The Generalized Periodic Ultradiscrete KdV Equation and Its Background Solutions

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Abstract. We investigate the ultradiscrete KdV equation with periodic boundary conditions where the two parameters (capacity of the boxes and that of the carrier) are arbitrary integers. We give a criterion to allow a periodic boundary condition when initial states take arbitrary integer values. Conserved quantities are constructed for the periodic systems. Construction of background solutions of the periodic ultradiscrete KdV equation from the Jacobi theta function is also presented.

1. Preface

In this paper we investigate the boundary conditions and the background solutions of a cellular automaton which is called the Box Ball System (BBS).

The content of this paper is as follows. The BBS is derived from the discrete KdV equation by a limiting procedure called 'ultradiscretization', which we will explain briefly in section 2. In section 3, we give a criterion for the BBS and the BBS with a Carrier (BBSC) to allow a periodic boundary condition even when the system allows 'negative solitons'. We also define the background solution of the BBSC using the conserved quantities of the BBSC. In section 4, we introduce a gauge transformed discrete KdV equation whose solutions converge to the upward-shifted solutions of the BBSC by ultradiscretization. We then investigate the relation between the Jacobi theta solutions of the discrete KdV equation and the background solutions of the BBS in section 5. In section 6, we present an example of multi-soliton solutions with the boundary condition in which the value in $n \to \infty$ and that in $n \to -\infty$ differs from each other.

2.   Discrete KdV Equation and BBS

2.1.   Ultradiscretization

The discrete KdV equation is defined as

$$(2.1) \qquad \frac{1}{w_{n+1}^{t+1}} - \frac{1}{w_n^t} + \frac{\delta}{1+\delta}(w_n^{t+1} - w_{n+1}^t) = 0,$$

where n and t take only integer values.   The discrete KdV equation is transformed to the bilinear form

$$(2.2) \qquad (1+\delta)\sigma_{n+1}^{t+1}\sigma_n^{t-1} = \delta\sigma_{n+1}^{t-1}\sigma_n^{t+1} + \sigma_n^t\sigma_{n+1}^t,$$

by putting

$$w_n^t = \frac{\sigma_n^t\sigma_{n+1}^{t-1}}{\sigma_{n+1}^t\sigma_n^{t-1}}.$$

The ultradiscretization is a limiting procedure in which the dependent variables of the discrete equations become also discretized [1].   The ultradiscretization transforms discrete equations into piecewise linear equations. First we use the following lemma to ultradiscretize (2.1).

LEMMA 2.1.   *Under the boundary condition* $\lim_{n\to-\infty} w_n^t = 1$, *the discrete KdV equation* (2.1) *is turned into*

$$(2.3) \qquad w_{n+1}^{t+1} = \left(\delta w_{n+1}^t + (1-\delta)\prod_{k=-\infty}^{n}\frac{w_k^{t+1}}{w_k^t}\right)^{-1}.$$

Then we put $w_n^t = \exp\left(\frac{U_n^t}{\varepsilon}\right)$, $\delta = \exp\left(-\frac{L}{\varepsilon}\right)$ $(\varepsilon > 0)$ and take the limit $\varepsilon \to +0$ to obtain

$$(2.4) \qquad U_{n+1}^{t+1} = \min\left(L - U_{n+1}^t, \sum_{k=-\infty}^{n}(U_k^t - U_k^{t+1})\right),$$

under the boundary condition $\lim_{n\to-\infty} U_n^t = 0$.

We define the ultradiscrete KdV equation by the equation (2.4) [2].   It is also equivalent to the time evolution equation of BBS with box capacity

L [3, 4]. Next we ultradiscretize (2.2). Putting $\sigma_n^t = e^{\tau_n^t/\varepsilon}$, $\delta = e^{-L/\varepsilon}$ and taking the limit $\varepsilon \to +0$, we obtain the bilinear form of the ultradiscrete KdV equation:

$$(2.5) \qquad \tau_{n+1}^{t+1} + \tau_n^{t-1} = \max[\tau_{n+1}^{t-1} + \tau_n^{t+1} - L, \tau_n^t + \tau_{n+1}^t].$$

The ultradiscretization preserves the solitonic nature of the continuous KdV equation.

Example 2.1. Let $L = 1$, then equation (2.4) is closed under $U_n^t \in \{0,1\}$. We give an example of the time evolution below (where a dot indicates a zero).

```
t=1:111...11...1.............
t=2:...111..11..1............
t=3:......11..11.11..........
t=4:........11..1..111........
t=5:..........11.1....111.....
```

2.2. BBS with a carrier

The time evolution of the BBS with a Carrier (BBSC) is expressed as follows [5]. We prepare the "carrier" which can carry at most l balls. At each time step in the evolution, the carrier moves from the left to the right. While the carrier passes the j-th box, the following action occurs. Assume that the carrier carries c $(0 \le c \le l)$ balls before it passes the j-th box, and also assume that there are U $(0 \le U \le L)$ balls in the j-th box. Then, when the carrier passes the box, the carrier puts $\min(c, L-U)$ balls into the box and receives $\min(U, l-c)$ balls from the box. That is to say, the carrier puts as many balls into the box as possible and simultaneously obtain as many balls from the box as possible. This rule can be expressed in the following formula

$$(2.6) \qquad U_n^{t+1} = \min\left(L - U_n^t, \sum_{k=-\infty}^{n-1}(U_k^t - U_k^{t+1})\right)$$
$$+ \max\left(0, \sum_{k=-\infty}^{n} U_k^t - \sum_{k=-\infty}^{n-1} U_k^{t+1} - l\right),$$

under the boundary condition

$$\lim_{n \to -\infty} U_n^t = 0.$$

3. Periodic BBSC That Allows Negative Solitons

3.1. Negative solitons

Negative solitons are non-solitonic trains of negative values at a speed of 1. These arise for example when we put negative initial values to the BBS(C) equation.

Example 3.1. An example of a negative soliton in a BBS with box capacity 1 is as follows (where a dot indicates a zero).

```
t=1: . .-1-1-1 . .-1 . . . . 1 1 . . . . .
t=2: . . .-1-1-1 . .-1 . . . . . 1 1 . . .
t=3: . . . .-1-1-1 . .-1 . . . . . . 1 1 .
```

While the sequence of 1's has a speed equal to its length, the sequences of -1 have speed 1 regardless of their lengths.

The pioneering works on negative solitons and background solutions are due to Hirota [6], and Willox et.al. [7]. In the previous work with Mada and Tokihiro [8], the author presented a way to construct the conserved quantities of equation (2.4) which can also be applied to the negative solitons, by using certain gauge transformation to the BBSC. We described the structure of the conserved quantities for the BBSC, in terms of arclines connecting balls and vacant boxes.

3.2. Coupled form of BBS(C)

Hereafter we consider the BBSC with box capacity L and carrier capacity l.

