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An Invariant of Embeddings of 3–Manifolds in 6–Manifolds and Milnor's Triple Linking Number

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Abstract. We give a simple axiomatic definition of a rational–valued invariant $\sigma(W, V, e)$ of triples (W, V, e), where $W \supset V$ are smooth oriented closed manifolds of dimensions 6 and 3, and e is a second rational cohomology class of the complement $W \setminus V$ satisfying a certain condition. The definition is stated in terms of cobordisms of such triples and the signature of 4-manifolds. When $W = S^6$ and V is a smoothly embedded 3–sphere, and when $e/2$ is the Poincaré dual of a Seifert surface of V, the invariant coincides with -8 times Haefliger's embedding invariant of (S^6, V). Our definition recovers a more general invariant due to Takase, and contains a new definition for Milnor's triple linking number of algebraically split 3–component links in $\mathbb{R}^3$ that is close to the one given by the perturbative series expansion of the Chern–Simons theory of links in $\mathbb{R}^3$.

1. Introduction and Main Results

Milnor [10] proved that the link homotopy classes of algebraically split 3–component links $L = K_1 \cup K_2 \cup K_3$ in the Euclidean 3–space $\mathbb{R}^3$ are classified by the triple linking number $\mu(L) \in \mathbb{Z}$. There are several definitions of $\mu(L)$, and one is given by the perturbative series expression of the Chern–Simons theory of links in $\mathbb{R}^3$ (Altschuler–Freidel [2], Bar-Natan–Vassiliev [3], Lescop [8], Thurston [17], etc.), more precisely, $\mu(L)$ is expressed as an integral over a manifold $(T^3 \times \mathbb{R}^3) \setminus \mathcal{L}$, where $T^3 = S^1 \times S^1 \times S^1$ is the 3–torus and $\mathcal{L} = T^3_1 \cup T^3_2 \cup T^3_3 \subset T^3 \times \mathbb{R}^3$ is the (disjoint) union of embedded 3–tori

$$T^3_i = \left\{(t_1, t_2, t_3, x) \in T^3 \times \mathbb{R}^3 \mid x = f_i(t_i)\right\},$$

and where $f_i \colon S^1 \to \mathbb{R}^3$ is a smooth embedding representing the knot K_i. On the other hand, Haefliger [6] [7] proved that the abelian group $\mathrm{Emb}(S^3, S^6)$ of the smooth isotopy classes of embeddings $S^3 \to S^6$, with

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the group structure given by the connected sum, is isomorphic to $\mathbb{Z}$. In this paper, we give an interpretation of $\mu(L)$ as an invariant of the embedding $\mathcal{L} \hookrightarrow T^3\times\mathbb{R}^3$ by generalizing Haefliger's construction. To this end, we will need to modify the manifold pair $(T^3\times\mathbb{R}^3, \mathcal{L})$ to make it fit into our settings. It will be replaced by $(T^3\times S^3, M_L)$, $M_L = \mathcal{L} \cup (-\mathcal{L}_0)$, so that the ambient manifold is closed and the submanifold is null–homologous, where $S^3 = \mathbb{R}^3 \cup \{\infty\}$, and $\mathcal{L}_0$ is a 3–submanifold of $T^3\times\mathbb{R}^3$ constructed from a 3–component unlink in $\mathbb{R}^3$ split from L in the same way as we construct $\mathcal{L}$ (so M_L is the union of 6 disjoint copies of T^3).

In this paper, we deal with triples $\beta = (Z, X, e)$, which we will call e-manifolds[1], consisting of (smooth, oriented, and compact) manifolds $Z \supset X$ of codimension 3 such that X is properly embedded in Z ($\partial X \subset \partial Z$, and X is transverse to ∂Z), and a cohomology class $e \in H^2(Z \setminus X; \mathbb{Q})$ such that

$$e|_{S(\nu_X)} = e(F_X)$$

over $\mathbb{Q}$, where $e(F_X) \in H^2(S(\nu_X); \mathbb{Z})$ is the Euler class of the vertical tangent subbundle $F_X \subset TS(\nu_X)$ of the total space of the normal sphere bundle $\rho_X\colon S(\nu_X) \to X$ of X, and where the normal bundle ν_X of X is identified with a tubular neighborhood of X so that we can regard $S(\nu_X)$ as a submanifold of $Z \setminus X$. Such a cohomology class e will be called an e-class of (Z, X) in this paper. The existence of an e-class implies the vanishing of the rational fundamental homology class of $(X, \partial X)$ in $(Z, \partial Z)$ (Proposition 6.1 (1)), but the converse is not true in general (Remark 6.1).

The cohomology class $e/2$ corresponds to the homology class of a Seifert surface of X (if it exists and its normal bundle is trivial over X) by the Poincaré duality (Corollary 7.1). In particular, if (Z, X) admits just one e-class, then $e/2$ represents the homomorphism $H_2(Z \setminus X; \mathbb{Z}) \to \mathbb{Q}$, $y \mapsto \mathrm{lk}(y, X)$, where $\mathrm{lk}(y, X)$ is the linking number of y with X.

Precise definitions of e-class, e-manifold, isomorphism (denoted by $\cong$) and cobordism of e-manifolds, etc. will be given in Section 2. We will also introduce notions of quasi e-class and quasi e-manifold. These are slight generalizations of e-class and e-manifold, and easier to handle as we will explain in Remark 1.3. We remark that, for (quasi) e-manifolds $\beta = (Z, X, e)$ and β' (of the same dimension), the disjoint sum $\beta \amalg \beta'$, the boundary $\partial\beta = (\partial Z, \partial X, e|_{\partial Z\setminus\partial X})$, and the reversing the orientation

[1] "e" is the first letter of "Euler class".

$-\beta = (-Z, -X, e)$ are defined in natural ways. By definition, an e-class is a quasi e-class, and the boundary of a quasi e-manifold is an e-manifold (not just a quasi e-manifold).

There are two main purposes of the present paper. One is to show the existence and the uniqueness of a rational–valued invariant $\sigma(\alpha)$ of the isomorphism classes of closed 6–dimensional e-manifolds $\alpha = (W, V, e)$ ($\partial W = \partial V = \emptyset$, $\dim W = 6$) which is uniquely characterized by the following two axioms (Theorem 1.2):

Axiom 1. *The invariant σ is additive. Namely, for 6–dimensional closed e-manifolds α and α',*

$$\sigma(-\alpha) = -\sigma(\alpha),$$
$$\sigma(\alpha \amalg \alpha') = \sigma(\alpha) + \sigma(\alpha').$$

Axiom 2. *If a 6–dimensional closed e-manifold α bounds a 7–dimensional e-manifold (Z, X, e) ($\partial(Z, X, e) \cong \alpha$), then*

$$\sigma(\alpha) = \operatorname{Sign} X.$$

Here, $\operatorname{Sign} X \in \mathbb{Z}$ is the signature of a 4–manifold X. The second purpose is to show that σ can detect both Haefliger's invariant (Theorem 1.4) and Milnor's triple linking number (Theorem 1.5).

The author's initial motivition to define the invariant σ is to provide another application to 3–dimensional topology [12]. If an integral homology 3–sphere M is amphichiral (namely, M admits a self–homeomorphism reversing the orientation), then the Rokhlin invariant of M vanishes. This fact is a natural consequence of fundamental properties of the Casson invariant [1]. In our future paper [12], we give a new direct proof of this vanishing property.

Remark 1.1. In [12], the invariant σ is applied to a manifold pair $(M \times M, (M \times p) \cup (p \times M) \cup M_\Delta)$ (where $p \in M$ is a fixed point, and $M_\Delta \subset M \times M$ is the diagonal submanifold) after performing a resolution of the triple intersection at $(p, p) \in M \times M$ of the submanifold. It will appears that σ is congluent to the Rokhlin invariant of M (modulo some constant) and that σ vanishes if M is amphichiral. Our approach is more direct in the sense that we only consider the signature of 4–dimensional manifolds or related characteristic classes, and we do not count numbers of irreducible $SU(2)$–representations of the fundamental group of M.

The rest of this section describes our main results.

1.1. Existence and uniqueness theorem

Throughout this paper, all manifolds are assumed to be smooth, oriented, and compact unless otherwise stated. The following fundamental theorem on 6–dimensional e-manifolds is the key to proving the existence and the uniqueness of the invariant σ, and the proof will be given in Section 4.

THEOREM 1.1. *Any 6–dimensional closed e-manifold is rationally null–cobordant.*

The statement means that, for any 6–dimensional closed e-manifold α, there exists a positive integer m and a 7–dimensional e-manifold β such that $\partial\beta \cong \amalg^m \alpha$ (union of m disjoint copies of α). This is a direct consequence of the fact that the cobordism group Ω_6^e of 6–dimensional e-manifolds (Section 4) is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ (Theorem 4.1), and m can be chosen to be the order of the cobordism class $[\alpha] \in \Omega_6^e$ of α.

The following is the existence and the uniqueness theorem of the invariant σ.

THEOREM 1.2. *Let α be a 6–dimensional closed e-manifold. Take any 7–dimensional e-manifold $\beta = (Z, X, e)$ such that $\partial\beta \cong \amalg^m \alpha$ for some positive integer m (such β and m exist by* Theorem 1.1*). Then, the rational number*

$$\sigma(\alpha) \stackrel{def}{=} \frac{\operatorname{Sign} X}{m}$$

depends only on the isomorphism class of α. Moreover, the invariant σ has the following properties:

(1) *The invariant σ satisfies* Axiom 1 *and* Axiom 2.

(2) *The invariant σ is unique. That is, if an invariant σ' of 6–dimensional closed e-manifolds satisfies* Axiom 1 *and* Axiom 2*, then $\sigma' = \sigma$.*

(3) *If a 6–dimensional closed e-manifold α bounds a 7–dimensional quasi e-manifold $\beta = (Z, X, e)$, then*

$$\sigma(\alpha) = \operatorname{Sign} X - 4\Lambda(\beta).$$

Here, $\Lambda(\beta) \in \mathbb{Q}$ is the self–linking number of a 7–dimensional quasi e-manifold β, and it will be defined in Section 2.3. The proof of Theorem 1.2 will be given in Section 5.3.

Remark 1.2. If β is a 7–dimensional e-manifold, then the formulas in Axiom 2 and Theorem 1.2 (3) are the same, since $\Lambda(\beta) = 0$ by the definition of Λ.

Remark 1.3. If we use only the axioms to compute $\sigma(\alpha)$, we need to find (or construct) an e-manifold β such that $\partial\beta \cong \amalg^m \alpha$ and that the signature of the submanifold is computable, but that may not always be easy. However, sometimes finding a simple quasi e-manifold bounded by α may be much easier. In such cases, the formula in Theorem 1.2 (3) gives us an alternative and effective way to compute $\sigma(\alpha)$. In fact, this formula will be used when we explorer the relationship between our invariant and Haefliger's invariant (Section 8), or Milnor's triple linking number (Section 9).

The essential reason why the rational number $\sigma(\alpha)$ is independent of the choices of β is that if a 7–dimensional e-manifold $\beta = (Z, X, e)$ is closed, then $\operatorname{Sign} X = 0$ (Corollary 5.1), and that the signature is additive with respect to the decompositions of closed manifolds (Novikov additivity).

More generally, if β is a closed 7–dimensional quasi e-manifold, then the equality $\operatorname{Sign} X = 4\Lambda(\beta)$ holds (Proposition 5.1), and $\Lambda(\beta)$ is also additive with respect to the decompositions of closed quasi e-manifolds (see the proof of Proposition 5.2). These are the main reasons why Theorem 1.2 (3) holds.

1.2. An invariant of smooth embeddings

Two manifold pairs (Z, X) and (Z', X') are *isomorphic* if there exists an orientation preserving diffeomorphism $f\colon Z \to Z'$ such that $f(X) =$

X' as oriented submanifolds. The rational number $\sigma(W, V, e)$ defined in Theorem 1.2 is not an invariant of the isomorphism class of (W, V) in general, since it may depend on the choice of e. However, if we put all e-classes together, we obtain an invariant of (W, V) as follows. Let

$$\mathcal{E}_{W,V} = \left\{ e \in \ H^2(W \setminus V; \mathbb{Q}) \;\middle|\; e \text{ is an } e\text{-class of } (W, V) \right\}$$

be the set of all e-classes of (W, V). For example, if V is empty, then $\mathcal{E}_{W,\emptyset} = H^2(W; \mathbb{Q})$ by definition, and if $V \neq \emptyset$, then $\mathcal{E}_{W,V}$ is empty or an affine subspace of $H^2(W \setminus V; \mathbb{Q})$ which misses the origin. Let

$$\sigma_{W,V} \colon \mathcal{E}_{W,V} \to \mathbb{Q}$$

be the function defined by $\sigma_{W,V}(e) = \sigma(W, V, e)$ for $e \in \ \mathcal{E}_{W,V}$. The following is a corollary of Theorem 1.2.

COROLLARY 1.1. *For a pair (W, V) of closed manifolds of dimensions 6 and 3, the function $\sigma_{W,V} \colon \mathcal{E}_{W,V} \to \mathbb{Q}$ is an invariant of the isomorphism class of (W, V).*

The statement means that if there is an isomorphism $f \colon (W', V') \to (W, V)$ of pair of manifolds, then the pull–back $f^* \colon H^2(W \setminus V; \mathbb{Q}) \to H^2(W' \setminus V'; \mathbb{Q})$ restricts to a bijection $f^* \colon \mathcal{E}_{W,V} \to \mathcal{E}_{W',V'}$, and the identity

$$\sigma_{W',V'}(f^* e) = \sigma_{W,V}(e)$$

holds for any $e \in \ \mathcal{E}_{W,V}$.

