1:1],[1:0]) \in Z_0 = \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Let $W = Z_0 \setminus \operatorname{supp} D' \cup \operatorname{supp} D''$, and let $W^* = W \setminus \{x_{1,0}, x_{1,1}, x_{2,0}, x_{2,1}\}$. By Corollary 2, there exist no non-constant holomorphic maps from $\mathbb{C}$ to W^*.

Let $i : W^* \to W$ be the inclusion map, and let d_{W^*}, d_W be the Kobayashi pseudo distance of W^*, W (see Noguchi-Ochiai [4]). By Proposition 1.3.14. of [4], we have $i^* d_W = d_{W*}$. Therefore W^* is Brody hyperbolic but not Kobayashi hyperbolic.

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Graduate School of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914, Japan
E-mail: chiba@ms.u-tokyo.ac.jp