Lemma 3.1. *Equation* (2.6) *is equivalent to the following coupled equations*

$$(3.1) \qquad \begin{cases} U_i^{t+1} & = \min(c_i^t, L - U_i^t) + \max(0, U_i^t + c_i^t - l), \\ c_{i+1}^t & = U_i^t + c_i^t - U_i^{t+1}, \end{cases}$$
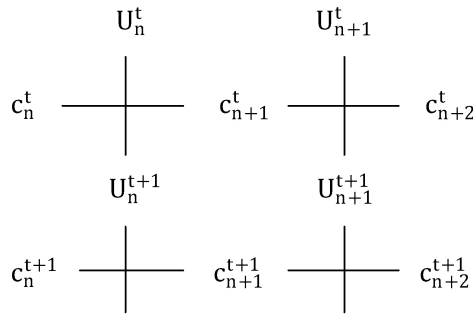
$$U_n^t \qquad\qquad U_{n+1}^t$$

$$c_n^t \quad\underline{\quad\quad}\quad c_{n+1}^t \quad\underline{\quad\quad}\quad c_{n+2}^t$$

$$U_n^{t+1} \qquad\qquad U_{n+1}^{t+1}$$

$$c_n^{t+1} \quad\underline{\quad\quad}\quad c_{n+1}^{t+1} \quad\underline{\quad\quad}\quad c_{n+2}^{t+1}$$

Fig. 1.   The time evolution of the coupled BBSC equation.

with the boundary condition

$$\lim_{i\to-\infty} c_i^t = \lim_{i\to-\infty} U_i^t = 0.$$

PROOF.   If we determine c_i^t by $c_i^t = \sum_{k=-\infty}^{i-1}(U_k^t - U_k^{t+1})$ in (2.6), we have (3.1). $\square$

On the contrary, we have $c_i^t = \sum_{k=-\infty}^{i-1}(U_k^t - U_k^{t+1})$ from the last equation in (3.1) with the boundary condition. Then the first equation gives (2.6).

REMARK 3.1.   The equation (3.1) is also called an "ultradiscrete Yang-Baxter map".

COROLLARY 3.1.   *The coupled equations of the BBS are expressed as*

(3.2)
$$\begin{cases} U_i^{t+1} & = \min(c_i^t, L - U_i^t), \\ c_{i+1}^t & = U_i^t + c_i^t - U_i^{t+1}. \end{cases}$$

PROOF.   We have only to set $l = +\infty$ in (3.1). $\square$

3.3.   Periodic BBS(C)

We consider the time evolution of the BBS(C) with periodic boundary condition. For a detailed discussion of the periodic BBS with U_i, c_i being non-negative, see [9] and [10]. We extend the results in [9, 10] to the periodic BBSC that allows negative solitons.

DEFINITION 3.1.   We say that the set of initial values $U_1^t, U_2^t, \cdots U_N^t$ evolves as a periodic BBSC of size N, if and only if there exists at least one c_1^t such that $c_{N+1}^t = c_1^t$ and such that $U_1^{t+1}, U_2^{t+1}, \cdots U_N^{t+1}$ are defined uniquely, independent of the choice of possible c_1^t's.

REMARK 3.2.   For some initial values $\{U_i^t\}_{i=1}^N$, there may exist more than one c_1^t's such that $c_{N+1}^t = c_1^t$ and such that $\{U_i^{t+1}\}$ depend on c_1^t (Example 3.3).   The time evolution of the BBSC is not unique in these cases, and we exclude such cases for simplicity in this paper.

3.3.1   Periodic BBS
First, we deal with the periodic BBS (3.2).

LEMMA 3.2.

$$c_{N+1}^t = \max[c_1^t, v_c^t] + \tilde{N}$$

Here,

$$
\begin{aligned}
v_c^t &= \max_{i \in \{1,2,\cdots,N\}} [iL - 2(U_1^t + \cdots + U_{i-1}^t) - U_i^t], \\
\tilde{N} &= 2(U_1^t + U_2^t + \cdots U_N^t) - NL.
\end{aligned}
$$

(See figure 2.)

PROOF.   First note that $\tilde{N}$ does not change under the time evolution of the periodic BBSC. We omit the superscript $(\cdot)^t$ of c_i^t's and v_c^t for convenience and consider $c_{N+1} =: f(c_1)$ as a function of c_1.

Fix one c_1 such that $c_1 \geq v_c$.

Since $c_1 \geq v_c \geq L - U_1$,

$$
\begin{aligned}
U_1^{t+1} &= \min(c_1, L - U_1) = L - U_1, \\
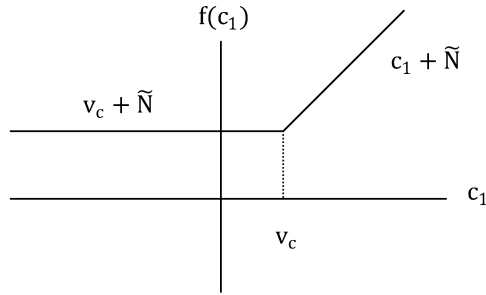c_2 &= c_1 + 2U_1 - L.
\end{aligned}
$$

Fig. 2. The graph of $c_{N+1} = f(c_1)$.

The inequality $c_1 \geq v_c \geq 2L - 2U_1 - U_2$ gives

$$
\begin{aligned}
U_2^{t+1} &= L - U_2, \\
c_3 &= c_2 + 2(U_1 + U_2) - 2L.
\end{aligned}
$$

Using $v_c \geq iL - 2(U_1 + \cdots + U_{i-1}) - U_i$, repeated calculations lead to

$$
\begin{aligned}
U_i^{t+1} &= L - U_i, \\
c_{i+1} &= c_i + 2(U_1 + U_2 + \cdots + U_i) - iL.
\end{aligned}
$$

for $i = 1, 2, \cdots, N$. (This is easily seen by induction.)

Thus we obtain $c_{N+1} = f(c_1) = c_1 + \tilde{N}$ if $c_1 \geq v_c$.

If $c_1 < v_c$, on the other hand, there exists $1 \leq J \leq N$ such that

$$
\begin{aligned}
U_J^{t+1} &= c_J, \\
c_{J+1} &= U_J + c_J - U_J^{t+1} = U_J.
\end{aligned}
$$

holds. Hence (U_i, c_i) does not depend on c_1 if $i \geq J+1$. In fact (U_i, c_i) will be a constant that only depends on the initial values $(U_1, \cdots, U_N)$. Therefore, $c_{N+1} = f(c_1)$ is also a constant if $c_1 < v_c$. It is easily seen from (3.2) that $f(c_1 + 1) - f(c_1) = 0$ or 1, and we already know that $f(c_1 + 1) - f(c_1) = 1$ is equivalent to $c_1 \geq v_c$. Thus we obtain $f(c_1) = v_c + \tilde{N}$ if $c_1 < v_c$. $\square$

THEOREM 3.1. *The BBS (3.2) evolves as a periodic system if and only if*

$$
\tilde{N} \leq 0
$$

holds for the initial values $\{U_i^0\}_{i=1}^N$.

PROOF. From the lemma 3.2

- If $\tilde{N} > 0$, no $c_1^0 \in \mathbb{Z}$ satisfies $f(c_1^0) = c_1^0$.

- If $\tilde{N} = 0$, we have $f(c_1^0) = c_1^0$ for all $c_1^0 \geq v_c$.

- If $\tilde{N} < 0$, the only c_1^0 that satisfies $f(c_1^0) = c_1^0$ is $c_1^0 = v_c + \tilde{N}$.

In the second case, U_i^1 is independent of the choice of c_1^0 because we know from the previous lemma that $U_i^1 = 1 - U_i^0$ $(1 \leq i \leq N)$ for any $c_1^0 \geq v_c$. The evolutions are the same in $t = 2, 3, \cdots$. $\square$

REMARK 3.3. Although normally we treat the case where $U_i^t,\ c_i^t \in \mathbb{Z}$, this proposition is also valid for arbitrary real values of $U_i^t,\ c_i^t$.

3.3.2 Periodic BBSC
Next we deal with the periodic BBSC (3.1). As we are investigating whether the system evolves from the time t to $t+1$, we can omit the superscript $(\cdot)^t$ of the variables below.