In a special case, we can obtain a rational–valued invariant of the isomorphism class of (W, V), rather than a function–valued invariant, as follows.

DEFINITION 1.1. A pair (Z, X) of manifolds of codimension 3 is *simple* if it admits at least one e-class and the restriction $H^2(Z; \mathbb{Q}) \to H^2(X; \mathbb{Q})$ is injective.

THEOREM 1.3. *If a pair (W, V) of closed manifolds of dimensions 6 and 3 is simple, then the rational number*

$$\sigma(W, V) \overset{def}{=} \sigma_{W,V}(e) = \sigma(W, V, e), \quad e \in \ \mathcal{E}_{W,V}.$$

is an invariant of the isomorphism class of (W, V).

The proof will be given at the end of Section 6, and is easy. The essential part is that (W, V) is simple if, and only if, (W, V) admits just one e-class (Proposition 6.1 (4)).

1.3. Haefliger's invariant

Let $H\colon \mathrm{Emb}(S^3, S^6) \to \mathbb{Z}$ be Haefliger's isomorphism. A short review of the definition of H will be given in Section 8.1. Let $f\colon S^3 \to S^6$ be a smooth embedding, and write $M_f = f(S^3)$.

There is an easy–to–check condition for the simplicity of pairs of manifolds as follows.

PROPOSITION 1.1. *Let* (Z, X) *be a pair of manifolds of codimension* 3*, and assume that the restriction* $H^2(Z; \mathbb{Q}) \to H^2(X; \mathbb{Q})$ *is an isomorphism. Then,* (Z, X) *is simple if, and only if,* $(X, \partial X)$ *is rationally null–homologous in* $(Z, \partial Z)$.

The proof will be given in Section 6. By Proposition 1.1, the pair (S^6, M_f) is simple, and the rational number $\sigma(S^6, M_f)$ is well–defined by Theorem 1.3. The relationship between Haefliger's invariant $H(f)$ and our invariant $\sigma(S^6, M_f)$ is the following one.

THEOREM 1.4. *For a smooth embedding* $f\colon S^3 \to S^6$*, we have*

$$\sigma(S^6, M_f) = -8H(f).$$

The proof will be given in Section 8.3, and it will turn out that our invariant σ is a natural generalization of Haefliger's invariant H.

There are some generalizations of Haefliger's invariant due to Takase [16] and Skopenkov [13]. Takase [16] [15] proved that there is a bijection $\Omega\colon \mathrm{Emb}(M, S^6) \to \mathbb{Z}$ for any integral homology 3–sphere M such that if $M = S^3$ then $\Omega = H$. Our invariant recovers Takase's invariant too (Corollary 8.1), and that is a direct consequence of the geometric formula for $\sigma(W, V, e)$ (Theorem 8.1):

$$\sigma(W, V, e) = \mathrm{Sign}\, S - \int_S e(\nu_S)^2$$

Here, $S \supset W$ is a Seifert surface of V which Poincaré dual is $e/2$, and which rational normal Euler class $e(\nu_S) \in H^2(S;\mathbb{Q})$ is trivial over ∂S. If $W = S^6$ and V is an integral homology 3–sphere, then the right–hand side is nothing but (-8 times) the definition of Ω.

Recently, Skopenkov [13] proved a classification theorem of elements in $\mathrm{Emb}(M, S^6)$ for any oriented connected closed 3–manifolds M. When $M = S^3$, his invariant $\mu\colon \mathrm{Wh}^{-1}(0) \to \mathbb{Z}$ (where $\mathrm{Wh}^{-1}(0)$ is a subset[2] of $\mathrm{Emb}(M, S^6)$, and μ is called the Kreck invariant in his paper) coincides with H. The invariants μ and σ seem to be closely related, and possibly identical (up to multiplication by a constant) for any M.

We remark that Zhubr [18] [19] had studied on classification of 6–dimensional manifolds and of 3–dimensional knots in 6–dimensional manifolds. Section 3.5 and 3.6 in [18] and Section 5 in [19] are closely related to our article.

1.4. Milnor's triple linking number

Let L be an oriented algebraically split 3–component link in $\mathbb{R}^3$, and let $(T^3 \times S^3, M_L)$ be the manifold pair defined as before. In Section 9, we will prove that $(T^3 \times S^3, M_L)$ is simple (Proposition 9.1), and consequently, the rational number $\sigma(T^3 \times S^3, M_L)$ is well–defined by Theorem 1.3.

REMARK 1.4. It is easy to see that $\sigma(T^3 \times S^3, M_L)$ is a link homotopy invariant of L without explicit computations, in fact, we can see that the isotopy class of the submanifold M_L depends only on the link homotopy type of L as follows. Suppose that two algebraically split 3–component links L and L' in $\mathbb{R}^3$ have the same link homotopy type, and let $\{L(t)\}_{t \in [0,1]}$ be a smooth link homotopy[3] from L to L'. For each $t \in [0,1]$, we can construct a smoothly embedded 3–submanifold $M_{L(t)} \subset T^3 \times S^3$, in exactly the same way as we construct M_L. The obtained family $\{M_{L(t)}\}_{t \in [0,1]}$ is a smooth isotopy from M_L to $M_{L'}$.

The relationship between Milnor's triple linking number $\mu(L)$ and our invariant $\sigma(T^3 \times S^3, M_L)$ is the following one.

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[2] The subset $\mathrm{Wh}^{-1}(0) \subset \mathrm{Emb}(M, S^3)$ consists of elements which Whitney invariant $\mathrm{Wh}\colon \mathrm{Emb}(M, S^6) \to H_1(M;\mathbb{Z})$ [13] vanish.

[3] Each connected component $K_i(t)$ of each intermediate link $L(t) = K_1(t) \cup K_2(t) \cup K_3(t)$ may intersects itself, but no other components $K_j(t)$ $(i \neq j)$.

THEOREM 1.5. *For an oriented algebraically split 3–component link L in S^3, we have*

$$\sigma(T^3 \times S^3, M_L) = -8\mu(L).$$

The proof will be given in Section 9.

Now, here is the plan of the paper. In Section 2, we introduce definitions and notation which are necessary to understand the main theorems given in this section. In Section 3, we study some elementary facts on the low–dimensional oriented cobordism groups $\Omega_*(K(\mathbb{Q}, 2))$ and $\Omega_*(BSO(3))$ of the Eilenberg–MacLane space $K(\mathbb{Q}, 2)$ of type $(\mathbb{Q}, 2)$ and the classifying space $BSO(3)$ of the Lie group $SO(3)$, and this is a preliminary to the next section. In Section 4, we show that there is an isomorphism $\Omega_6^e \cong (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ (Theorem 4.1), and we prove Theorem 1.1 as a consequence of this isomorphism. The isomorphism is given by a short exact sequence

$$0 \to \Omega_4(BSO(3)) \to \Omega_6(K(\mathbb{Q}, 2)) \to \Omega_6^e \to 0,$$

which is isomorphic to $0 \to \mathbb{Z}^{\oplus 2} \to \mathbb{Q}^{\oplus 2} \to (\mathbb{Q}/\mathbb{Z})^{\oplus 2} \to 0$ (see the proof of Theorem 4.1). Section 5 is devoted to the proof of Theorem 1.2, roughly speaking, which relies on the two properties of e-manifolds as follows:

(1) Theorem 1.1 implies the existence and the uniqueness of the invariant σ.

(2) The formula $\operatorname{Sign} X = \Lambda(\beta)$ (Proposition 5.1) implies that σ is well–defined, and that Theorem 1.2 (3) holds.

Section 6 is the study of necessary and sufficient conditions ensuring the existence and uniqueness of e-classes. In particular, we prove that a pair (W, V) is simple if, and only if, it is admits just one e-class (Proposition 6.1 (4)). The proof of Theorem 1.3 is given at the end of the section. In Section 7, we study the relationship between Seifert surfaces and e-classes. In Section 8, we prove Theorem 1.4, and it will turn out that our invariant σ is a natural generalization of the Haefliger's invariant H. We also prove the geometric formula (Theorem 8.1) for $\sigma(W, V, e)$ when $e/2$ is represented by a Seifert surface of V, and as a direct consequence, we prove that our invariant also recovers Takase's invariant (Corollary 8.1). In Section 9, we prove Theorem 1.5.

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2. Preliminaries

2.1. Notation

We use the "outward normal first" convention for boundary orientation of manifolds. For an oriented real vector bundle E of rank 3 over a manifold X, we denote the associated unit sphere bundle by $\rho_E\colon\ S(E) \to X$, and let $F_E \subset TS(E)$ denote the vertical tangent subbundle of $S(E)$ with respect to ρ_E. The orientations of F_E and $S(E)$ are given by the isomorphisms $\rho_E^* E \cong \mathbb{R}_E \oplus F_E$ and $TS(E) \cong \rho_E^* TX \oplus F_E$, where $\mathbb{R}_E \subset \rho_E^* E$ is the tautological real line bundle of E over $S(E)$. Consequently, the Euler class

$$e(F_E) \in \ H^2(S(E);\mathbb{Z})$$

of F_E is defined.

Next, let (Z, X), $Z \supset X$, be a pair of manifolds, and we assume that X is properly embedded in Z and the codimension is 3. Throughout this paper, we always impose these assumptions for all pairs of manifolds. In particular, when we write (W, V) or (Z, X), we always mean a pair of manifolds of codimension 3 such that W and V are closed and $\dim W = 6$, and that Z and X may have boundaries and $\dim Z$ can be any (mainly assumed to be 7 or 6). Denote by ν_X the normal bundle of X, which can be identified with a tubular neighborhood of X so that $X \subset \nu_X \subset Z$. For simplicity, we will write

$$\hat{X} = S(\nu_X), \qquad \rho_X = \rho_{\nu_X}\colon\ \hat{X} \to X, \qquad F_X = F_{\nu_X}.$$

Let us write $(W, V) = \partial(Z, X)$ for the boundary pair of (Z, X) for a moment. We can define ν_V, F_V, $\hat{V}$, $\rho_V\colon\ \hat{V} \to V$, etc. in exactly the same way as above. Let $((0,1]\times W, (0,1]\times V)$ be the pair of collar neighborhoods of the boundary pair $(\{1\}\times W, \{1\}\times V) = (W, V)$. Without loss of generality, we shall always assume $\nu_X|_{(0,1]\times V} = (0,1]\times\nu_V$ as tubular neighborhoods of $(0,1]\times V$ in $(0,1]\times W$. Consequently, we have

$$\partial\hat{X} = \hat{V}, \qquad e(F_X)|_{\hat{V}} = e(F_V).$$

2.2. e-classes and e-manifolds

Here are the definitions of e-class and quasi e-class.

DEFINITION 2.1. Let (Z, X) be a manifold pair of dimensions n and $n-3$.

(1) A cohomology class $e \in H^2(Z \setminus X; \mathbb{Q})$ is called an *e-class* of (Z, X) if

 (a) $e|_{\hat{X}} = e(F_X)$ over $\mathbb{Q}$.

 We call (Z, X, e) an n–dimensional e-manifold.

(2) A cohomology class $e \in H^2(Z \setminus X; \mathbb{Q})$ is called a *quasi e-class* of (Z, X) if

 (a) $e|_{\partial Z \setminus \partial X}$ is an e-class of $\partial(Z, X)$, and
 (b) $\langle [S_p^2], e\rangle = 2$ for all $p \in X$.

 We call (Z, X, e) an n–dimensional quasi e-manifold.

Here, $S_p^2 = \rho_X^{-1}(p) \subset \hat{X}$ is the fiber of ρ_X at p, and the bracket $\langle\ ,\ \rangle$ denotes the pairing of a homology class and a cohomology class. Note that any e-class is a quasi e-class, since $\langle [S_p^2], e(F_X)\rangle = 2$, which is the Euler characteristic of the 2–sphere. Also note that the boundary of a quasi e-manifold is an e-manifold by definition.

For (quasi) e-manifolds $\beta = (Z, X, e)$ and $\beta' = (Z', X', e')$, if there exists an isomorphism $f\colon (Z', X') \to (Z, X)$ of pairs of manifolds such that $f^*e = e'$, then we say β and β' are *isomorphic* (denoted by $\beta \cong \beta'$). The empty e-manifold $(\emptyset, \emptyset, 0)$, where $0 \in H^2(\emptyset \setminus \emptyset; \mathbb{Q})$, will be simply denoted by $\emptyset$. If $\partial\beta \cong \emptyset$, then we say β is *closed*. If a closed e-manifold α bounds an e-manifold β, i.e. $\partial\beta \cong \alpha$, then we say α is *null–cobordant*. If $\alpha \amalg (-\alpha')$ is null–cobordant, then we say α and α' are *cobordant*.

2.3. Self–linking form and self–linking number

For a quasi e-manifold $\beta = (Z, X, e)$, we define the *self–linking form* $\gamma \in H^2(X; \mathbb{Q})$ and the *self–linking number* $\Lambda(\beta) \in \mathbb{Q}$ as follows. Here, $\Lambda(\beta)$ is defined only when $\dim \beta = 7$, i.e. $\dim X = 4$.

By the Thom–Gysin exact sequence

$$\cdots \to 0 \to H^2(X;\mathbb{Q}) \xrightarrow{\rho_X^*} H^2(\hat{X};\mathbb{Q}) \xrightarrow{\rho_{X!}} H^0(X;\mathbb{Q}) \to \cdots$$

of ν_X, the cohomology class $e|_{\hat{X}} - e(F_X) \in H^2(\hat{X};\mathbb{Q})$ belongs to the image of the pull–back ρ_X^* which is injective. We define the self–linking form $\gamma \in H^2(X;\mathbb{Q})$ of β to be the unique cohomology class such that

$$e|_{\hat{X}} = e(F_X) + 2\,\rho_X^*\gamma.$$

Since $\partial\beta$ is an e-manifold, we have $\gamma|_{\partial X} = 0$. There exists $\tilde{\gamma} \in H^2(X,\partial X;\mathbb{Q})$ such that the homomorphism $H^2(X,\partial X;\mathbb{Q}) \to H^2(X;\mathbb{Q})$ maps $\tilde{\gamma}$ to γ. When $\dim\beta = 7$, the self–linking number $\Lambda(\beta)$ of β is defined by

$$\Lambda(\beta) = \int_X \tilde{\gamma}^2.$$

It is easy to check that $\Lambda(\beta)$ does not depend on the choice of $\tilde{\gamma}$. Note that $\gamma = 0$ if β is an e–manifold, so $\Lambda(\beta) = 0$ if β is a 7–dimensional e-manifold.

In Proposition 7.1, we will give an interpretation of γ by using a Seifert surface of X.

3. Oriented Cobordism Groups of $BSO(3)$ and $K(\mathbb{Q},2)$

Let $K(\mathbb{Q},2)$ be the Eilenberg–MacLane space of type $(\mathbb{Q},2)$, i.e. $\pi_2(K(\mathbb{Q},2)) \cong \mathbb{Q}$ and $\pi_i(K(\mathbb{Q},2)) = 0$ for $i \neq 2$, and $BSO(3)$ the classifying space of the Lie group $SO(3)$. We can assume that $BSO(3)$ and $K(\mathbb{Q},2)$ have structures of CW–complexes. Let $\Omega_*(Y)$ denote the oriented cobordism group of a CW–complex Y. As a preparation for the next section, in this section we study some relationship between $\Omega_*(BSO(3))$ and $\Omega_*(K(\mathbb{Q},2))$.

3.1. Homology groups

We begin by recalling some elementary facts on the homology groups of $K(\mathbb{Q},2)$ and $BSO(3)$. The homotopy class of a map $\mathbb{C}P^\infty \to K(\mathbb{Q},2)$ from the infinite dimensional complex projective space $\mathbb{C}P^\infty$ ($\simeq K(\mathbb{Z},2)$), corresponding to the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ on the second homotopy groups, provides an isomorphism between the reduced homology groups (cf. [5]):

$$\begin{aligned} \tilde{H}_k(K(\mathbb{Q},2);\mathbb{Z}) &\cong \tilde{H}_k(\mathbb{C}P^\infty;\mathbb{Q}) \\ &\cong \begin{cases} \mathbb{Q} & \text{if } k \text{ is positive even} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{3.1}$$

Let $a_1 \in H^2(K(\mathbb{Q},2);\mathbb{Q}) \cong \mathbb{Q}$ denote the element dual to $1 \in \pi_2(K(\mathbb{Q},2)) \cong H_2(K(\mathbb{Q},2);\mathbb{Q})$, then the k–th power a_1^k of a_1 generates $H^{2k}(K(\mathbb{Q},2);\mathbb{Q})$ over $\mathbb{Q}$ for $k \geq 1$.

It is easy to check that the low–dimensional homology groups of $BSO(3)$ are given as follows:

(3.2)

k	0	1	2	3	4	5
$H_k(BSO(3);\mathbb{Z})$	$\mathbb{Z}$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	$\mathbb{Z}/2$

This table, for example, is obtained by use of the Serre spectral sequence of the universal principal $SO(3)$-bundle, and by the fact that the cohomology ring $H^*(BSO(3);\mathbb{Z}/2)$ is a free polynomial algebra generated by the second and third Stiefel-Whitney classes over $\mathbb{Z}/2$. The group $H_4(BSO(3);\mathbb{Z})$ is generated by the dual element of the first Pontryagin class $p_1 \in H^4(BSO(3);\mathbb{Z})$.

3.2. Cobordism groups

The next step is the study of the low–dimensional oriented cobordism groups of $K(\mathbb{Q},2)$ and $BSO(3)$. In low–dimensions, the cobordism group $\Omega_* = \Omega_*(pt)$ of one point pt is given as follows (cf. [11, Section 17]):

(3.3)

k	0	1	2	3	4	5	6
Ω_k	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	0

Here, the isomorphism $\Omega_4 \cong \mathbb{Z}$ is given by the signature of 4–manifolds.

In general, for any CW–complex Y, the Atiyah–Hirzebruch spectral sequence $E^n_{p,q}(Y)$ for $\Omega_*(Y)$ converges (cf. [14, Theorem 15.7]):

$$E^2_{p,q}(Y) = H_p(Y;\Omega_q) \implies \Omega_{p+q}(Y)$$

The following lemma is an easy application of the Atiyah–Hirzebruch spectral sequence.

Lemma 3.1. *The following isomorphisms hold*:

$$\Omega_6(K(\mathbb{Q},2)) \cong \mathbb{Q}^{\oplus 2}, \qquad \Omega_3(BSO(3)) = 0, \qquad \Omega_4(BSO(3)) \cong \mathbb{Z}^{\oplus 2}.$$

Proof. We use the isomorphism (3.1) and the tables (3.2) and (3.3) to prove this lemma. The Atiyah–Hirzebruch spectral sequence $E^n_{p,q} = E^n_{p,q}(K(\mathbb{Q},2))$ converges on the E^2-stage within the range $p+q \leq 6$, and so $E^\infty_{p,q} \cong E^2_{p,q}$ in the same range. Consequently, we have

$$E^\infty_{p,6-p} \cong \begin{cases} \mathbb{Q} & \text{if } p = 6, 2, \\ 0 & \text{otherwise,} \end{cases}$$

and therefore, $\Omega_6(K(\mathbb{Q},2)) \cong \mathbb{Q}^{\oplus 2}$.

Similarly, the spectral sequence $F^n_{p,q} = E^n_{p,q}(BSO(3))$ converges on the F^2-stage in the range $p+q \leq 4$, and

$$F^\infty_{p,4-p} \cong \begin{cases} \mathbb{Z} & \text{if } p = 4, 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, $\Omega_4(BSO(3)) \cong \mathbb{Z}^{\oplus 2}$. The vanishing of $\Omega_3(BSO(3))$ follows from $F^2_{p,3-p} = 0$ for all p. □

A pair (W,e) of a closed 6–manifold W and a cohomology class $e \in H^2(W;\mathbb{Q})$ represents a cobordism class $[W,e] \in \Omega_6(K(\mathbb{Q},2))$. Here, we identify e with the homotopy class of a map $f\colon W \to K(\mathbb{Q},2)$ such that $f^*a_1 = e$. Define a homomorphism $\chi\colon \Omega_6(K(\mathbb{Q},2)) \to \mathbb{Q}^{\oplus 2}$ by

$$\begin{aligned} \chi([W,e]) &= (\chi_1(W,e), \chi_2(W,e)), \\ \chi_1(W,e) &= \frac{1}{6}\int_W p_1(TW)\, e - e^3 \in \mathbb{Q}, \\ \chi_2(W,e) &= \frac{1}{2}\int_W e^3 \in \mathbb{Q}. \end{aligned}$$

Similarly, a pair (X,E) of a closed 4–manifold X and an oriented vector bundle E of rank 3 over X represents a cobordism class $[X,E] \in \Omega_4(BSO(3))$. Here, we identify the isomorphism class of E with the homotopy class of the classifying map $X \to BSO(3)$ of E. Define a homomorphism $\xi\colon \Omega_4(BSO(3)) \to \mathbb{Z}^{\oplus 2}$ by

$$\xi([X,E]) = \left(\operatorname{Sign} X, \int_X p_1(E))\right).$$

We will see soon that the homomorphisms χ and ξ are isomorphic (Lemma 3.3).

3.3. Homomorphism $\Omega_4(BSO(3)) \to \Omega_6(K(\mathbb{Q},2))$

Let us consider the homomorphism

$$\upsilon\colon \Omega_4(BSO(3)) \to \Omega_6(K(\mathbb{Q},2))$$

defined by $\upsilon([X,E]) = [S(E), e(F_E)]$ for $[X,E] \in \Omega_4(BSO(3))$. For a pair (X,E) representing an element in $\Omega_4(BSO(3))$, the characteristic classes of the vector bundles E, F_E, TX, and $TS(E)$ satisfy the following relations:

$$e(F_E)^2 = p_1(F_E) = \rho_E^* p_1(E) \tag{3.4}$$

$$\equiv p_1(TS(E)) - \rho_E^* p_1(TX) \quad (\text{mod. 2–torsion elements}), \tag{3.5}$$

$$\rho_{E!} e(F_E) = 2 \tag{3.6}$$

Here, $\rho_{E!}\colon H^2(S(E);\mathbb{Z}) \to H^0(X;\mathbb{Z})$ is the Gysin homomorphism of ρ_E, and $2 \in H^0(X;\mathbb{Z})$ denotes the element given by the constant function on X with the value 2. The Hirzebruch signature theorem states that

$$\operatorname{Sign} X = \frac{1}{3}\int_X p_1(TX). \tag{3.7}$$

The next two lemmas are easy to prove.

Lemma 3.2. $\chi\upsilon = \xi$. *Namely, for any pair (X,E) of closed 4–manifold X and an oriented vector bundle E of rank 3 over X, we have*

$$\chi([S(E), e(F_E)]) = \left(\operatorname{Sign} X, \int_X p_1(E)\right).$$

Proof. This follows from the formulas (3.4), (3.5), (3.6), and (3.7). In fact, these imply

$$p_1(TS(E))e(F_E) - e(F_E)^3 = \rho_E^* p_1(TX)\, e(F_E)$$

over $\mathbb{Q}$, and

$$\chi_1(S(E), e(F_E)) = \frac{1}{6}\int_{S(E)} \rho_E^* p_1(TX)\, e(F_E) = \frac{1}{3}\int_X p_1(TX) = \operatorname{Sign} X.$$

The equality $\chi_2(S(E), e(F_E)) = \int_X p_1(E)$ can be obtained in a similar way. □

Lemma 3.3. *The homomorphisms* $\chi\colon \Omega_6(K(\mathbb{Q},2)) \to \mathbb{Q}^{\oplus 2}$ *and* $\xi\colon \Omega_4(BSO(3)) \to \mathbb{Z}^{\oplus 2}$ *are isomorphisms.*

Proof. For $k = 0, 1$, let F_k be an oriented vector bundle of rank 2 over $\mathbb{C}P^2$ such that $\langle [\mathbb{C}P^1], e(F_k)\rangle = k$, and we set $u_k = [\mathbb{C}P^2, F_k \oplus \mathbb{R}] \in \Omega_4(BSO(3))$. Then, two elements $\xi(u_0) = (1, 0)$ and $\xi(u_1) = (1, 1)$ form a basis of the abelian group $\mathbb{Z}^{\oplus 2}$. Therefore, ξ is a surjective homomorphism from $\Omega_4(BSO(3)) \cong \mathbb{Z}^{\oplus 2}$ (Lemma 3.1) to $\mathbb{Z}^{\oplus 2}$. This means that ξ is an isomorphism.

Similarly, we have $\chi(\upsilon(u_0)) = (1, 0)$ and $\chi(\upsilon(u_1)) = (1, 1)$ by Lemma 3.2, and these two elements form a basis of the vector space $\mathbb{Q}^{\oplus 2}$. Therefore, χ is a linear homomorphism from $\Omega_6(K(\mathbb{Q}), 2) \cong \mathbb{Q}^{\oplus 2}$ (Lemma 3.1) to $\mathbb{Q}^{\oplus 2}$ of rank 2. This means that χ is an isomorphism. □

The following proposition is the goal of this section.

Proposition 3.1. *The sequence of abelian groups*

$$0 \to \Omega_4(BSO(3)) \xrightarrow{\upsilon} \Omega_6(K(\mathbb{Q}, 2)) \xrightarrow{\chi'} (\mathbb{Q}/\mathbb{Z})^{\oplus 2} \to 0$$

is exact, where $\chi' = \chi \mod \mathbb{Z}^{\oplus 2}$.

Proof. This follows from that, the diagram

$$\begin{array}{ccc} \Omega_4(BSO(3)) & \xrightarrow{\upsilon} & \Omega_4(K(\mathbb{Q}, 2)) \\ \xi\downarrow\cong & & \chi\downarrow\cong \\ \mathbb{Z}^{\oplus 2} & \xrightarrow{\text{inclusion}} & \mathbb{Q}^{\oplus 2} \end{array}$$

commutes (Lemma 3.2) and the vertical arrows are isomorphic (Lemma 3.3). □

4. Cobordism Group of 6–Dimensional e-Manifolds

We define Ω_6^e to be the cobordism group of 6–dimensional e-manifolds, namely, it is an abelian group consisting of the cobordism classes $[\alpha]$ of 6–dimensional closed e-manifolds α, with the group structure given by the disjoint sum. Note that $[\alpha] + [\alpha'] = [\alpha \amalg \alpha']$, $-[\alpha] = [-\alpha]$, and $0 = [\emptyset]$. In this section, we prove that Ω_6^e is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ (Theorem 4.1), and then we prove Theorem 1.1.

We begin by preparing some notation as follows. For a pair (Z, X), we will write

$$Z_X = Z \setminus U_X, \tag{4.1}$$

where U_X is the total space of the open unit disk bundle of ν_X. If $(W, V) = \partial(Z, X)$ denotes the boundary pair, then the manifold $W_V = W \setminus U_V$ can be defined in the same way as above. In line with our orientation conventions, the boundaries of Z_X and W_V are given as follows:

$$\partial Z_X = W_V \cup (\mp \hat{X}), \qquad \partial W_V = \pm \hat{V} \tag{4.2}$$

Here, the symbols $\mp = (-1)^{\dim Z}$ and $\pm = (-1)^{\dim W}$ are the signs of orientations. Note that Z_X have the corner $\hat{V}$ which is empty when X is closed.