THEOREM 3.2. *Let,*

$$
\begin{aligned}
M &= 2(U_1 + \cdots + U_N) - NL, \\
c_L &= \max_{i \in \{0,1,\cdots N-1\}}[L - 2(U_1 + \cdots + U_i) - U_{i+1} + iL], \\
c_R &= \min_{i \in \{0,1,\cdots N-1\}}[l - 2(U_1 + \cdots + U_i) - U_{i+1} + iL], \\
N_{odd} &= \{i|\ 1 \leq i \leq N,\ i : odd\}, \\
N_{even} &= \{i|\ 1 \leq i \leq N,\ i : even\}, \\
d_L &= \max\left[\max_{i \in N_{odd}}(l - U_i),\ \max_{i \in N_{even}}(U_i - L + l + 1)\right], \\
d_R &= \min\left[\min_{i \in N_{odd}}(L - U_i - 1),\ \min_{i \in N_{even}}(U_i)\right] + 1.
\end{aligned}
$$

When we put $c_{N+1} = f(c_1)$, the following cases occur.

- *If $\infty > l > L$:*

- If $M \neq 0$, there exists a unique $c_1 \in \mathbb{Z}$ such that $f(c_1) = c_1$ holds.
- If $M = 0$ and $c_L < c_R$, we have $f(c_1) = c_1$ for all c_1 with $c_L \leq c_1 \leq c_R$.
- If $M = 0$ and $c_L \geq c_R$, there exists a unique c_1 such that $f(c_1) = c_1$ holds.

- **If $l < L$:**

  - If N is an odd number:
    There exists a unique $c_1 \in \frac{1}{2}\mathbb{Z}$ such that $f(c_1) = c_1$ holds.
    (For $c_1 \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}$, we do not have a normal box ball interpretation. However, $\forall i\ U_i^t \in \mathbb{Z} \rightarrow \forall i\ U_i^{t+1} \in \mathbb{Z}$ holds, which means the number of balls in the box is an integer all the same.) $\cdots (**)$

  - If N is an even number,
    * For $d_L \geq d_R$, there is a unique $c_1 \in \mathbb{Z}$ that satisfy $f(c_1) = c_1$.
    * For $d_L < d_R$, we have $f(c_1) = c_1$ for all c_1 such that $d_L \leq c_1 \leq d_R$, but $\{U_i^{t+1}\}$ depends on c_1.
      (This is the only case where we cannot determine the time evolution uniquely.) $\cdots (***)$

- **If $l = L$, there is a unique c_1 that satisfies $f(c_1) = c_1$.**

PROOF.
If $l > L$:
We regard c_{i+1} as a function of c_i to find

$$c_{i+1}(c_i + 1) - c_{i+1}(c_i) = \begin{cases} 1 & (L - U_i \leq c_i < l - U_i), \\ 0 & (\text{otherwise}), \end{cases}$$

which leads to

$$f(c_1 + 1) - f(c_1) = \begin{cases} 1 & (\forall i,\ L - U_i \leq c_i < l - U_i), \\ 0 & (\text{otherwise}). \end{cases}$$

If $L - U_i \leq c_i < l - U_i$ for all i's then, by induction we have

$$c_{i+1} = c_1 + 2(U_1 + \cdots U_i) - iL \quad (i = 1, \cdots, N).$$

Thus the condition $(\forall i, \ L - U_i \leq c_i < l - U_i)$ is equivalent to the following:

(3.3)
$$(\forall i) \ L - 2(U_1 + \cdots + U_i) - U_{i+1} + iL$$
$$\leq c_1 < l - 2(U_1 + \cdots + U_i) - U_{i+1} + iL.$$

The inequality (3.3) can be expressed as

$$c_L \leq c_1 \text{ and } c_1 < c_R,$$

and we have

$$c_{N+1} = f(c_1) = \begin{cases} f(c_L) & (c_1 \leq c_L), \\ c_1 + M & (c_L \leq c_1 \text{ and } c_1 \leq c_R), \\ f(c_R) & (c_R \leq c_1). \end{cases}$$

(Note that c_L may be larger than c_R.)

By examining the intersection of $y = f(c_1)$ and $y = c_1$, we obtain the desired result. (When $M = 0$ and $c_L < c_R$, possible c_1's are not unique, but we have $U_i^{t+1} = L - U_i^t$ regardless of the choice of c_1 which makes the time evolution of U_i^t unique.)

We conclude from the above argument that the system uniquely evolves in time if $l > L$.

If $l < L$:

We have

$$c_{i+1}(c_i + 1) - c_{i+1}(c_i) = \begin{cases} -1 & (l - U_i \leq c_i < L - U_i), \\ 0 & (\text{otherwise}). \end{cases}$$

Hence,

$$f(c_1 + 1) - f(c_1) = \begin{cases} 1 & (\forall i, \ l - U_i \leq c_i < L - U_i) \text{ and } (N : \text{even}), \\ -1 & (\forall i, \ l - U_i \leq c_i < L - U_i) \text{ and } (N : \text{odd}), \\ 0 & (\text{otherwise}). \end{cases}$$

If $l - U_i \leq c_i < L - U_i$ for all i, we have

$$\begin{aligned} U_i^{t+1} &= 2c_i + U_i^t - l, \\ c_{i+1} &= l - c_i, \end{aligned}$$
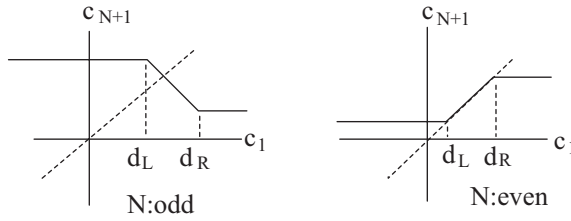
Fig. 3. Graph of $c_{N+1} = f(c_1)$ of BBSC under the condition $l < L$.

for each i. Thus we have

$$c_i = \begin{cases} c_1 & (i \in N_{\text{odd}}), \\ l - c_1 & (i \in N_{\text{even}}). \end{cases}$$

Therefore, the condition $(\forall i, \ l - U_i \le c_i < L - U_i)$ is equivalent to the following set of inequalities:

$$(\forall i) \begin{cases} l - U_i \le c_1 < L - U_i & (i \in N_{\text{odd}}), \\ U_i - L + l < c_1 \le U_i & (i \in N_{\text{even}}). \end{cases}$$

Thus if N is an odd number we obtain

$$f(c_1) = \begin{cases} f(d_L) & (c_1 < d_L), \\ l - c_1 & (d_L \le c_1 \text{ and } c_1 \le d_R), \\ f(d_R) & (d_R < c_1). \end{cases}$$

(See the left graph of figure 3.)

Therefore, there is a unique c_1 such that $f(c_1) = c_1$ $(c_1 \in \frac{1}{2}\mathbb{Z})$. Even though c_1 may not be an integer, as long as $\{U_i^t\}$ are all integers, $\{U_i^{t+1}\}$ are closed in $\mathbb{Z}$. In particular if $d_L \ge d_R$, $f(c_1)$ is constant in c_1.

If N is an even number, we obtain:

$$f(c_1) = \begin{cases} f(d_L) & (d_1 < d_L), \\ c_1 & (d_L \le c_1 \text{ and } c_1 \le d_R), \\ f(d_R) & (d_R < c_1). \end{cases}$$

(See the right graph of figure 3.)

Therefore if $d_L \geq d_R$, $f(c_1)$ is constant in c_1, which renders the time evolution unique.

If $d_L < d_R$, by fixing c_1 such that $d_L \leq c_1 \leq d_R$ we obtain

$$U_i^{t+1} = U_i^t + (-1)^i(l - 2c_1).$$

This indicates that the time evolution is dependent on c_1. We do not have a unique periodic BBSC in this case. $\square$

COROLLARY 3.2.   *The time evolution of the BBSC* (3.1) *with the periodic boundary condition is* <u>*not*</u> *well-defined if and only if the three conditions*

$$\begin{cases} l < L, \\ \text{the system size } N \text{ is an even number,} \\ d_L < d_R, \end{cases}$$

hold simultaneously.

Example 3.2.   We show an example of the periodic BBSC. We express the time evolution

$$\begin{cases} V = \min(c, L - U) + \max(0, U + c - l), \\ d = U + c - V, \end{cases}$$

as follows:

```
  U
c-|-d.
  V
```

Let $N = 8$, $L = 1$ and $l \gg 1$.

```
  0    3   0   0   0   0   0   0
0-|-0-|-5-|-4-|-3-|-2-|-1-|-0-|-0
  0    a   1   1   1   1   1   0
4-|-3-|-a-|-1-|-2-|-3-|-4-|-5-|-4
  1    3   a   0   0   0   0   1
1-|-2-|-7-|-2-|-1-|-0-|-0-|-0-|-1
```

```
   0   a   3   1   1   0   0   0
2-|-1-|-a-|-3-|-4-|-5-|-4-|-3-|-2
   1   1   a   0   0   1   1   1
3-|-4-|-5-|-0-|-0-|-0-|-1-|-2-|-3
   0   0   3   0   0   0   0   0
```

Here $a = -2$.

Example 3.3. We present two irregular cases in defining periodic BBSC. Here is an example of the case $(**)$ in proposition 3.2. Let $N = 7$, $L = 5$, $l = 1$ and $b = \frac{1}{2}$.

```
   1   4   1   4   1   4   1
b-|-b-|-b-|-b-|-b-|-b-|-b-|-b
   1   4   1   4   1   4   1
```

Next we show an example when the periodic BBSC is not well-defined. Let $N = 4$, $L = 5$ and $l = 2$.

```
   2   3   2   3            2   3   2   3
1-|-1-|-1-|-1-|-1  ,     2-|-0-|-2-|-0-|-2
   2   3   2   3            4   1   4   1
```

We have more than one types of time evolutions depending on the choice of c_1.

REMARK 3.4. The BBSC is an invertible system, which is true even when some of the variables $\{U_i^t, c_i^t\}$ take negative values.

3.4. Periodic BBS with K kinds of balls

We consider the extended BBS where we have K kinds of balls distinguished by integers $1 \le k \le K$. We call this system BBS_K. Here we consider the case where the capacity of the box is 1 in each box. Let $U_{i,k}^t$ be the number of balls k in the i-th box and $c_{i,k}^t$ the number of balls k in the carrier when the carrier is located between the $(i-1)$-th box and the i-th one. The time evolution rule of the BBS_K is

$$
\begin{aligned}
U_{i,k}^t &= \min\left(1 - \sum_{j=1}^{k-1} U_{i,j}^{t+1} - \sum_{j=k}^{K} U_{i,j}^t, \; c_{i,k}^t\right), \\
c_{i+1,k}^t &= U_{i,k}^t + c_{i,k}^t - U_{i,k}^{t+1},
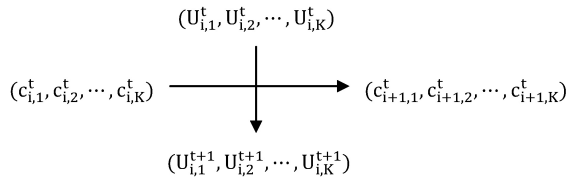\end{aligned}
$$

$$(U_{i,1}^{t}, U_{i,2}^{t}, \cdots, U_{i,K}^{t})$$

$$(c_{i,1}^{t}, c_{i,2}^{t}, \cdots, c_{i,K}^{t}) \longrightarrow (c_{i+1,1}^{t}, c_{i+1,2}^{t}, \cdots, c_{i+1,K}^{t})$$

$$(U_{i,1}^{t+1}, U_{i,2}^{t+1}, \cdots, U_{i,K}^{t+1})$$

Fig. 4.   Time evolution of BBS_K.

where $k = 1, 2, \cdots, K$ (See figure 4).

DEFINITION 3.2.   The BBS_K can be defined as a periodic system if there exists at least one set of

$$(c_{1,1}^{t}, c_{1,2}^{t}, \cdots, c_{1,K}^{t}) \in \mathbb{R}^{K}$$

such that $c_{N+1,j}^{t} = c_{1,j}^{t}$ hold for all $j \in \{1, \cdots, K\}$ and the $U_{i,k}^{t+1}$'s are determined uniquely independent of the choice of possible $c_{1,k}^{t}$'s.

THEOREM 3.3.   *The periodic BBS_K is well-defined if and only if*

$$\sum_{j=1}^{N} U_{j,l}^{0} \leq \frac{1}{2} \left(N - \sum_{j=1}^{N} \sum_{\substack{k=1 \\ k \neq l}}^{K} U_{j,k}^{0} \right)$$

holds for all $l \in \{1, 2, \cdots, N\}$.

PROOF.   First note that we only have to consider the case where $t = 0$ because $\sum_{j=1}^{N} U_{j,l}^{t}$ is independent of t. We know from the time evolution rule that $c_{N+1,k}^{0}$ depends only on $c_{1,1}^{0}, c_{1,2}^{0}, \cdots, c_{1,k}^{0}$ and does not depend on $c_{1,k+1}^{0}, c_{1,k+2}^{0}, \cdots, c_{1,K}^{0}$. We omit the superscript $(\cdot)^{0}$ for convenience hereafter.

For $l = 1, 2, \cdots, N$ let

$$N_l = \sum_{j=1}^{N} \sum_{k=1}^{K} U_{j,k} + \sum_{j=1}^{N} U_{j,l} - N.$$

The condition in the proposition is rewritten as $N_l \leq 0$ for all l.

We first determine the cases that allow for a $c_{1,1}$ such that satisfies $c_{N+1,1} = c_{1,1}$. We calculate in the same way as in lemma 3.2 to obtain

$$
\begin{aligned}
c_{N+1,1} &= \max[c_{1,1}, \gamma_1] + N_1, \\
\gamma_1 &= \max_{i \in \{1, \cdots, N\}} \left[i - \sum_{j=1}^{i-1} U_{j,1} - \sum_{j=1}^{i} \sum_{k=1}^{K} U_{j,k} \right].
\end{aligned}
$$

Thus there exists $c_{1,1}$ such that $c_{N+1,1} = c_{1,1}$ if and only if $N_1 \leq 0$.