4.1. Extension of the cobordism group of 6–dimensional e-manifolds

In this subsection, we show that any element in Ω_6^e can be represented by a 6–dimensional closed e-manifold with empty submanifold. More precisely, let

$$\pi \colon \Omega_6(K(\mathbb{Q}, 2)) \to \Omega_6^e$$

be the homomorphism defined by $\pi([W, e]) = [W, \emptyset, e]$ for $[W, e] \in \Omega_6(K(\mathbb{Q}, 2))$, and we prove that π is surjective (Proposition 4.1).

For a 6–dimensional closed e-manifold $\alpha = (W, V, e)$, we construct a cobordism class $[W', e'] \in \Omega_6(K(\mathbb{Q}, 2))$ such that $\pi([W', e']) = [\alpha]$ as follows. Since the oriented cobordism group $\Omega_3(BSO(3))$ vanishes by Lemma 3.1, there exists a pair (X, E) of a 4–manifold X and an oriented vector bundle E of rank 3 over X equipped with fixed identifications $\partial X = V$

and $E|_V = \nu_V$. Two pairs $(S(E), e(F_E))$ and $(W_V, e|_{W_V})$ have the common boundary

$$\partial(S(E), e(F_E)) = (\hat{V}, e(F_V)) = \partial(W_V, e|_{W_V}).$$

Let us consider the closed 6–manifold

$$W' = W_V \cup_{\hat{V}} (-S(E))$$

obtained from W_V and $-S(E)$ by gluing along the common boundaries (namely, W' is constructed by performing a kind of surgery along V, replacing the tubular neighborhood of V with $-S(E)$). There exists $e' \in H^2(W'; \mathbb{Q})$ such that $e'|_{W_V} = e|_{W_V}$ and $e'|_{S(E)} = e(F_E)$, and we have a cobordism class

$$[W', e'] \in \Omega_6(K(\mathbb{Q}, 2)).$$

PROPOSITION 4.1. *We have $\pi([W', e']) = [\alpha]$ in Ω_6^e. Consequently, the homomorphism $\pi\colon \Omega_6(K(\mathbb{Q}, 2)) \to \Omega_6^e$ is surjective.*

PROOF. We only need to show the existence of a 7–dimensional e-manifold β bounded by $\alpha \amalg (-\alpha')$, where $\alpha' = (W', \emptyset, e')$. Let $I = [0, 1]$ be the interval. In this proof, for a subset $A \subset W$, we write $A_t = \{t\} \times A \subset I \times W$ for $t = 0, 1$. The boundaries of the 7–manifold $I \times W$ and the closed unit disk bundle $D(E)$ of E are given as follows:

$$\partial(I \times W) = (-W_0) \amalg W_1$$
$$\partial D(E) = S(E) \cup D(\nu_V)$$

Gluing the manifolds $I \times W$ and $D(E)$ along $D(\nu_V)_0 \subset W_0$ and $D(\nu_V) \subset \partial D(E)$ by the identity map, we obtain a 7–manifold

$$Z = D(E) \cup_{D(\nu_V)_0} (I \times W)$$

with the boundary

$$\partial Z = W_1 \amalg \Big(S(E) \cup_{\hat{V}_0} (-(W_V)_0)\Big)$$
$$\cong W \amalg (-W'),$$

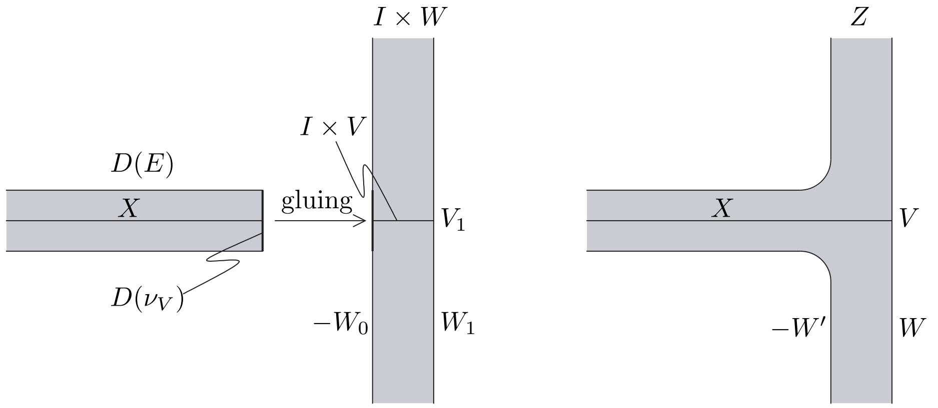

Fig. 1. Gluing $D(E)$ and $I{\times}W$, and the obtained manifold pair (Z, X).

and we shall assume that ∂Z is smooth after the corner $\hat{V}_0$ is rounded, see Figure 1.

The 4–submanifold

$$X \cup_{V_0} (I{\times}V) \subset Z$$

(where X is identified with the image of the zero–section of E so that $X \subset D(E)$) is properly embedded in Z, and is bounded by V_1. We will rewrite $X \cup_{V_0} (I{\times}V)$ as X and identify ∂Z with $W \amalg (-W')$, so that

$$\partial(Z, X) = (W, V) \amalg (-W', \emptyset).$$

Now, all that is left to do is to show the existence of an e-class of (Z, X) restricting to e and e' on the boundary components. Since the inclusion $W' \hookrightarrow Z \setminus X$ is homotopy equivalence, there exists a cohomology class $\tilde{e} \in H^2(Z \setminus X; \mathbb{Q})$ of (Z, X) such that $\tilde{e}|_{W'} = e'$. By construction, $\tilde{e}$ is an e-class of (Z, X) and $\tilde{e}|_{W \setminus V} = e$. Hence, we obtain a 7–dimensional e-manifold $\beta = (Z, X, \tilde{e})$ bounded by

$$\partial\beta = (W, V, \tilde{e}|_{W \setminus V}) \amalg (-W', \emptyset, \tilde{e}|_{W'}) = \alpha \amalg (-\alpha'). \ \Box$$

4.2. Proof of Theorem 1.1

We define a homomorphism

$$\Phi\colon\ \Omega_6^e \to (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$$

as follows. By Proposition 4.1, any element in Ω_6^e is represented by an e-manifold of the form $(W, \emptyset, e)$, where W is a closed 6–manifold and $e \in\ H^2(W;\mathbb{Q})$. We then define

$$\begin{aligned}\Phi([W,\emptyset,e]) &= \chi'([W,e]) \\ &\equiv \left(\frac{1}{6}\int_W p_1(TW)\,e - e^3,\ \frac{1}{2}\int_W e^3\right) \quad \bmod \mathbb{Z}^{\oplus 2}.\end{aligned}$$

The rest of this section is devoted to proving that Φ is an isomorphism. The first thing we have to do is to show that $\Phi([W,\emptyset,e])$ is independent of the representative $(W,\emptyset,e)$ of $[W,\emptyset,e]$.

LEMMA 4.1. *The homomorphism* $\Phi\colon\ \Omega_6^e \to (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ *is well–defined.*

PROOF. Let $\alpha = (W,\emptyset,e)$ and $\alpha' = (W',\emptyset,e')$ be any 6–dimensional closed e-manifolds representing the same cobordism class in Ω_6^e, and we prove that the difference $\chi([W,\emptyset,e]) - \chi([W',\emptyset,e'])$ belongs to $\mathbb{Z}^{\oplus 2}$, which implies $\chi'([W,\emptyset,e]) = \chi'([W',\emptyset,e'])$.

There exists a 7–dimensional e-manifold $\beta = (Z, X, \tilde{e})$ such that $\partial\beta \cong \alpha \amalg (-\alpha')$, in particular, X is closed and embedded in the interior of Z. Thus, the manifold Z_X (see (4.1)) has the smooth boundary

$$\partial Z_X = \partial Z \amalg (-\hat{X}) \cong W \amalg (-W') \amalg (-\hat{X}).$$

Since $\tilde{e}|_{\hat{X}} = e(F_X)$, we can write

$$\partial(Z_X, \tilde{e}|_{Z_X}) \cong (W,e) \amalg (-W',e') \amalg (-\hat{X}, e(F_X)),$$

and this implies $[W,e] - [W',e'] = [\hat{X}, e(F_X)]$ in $\Omega_6(K(\mathbb{Q},2))$. We have

$$\chi([W,e]) - \chi([W',e']) = \left(\operatorname{Sign} X,\ \int_X p_1(\nu_X)\right)$$

by Lemma 3.2, and the right–hand side belongs to $\mathbb{Z}^{\oplus 2}$. □

The following theorem is the goal of this section.

THEOREM 4.1. *The homomorphism $\Phi\colon \Omega_6^e \to (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ is an isomorphism.*

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_4(BSO(3)) & \xrightarrow{\upsilon} & \Omega_6(K(\mathbb{Q},2)) & \xrightarrow{\pi} & \Omega_6^e & \longrightarrow & 0 \\ & & \| & & \| & & \Phi\downarrow & & \\ 0 & \longrightarrow & \Omega_4(BSO(3)) & \xrightarrow{\upsilon} & \Omega_6(K(\mathbb{Q},2)) & \xrightarrow{\chi'} & (\mathbb{Q}/\mathbb{Z})^{\oplus 2} & \longrightarrow & 0 \end{array}$$

The lower horizontal sequence is exact by Proposition 3.1, and the homomorphism π is surjective by Proposition 4.1. To complete the proof, we only have to show that the upper horizontal sequence is exact, more specifically, $\operatorname{Im}\upsilon = \operatorname{Ker}\pi$. We prove this in two steps as follows.

CLAIM 1. $\operatorname{Im}\upsilon \subset \operatorname{Ker}\pi$. Let $[X, E] \in \Omega_4(BSO(3))$ be any element, then $\pi(\upsilon([X, E])) = [S(E), \emptyset, e(F_E)]$. Regard X as the image of the zero–section of E so that $X \subset \operatorname{Int} D(E)$. The cohomology class $e(F_E)$ is an e-class of $(S(E), \emptyset) = \partial(D(E), X)$, and it uniquely extends to an e-class e_E of $(D(E), X)$. The obtained e-manifold $(D(E), X, e_E)$ is bounded by $(S(E), \emptyset, e(F_E))$, and hence, we have $\pi(\upsilon([X, E])) = 0$.

CLAIM 2. $\operatorname{Im}\upsilon \supset \operatorname{Ker}\pi$. Next, we prove the opposite inclusion. Let $[W, e] \in \operatorname{Ker}\pi$ be any element, then $\alpha = (W, \emptyset, e)$ bounds a 7–dimensional e-manifold $\beta = (Z, X, \tilde{e})$, namely $\partial\beta \cong \alpha$. In particular, we have $\tilde{e}|_{\hat{X}} = e(F_X)$. Since

$$\partial(Z_X, \tilde{e}|_{Z_X}) \cong (W, e) \amalg (-\hat{X}, e(F_X)),$$

we have

$$[W, e] = [\hat{X}, e(F_X)] = \upsilon([X, \nu_X])$$

in $\Omega_6(K(\mathbb{Q}, 2))$, where ν_X is the normal bundle of X. Therefore, $[W, e]$ belongs to $\operatorname{Im}\upsilon$. This completes the proof. $\square$

Theorem 1.1 can be proved very easily from Theorem 4.1 as follows.

Proof of Theorem 1.1. For a 6–dimensional closed e-manifold α, its cobordism class $[\alpha] \in \Omega_6^e \cong (\mathbb{Q}/\mathbb{Z})^{\oplus 2}$ (Theorem 4.1) has a finite order, say m. The meaning of $m[\alpha] = 0$ is that there exists a 7–dimensional e-manifold β such that $\partial\beta \cong \amalg^m \alpha$. □

5. Proof of Theorem 1.2

In this section, we give a proof of Theorem 1.2.

5.1. Signature of 4–manifolds in 7–manifolds

For a pair (Z, X), let us consider the Mayer–Vietoris exact sequence of $(Z; Z \setminus X, U_X)$:

$$\cdots \to H^2(Z;\mathbb{Q}) \xrightarrow{(j_X^*, i_X^*)} H^2(Z \setminus X;\mathbb{Q}) \oplus H^2(X;\mathbb{Q}) \xrightarrow{\iota_X^* - \rho_X^*} H^2(\hat{X};\mathbb{Q}) \xrightarrow{\delta^*} H^3(Z;\mathbb{Q}) \to \cdots \tag{5.1}$$

Here, we identified $H^2(U_X;\mathbb{Q})$ with $H^2(X;\mathbb{Q})$, and $H^2(U_X \setminus X;\mathbb{Q})$ with $H^2(\hat{X};\mathbb{Q})$ via the isomorphisms given by the homotopy equivalences $U_X \simeq X$ and $U_X \setminus X \simeq \hat{X}$, and here, the maps

$$i_X\colon X \hookrightarrow Z, \qquad \iota_X\colon \hat{X} \hookrightarrow Z \setminus X, \qquad j_X\colon Z \setminus X \hookrightarrow Z$$

denote the inclusions. Denote by

$$t_X \in H^3(Z;\mathbb{Q})$$

the fundamental cohomology class of X (the Poincaré dual of the fundamental homology class $[X, \partial X] \in H_{\dim X}(Z, \partial Z;\mathbb{Q})$). Since $\rho_{X!} e(F_X) = 2$ (see (3.6)), we have

$$\delta^* e(F_X) = 2t_X. \tag{5.2}$$

The following lemma states that the existence of a quasi e-class of (Z, X) is almost equivalent to $[X, \partial X] = 0$ (exactly equivalent if the second betti–number of ∂X vanishes).