Next we determine $c_{N+1,2}$ from $c_{1,1}$ and $c_{1,2}$. We obtain

$$
\begin{aligned}
c_{N+1,2} &= \max[c_{1,2}, \gamma_2] + N_2', \\
N_2' &= \sum_{j=1}^{N} \left(\tilde{U}_{j,1} + \sum_{k=2}^{K} U_{j,k} \right) + \sum_{j=1}^{N} U_{j,2} - N,
\end{aligned}
$$

and γ_2 is a constant determined by the $U_{i,k}$'s only. Therefore we have $c_{1,2}$ such that $c_{N+1,2} = c_{1,2}$ if and only if $N_2' \leq 0$. We know from $\sum_{i=1}^{N} \tilde{U}_{i,1} = \sum_{i=1}^{N} U_{i,1}$ that $N_2' = N_2$. By repeating the same calculations for $c_{N+1,k}$ ($k \geq 3$) we obtain the proposition. If $N_j = 0$ for some j there exist more than one $c_{1,j}$ such that $c_{N+1,j} = c_{1,j}$. The time evolution of $U_{i,k}$ is still unique in this case. $\square$

3.5.   Some elaborations on conserved quantities

PROPOSITION 3.1.   *The invertible transformation*

$$
\begin{cases}
\tilde{U}_i^t &= U_i^t + m, \\
\tilde{c}_i^t &= c_i^t + m,
\end{cases}
$$

deforms BBSC (3.1) to

(3.4) $\begin{cases} \tilde{U}_i^{t+1} &= \min(\tilde{c}_i^t, (L+2m) - \tilde{U}_i^t) + \max(0, \tilde{U}_i^t + \tilde{c}_i^t - (l+2m)), \\ \tilde{c}_{i+1}^t &= \tilde{U}_i^t + \tilde{c}_i^t - \tilde{U}_i^{t+1}. \end{cases}$

Here m is an arbitrary real number.

PROOF.   We immediately obtain (3.4) by direct calculation [11]. $\square$

When $m > 0$, in particular, we call this transformation an "upward-shift translation". If some of the variables take negative values in the initial conditions, by putting $m > 0$ large enough, all U_i^t's and c_i^t's are transformed to be positive. Note that the capacity of the boxes and that of the carrier increase by $2m$, and that the boundary conditions will change: U_i^t is not 0 at $i \to \pm\infty$ anymore, but it becomes some nonzero constant at $i \to \pm\infty$. Also note that the value $[2(U_1^0 + U_2^0 + \cdots U_N^0) - NL]$ does not change through this shift. Hence, treating the negative solitons is equivalent to treating the non-negative BBS's with the boundary conditions that the solutions U_n^t have some positive constant value at $n \to \pm\infty$.

Remark 3.5. In this transformation, the values U_i^t at $i \to +\infty$ and those at $i \to -\infty$ have to be the same. In section 5, we present a way to deal with the boundary condition such that $\lim_{i\to-\infty} U_i^t \neq \lim_{i\to+\infty} U_i^t$.

The construction of conserved quantities found in [8] can also be performed for a periodic BBSC with general L and l. We depict by arclines the movement of the balls from the box to the other box according to the time evolution of BBSC. The operation of drawing arclines when the capacity of the carrier is l is denoted by $(OP)_l$.

$\underline{(\mathbf{OP})_l}$

- We take out balls from the boxes from the left to the right according to the evolution of BBSC. We distinguish each ball and when two or more balls are taken out from the same box at each step, the one at a lower position is taken out first. The carrier can hold at most l balls.

- When two or more balls are passed from the carrier to a box at the same step, the one taken by the carrier last will be the first to go back to a box.

- We depict by arclines the movement of the balls from the box to the carrier to the other box.

We can see some structures in the set of arclines connecting the balls and the empty boxes.

Theorem 3.4 (Kanki-Mada-Tokihiro [8]). *On the periodic BBSC upward-shifted by m, let*

$$\tilde{C}_l := \#\{arclines\ drawn\ at\ (OP)_l\} - \#\{arclines\ drawn\ at\ (OP)_{l-1}\},$$

for the given initial condition. Then $(\tilde{C}_1, \tilde{C}_2, \cdots)$ *is a set of constants independent of the time evolution of the system. We also denote by* $(C_1^0, C_2^0, \cdots)$ *the set of these constants for the "vacuum state" where every box has exactly m balls. The difference of these two* $(\tilde{C}_1 - C_1^0, \tilde{C}_2 - C_2^0, \cdots)$ *is also a set conserved quantities of the BBSC. We rewrite it* $(C_1, C_2, \cdots)$ *and call it "the set of conserved quantities of the BBSC".*

We extend this theorem to the general periodic BBSC.

PROPOSITION 3.2. *For the periodic BBSC with parameters L and l and initial conditions* $\{U_i^0\}$ *that satisfy the following*

> *For both the initial conditions* $\{U_i^0\}$ *and the vacuum solution* $\{m\}$
> *we can define the periodic system for* $l \in [L, +\infty]$,

(3.5) *and also for all l, there exists* $c_1 \in \mathbb{Z}$ *such that* $f(c_1) = c_1$,

we can apply theorem 3.4 to construct the conserved quantities.

In other words, we consider the cases other than $(**)$ in proposition 3.2. Note that for l with $l < L$ we allow the case $(***)$ in proposition 3.2.

Example 3.4. See figure 5 for an example. We consider the periodic BBSC with a box capacity 1 and system size $N = 12$ and set the initial value to be 001$\underline{1}$01101000. By an upward-shift with $m = 1$, initial values are transformed into 112012212111. We have $(\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4) = (6, 5, 3, 1)$, $(C_1^0, C_2^0, C_3^0, C_4^0) = (6, 6, 0, 0)$ and $(C_1, C_2, C_3, C_4) = (0, -1, 3, 1)$.

3.6. Defining Background Solutions

We propose the following way to distinguish the background solutions from the positive solitons. Note that the balls connected in $(OP)_l$ are included in those connected in $(OP)_{l'}$ where $l' > l$.

DEFINITION 3.3. For the BBS upward-shifted by m, the balls connected in the process $(OP)_{2m}$ constitute negative soliton solutions. Remaining balls constitute positive solitons.

DEFINITION 3.4. The solution $\{U_i^t\}$ is a "background solution" if $C_k = 0$ for $\forall k \geq 2m + 1$.
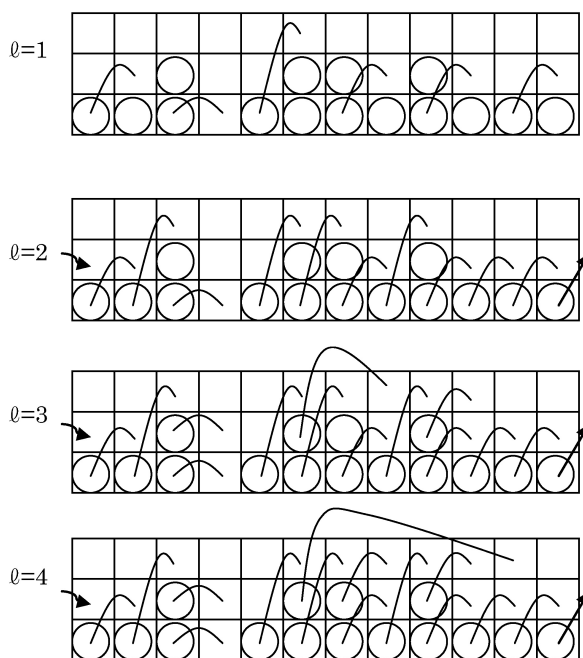
Fig. 5.   Drawing arclines to the periodic system of size $N = 12$.