LEMMA 5.1. *Let (Z, X) be a pair of manifolds of codimension 3. Then, $[X, \partial X] = 0$ if, and only if, there exist cohomology classes $e \in H^2(Z \setminus X; \mathbb{Q})$ and $\gamma \in H^2(X; \mathbb{Q})$ such that*

$$e|_{\hat{X}} = e(F_X) + 2\rho_X^* \gamma.$$

Moreover, if such e and γ exist and $\gamma|_{\partial X} = 0$, then e is a quasi e-class of (Z, X), and γ is the self–linking form of the quasi e-manifold (Z, X, e).

PROOF. The vanishing of $[X, \partial X]$ implies $e(F_X) \in \operatorname{Ker} \delta^*$ by (5.2), and thus, $e(F_X) = e|_{\hat{X}} - 2\rho_X^* \gamma$ holds for some elements $e \in H^2(Z \setminus X; \mathbb{Q})$ and $\gamma \in H^2(X; \mathbb{Q})$ by the exactness of the sequence (5.1). The converse also holds. The second half of the statement is obvious from Definition 2.1 (2). □

LEMMA 5.2. *Let (Z, X) be a pair of closed manifolds of dimensions 7 and 4. If X is rationally null–homologous in Z, then*

$$\chi([\hat{X}, e(F_X)]) = (\operatorname{Sign} X, -3 \operatorname{Sign} X).$$

PROOF. By Lemma 3.2, we have

$$\chi([\hat{X}, e(F_X)]) = \left(\operatorname{Sign} X, \int_X p_1(\nu_X) \right).$$

The Hirzebruch signature theorem (3.7) and the vanishing of the homology class of X imply

$$\int_X p_1(\nu_X) = \int_X p_1(TZ) - \int_X p_1(TX) = -\int_X p_1(TX) = -3 \operatorname{Sign} X. \ \square$$

PROPOSITION 5.1. *Let (Z, X) be a pair of closed manifolds of dimensions 7 and 4. If e is a quasi e-class of (Z, X), then we have*

$$\operatorname{Sign} X = 4\, \Lambda(Z, X, e).$$

Proof. Let $\gamma \in H^2(X;\mathbb{Q})$ be the self–linking form of $\beta = (Z, X, e)$, namely $e|_{\hat{X}} = e(F_X) + 2\,\rho_X^*\gamma$. Then, we can write

$$\begin{aligned} e^3|_{\hat{X}} &= e(F_X)^3 + 6e(F_X)^2\rho_X^*\gamma + 12e(F_X)\rho_X^*\gamma^2 + 8\rho_X^*\gamma^3 \\ &= e(F_X)^3 + 6\rho_X^*\left(p_1(\nu_X)\gamma\right) + 12e(F_X)\rho_X^*\gamma^2 + 8\rho_X^*\gamma^3. \end{aligned}$$

Here, we used the relation (3.4) (where $E = \nu_X$) on the second term of the right–hand side. Since $\dim X < 6$, the second and the last terms of the right–hand side vanish. Integrating the both sides over $\hat{X}$, we obtain

$$\int_{\hat{X}} e^3 = 2\chi_2(\hat{X}, e(F_X)) + 24\Lambda(\beta).$$

The left–hand side vanishes by Stokes' theorem, since $\partial(Z_X, e^3|_{Z_X}) = (-\hat{X}, e^3|_{\hat{X}})$. Thus, we obtain $\chi_2(\hat{X}, e(F_X)) = -12\Lambda(\beta)$.

On the other hand, the existence of a quasi e-class of (Z, X) implies that X is rationally null–homologous in Z by Lemma 5.1, and so (Z, X) satisfies the assumption of Lemma 5.2. Hence, we obtain

$$\operatorname{Sign} X = -\frac{1}{3}\chi_2(\hat{X}, e(F_X)) = 4\Lambda(\beta). \ \Box$$

As a corollary, we obtain the following vanishing property of the signature.

Corollary 5.1. *If a pair (Z, X) of closed manifolds of dimensions 7 and 4 admits an e-class, then* $\operatorname{Sign} X = 0$.

Proof. Let e be an e-class of (Z, X), then $\Lambda(Z, X, e) = 0$ by the definition of Λ. This implies $\operatorname{Sign} X = 0$ by Proposition 5.1. □

5.2. Definition of the invariant $\sigma(\alpha)$

Let α be a 6–dimensional closed e-manifold. We first review and generalize the definition of the invariant $\sigma(\alpha)$ as follows. By Theorem 1.1, there exists a 7–dimensional e-manifold $\beta = (Z, X, \tilde{e})$ such that $\partial\beta \cong \amalg^m \alpha$ for some positive integer m. More generally, we shall assume that β is a quasi e-manifold. We then define

$$\sigma(\alpha) = \frac{\operatorname{Sign} X - 4\Lambda(\beta)}{m} \in \mathbb{Q}. \tag{5.3}$$

In this section, we use this definition instead of the one given in Theorem 1.2, because (5.3) includes the formula in Theorem 1.2 (3).

PROPOSITION 5.2. *For a 6–dimensional closed e-manifold α, the rational number $\sigma(\alpha)$, defined as in* (5.3), *depends only on the isomorphism class of α.*

PROOF. Let $\mathcal{N}$ be the set of the isomorphism classes of 6–dimensional null–cobordant closed e-manifolds. Note that if $\alpha \in \mathcal{N}$, then $\amalg^m \alpha \in \mathcal{N}$ for any positive integer m. We prove the statement in three steps as follows. The first step is the most important, and the rests are proved in a formal way.

CLAIM 1. If $\alpha \in \mathcal{N}$, then $\sigma(\alpha)$ is well–defined. Assume $\alpha \in \mathcal{N}$, and take any 7–dimensional quasi e-manifolds $\beta_0 = (Z_0, X_0, e_0)$ and $\beta_1 = (Z_1, X_1, e_1)$ equipped with fixed identifications $\partial\beta_0 = \alpha = \partial\beta_1$. We need to show the equality

$$\operatorname{Sign} X_0 - 4\Lambda(\beta_0) = \operatorname{Sign} X_1 - 4\Lambda(\beta_1). \tag{5.4}$$

For that, we consider the pair of closed manifolds

$$(Z, X) = (Z_0 \cup (-Z_1), X_0 \cup (-X_1))$$

obtained from (Z_0, X_0) and $(-Z_1, -X_1)$ by gluing along the identified boundaries, and write $(W, V) = (\partial Z_0, \partial X_0) \subset (Z, X)$. Since the quasi e-classes e_0 and e_1 restrict to the same e-class of (W, V), there exists a quasi e-class e of (Z, X) such that $e|_{Z_i \setminus X_i} = e_i$ for $i = 0, 1$. Thus, we obtain a 7–dimensional closed quasi e-manifold $\beta = (Z, X, e)$.

The self–linking form $\gamma \in H^2(X; \mathbb{Q})$ of β is trivial on V, and thus, there exists an element

$$\tilde{\gamma} = (\tilde{\gamma}_0, \tilde{\gamma}_1) \in H^2(X, V; \mathbb{Q}) = H^2(X_0, \partial X_0; \mathbb{Q}) \oplus H^2(X_1, \partial X_1; \mathbb{Q})$$

such that the homomorphism $H^2(X, V; \mathbb{Q}) \to H^2(X; \mathbb{Q})$ maps $\tilde{\gamma}$ to γ. Similarly, the homomorphism $H^2(X_i, \partial X_i; \mathbb{Q}) \to H^2(X_i; \mathbb{Q})$ maps $\tilde{\gamma}_i$ to the self–linking form of β_i, and thus

$$\Lambda(\beta_i) = \int_{X_i} \tilde{\gamma}_i^2.$$

Hence, we have

$$\Lambda(\beta) = \int_X \gamma^2 = \int_{X_0} \tilde{\gamma}_0^2 - \int_{X_1} \tilde{\gamma}_1^2 = \Lambda(\beta_0) - \Lambda(\beta_1).$$

(Namely, the self–linking number Λ is additive with respect to the decompositions of closed quasi e-manifolds.) By the additive properties of Λ and the signature, and by Proposition 5.1, we have $\operatorname{Sign} X_0 - \operatorname{Sign} X_1 = 4\Lambda(\beta_0) - 4\Lambda(\beta_1)$, and so (5.4) holds.

Claim 2. If $\alpha \in \mathcal{N}$, then $\sigma(\amalg^m \alpha) = m\sigma(\alpha)$ for any positive integer m. Let $\beta = (Z, X, e)$ be a 7–dimensional e-manifold such that $\partial\beta \cong \alpha$, then $\amalg^m \alpha$ bounds $\amalg^m \beta$. The rational number $\sigma(\amalg^m \alpha)$ is well–defined by Claim 1, and we have

$$\sigma(\amalg^m \alpha) = \operatorname{Sign}(\amalg^m X) = m \operatorname{Sign} X = m\,\sigma(\alpha).$$

Claim 3. $\sigma(\alpha)$ is well–defined for any 6–dimensional closed e-manifold α. Let α be a closed e-manifold, and m a positive integer such that $\amalg^m \alpha \in \mathcal{N}$ (such m exists by Theorem 1.1). The rational number $\sigma(\amalg^m \alpha)$ is well–defined by Claim 1. We can show that the rational number $\sigma(\amalg^m \alpha)/m$ does not depend on the choice of m as follows.

If $\amalg^m \alpha \in \mathcal{N}$ and $\amalg^{m'} \alpha \in \mathcal{N}$ for some positive integers m and m', then $\amalg^{mm'} \alpha \in \mathcal{N}$. Thus, the rational numbers $\sigma(\amalg^m \alpha)$, $\sigma(\amalg^{m'} \alpha)$, and $\sigma(\amalg^{mm'} \alpha)$ are well–defined by Claim 1. Since $\amalg^{m'}(\amalg^m \alpha) = \amalg^{mm'} \alpha = \amalg^m(\amalg^{m'} \alpha)$, we have

$$\frac{\sigma(\amalg^m \alpha)}{m} = \frac{\sigma(\amalg^{mm'} \alpha)}{mm'} = \frac{\sigma(\amalg^{m'} \alpha)}{m'}$$

by Claim 2. This implies that $\sigma(\amalg^m \alpha)/m$ does not depend on the choice of m. □

5.3. Proof of Theorem 1.2

By using the results we have obtained so far, we prove Theorem 1.2.

Proof of Theorem 1.2. By Proposition 5.2, the rational number $\sigma(\alpha)$ is well–defined for any 6–dimensional closed e-manifold α. Axiom 2 and Theorem 1.2 (3) are obvious from the definition (5.3).

We prove that σ satisfies Axiom 1 as follows. Let α be a 6–dimensional closed e-manifold, and $\beta = (Z, X, e)$ a 7–dimensional e-manifold such that $\partial\beta \cong \amalg^m \alpha$ for some positive integer m. Then, $\partial(-\beta) \cong \amalg^m(-\alpha)$, and the definition of σ implies

$$\sigma(-\alpha) = \frac{\mathrm{Sign}\,(-X)}{m} = -\sigma(\alpha).$$

Next, let α' be another 6–dimensional closed e-manifold, and $\beta' = (Z', X', e')$ a 7–dimensional e-manifold such that $\partial\beta' \cong \amalg^{m'}\alpha$ for some positive integer m'. Then $(\amalg^{m'}\beta) \amalg (\amalg^m \beta')$ bounds $\amalg^{mm'}(\alpha \amalg \alpha')$, and we have

$$\begin{aligned}\sigma(\alpha \amalg \alpha') &= \frac{m'\mathrm{Sign}\,X + m\mathrm{Sign}\,X'}{mm'} \\ &= \frac{\mathrm{Sign}\,X}{m} + \frac{\mathrm{Sign}\,X'}{m'} \\ &= \sigma(\alpha) + \sigma(\alpha').\end{aligned}$$

Hence, Axiom 1 holds.

We prove Theorem 1.2 (2) (uniqueness of σ) as follows. Let σ' be an invariant of the isomorphism classes of 6–dimensional closed e-manifolds satisfying the axioms. Let us consider the difference

$$f(\alpha) = \sigma'(\alpha) - \sigma(\alpha) \in \mathbb{Q}.$$

If two 6–dimensional closed e-manifolds α_0 and α_1 are cobordant, that is, if there exists a 7–dimensional e-manifold $\beta = (Z, X, e)$ such that $\partial\beta \cong \alpha_0 \amalg (-\alpha_1)$, then

$$f(\alpha_0) - f(\alpha_1) = \sigma'(\partial\beta) - \sigma(\partial\beta) = \mathrm{Sign}\,X - \mathrm{Sign}\,X = 0$$

by the axioms. Thus, we can regard f as a function on Ω_6^e:

$$f\colon\ \Omega_6^e \to \mathbb{Q}$$

Moreover, Axiom 1 implies that f is a homomorphism.

On the other hand, any homomorphism $\Omega_6^e \to \mathbb{Q}$ is trivial by Theorem 1.1, in particular, f must be trivial. Namely, $\sigma' = \sigma$. □

This completes the proof of Theorem 1.2.

6. Existence and Uniqueness of e-Classes

As in Section 1.2, we denote by $\mathcal{E}_{Z,X} \subset H^2(Z \setminus X;\mathbb{Q})$ the set of all e-classes of a manifold pair (Z,X) of codimension 3. In this short section, we study some elementary properties of $\mathcal{E}_{Z,X}$, and then we give a proof of Theorem 1.3.