Example 3.5.   For the BBS with the box capacity one, $\cdots 0\underline{111}0 \cdots$ and $\cdots 0\underline{121}0 \cdots$ are both stationary solitary waves moving at speed one. We can distinguish these two waves by constructing conserved quantities. The former is a background solution <u>without</u> positive solitons, and the latter can be interpreted as a background solution with a soliton of length one superimposed. We shift both systems upward by $m = 1$, and find that the former has the conserved quantity $(C_1, C_2) = (0, -1)$, $C_k = 0(k \geq 3)$. On the other hand the latter has $(C_1, C_2, C_3) = (0, -1, 1)$. (See figure 6)

4.   Relation to the Discrete System

We consider the following transformation

$$\tilde{\sigma}_n^t = \delta^{-(t-n)^2/2L} \sigma_n^t \tag{4.1}$$

| $\widetilde{U_n^t}$ | 1 | 0 | 2 | 0 | 1 | 1 |
| U_n^t | 0 | 1 | 1 | 1 | 0 | 0 |

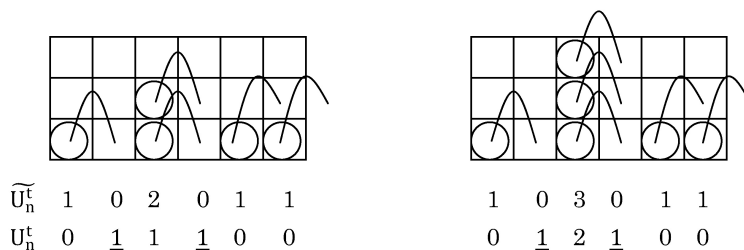| | 1 | 0 | 3 | 0 | 1 | 1 |
| | 0 | 1 | 2 | 1 | 0 | 0 |

Fig. 6.   (Example 3.5):Distinguishing between -11-1 and -12-1.

for the solution σ_n^t of the bilinear discrete KdV equation.

PROPOSITION 4.1.   *The transformed function $\tilde{\sigma}_n^t$ satisfies the following gauge transformed discrete KdV equation*

$$(4.2) \qquad (1+\delta)\tilde{\sigma}_{n+1}^{t+1}\tilde{\sigma}_n^{t-1} = \delta^{(1+2/L)}\tilde{\sigma}_{n+1}^{t-1}\tilde{\sigma}_n^{t+1} + \tilde{\sigma}_n^t\tilde{\sigma}_{n+1}^t.$$

This statement is proved by a direct calculation. We consider the ultradiscrete limit of the equation (4.2). We let $\delta = e^{-L/\varepsilon}$, $\sigma_n^t = e^{\tau_n^t/\varepsilon}$ and $\tilde{\sigma}_n^t = e^{\tilde{\tau}_n^t/\varepsilon}$, and the following proposition holds.

PROPOSITION 4.2.   *For U_n^t and $\tilde{U}_n^t$ defined as $U_n^t = \tau_n^t + \tau_{n-1}^{t+1} - \tau_n^{t+1} - \tau_{n-1}^t$ and $\tilde{U}_n^t = \tilde{\tau}_n^t + \tilde{\tau}_{n-1}^{t+1} - \tilde{\tau}_n^{t+1} - \tilde{\tau}_{n-1}^t$, the relation (4.1) is transformed into*

$$\tilde{U}_n^t = U_n^t + 1.$$

in the limit $\varepsilon \to +0$.

PROOF.   We take the limit $\varepsilon \to +0$ to obtain $\tilde{\tau}_n^t = \tau_n^t + \frac{1}{2}(t-n)^2$. Therefore we have $\tilde{U}_n^t = U_n^t + 1$. $\square$

REMARK 4.1.   The transformation (4.1) of the discrete KdV equation corresponds to the '*1*' upward-shift to the BBSC in the ultradiscrete limit. Thus the solution of the equation (4.2) goes to a '*1*' upward-shifted solution of the BBSC in the ultradiscrete limit.

5. Background Solutions from Jacobi Theta Functions

We define Jacobi theta function by

$$\vartheta_3(v) = \vartheta_3(v, \eta) = \sum_{n=-\infty}^{\infty} q^{n^2} z^{2n},$$

where $q = e^{i\pi\eta}$, $z = e^{i\pi v}$ and $i = \sqrt{-1}$. We suppose $\mathrm{Im}(\eta) > 0$. The Jacobi theta functions are quasi-doubly periodic functions, that is, we have

$$\begin{aligned} \{\vartheta_3(v+1)\}^2 &= \{\vartheta_3(v)\}^2, \\ \{\vartheta_3(v+\eta)\}^2 &= e^{-2\pi i(2v+\eta)}\{\vartheta_3(v)\}^2. \end{aligned}$$

PROPOSITION 5.1.   *Let* $v = \eta(n_1 - n_2 - n_3) + \eta_0$. *Then the function* $\tau(n_1, n_2, n_3) := \vartheta_3(v)$ *is a solution of the following equation, which is a gauge-transformed discrete KP equation*

$$(5.1) \qquad\qquad (s\alpha)\tau_1\tau_{23} - \tau_2\tau_{13} + (1-s)\tau_3\tau_{12} = 0.$$

Here, $\alpha = e^{4i\pi\eta}(= q^4)$ *and* s *is an arbitrary complex number. The lower script 'i' of the function* τ *denotes a '+1' shift in the variable* n_i, *i.e.,* $\tau_1 = \tau(n_1 + 1, n_2, n_3)$, $\tau_{23} = \tau(n_1, n_2 + 1, n_3 + 1)$, *etc* $\cdots$.

PROOF.   From the quasi-doubly periodicity we have

$$\begin{aligned} \tau_1 &= \vartheta_3(v+\eta) = e^{-i\pi(2v+\eta)}\vartheta_3(v), \\ \tau_{23} &= \vartheta_3(v-2\eta) = e^{i\pi(4v-4\eta)}\vartheta_3(v), \end{aligned}$$

which lead to

$$(5.2) \qquad\qquad \tau_1\tau_{23} = e^{i\pi(2v-5\eta)}(\vartheta_3(v))^2.$$

We also obtain

$$(5.3) \qquad\qquad \tau_2\tau_{13} = e^{i\pi(2v-\eta)}(\vartheta_3(v))^2,$$
$$(5.4) \qquad\qquad \tau_3\tau_{12} = e^{i\pi(2v-\eta)}(\vartheta_3(v))^2.$$

Thus we have the result. □

REMARK 5.1.   If a function $\alpha(n_1, n_2, n_3)$ satisfies the relation $\alpha\tau_1\tau_{23} = \tau_2\tau_{13} = \tau_3\tau_{12}$, then $\tau(n_1, n_2, n_3)$ satisfies the same equation (5.1). The discussions in this section are therefore equally valid for such functions α.

5.1. Reduction

Since the parameter s is arbitrary, we can put s to be $s = e^{i\pi\eta\cdot\xi}$ $(\xi > 0)$. The parameter ξ is related to the box capacity of the BBS. From the way v is chosen, we have $\tau = \tau_{12}$, which is the reduction condition from the KP equation to the KdV equation. If we rewrite $\tau_n^t := \tau(t, 0, n)$ the following lemma holds.

LEMMA 5.1. *The gauge-transformed discrete KP equation* (5.1) *is reduced to the following gauge-transformed discrete KdV equation*

$$(5.5) \qquad s\alpha\tau_n^{t+1}\tau_{n+1}^{t-1} + (1-s)\tau_{n+1}^t\tau_n^t = \tau_n^{t-1}\tau_{n+1}^{t+1}.$$

REMARK 5.2. We have the following relation with the upward shifted discrete KdV equation (4.2):

$$\begin{aligned}
s &= \frac{\delta}{1+\delta}, \\
\alpha &= \delta^{2/L}.
\end{aligned}$$

Although the coefficients in equation (5.5) are different from the bilinear form of the normal discrete KdV equation, we find from the following proposition that this does not lead to the loss of generality.

PROPOSITION 5.2 (Tsujimoto-Hirota [2]). *By change of variable*

$$w_n^t = \frac{\tau_n^t\tau_{n+1}^{t-1}}{\tau_{n+1}^t\tau_n^{t-1}},$$

the equation (5.5) *is transformed into the discrete KdV equation*

$$\frac{1}{w_{n+1}^{t+1}} - \frac{1}{w_n^t} = \delta'(w_{n+1}^t - w_n^{t+1}).$$

where $\delta' = s\alpha$.

PROOF. From the following identity

$$\begin{aligned}
&(\text{LHS of } (5.5))_{n\to n+1} \times \tau_n^t - (\text{LHS of } (5.5)) \times \tau_{n+2}^t \\
&= (\text{RHS of } (5.5))_{n\to n+1} \times \tau_n^t - (\text{RHS of } (5.5)) \times \tau_{n+2}^t,
\end{aligned}$$

we have

$$\tau_{n+2}^{t+1}\tau_n^t\tau_{n+1}^{t-1} - \tau_{n+1}^{t+1}\tau_{n+2}^t\tau_n^{t-1} = \delta'\tau_n^t\tau_{n+1}^{t+1}\tau_{n+2}^{t-1} - \delta'\tau_{n+2}^t\tau_{n+1}^{t-1}\tau_n^{t+1}.$$

Multiplying

$$\frac{\tau_{n+1}^t}{\tau_{n+1}^{t+1}\tau_{n+2}^t\tau_n^t\tau_{n+1}^{t-1}}$$

on both sides, we have the result. □

5.2. Coexistence of Solitons and Jacobi theta backgrounds

If we ultradiscretize the solutions of the equation (5.5), we will obtain the solutions of the BBS. We will ultradiscretize the equation and the solutions. To do this we first assume that η and η_0 are both purely imaginary and then replace $i\pi\eta$ with $-L/\varepsilon$. (This choice of ε is empirical.) Then we take logarithms on both sides and take the limit $\varepsilon \to +0$.

Note that our paper is not the first to ultradiscretize the theta functions. For example in [12], Iwao and Tokihiro ultradiscretized the theta function solution of the periodic Toda equation. Our method is unique in that we connected the ultradiscrete theta function to the background solution of the BBS.

LEMMA 5.2. *The ultradiscretization of the solution $\tau_n^t = \vartheta_3(v)$ is the tau function Θ_n^t of the constant solution which takes $U_n^t = 2L$ for $n \in (-\infty, \infty)$. Here we have the following relation between U_n^t and Θ_n^t.*

$$U_n^t = \Theta_n^t + \Theta_{n-1}^{t+1} - \Theta_n^{t+1} - \Theta_{n-1}^t.$$

PROOF.

$$
\begin{aligned}
\vartheta_3(v) &= \sum_{k=-\infty}^{\infty} e^{i\pi\eta k^2} e^{2i\pi k \cdot v} \\
&= \sum_{k=-\infty}^{\infty} \exp\left[-i\pi\eta k\{2(t-n) - k + 2\eta_0/\eta\}\right].
\end{aligned}
$$

From the first line to the second line we replaced k with $-k$. We then transform $i\pi\eta$ into $-\frac{L}{\varepsilon}$, and affect on both sides $\lim_{\varepsilon\to+0}\varepsilon\log(\cdot)$. If we write the left hand side Θ_n^t after the ultradiscretization, we have

$$\Theta_n^t = \max_{k\in\mathbb{Z}}[Lk(2(t-n) - k + \eta')].$$

Here we put $\eta' = 2\eta_0/\eta$. The solution U_n^t constructed from Θ_n^t takes constant value $2L$ on $n \in (-\infty, \infty)$. $\square$

We now have the background state $\vartheta_3(v)$. We can add N-soliton solutions onto the background state $\vartheta_3(v)$. If we suppose that $\tau_n^t := \vartheta_3(v)\phi_n^t$ is also a solution of (5.5), then ϕ_n^t satisfies the ordinary discrete KdV equation

$$s\phi_n^{t+1}\phi_{n+1}^{t-1} + (1-s)\phi_{n+1}^t\phi_n^t = \phi_n^{t-1}\phi_{n+1}^{t+1}.$$

We can take ϕ_n^t to be the N-soliton solution of the discrete KdV equation. We denote the ultradiscrete limit of ϕ_n^t by Φ_n^t and define V_n^t by $V_n^t = \Phi_n^t + \Phi_{n-1}^{t+1} - \Phi_n^{t+1} - \Phi_{n-1}^t$. Since $s = e^{i\pi\eta\xi} = e^{-L\xi/\varepsilon}$, the solution V_n^t corresponds to the solution of BBS with box capacity $L\xi$.

PROPOSITION 5.3. *We have the BBS with box capacity $(4+\xi)L$ from the equation* (5.5) *through ultradiscretization.*

PROOF. The proof is based on Takahashi and Hirota [13]. If we put $\delta' = se^{4i\pi\eta}$, then (5.5) is transformed into

$$\frac{1}{w_{n+1}^{t+1}} - \frac{1}{w_n^t} = \delta'(w_{n+1}^t - w_n^{t+1})$$

from the proposition 5.2. Thus we have

(5.6) $$\frac{w_{n+1}^{t+1}}{w_n^t} = \frac{1 - \delta' w_{n+1}^t w_{n+1}^{t+1}}{1 - \delta' w_n^t w_n^{t+1}}.$$

From (5.6) we obtain for any $M < 0$

$$\prod_{k=-M}^{n} \frac{w_{k+1}^{t+1}}{w_k^t} = \frac{1 - \delta' w_{n+1}^t w_{n+1}^{t+1}}{1 - \delta' w_{-M}^t w_{-M}^{t+1}}.$$

Here we have

$$w_n^t = \frac{\tau_n^t \tau_{n+1}^{t+1}}{\tau_{n+1}^t \tau_n^{t-1}} = \frac{\vartheta_n^t \vartheta_{n+1}^{t+1}}{\vartheta_{n+1}^t \vartheta_n^{t-1}} \frac{\phi_n^t \phi_{n+1}^{t+1}}{\phi_{n+1}^t \phi_n^{t-1}},$$

where

$$\frac{\vartheta_n^t \vartheta_{n+1}^{t+1}}{\vartheta_{n+1}^t \vartheta_n^{t-1}} = \frac{\vartheta_3(v)\vartheta_3(v-2\eta)}{(\vartheta_3(v-\eta))^2} = e^{-2i\pi\eta}.$$

and the positive solitons satisfy $\lim_{n\to-\infty} \phi_n^t = 1$.

Thus we obtain $\lim_{n\to-\infty} w_n^t = e^{-2i\pi\eta}$. Therefore in the limit $M \to +\infty$ we have

$$\prod_{k=-\infty}^{n} \frac{w_{k+1}^{t+1}}{w_k^t} = \frac{1}{1-s}(1 - \delta' w_{n+1}^t w_{n+1}^{t+1}),$$

which is equivalent to

$$w_{n+1}^{t+1} = \left(\delta' w_{n+1}^t + (1-s) \prod_{k=-\infty}^{n} \frac{w_k^{t+1}}{w_k^t} \right)^{-1}.$$

After replacing $i\pi\eta$ with $-\frac{L}{\varepsilon}$, and w_n^t with $e^{U_n^t/\varepsilon}$, we take the limit $\varepsilon \to 0$ to obtain

$$U_{n+1}^{t+1} = -\max[U_{n+1}^t - (4+\xi)L, \sum_{k=-\infty}^{n}(U_k^{t+1} - U_k^t)].$$

From $-\max(a,b) = \min(-a,-b)$, the equation is the BBS with the box capacity $(4+\xi)L$. $\square$

These results can be summed up to the following theorem.