For elements $e \in \mathcal{E}_{Z,X}$ and $a \in \operatorname{Ker} i_X^*$, where

$$i_X^*\colon\ H^2(Z;\mathbb{Q}) \to H^2(X;\mathbb{Q})$$

is the restriction, the cohomology class

$$g_e(a) \overset{def}{=} e + a|_{Z\setminus X} \in\ H^2(Z \setminus X;\mathbb{Q})$$

belongs to $\mathcal{E}_{Z,X}$, because $a|_{\hat{X}} = 0$. Thus, we obtain an affine homomorphism

$$g_e\colon\ \operatorname{Ker} i_X^* \to \mathcal{E}_{Z,X}.$$

Note that g_e is defined only when $\mathcal{E}_{Z,X} \neq \emptyset$. Let $\#A \in\ \mathbb{N} \cup \{\infty\}$ denote the number of elements in a set A, and let $[X,\partial X] \in\ H_{\dim X}(Z,\partial Z;\mathbb{Q})$ be the fundamental homology class of $(X,\partial X)$.

PROPOSITION 6.1. *The following statements hold*:

(1) *If* $[X,\partial X] \neq 0$, *then* $\mathcal{E}_{Z,X} = \emptyset$.

(2) *For any* $e \in\ \mathcal{E}_{Z,X}$, *the map* $g_e\colon \operatorname{Ker} i_X^* \to \mathcal{E}_{Z,X}$ *is an affine isomorphism.*

(3) *Assume that* i_X^* *is surjective. Then,* $\mathcal{E}_{Z,X} \neq \emptyset$ *if, and only if,* $[X,\partial X] = 0$.

(4) $\#\mathcal{E}_{Z,X} = 1$ *if, and only if,* (Z,X) *is simple.*

(5) Proposition 1.1 *holds. Namely, when* i_X^* *is an isomorphism,* (Z,X) *is simple if, and only if,* $[X,\partial X] = 0$.

PROOF. (1) is a direct consequence of Lemma 5.1. In fact, if $[X,\partial X] \neq 0$, then (Z,X) does not even admit a quasi e-class.

(2) Since two elements in $\mathcal{E}_{Z,X}$ differ by an element in the kernel $\operatorname{Ker} \iota_X^*$ of the homomorphism $\iota_X^*\colon\ H^2(Z\setminus X;\mathbb{Q}) \to H^2(\hat{X};\mathbb{Q})$, we only need to show

that the homomorphism $j_X^*\colon\ H^2(Z;\mathbb{Q}) \to H^2(Z\setminus X;\mathbb{Q})$ restricts a bijection $\operatorname{Ker} i_X^* \to \operatorname{Ker} \iota_X^*$, and this is immediate from the following commutative diagram:

$$\begin{array}{ccccc} H^2(Z\setminus X,\hat{X};\mathbb{Q}) & \longrightarrow & H^2(Z\setminus X;\mathbb{Q}) & \xrightarrow{\iota_X^*} & H^2(\hat{X};\mathbb{Q}) \\ {\scriptstyle f}\uparrow{\scriptstyle \cong} & & {\scriptstyle j_X^*}\uparrow & & \\ H^2(Z,X;\mathbb{Q}) & \longrightarrow & H^2(Z;\mathbb{Q}) & \xrightarrow{i_X^*} & H^2(X;\mathbb{Q}) \end{array}$$

Here, the horizontal sequences are the long exact sequences of $(Z\setminus X,\hat{X})$ and (Z,X), and f is the excision isomorphism. Note that the vanishing of $H^2(Z,Z\setminus X;\mathbb{Q})$ follows the injectivity of j_X^*.

(3) We only give a proof of that the vanishing of $[X,\partial X]$ implies $\mathcal{E}_{Z,X}\neq\emptyset$, since the converse is given by (1). Assume that i_X^* is surjective and $[X,\partial X]=0$. By Lemma 5.1, there exist $e\in\ H^2(Z\setminus X;\mathbb{Q})$ and $\gamma\in\ H^2(X;\mathbb{Q})$ such that $e|_{\hat{X}}=e(F_X)+2\rho_X^*\gamma$. By the assumption, there exists $\varepsilon\in\ H^2(Z;\mathbb{Q})$ such that $\varepsilon|_X=\gamma$. The homotopy equivalence $j_X\iota_X\simeq i_X\rho_X\colon\ \hat{X}\to Z$ implies $\varepsilon|_{\hat{X}}=\rho_X^*\gamma$, and the cohomology class $e'=e-2\varepsilon|_{Z\setminus X}\in\ H^2(Z\setminus X;\mathbb{Q})$ satisfies

$$e'|_{\hat{X}}=(e(F_X)+2\rho_X^*\gamma)-2\rho_X^*\gamma=e(F_X).$$

This means $e'\in\ \mathcal{E}_{Z,X}\neq\emptyset$.

(4) By (2), $\#\mathcal{E}_{Z,X}=1$ implies $\operatorname{Ker} i_X^*=\{0\}$ which means that i_X^* is injective, and thus, (Z,X) is simple. Conversely, let us assume that (Z,X) is simple, that is, i_X^* is injective and $\mathcal{E}_{Z,X}\neq\emptyset$. Fix any element $e\in\ \mathcal{E}_{Z,X}$, then the map $g_e\colon\ \operatorname{Ker} i_X^*=\{0\}\to\mathcal{E}_{Z,X}$ is a bijection by (2), and thus, $\#\mathcal{E}_{Z,X}=\#\{e\}=1$.

(5) This is a combination of (3) and (4). □

If we drop the surjectivity assumption of i_X^* in Proposition 6.1 (3), then the statement does not hold anymore, and a counterexample is given in the following.

Remark 6.1. It is known that any oriented closed smooth 4–manifold X can be smoothly embedded in S^7 (c.f. [4, 9.1.23, 9.1.24]). Let us assume $\operatorname{Sign} X\neq 0$ (for example, $\operatorname{Sign}\mathbb{C}P^2=1$). Obviously, X is null–homologous in S^7, but (S^7,X) does not admit any e-class by Corollary 5.1.

Now, Theorem 1.3 is proved as follows.

PROOF OF THEOREM 1.3. Let us assume that (W, V) is simple. By Corollary 1.1, the image $\operatorname{Im} \sigma_{W,V} \subset \mathbb{Q}$ of the function $\sigma_{W,V} \colon \mathcal{E}_{W,V} \to \mathbb{Q}$ depends only on the isomorphism class of (W, V). The simplicity of (W, V) implies that $\mathcal{E}_{W,V}$ consists of just one element, say $\mathcal{E}_{W,V} = \{e\}$, and therefore, $\operatorname{Im} \sigma_{W,V} = \{\sigma(W, V, e)\}$. Thus, $\sigma(W, V, e)$ is an invariant of the isomorphism class of (W, V). □

7. Seifert Surfaces

In this section, we establish the relationship between Seifert surfaces and quasi e–classes. When we consider the intersection of submanifolds, we will always assume that the submanifolds are in general positions (by deforming them slightly if necessary) so that the intersections becomes smooth manifolds.

Let (X, E) be a pair of a manifold X and an oriented vector bundle E of rank 3 over X. We denote by

$$\tau \colon E \to E, \quad v \mapsto -v$$

the involution given by the multiplication by a scalar -1. There is a direct sum decomposition

$$H^2(S(E); \mathbb{Q}) = H_{+1} \oplus H_{-1}, \tag{7.1}$$

where $H_{\pm 1}$ is the eigenspace of the involution $\tau^* \colon H^2(S(E); \mathbb{Q}) \to H^2(S(E); \mathbb{Q})$ with the eigenvalue ± 1. The subspace H_{+1} is the image of the pull–back $\rho_E^* \colon H^2(X; \mathbb{Q}) \to H^2(S(E); \mathbb{Q})$, and H_{-1} is the subspace spanned by the Euler class class $e(F_E)$ of F_E. These facts are proved by using the Thom–Gysin exact sequence of E.

The Euler class $e(F_E)$ of F_E is algebraically characterized as follows.

LEMMA 7.1. *Let (X, E) be as above. If a cohomology class $a \in H^2(S(E); \mathbb{Q})$ satisfies two conditions*

(1) $\tau^* a = -a$,

(2) $\rho_{E!} a = 2$,

then $a = e(F_E)$ *over* $\mathbb{Q}$. *Here,* $\rho_{E!}\colon H^2(S(E);\mathbb{Q}) \to H^0(X;\mathbb{Q})$ *is the Gysin homomorphism of the associated sphere bundle* $\rho_E\colon S(E) \to X$.

PROOF. The property (1) implies $a \in H_-$, and (2) implies $a = e(F_E)$. □

For a manifold pair (Z, X) of codimension 3, a *Seifert surface* of X is a proper (oriented) submanifold Y of Z_X (see (4.1)) such that $Y \cap \hat{X} = s(X)$ for some section $s\colon X \to \hat{X}$, and such that the natural isomorphism $\nu_Y|_{s(X)} \cong F_X|_{s(X)}$ preserves the orientation. Note that Y may have the corner $s(\partial X)$ (see also (7.4) below). More generally, if $Y \setminus U_{s(X)}$ is immersed in $Z_X \setminus U_{\hat{X}}$ for some open neighborhoods $U_{s(X)}$ of $s(X)$ and $U_{\hat{X}}$ of $\hat{X}$ (and so Y has no multiple points on $U_{s(X)}$), then we say Y is an *immersed* Seifert surface.

Let Y be such an immersed Seifert surface of X in Z. Define

$$F(s) \overset{def}{=} s^* F_X \cong s^*(\nu_Y|_{s(X)}), \tag{7.2}$$

which is an oriented vector bundle of rank 2 over X, and so its Euler class

$$e(F(s)) \in H^2(X;\mathbb{Z})$$

is defined. Unless otherwise stated, we do not assume that $F(s)$ is trivial in this paper. Let

$$t_Y \in H^2(Z \setminus X;\mathbb{Q}) \cong H^2(Z_X;\mathbb{Q})$$

be the fundamental cohomology class of Y, then we have

$$s^*(t_Y|_{\hat{X}}) = e(F(s)), \qquad s^*\tau^*(t_Y|_{\hat{X}}) = 0. \tag{7.3}$$

Now, let us write

$$(W, V) = \partial(Z, X), \quad S = Y \cap W_V$$

for a moment. Then, S is an immersed Seifert surface of V in W with respect to the section $s|_V\colon V \to \hat{V}$ (namely $S \cap \hat{V} = s(V)$). Note that

$$t_S = t_Y|_{W\setminus V}, \qquad F(s|_V) = F(s)|_V, \qquad \nu_S = \nu_Y|_S$$

by definition. In line with our orientation conventions, the (oriented) boundaries of Y and S are given as follows:

$$\partial Y = (\pm S) \cup (\mp s(X)), \qquad \partial S = \pm s(V) \tag{7.4}$$

Here, $\pm = (-1)^{\dim W}$ and $\mp = (-1)^{\dim Z}$.

Remark 7.1. Let $Y' \subset Z$ be an (immersed) submanifold with the boundary $\partial Y' = (\pm(\partial Y' \cap W)) \cup (\mp X)$ and with the corner $\angle Y = \partial X$ such that Y' intersects W transversely, and we assume that a neighborhood of $X \subset Y$ has no multiple points. Then, $Y = Y' \cap Z_X$ is an (immersed) Seifert surface of X in the sense described above. In order to avoid introducing too much notation, we will also call Y' an (immersed) Seifert surface of X. The corresponding section $s\colon X \to \hat{X}$ is defined such that $Y' \cap \hat{X} = s(X)$, and so $F(s) \cong \nu_{Y'}|_X$. The cohomology class $t_Y \in H^2(Z \setminus X; \mathbb{Q})$ is nothing but the fundamental cohomology class of $Y' \setminus X$ (the Poincaré dual of the locally finite fundamental homology class $[Y' \setminus X] \in H^{lf}_{\dim Y'}(Z \setminus X; \mathbb{Q})$ of $Y' \setminus X$).

The following proposition states that an immersed Seifert surface implies a quasi e-class.

Proposition 7.1. *Let (Z, X) be a pair of manifolds of codimension 3, and Y an immersed Seifert surface of X with respect to a section $s\colon X \to \hat{X}$ such that $e(F(s))|_{\partial X} = 0$ over $\mathbb{Q}$. Then, $2t_Y$ is a quasi e-class of (Z, X), and $e(F(s))/2$ is the self–linking form of the quasi e-manifold $(Z, X, 2t_Y)$.*

Proof. We write $b = t_Y|_{\hat{X}} \in H^2(\hat{X}; \mathbb{Q})$ in this proof. By Lemma 7.1, we have

$$b - \tau^* b = e(F_X). \tag{7.5}$$

Since $b + \tau^* b$ belongs to $H_+ = \operatorname{Im} \rho_X^*$ (see (7.1), where $E = \nu_X$), there exists $c \in H^2(X; \mathbb{Q})$ such that $b + \tau^* b = \rho_X^* c$. The pull–back $s^*\colon H^2(\hat{X}; \mathbb{Q}) \to H^2(X; \mathbb{Q})$ is a left–inverse of ρ_X^*, and so

$$c = s^*(\rho_X^* c) = s^* b + s^* \tau^* b = e(F(s))$$

by (7.3), and thus

$$b + \tau^* b = \rho_X^* e(F(s)). \tag{7.6}$$

By (7.5) and (7.6), we have

$$2t_Y|_{\hat{X}} = e(F_X) + 2\left(\rho_X^* e(F(s))/2\right).$$

Since $e(F(s))|_{\partial X} = 0$, $2t_Y$ is a quasi e-class of (Z, X), and $e(F(s))/2$ is the self–linking form of $(Z, X, 2t_Y)$. □

The following is a direct consequence of Proposition 7.1.

COROLLARY 7.1. *Let* (Z, X) *and* Y *be as in* Proposition 7.1. *If* $e(F(s)) = 0$ *over* $\mathbb{Q}$, *then* $2t_Y$ *is an* e*-class of* (Z, X).

PROOF. By Proposition 7.1, the self–linking form of $(Z, X, 2t_Y)$ vanishes, in other words, $2t_Y$ is an e-class by definition. □

8. Haefliger's Invariant

In this section, we prove Theorem 1.4. We also prove the geometric formula for σ, and as a corollary, we obtain more general results (Corollary 8.1) establishing the relationship between Takase's invariant Ω and our invariant σ.

8.1. Review of Haefliger's invariant

We begin by reviewing Haefliger's results [6] [7] on the classification of smooth 3–knots in S^6. He first showed that the set $\mathrm{Emb}(S^3, S^6)$ of the isotopy classes of smooth embeddings $f\colon S^3 \to S^6$ is an abelian group with the group structure given by the connected sum. Write $M_f = f(S^3)$. He showed the existence of an oriented proper framed 4–submanifold $X \subset D^7$ such that $\partial X = M_f$ and $\mathrm{Sign}\, X = 0$. Here, a *framing* of X is the homotopy class of a triple (s_1, s_2, s_3), $s_i\colon X \to \nu_X$, of linearly independent sections of the normal bundle ν_X of X. We shall assume $s_i(X) \subset \hat{X}$. The homomorphism

$$H^2(X, \partial X; \mathbb{Q}) \xrightarrow{\cong} H^2(X; \mathbb{Q})$$

is an isomorphism, and we will identify these two groups. For a 2–cycle c of X, the linking number $\mathrm{lk}(s_1(c), X) \in \mathbb{Q}$ of $s_1(c)$ with X is well–defined, and it depends only on the homology class $[c] \in H_2(X;\mathbb{Z})$ of c. Thus, we obtain a homomorphism

$$\lambda\colon\ H_2(X;\mathbb{Z}) \to \mathbb{Q}, \quad \lambda([c]) = \mathrm{lk}(s_1(c), X),$$

which gives a cohomology class $\lambda \in H^2(X, \partial X;\mathbb{Q})$. He proved that the integral

$$H(f) = \frac{1}{2}\int_X \lambda^2 \tag{8.1}$$

is an integer and depends only on the isotopy class of f, and that the induced map $H\colon\ \mathrm{Emb}(S^3, S^6) \to \mathbb{Z}$ is an isomorphism of abelian group.

8.2. Invariant $\sigma(S^6, M_f)$

Let M_f and X be as before. The pair (S^6, M_f) is simple by Proposition 1.1, and we can define the invariant $\sigma(S^6, M_f) \in \mathbb{Q}$ by Theorem 1.3. By Lemma 5.1, there exist $e \in H^2(D^7 \setminus X;\mathbb{Q})$ and $\gamma \in H^2(X;\mathbb{Q})$ such that

$$e|_{\hat{X}} = e(F_X) + 2\rho_X^*\gamma.$$

Since $\gamma|_{M_f} = 0 \in H^2(M_f;\mathbb{Q}) = 0$, e is a quasi e-class of (D^7, X), and γ is the self–linking form of the quasi e-manifold (D^7, X, e). In particular, $e_f = e|_{S^6\setminus M_f} \in H^2(S^6 \setminus M_f;\mathbb{Q})$ is the unique e-class of (S^6, M_f), and

$$\partial(D^7, X, e) = (S^6, M_f, e_f)$$

as e-manifolds. By Theorem 1.2 (3), we obtain a formula

$$\sigma(S^6, M_f) = -4\int_X \gamma^2. \tag{8.2}$$

8.3. Proof of Theorem 1.4

The essential part of the proof of Theorem 1.4 is that the cohomology classes λ and γ, we defined in this section, are actually the same.

PROPOSITION 8.1. *We have $\lambda = \gamma$ in $H^2(X, \partial X;\mathbb{Q})$.*

Proof. Fix a 2-cycle c of X, and let $\omega \in H^2(X, \partial X; \mathbb{Q})$ be the Poincaré dual of the homology class $[c] \in H_2(X; \mathbb{Q})$ of c. Let us write $\tilde{c} = s_1(c)$ and $\tilde{X} = s_1(X)$ which are cycles of X. Note that the Poincaré dual of $[\tilde{X}, \partial\tilde{X}] \in H_4(\hat{X}; \mathbb{Q})$ is $e(F_X)/2$.

Since $\tilde{c} = \tilde{X} \cap \rho_X^{-1}(c)$, its Poincaré dual is

$$\frac{1}{2} e(F_X)\, \rho_X^* \omega \in H^4(\hat{X}, \partial\hat{X}; \mathbb{Q}).$$

The homomorphism

$$H^2(D^7 \setminus X; \mathbb{Q}) \to H^3(D^7, D^7 \setminus X; \mathbb{Q})$$

given by the pair $(D^7, D^7 \setminus X)$ maps $e/2$ to the Thom class of ν_X, and thus, we have

$$\mathrm{lk}(\tilde{c}, X) = \langle [\tilde{c}], e/2 \rangle = \frac{1}{4} \int_{\hat{X}} e(F_X)\, e\, \rho_X^* \omega.$$

Therefore, we have

$$\lambda([c]) = \frac{1}{4} \int_{\hat{X}} e(F_X)\, e\, \rho_X^* \omega = \frac{1}{2} \int_{\hat{X}} e(F_X)\, \rho_X^* (\gamma\, \omega) = \int_X \gamma\, \omega = \langle [c], \gamma \rangle,$$

where we used the relation $e|_{\hat{X}} = e(F_X) + 2\rho_X^* \gamma$ and the vanishing

$$\int_{\hat{X}} e(F_X)^2 \rho_X^* \omega = \int_{\hat{X}} \rho_X^* \left(p_1(\nu_X) \omega\right) = 0$$

following from (3.4) and $\dim X < 6$. Hence, $\lambda([c]) = \langle [c], \gamma \rangle$ for any 2–cycle c of X, and thus, $\lambda = \gamma \in H^2(X, \partial X; \mathbb{Q})$. □

Theorem 1.4 is now quite easy to proof.

Proof of Theorem 1.4. By (8.1), (8.2), and Proposition 8.1, we have

$$\sigma(S^6, M_f) = -4 \int_X \gamma^2 = -4 \int_X \lambda^2 = -8H(f). \; \square$$

We shall say that σ is a natural generalization of Haefliger's invariant H.

8.4. Geometric formula

Let (W, V) be a pair of closed manifolds of dimensions 6 and 3, and $S \subset W_V$ a Seifert surface of V with respect to a section $s\colon V \to \hat{V}$ (namely $\partial S = s(V)$) such that $e(F(s)) = 0$ in $H^2(V; \mathbb{Q})$. By Corollary 7.1, $2t_S$ is an e-class of (W, V). In this subsection, we prove the geometric formula for $\sigma(W, V, 2t_S)$.

The vanishing $e(F(s)) = 0$ implies $e(\nu_S)|_{\partial S} = 0$ by (7.2), and the integral

$$\int_S e(\nu_S)^2 \in \mathbb{Q} \tag{8.3}$$

is well–defined. (More precisely, (8.3) means the integral $\int_S a^2$, where $a \in H^2(S, \partial S; \mathbb{Q})$ is an element such that the homomorphism $H^2(S, \partial S; \mathbb{Q}) \to H^2(S; \mathbb{Q})$ maps a to $e(\nu_S)$.)

The following is the geometric formula for $\sigma(W, V, 2t_S)$.

THEOREM 8.1. *Let (W, V) be a pair of closed manifolds of dimensions 6 and 3, and S a Seifert surface of V with respect to a section $s\colon V \to \hat{V}$ such that $e(F(s)) = 0$ over $\mathbb{Q}$. Then, we have*

$$\sigma(W, V, 2t_S) = \operatorname{Sign} S - \int_S e(\nu_S)^2. \tag{8.4}$$

REMARK 8.1. The formula holds only for embedded Seifert surfaces, and not for immersed Seifert surfaces.

PROOF. Set $Z = [-1, 1] \times W$ which is a 7–manifold, and let us consider the submanifolds $X, Y \subset Z$ of dimensions 4, 5 defined by

$$X = ([0, 1] \times V) \cup_{\{0\} \times V} (\{0\} \times S)$$
$$Y = [0, 1] \times S \subset Z.$$

Here, we shall assume that X is a smooth proper 4–submanifold such that $\partial X = \{1\} \times V$, after "non–smooth part" $\{0\} \times V$ is rounded in a standard fashion, and that Y has the smooth boundary $\partial Y = S \cup (-X)$ and the corner $\{1\} \times V$. Thus, Y is a Seifert surface of X in Z with respect to the section $\tilde{s}\colon X \to \hat{X}$ such that $\tilde{s}(X) = \hat{X} \cap Y$. By Proposition 7.1,

$2t_Y \in H^2(Z \setminus X; \mathbb{Q})$ is a quasi e-class of (Z, X), and $e(F(\tilde{s}))/2$ is the self–linking form of the quasi e-manifold $(Z, X, 2t_Y)$. By the construction, the boundary of $(Z, X, 2t_Y)$ is

$$\partial(Z, X, 2t_Y) \cong (W, V, 2t_S) \amalg (-(W, \emptyset, 0)).$$

The oriented cobordism group Ω_6 vanishes (see (3.3)), and so $(W, \emptyset, 0)$ bounds an e-manifold of the form $(Z', \emptyset, 0)$. That implies $\sigma(W, \emptyset, 0) = 0$ by Axiom 2. Since $X \cong S$, we have $\operatorname{Sign} X = \operatorname{Sign} S$. By Theorem 1.2 (3) and by the definition of Λ, we have

$$\sigma(W, V, 2t_S) = \operatorname{Sign} X - 4 \int_X \frac{e(F(\tilde{s}))^2}{4} = \operatorname{Sign} S - \int_X e(F(\tilde{s}))^2.$$

By the Stokes' theorem,

$$\int_X e(F(\tilde{s}))^2 = \int_S e(\nu_S)^2 - \int_{\partial Y} e(\nu_Y)^2 = \int_S e(\nu_S)^2,$$

where note that $\nu_Y|_S = \nu_S$ and $\nu_Y|_X = F(\tilde{s})$. Hence, the formula (8.4) holds. □

8.5. Takase's invariant

Let M be an integral homology 3–sphere, and $f\colon M \to S^6$ a smooth embedding. We write $M_f = f(M)$ as before. Since (S^6, M_f) is simple by Proposition 1.1, the invariant $\sigma(S^6, M_f) \in \mathbb{Q}$ is well–defined by Theorem 1.3. It is not difficult to show that there is a Seifert surface S of M_f with respect to some section $s\colon M_f \to \hat{M}_f$ (cf. [16, Proposition 2.5]). Since $e(F(s)) = 0 \in H^2(M_f; \mathbb{Q}) = 0$, the cohomology class $2t_S$ is the unique e-class of (S^6, M_f) by Corollary 7.1, and so $\sigma(S^6, M_f) = \sigma(S^6, M_f, 2t_S)$. By Theorem 8.1, the geometric formula

$$\sigma(S^6, M_f) = \operatorname{Sign} S - \int_S e(\nu_S)^2$$

holds. The right–hand side is nothing but (-8 times) the definition of Takase's invariant $\Omega(f)$ [16, Proposition 4.1], and thus, we obtain the following immediate corollary.

Corollary 8.1. *For a smooth embedding $f\colon M \to S^6$ of an integral homology 3–sphere M, we have*

$$\sigma(S^6, M_f) = -8\Omega(f).$$

Since $\Omega(f) = H(f)$ when $M = S^3$ [15, Corollary 6.5], and so again we obtain Theorem 1.4 as a direct consequence of Corollary 8.1.

9. Milnor's Triple Linking Number

In this section, we prove Theorem 1.5. We begin by reviewing the definition of the triple linking number $\mu(L) \in \mathbb{Z}$ of oriented algebraically split 3–component links $L = K_1 \cup K_2 \cup K_3$ in $\mathbb{R}^3$ by using Seifert surfaces.

9.1. Review of the triple linking number

The letters i and j will denote elements in $\{1, 2, 3\}$. Since the linking number $\mathrm{lk}(K_i, K_j)$ vanishes $(i \neq j)$, K_i has a Seifert surface $\Sigma'_i \subset \mathbb{R}^3$, $\partial\Sigma'_i = K_i$, such that $\Sigma'_i \cap K_j = \emptyset$ $(i \neq j)$. The triple linking number $\mu(L)$ is defined to be the algebraic intersection number $\mu(L) = \#\,(\Sigma'_1 \cap \Sigma'_2 \cap \Sigma'_3)$. In other words, regarding the intersection $C'_{i,j} = \Sigma'_i \cap \Sigma'_j$ $(i < j)$ as an oriented 1–dimensional closed submanifold of Σ'_i, we can write

$$\mu(L) = \#(C'_{1,2} \cap C'_{1,3}).$$

For the proof of Theorem 1.5, we introduce a slightly different (but essentially the same) definition of $\mu(L)$ as the following. Let $L_0 = K_{1,0} \cup K_{2,0} \cup K_{3,0}$ be a 3–component unlink in $\mathbb{R}^3$ split from L. Then the link $L_i = K_i \cup (-K_{i,0})$ has a connected Seifert surface $\Sigma_i \subset \mathbb{R}^3$, $\partial\Sigma_i = L_i$, such that $\Sigma_i \cap L_j = \emptyset$ $(i \neq j)$, and $\mu(L)$ is defined to be

$$\mu(L) = \#(C_{1,2} \cap C_{1,3}), \tag{9.1}$$

where $C_{i,j} = \Sigma_i \cap \Sigma_j \subset \Sigma_i$ $(i < j)$ which is an oriented 1–dimensional closed submanifold of Σ_i.

From now on, we regard L and L_0 as links in $S^3 = \mathbb{R}^3 \cup \{\infty\}$.