THEOREM 5.1. *As solutions of the gauge-transformed discrete KdV equation (5.5), we have the following type of solutions τ_n^t constructed from the Jacobi theta functions.*

$$\tau_n^t = \underbrace{\vartheta_3(v)}_{(Background\ solution)} \times \underbrace{\phi_n^t}_{(N\text{-}soliton\ solution)}.$$
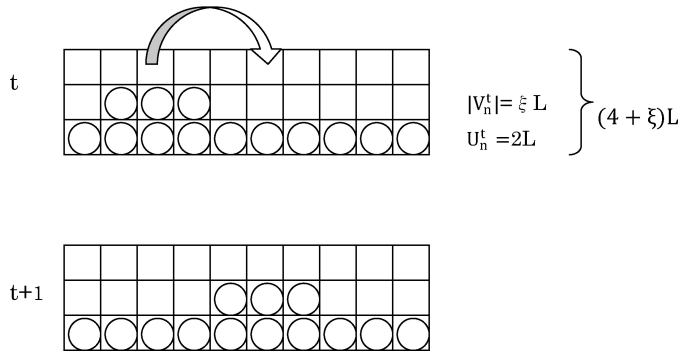
Fig. 7.   Example of background and *1*-soliton solution constructed from Jacobi theta function.

The ultradiscretization of this solution is a solution of the BBS with box capacity $(4 + \xi)L$ and is $2L + V_n^t$.

Example 5.1.   For example when $L = 1/2, \xi = 2$, we have a solution of the BBS with box capacity 3 as shown in figure 7.

We successfully obtained the positive solitons V_n^t superimposed over the background solutions U_n^t. However, to obtain negative solitons we have to consider solutions with holes in the background solutions, which needs a more detailed discussion [8].

6.   BBS with Irregular Boundary Conditions

Finally we present the BBS with a boundary condition where

$$\lim_{n \to +\infty} U_n^t \neq \lim_{n \to -\infty} U_n^t.$$

LEMMA 6.1.   *The τ-function of the N-soliton solution of the BBS is as follows*

$$(6.1) \qquad \tau_n^t = \max_{J \subset [N]} \left[\left(\sum_{i \in J} P_i \right) t - \left(\sum_{i \in J} \min(L, P_i) \right) n \right.$$

$$\left. + \left(\sum_{i \in J} \theta_i \right) - \sum_{\substack{i > j \\ i,j \in J}} 2 \min(P_i, P_j) \right],$$

where $\forall i \ P_i > 0$ and $L_i \in \mathbb{R}$.

We take $L = 3$ and consider the soliton solution made up of solitons $(P_i, \theta_i) = (2, 1) \ (i = 1, 2, \cdots, N_0)$. It has the expression

$$\tau(x) = \max_{0 \leq |J| \leq N_0} [2|J|x + |J| - 2|J|(|J| - 1)]. \quad (x = t - n)$$

In the limit $N_0 \to \infty$, the N_0-soliton solution converges to the following background solution:

$$(6.2) \qquad \tau_{bg}(x) = \max_{k \geq 0} \left[2k \left(x + \frac{1}{2} \right) - 2k(k - 1) \right].$$

Values of $U_n^t = \tau_{bg}(x) + \tau_{bg}(x + 2) - 2\tau_{bg}(x + 1)$ at integer points at time $t = 0$ are as follows.

n	$\cdots$	0	1	2	3	4	5	$\cdots$
U_n^t	1	1	1	1	0	0	0	0

The background solution (6.2) has the new boundary condition where the solution tends to 1 in $n \to -\infty$ and to 0 in $n \to \infty$. We can add positive solitons to (6.2).

Example 6.1.   We construct the following 2-soliton solution travelling in the background state (6.2). Let the width and the phases of the two solitons to be added be

$$\hat{P}_1 = 4, \hat{\theta}_1 = -8; \ \hat{P}_2 = 5, \hat{\theta}_2 = -12,$$

and we denote these solitons by soliton A and B respectively. The general form of the solution $\tau(x)$ is given as

$$\tau(x) = \max[\tau_{bg}(x),\ 4t - 3n - 8 + \tau_{bg}(x - 2),$$
$$5t - 3n - 12 + \tau_{bg}(x - 2),\ 9t - 6n - 28 + \tau_{bg}(x - 4)].$$

Construction of the coexisting state of solitons and negative solitons is based on [8]. What happens during the time evolution of this system is as follows. (See also the time evolution pattern of the system at the end of this section.)

- $0 \le t \le 5$

  We observe that the soliton A travels at speed 2 and the soliton B travels at speed 3 on the background 1. The soliton B takes over the soliton A in the same way as in BBS with box capacity 1.

- $t \sim 6$

  The soliton B climbs down the discontinuity of the background state from 1 to 0. The phase of the background is shifted 2 to the left. The soliton B travels at speed $5/3$ from now on.

- $t \sim 12$

  The same incident happens to the soliton A and it travels at speed $4/3$ hereafter. The phase shift of the background is also -2.

REMARK 6.1. The BBS may take different boundary values at $n \to \infty$ and $n \to -\infty$ like in this example, which is not treated as a periodic BBS. Other irregular boundary conditions can also be considered and general solutions are calculated in the same way. In this section we have only dealt with the case where $L = 3$. We can consider general L to obtain the boundary condition in which the left side and the right side take arbitrary integer values.

```
t= 0 ::222:::::22::::::::::..................
t= 1 ::::222::::22::::::::::..................
t= 2 ::::::222:::22::::::::..................
t= 3 :::::::::222::22:::::::..................
t= 4 :::::::::22::222:::::..................
t= 5 ::::::::::22:::222::..................
```

```
t= 6 :::::::::::::22::::222.................
t= 7 :::::::::::::22::::131.................
t= 8 :::::::::::::22::::.23.................
t= 9 :::::::::::::22::...32...............
t=10 ::::::::::::::22:..131..............
t=11 ::::::::::::::22...23.............
t=12 :::::::::::::::3....32.............
t=13 :::::::::::::::.31...131............
t=14 :::::::::::::::.22....23............
t=15 :::::::::::::::.13.....32...........
t=16 ::::::::::::::::.31....131...........
t=17 ::::::::::::::::.22.....23..........
t=18 ::::::::::::::::.13......32..........
t=19 :::::::::::::::::.31.....131.........
t=20 :::::::::::::::::.22......23.........
```

The time evolution of 2-solitons on an irregular background is shown above.
Here the frame itself moves along with the background state so that the
discontinuity in the background solution seems fixed. Note that the back-
ground state itself is moving to the right at speed 1. The symbol ':' indicates
a background '1' and '.' a zero respectively. We see that the phase shift of
the background after colliding each soliton is −2.

7.   Concluding Remarks

   We first discussed the conditions under which the periodic BBSC with
general box and carrier capacities is well-defined. We extended the construc-
tion of the conserved quantities to general BBSC. Conserved quantities are
useful in distinguishing background solutions from positive soliton solutions.
We then showed that the Jacobi theta function is a solution of the gauge
transformed discrete KdV equation, and that the ultradiscretization of this
solution corresponds to the background solution of the BBS. The author
wishes to extend this method to more general functions in order to deal
with negative solitons. Finally, the BBS with irregular boundary conditions
has been constructed. To obtain the solution of the system with general L
is a future problem.

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