9.2. Seifert surface of M_L

We construct an immersed Seifert surface S of M_L as follows. First of all, let us recall the definition of 3–submanifold M_L of $T^3 \times S^3$:

$$\begin{aligned} T_i^3 &= \left\{(t_1, t_2, t_3, x) \in T^3 \times S^3 \mid f_i(t_i) = x\right\}, \qquad \mathcal{L} = T_1^3 \cup T_2^3 \cup T_3^3, \\ T_{i,0}^3 &= \left\{(t_1, t_2, t_3, x) \in T^3 \times S^3 \mid f_{i,0}(t_i) = x\right\}, \quad \mathcal{L}_0 = T_{1,0}^3 \cup T_{2,0}^3 \cup T_{3,0}^3, \\ M_L &= \mathcal{L} \cup (-\mathcal{L}_0). \end{aligned}$$

Here, $f_i \colon S^1 \to S^3$ and $f_{i,0} \colon S^1 \to S^3$ are smooth embeddings representing K_i and $K_{i,0}$ respectively, and T^3 is the 3–torus with coordinates (t_1, t_2, t_3) such that $t_i \in S^1 = \mathbb{R}/\mathbb{Z}$. For convenience, we also write

$$\mathcal{L}_i = T_i^3 \cup (-T_{i,0}^3)$$

so that $M_L = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$.

Since Σ_i is connected, there exists a map

$$p_i \colon \Sigma_i \to S^1,$$

such that $p_i\, f_i = p_i\, f_{i,0} = \text{identity} \colon S^1 \to S^1$. Let us consider the following smooth embeddings:

$$\begin{aligned} F_1 &\colon \Sigma_1 \times S^1 \times S^1 \to T^3 \times S^3, \qquad (x, t_2, t_3) \mapsto (p_1(x), t_2, t_3, x) \\ F_2 &\colon S^1 \times \Sigma_2 \times S^1 \to T^3 \times S^3, \qquad (t_1, x, t_3) \mapsto (t_1, p_2(x), t_3, x) \\ F_3 &\colon S^1 \times S^1 \times \Sigma_3 \to T^3 \times S^3, \qquad (t_1, t_2, x) \mapsto (t_1, t_2, p_3(x), x) \end{aligned}$$

The image $S_i \subset T^3 \times S^3$ of F_i is a Seifert surface of $\mathcal{L}_i$, such that

$$S_i \cap \mathcal{L}_j = \emptyset \quad (i \neq j), \tag{9.2}$$

and the union

$$S = S_1 \cup S_3 \cup S_3 \tag{9.3}$$

is an immersed Seifert surface of M_L (S_i may intersects the other components S_j $(i \neq j)$).

The intersection

$$\Sigma_{i,j} = S_i \cap S_j \subset S_i \quad (i \neq j),$$

is a $\Sigma_{i,j}$ as a 2–dimensional closed submanifold of S_i, and the intersection number $\#(\Sigma_{1,2} \cap \Sigma_{1,3}) \in \mathbb{Z}$ is defined. The following lemma will be used to prove Theorem 1.5 in Section 9.4

Lemma 9.1. *$\#(\Sigma_{1,2} \cap \Sigma_{1,3}) = \mu(L)$.*

Proof. Let $h\colon S_1 \to \Sigma_1 \times S^1 \times S^1$ be the diffeomorphism defined by

$$h(p_1(x), t_2, t_3, x) = (x, t_2 - p_2(x), t_3 - p_3(x))$$

for $(p_1(x), t_1, t_3, x) \in S_1$, then we have

$$h(\Sigma_{1,2}) = C_{1,2} \times \{0\} \times S^1,$$
$$h(\Sigma_{1,3}) = C_{1,3} \times S^1 \times \{0\},$$

and therefore,

$$\begin{aligned}\#(\Sigma_{1,2} \cap \Sigma_{1,3}) &= \#((C_{1,2} \times \{0\} \times S^1) \cap (C_{1,3} \times S^1 \times \{0\})) \\ &= \#(C_{1,2} \cap C_{1,3}) \\ &= \mu(L)\end{aligned}$$

by the definition (9.1). □

9.3. $(T^3 \times S^3, M_L)$ **is simple**

By using the immersed Seifert surface S of M_L constructed in the previous subsection, we prove that $(T^3 \times S^3, M_L)$ is simple.

Lemma 9.2. *The normal bundle ν_{S_i} of $S_i \subset T^3 \times S^3$ is trivial.*

Proof. The vector field $\partial/\partial t_i$ on $T^3 \times S^3$ is transverse to the submanifold S_i, and this gives a non–vanishing section of ν_{S_i}. Since the rank of ν_{S_i} is 2, ν_{S_i} is trivial. □

Let $t_S \in H^2((T^3 \times S^3) \setminus M_L; \mathbb{Q})$ be the fundamental cohomology class of S, and we define

$$e_L = 2t_S.$$

PROPOSITION 9.1. *The manifold pair $(T^3 \times S^3, M_L)$ has an e-class e_L, and is simple.*

PROOF. Since the normal bundle ν_{S_i} is trivial by Lemma 9.2, the Euler class $e(F(s)) \in H^2(M_L;\mathbb{Q})$ vanishes, where $s\colon M_L \to \hat{M}_L$ is the section such that $s(M_L) = \hat{M}_L \cap S$. Thus, e_L is an e-class of $(T^3 \times S^3, M_L)$ by Corollary 7.1. Since the restriction $H^2(T^3 \times S^3;\mathbb{Q}) \to H^2(M_L;\mathbb{Q})$ is injective, $(T^3 \times S^3, M_L)$ is simple. □

By Theorem 1.3 and Proposition 9.1, we can define the invariant

$$\sigma(T^3 \times S^3, M_L) = \sigma(T^3 \times S^3, M_L, e_L) \in \mathbb{Q}$$

of $(T^3 \times S^3, M_L)$. As we explained in Remark 1.4, this is a link homotopy invariant of L.

9.4. Proof of Theorem 1.5

In this subsection, we prove Theorem 1.5 by using the formula in Theorem 1.2 (3).

We begin by constructing a proper 4–submanifold $X \subset T^3 \times D^6$ such that $\partial X = M_L$ and $X \cong S_1 \amalg S_2 \amalg S_3$. Pushing $\operatorname{Int} S_i$ into the inside of $T^3 \times D^4$ (∂S_i is fixed on the boundary $T^3 \times S^3$), we obtain a proper 4–submanifold $X_i \subset T^3 \times D^4$ such that $X_i \cong S_i$ and $\partial X_i = \mathcal{L}_i$, and we assume that the depth of X_i is shallower than X_{i+1} so that $X_i \cap X_j = \emptyset$ if $i \neq j$, see Figure 2. We then define

$$X = X_1 \cup X_2 \cup X_3.$$

The natural isotopy of sinking S_i down onto X_i gives a 5–submanifold

$$Y_i \subset T^3 \times D^4$$

with the boundary $\partial Y_i = S_i \cup (-X_i)$ and the corner $\mathcal{L}_i$, and it is a Seifert surface of X_i in $T^3 \times D^4$ with respect to the section

$$s_i\colon\ X_i \to \hat{X}_i$$

such that $s_i(X_i) = Y_i \cap \hat{X}_i$. Let $t_{Y_i} \in H^2((T^3 \times D^4)\setminus X_i;\mathbb{Q})$ be the fundamental cohomology class of Y_i. We define a cohomology class $\tilde{e} \in H^2((T^3 \times D^4)\setminus X;\mathbb{Q})$ by

$$\tilde{e} = 2(t_{Y_1} + t_{Y_2} + t_{Y_3}).$$

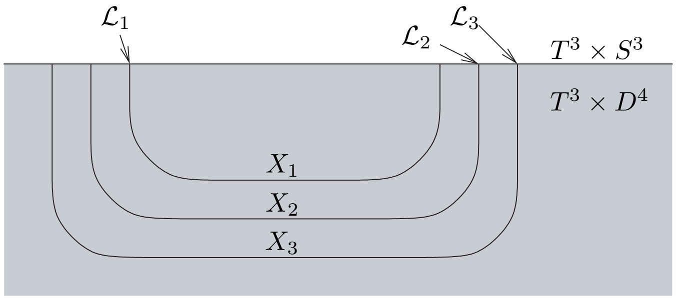

Fig. 2. Submanifold $X = X_1 \cup X_2 \cup X_3$.

Note that

$$\tilde{e}|_{(T^3\times S^3)\setminus M_L} = e_L \tag{9.4}$$

by definition. We will see soon that $\tilde{e}$ is a quasi e-class of $(T^3 \times D^4, X)$ (Proposition 9.2).

Remark 9.1. The union $Y = Y_1 \cup Y_2 \cup Y_3$ may not be an immersed Seifert surface of $X \subset T^3 \times D^4$ in our sense, because $X_i \cap Y_j$ may not be empty if $i < j$. Thus, we cannot apply Proposition 7.1 to Y to prove that $\tilde{e}$ is a quasi e-class.

Let us consider the intersection

$$\Sigma'_{i,j} = X_i \cap Y_j \subset X_i \quad (i < j)$$

which is an oriented 2–submanifold of X_i such that

$$\Sigma'_{i,j} \subset \operatorname{Int} X_i, \tag{9.5}$$

$$\Sigma'_{1,2} \cap \Sigma'_{2,3} = \Sigma'_{1,3} \cap \Sigma'_{2,3} = \emptyset. \tag{9.6}$$

Let ν_{Y_i} and $\nu_{\Sigma'_{i,j}}$ be the normal bundles of $Y_i \subset T^3 \times D^4$ and $\Sigma'_{i,j} \subset X_i$ respectively.

Lemma 9.3. *The normal bundles ν_{Y_i} and $\nu_{\Sigma'_{i,j}}$ are trivial.*

Proof. The triviality of ν_{Y_i} follows from the definition of Y_i and Lemma 9.2. Since $\nu_{\Sigma'_{i,j}}$ is isomorphic to $\nu_{Y_j}|_{\Sigma'_{i,j}}$ (where we regard $\Sigma'_{i,j} \subset Y_j$), this is trivial too. □

Lemma 9.4. *The cohomology class $2t_{Y_i}$ is an e–class of $(T^3 \times D^4, X_i)$.*

Proof. Since ν_{Y_i} is trivial by Lemma 9.3, the cohomology class $e(F(s_i)) \in H^2(X_i; \mathbb{Q})$ vanishes. Thus, $2t_{Y_i}$ is an e-class by Corollary 7.1. □

Let $\gamma_{i,j} \in H^2(X_i; \mathbb{Q})$ be the Poincaré dual of $\Sigma_{i,j}$, and we write

$$\gamma = \gamma_{1,2} + \gamma_{1,3} + \gamma_{2,3} \in H^2(X; \mathbb{Q}).$$

By the definitions of X_i and Y_j, we have $\hat{X}_i \cap Y_j = \rho_{X_i}^{-1}(\Sigma'_{i,j})$ $(i < j)$, and this implies

$$t_{Y_j}|_{\hat{X}_i} = \rho^*_{X_i}\gamma_{i,j} \quad (i < j) \tag{9.7}$$

in $H^2(\hat{X}_i; \mathbb{Q})$ by the Poincaré duality.

We obtain the following proposition.

Proposition 9.2. *The triple $(T^3 \times D^4, X, \tilde{e})$ is a quasi e-manifold with the boundary $(T^3 \times S^3, M_L, e_L)$ and with the self–linking form γ.*

Proof. By Lemma 9.4 and (9.7), we have

$$\begin{aligned} \tilde{e}|_{\hat{X}} &= \sum_{i=1}^{3} e(F_{X_i}) + 2(\rho^*_{X_1}\gamma_{1,2} + \rho^*_{X_1}\gamma_{1,3} + \rho^*_{X_2}\gamma_{2,3}) \\ &= e(F_X) + 2\rho^*_X\gamma \end{aligned}$$

in $H^2(\hat{X}; \mathbb{Q})$. It follows from (9.5) that $\gamma|_{\partial X} = 0$. Thus, $\tilde{e}$ is a quasi e-class of $(T^3 \times D^4, X)$ with the self–linking form γ. By (9.4), we have $\partial(T^3 \times D^4, X, \tilde{e}) = (T^3 \times S^3, M_L, e_L)$. □

This is the proof of Theorem 1.5.

PROOF OF THEOREM 1.5. By Theorem 1.2 (3) and Proposition 9.2, we have given as follows.

$$\sigma(T^3\times S^3, M_L) = \operatorname{Sign} X - 4\int_X \gamma^2.$$

Since $X_i \cong S_i \cong \pm\Sigma\times S^1\times S^1$, we have $\operatorname{Sign} X = 0$. We have $\gamma_{i,j}^2 = 0$ $(i \neq j)$ by Lemma 9.3, and $\gamma_{1,j}\gamma_{2,3} = 0$ $(j = 2, 3)$ by (9.6). Thus,

$$\int_X \gamma^2 = 2\int_X \gamma_{1,2}\gamma_{1,3} = 2\#(\Sigma'_{1,2}\cap\Sigma'_{1,3}).$$

Since the intersection $C_{1,2,3} = Y_1 \cap Y_2 \cap Y_3$ is a 1–dimensional oriented cobordism from $\Sigma_{1,2}\cap\Sigma_{1,3}$ to $\Sigma'_{1,2}\cap\Sigma'_{1,3}$, namely

$$\partial C_{1,2,3} = (\Sigma_{1,2}\cap\Sigma_{1,3}) \amalg (-(\Sigma'_{1,2}\cap\Sigma'_{1,3})),$$

we have

$$\#(\Sigma'_{1,2}\cap\Sigma'_{1,3}) = \#(\Sigma_{1,2}\cap\Sigma_{1,3}).$$

The right–hand side equals $\mu(L)$ by Lemma 9.1, and hence, we have

$$\sigma(T^3\times S^3, M_L) = -8\mu(M). \ \Box$$

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